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Schauder Estimates for Poisson Equations Associated with Non-local Feller Generators

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Abstract
We show how Hölder estimates for Feller semigroups can be used to obtain regularity results for solutions to the Poisson equation $Af = g$ associated with the (extended) infinitesimal generator of a Feller process. The regularity of f is described in terms of Hölder–Zygmund spaces of variable order and, moreover, we establish Schauder estimates. Since Hölder estimates for Feller semigroups have been intensively studied in the last years, our results apply to a wide class of Feller processes, e.g. random time changes of Lévy processes and solutions to Lévy-driven stochastic differential equations. Most prominently, we establish Schauder estimates for the Poisson equation associated with the fractional Laplacian of variable order. As a by-product, we obtain new regularity estimates for semigroups associated with stable-like processes.

Keywords Feller process · Infinitesimal generator · Regularity · Hölder space of variable order · Favard space

Mathematics Subject Classification (2010) Primary 60J25; Secondary 45K05 · 35B65 · 60J35 · 60J75

1 Introduction

Let $(X_t)_{t\geq 0}$ be an $\mathbb{R}^d$-valued Feller process with semigroup $P_t f(x) = \mathbb{E}^x f(X_t)$, $x \in \mathbb{R}^d$. In this paper, we study the regularity of functions in the abstract Hölder space

$$F_1 := \left\{ f \in \mathcal{B}_b(\mathbb{R}^d);\ \sup_{t\in(0,1)} \sup_{x\in\mathbb{R}^d} \left| \frac{P_t f(x) - f(x)}{t} \right| < \infty \right\},$$

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the so-called Favard space of order 1; cf. [9,14]. It is known that for any $f \in F_1$ the limit

$$A_e f(x) := \lim_{t\to 0} \frac{\mathbb{E}^x f(X_t) - f(x)}{t} \tag{1}$$

exists up to a set of potential zero (cf. [1]) and this gives rise to the extended infinitesimal generator A_e which maps the Favard space F_1 into the space of bounded Borel measurable functions $\mathcal{B}_b(\mathbb{R}^d)$; cf. Sect. 2 for details. It is immediate from Dynkin's formula that A_e extends the (strong) infinitesimal generator A of $(X_t)_{t\geq 0}$; in particular, F_1 contains the domain $\mathcal{D}(A)$ of the infinitesimal generator. We are interested in the following questions:

- What does the existence of limit (1) tell us about the regularity of $f \in F_1$? In particular: How smooth are functions in the domain of the infinitesimal generator of $(X_t)_{t\geq 0}$?
- If $f \in F_1$ is a solution to the equation $A_e f = g$ and g has a certain regularity, say g is Hölder continuous of order $\delta \in (0,1)$, then what additional information do we get on the smoothness of f?

Our aim is to describe the regularity of f in terms of Hölder spaces of variable order. More precisely, we are looking for a mapping $\kappa : \mathbb{R}^d \to (0,2)$ such that

$$f \in F_1 \implies f \in \mathcal{C}_b^{\kappa(\cdot)}(\mathbb{R}^d)$$

where $\mathcal{C}_b^{\kappa(\cdot)}(\mathbb{R}^d)$ denotes the Hölder–Zygmund space of variable order equipped with the norm

$$\|f\|_{\mathcal{C}_b^{\kappa(\cdot)}(\mathbb{R}^d)} := \|f\|_\infty + \sup_{x\in\mathbb{R}^d} \sup_{0<|h|\leq 1} \frac{|f(x+2h) - 2f(x+h) + f(x)|}{|h|^{\kappa(x)}},$$

cf. Sect. 2 for details. If $A_e f = g \in \mathcal{C}_b^\delta(\mathbb{R}^d)$ for some $\delta > 0$, then it is natural to expect that f "inherits" some regularity from g, i. e.

$$f \in F_1, A_e f = g \in \mathcal{C}_b^\delta(\mathbb{R}^d) \implies f \in \mathcal{C}_b^{\kappa(\cdot)+\varrho}(\mathbb{R}^d)$$

for some constant $\varrho = \varrho(\delta) > 0$. Moreover, we are interested in establishing Schauder estimates, i. e. estimates of the form

$$\begin{aligned} &\|f\|_{\mathcal{C}_b^{\kappa(\cdot)}(\mathbb{R}^d)} \leq C(\|f\|_\infty + \|A_e f\|_\infty) \quad \text{and} \\ &\|f\|_{\mathcal{C}_b^{\kappa(\cdot)+\varrho}(\mathbb{R}^d)} \leq C'(\|f\|_\infty + \|A_e f\|_{\mathcal{C}_b^\delta(\mathbb{R}^d)}). \end{aligned} \tag{2}$$

Let us mention that the results, which we present in this paper, do *not* apply to Feller semigroups with a roughening effect (see e.g. [16] for examples of such semigroups); we study exclusively Feller semigroups with a smoothing effect (see below for details).

The toy example, which we have in mind, is the stable-like Feller process $(X_t)_{t\geq 0}$ with infinitesimal generator A,

$$Af(x) = c_{d,\alpha(x)} \int_{y\neq 0} \left(f(x+y) - f(x) - y \cdot \nabla f(x) \mathbb{1}_{(0,1)}(|y|)\right) \frac{1}{|y|^{d+\alpha(x)}} \, \mathrm{d}y, \quad (3)$$

which is, roughly speaking, a fractional Laplacian of variable order, that is $A = -(-\Delta)^{\alpha(\bullet)/2}$. Intuitively, $(X_t)_{t\geq 0}$ behaves locally like an isotropic stable Lévy process, but its index of stability depends on the current position of the process. In view of the results in [27,30], it is an educated guess that any function $f \in \mathcal{D}(A)$ is "almost" locally Hölder continuous with Hölder exponent $\alpha(\cdot)$, in the sense that

$$|f(x+2h) - 2f(x+h) + f(x)| \leq C_{f,\varepsilon} |h|^{\alpha(x)-\varepsilon}, \quad x, h \in \mathbb{R}^d \quad (4)$$

for any small $\varepsilon > 0$. We will show that this is indeed true and, moreover, we will establish Schauder estimates for the equation $-(-\Delta)^{\alpha(\bullet)/2} f = g$ (cf. Theorem 4.1 and Corollary 4.3).

Let us comment on related literature. For some particular examples of Feller generators A, there are Schauder estimates for solutions to the integro-differential equation $Af = g$ available in the literature; for instance, Bass obtained Schauder estimates for a class of stable-like operators ($\nu(x, \mathrm{d}y) = c(x, y)|y|^{-d-\alpha}$ with $c : \mathbb{R}^2 \to (0, \infty)$ bounded and $\inf_{x,y} c(x, y) > 0$), and Bae and Kassmann [2] studied operators with functional order of differentiability ($\nu(x, \mathrm{d}y) = c(x, y)/(|y|^{\mathrm{d}} \varphi(y) \, \mathrm{d}y)$ for "nice" φ). The recent article [27] establishes Schauder estimates for a large class of Lévy generators using gradient estimate for the transition density p_t of the associated Lévy process. Moreover, we would like to mention the article [30] which studies a complementary question—namely, what are sufficient conditions for the existence of limit (1) in the space $C_\infty(\mathbb{R}^d)$ of continuous functions vanishing at infinity—and which shows that certain Hölder space of variable order is contained in the domain of the (strong) infinitesimal generator. Schauder estimates have interesting applications in the theory of stochastic differential equations (SDES): they can be used to obtain uniqueness results for solutions to SDEs driven by Lévy processes and to study the convergence of the Euler–Maruyama approximation (see e.g. [11,31,46] and the references therein).

This paper consists of two parts. In Sect. 3, we show how regularity estimates on Feller semigroups can be used to establish Schauder estimates (2) for functions f in the Favard space of a Feller process $(X_t)_{t\geq 0}$. Our first result, Proposition 3.1, states that if the semigroup $P_t u(x) := \mathbb{E}^x u(X_t)$ satisfies

$$\|P_t u\|_{\mathcal{C}_b^\kappa(\mathbb{R}^d)} \leq c t^{-\beta} \|u\|_\infty, \quad t \in (0, 1), \ u \in \mathcal{B}_b(\mathbb{R}^d)$$

for some $\beta \in [0, 1)$ and $\kappa > 0$, then $F_1 \subseteq \mathcal{C}_b^\kappa(\mathbb{R}^d)$ and

$$\|f\|_{\mathcal{C}_b^\kappa(\mathbb{R}^d)} \leq C \left(\|f\|_\infty + \|A_e f\|_\infty\right) \quad \text{for all } f \in F_1.$$

Proposition 3.1 has interesting applications, but, in general, it does not give optimal regularity results but rather a worst-case estimate on the regularity of $f \in F_1$; for instance, if $(X_t)_{t\geq 0}$ is an isotropic stable-like process with infinitesimal generator $A = -(-\Delta)^{\alpha(\bullet)/2}$ (cf. (3)), then an application of Proposition 3.1 shows

$$|f(x+2h) - 2f(x+h) + f(x)| \leq C_{f,\varepsilon}|h|^{\alpha_0-\varepsilon}, \quad x, h \in \mathbb{R}^d, \ f \in \mathcal{D}(A),$$

where $\alpha_0 := \inf_{x\in\mathbb{R}^d} \alpha(x)$, and this is much weaker than regularity (4) which we would expect. Our main result in Sect. 3 is a “localized” version of Proposition 3.1 which takes into account the local behaviour of the Feller process $(X_t)_{t\geq 0}$ and which allows us to describe the local regularity of a function $f \in F_1$ (cf. Theorem 3.2 and Corollary 3.4). As an application, we obtain a regularity result for solutions to the Poisson equation $A_e f = g$ with $g \in \mathcal{C}_b^\delta(\mathbb{R}^d)$ (cf. Theorem 3.5).

In the second part of the paper, Sect. 4, we illustrate the results from Sect. 3 with several examples. Applying the results to isotropic stable-like processes, we establish Schauder estimates for the Poisson equation $-(-\Delta)^{\alpha(\bullet)/2} f = g$ associated with the fractional Laplacian of variable order (cf. Theorem 4.1 and Corollary 4.3). Schauder estimates of this type seem to be a novelty in the literature. As a by-product of the proof, we obtain Hölder estimates for semigroups of isotropic stable-like processes which are of independent interest (see Sect. 6.1). Furthermore, we present Schauder estimates for random time changes of Lévy processes (Proposition 4.5) and solutions to Lévy-driven SDEs (Proposition 4.7) and discuss possible extensions.

2 Basic Definitions and Notation

We consider the Euclidean space $\mathbb{R}^d$ with the scalar product $x \cdot y := \sum_{j=1}^d x_j y_j$ and the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ generated by the open balls $B(x,r)$ and closed balls $\overline{B(x,r)}$. As usual, we set $x \wedge y := \min\{x,y\}$ and $x \vee y := \max\{x,y\}$ for $x, y \in \mathbb{R}$. If f is a real-valued function, then supp f denotes its support, ∇f the gradient and $\nabla^2 f$ the Hessian of f. For two stochastic processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ we write $(X_t)_{t\geq 0} \stackrel{d}{=} (Y_t)_{t\geq 0}$ if $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ have the same finite-dimensional distributions.

Function spaces: $\mathcal{B}_b(\mathbb{R}^d)$ is the space of bounded Borel measurable functions $f : \mathbb{R}^d \to \mathbb{R}$. The smooth functions with compact support are denoted by $C_c^\infty(\mathbb{R}^d)$, and $C_\infty(\mathbb{R}^d)$ is the space of continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ vanishing at infinity. Superscripts $k \in \mathbb{N}$ are used to denote the order of differentiability, e.g. $f \in C_\infty^k(\mathbb{R}^d)$ means that f and its derivatives up to order k are $C_\infty(\mathbb{R}^d)$-functions. For $U \subseteq \mathbb{R}^d$ and $\alpha : U \to [0,\infty)$ bounded we define Hölder–Zygmund spaces of variable order by

$$\mathcal{C}^{\alpha(\cdot)}(U) := \left\{ f \in C(U); \forall x \in U : \sup_{\substack{0<|h|\leq 1 \\ x\pm h\in U}} \frac{|\Delta_h^k f(x)|}{|h|^{\alpha(x)}} < \infty \right\}$$

and

$$\mathcal{C}_b^{\alpha(\cdot)}(U):=\left\{f\in C_b(U);\ \|f\|_{\mathcal{C}_b^{\alpha(\cdot)}(U)}:=\sup_{x\in U}|f(x)|+\sup_{\substack{x\in U,0<|h|\le 1\\ \overline{B(x,k|h|)}\subset U}}\frac{|\Delta_h^k f(x)|}{|h|^{\alpha(x)}}<\infty\right\},$$

where $k\in\mathbb{N}$ is the smallest number strictly larger than $\|\alpha\|_\infty$ and

$$\Delta_h f(x):=f(x+h)-f(x),\quad \Delta_h^m f(x):=\Delta_h\Delta_h^{m-1}f(x),\quad m\ge 2, \tag{5}$$

are the iterated difference operators. Moreover, we set

$$\mathcal{C}_b^{\alpha(\cdot)+}(U):=\bigcup_{\varepsilon>0}\mathcal{C}_b^{\alpha(\cdot)+\varepsilon}(U)\quad\text{and}\quad \mathcal{C}_b^{\alpha(\cdot)-}(U):=\bigcap_{\varepsilon>0}\mathcal{C}_b^{\max\{\alpha(\cdot)-\varepsilon,0\}}(U).$$

Clearly,

$$\mathcal{C}_b^{\alpha(\cdot)+}(U)\subseteq\mathcal{C}_b^{\alpha(\cdot)}(U)\subseteq\mathcal{C}_b^{\alpha(\cdot)-}(U)\quad\text{and}\quad\mathcal{C}_b^{\alpha(\cdot)}(U)\subseteq\mathcal{C}^{\alpha(\cdot)}(U).$$

If $\alpha(x)=\alpha$ is constant, then we write $\mathcal{C}^\alpha(U)$ and $\mathcal{C}_b^\alpha(U)$ for the associated Hölder–Zygmund spaces. For $U=\mathbb{R}^d$ and $\alpha\notin\mathbb{N}$, the Hölder–Zygmund space $\mathcal{C}_b^\alpha(\mathbb{R}^d)$ is the "classical" Hölder space $C_b^\alpha(\mathbb{R}^d)$ equipped with the norm

$$\|f\|_{C_b^\alpha(\mathbb{R}^d)}:=\|f\|_\infty+\sum_{j=0}^{\lfloor\alpha\rfloor}\sum_{\substack{\beta\in\mathbb{N}_0^d\\|\beta|=j}}\|\partial^\beta f\|_\infty+\max_{\substack{\beta\in\mathbb{N}_0^d\\|\beta|=\lfloor\alpha\rfloor}}\sup_{x\neq y}\frac{|\partial^\beta f(x)-\partial^\beta f(y)|}{|x-y|^{\alpha-\lfloor\alpha\rfloor}};$$

cf. [52, Section 2.7]. For $\alpha=1$, it is possible to show that $\mathcal{C}_b^1(\mathbb{R}^d)$ is strictly larger than the space of bounded Lipschitz continuous functions (cf. [51, p. 148]), which is in turn strictly larger than $C_b^1(\mathbb{R}^d)$.

Feller processes: A Markov process $(X_t)_{t\ge0}$ is a *Feller process* if the associated transition semigroup $P_tf(x):=\mathbb{E}^xf(X_t)$ is a *Feller semigroup* (see e.g. [6,19] for details). Without loss of generality, we may assume that $(X_t)_{t\ge0}$ has right-continuous sample paths with finite left-hand limits. Following [14, II.5.(b)], we call

$$F_1:=F_1^X:=\left\{f\in\mathcal{B}_b(\mathbb{R}^d);\ \sup_{t\in(0,1)}\left\|\frac{P_tf-f}{t}\right\|_\infty<\infty\right\} \tag{6}$$

the *Favard space of order 1*. The *(strong) infinitesimal generator* $(A,\mathcal{D}(A))$ is defined by

$$\mathcal{D}(A) := \left\{ f \in C_\infty(\mathbb{R}^d);\ \exists g \in C_\infty(\mathbb{R}^d) :\ \lim_{t \to 0} \left\| \frac{P_t f - f}{t} - g \right\|_\infty = 0 \right\},$$
$$Af := \lim_{t \to 0} \frac{P_t f - f}{t}, \quad f \in \mathcal{D}(A).$$

If $\mathcal{D}(A)$ is rich, in the sense that $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A)$, then a result by Courrège and von Waldenfels (see e.g. [6, Theorem 2.21]), shows that $A|_{C_c^\infty(\mathbb{R}^d)}$ is a pseudo-differential operator,

$$Af(x) = -q(x, D) f(x) := -\int_{\mathbb{R}^d} q(x, \xi) \mathrm{e}^{ix\cdot\xi} \hat{f}(\xi)\, \mathrm{d}\xi, \quad f \in C_c^\infty(\mathbb{R}^d), \tag{7}$$

where $\hat{f}(\xi) := (2\pi)^{-d} \int_{\mathbb{R}^d} \mathrm{e}^{-ix\cdot\xi} f(x)\, \mathrm{d}x$ is the Fourier transform of f and

$$q(x, \xi) = q(x, 0) - ib(x) \cdot \xi + \frac{1}{2} \xi \cdot Q(x) \xi + \int_{y \neq 0} \left(1 - \mathrm{e}^{iy\cdot\xi} + iy \cdot \xi \mathbb{1}_{(0,1)}(|y|) \right) \nu(x, \mathrm{d}y) \tag{8}$$

is a continuous negative definite *symbol*. If (7) holds, then we say that $(X_t)_{t\geq 0}$ is a Feller process with symbol q. We assume from now on that $q(x, 0) = 0$. For each $x \in \mathbb{R}^d$, $(b(x), Q(x), \nu(x, \mathrm{d}y))$ is a Lévy triplet, i.e. $b(x) \in \mathbb{R}^d$, $Q(x) \in \mathbb{R}^{d\times d}$ is symmetric positive semidefinite and $\nu(x, \cdot)$ is a measure on $\mathbb{R}^d \backslash \{0\}$ satisfying $\int_{y\neq 0} \min\{1, |y|^2\}\, \nu(x, \mathrm{d}y) < \infty$. The symbol q has *bounded coefficients* if

$$\sup_{x \in \mathbb{R}^d} \left(|b(x)| + |Q(x)| + \int_{y \neq 0} \min\{1, |y|^2\}\, \nu(x, \mathrm{d}y) \right) < \infty;$$

by [49, Lemma 6.2], q has bounded coefficients if, and only if,

$$\sup_{x \in \mathbb{R}^d} \sup_{|\xi| \leq 1} |q(x, \xi)| < \infty.$$

If $(X_t)_{t\geq 0}$ is a Feller process with symbol q, then

$$\mathbb{P}^x \left(\sup_{s \leq t} |X_s - x| > r \right) \leq ct \sup_{|y-x| \leq r} \sup_{|\xi| \leq r^{-1}} |q(y, \xi)|, \quad r > 0,\ t > 0,\ x \in \mathbb{R}^d \tag{9}$$

holds for an absolute constant $c > 0$; this maximal inequality goes back to Schilling [47] (see also [6, Theorem 5.1] or [22, Lemma 1.29]). If the symbol $q(\xi) = q(x, \xi)$ of a Feller process $(L_t)_{t\geq 0}$ does not depend on $x \in \mathbb{R}^d$, then $(L_t)_{t\geq 0}$ is a *Lévy process*. By [6, Theorem 2.6], this is equivalent to saying that $(L_t)_{t\geq 0}$ has stationary and independent increments. It is natural to ask whether for a given mapping q of form (8), there is a Feller process $(X_t)_{t\geq 0}$ with symbol q. In general, the answer is negative; see the monographs [6,19,22] for a survey on known existence results for Feller processes. In this article, we will frequently use an existence theorem from [22] which constructs Feller processes with symbol of the form $q(x,\xi) = \psi_{\alpha(x)}(\xi)$, where $\alpha : \mathbb{R}^d \to I$ is a Hölder continuous mapping and $\xi \mapsto \psi_\beta(\xi)$, $\beta \in I$, is a family of characteristic exponents of Lévy processes. For instance, it can be applied to the family $\psi_\beta(\xi) = |\xi|^\beta$, $\beta \in I = (0,2]$, to prove the existence of *isotropic stable-like* processes, i.e. Feller processes with symbol $q(x,\xi) = |\xi|^{\alpha(x)}$, where $\alpha : \mathbb{R}^d \to (0,2]$ is Hölder continuous and $\inf_{x\in\mathbb{R}^d} \alpha(x) > 0$ (cf. [22, Theorem 5.2]).

Later on, we will use that any Feller process $(X_t)_{t\geq 0}$ with infinitesimal generator $(A, \mathcal{D}(A))$ solves the $(A, \mathcal{D}(A))$-martingale problem, i.e.

$$M_t := f(X_t) - f(X_0) - \int_0^t Af(X_s)\,\mathrm{d}s$$

is a $\mathbb{P}^x$-martingale for any $x \in \mathbb{R}^d$ and $f \in \mathcal{D}(A)$. Our standard reference for Feller processes are the monographs [6,19]; for further information on martingale problems, we refer the reader to [15,18].

In the remaining part of this section, we define the extended infinitesimal generator and state some results which we will need later on. Following [44], we define the *extended (infinitesimal) generator* A_e in terms of the λ-potential operator R_λ, that is, $f \in \mathcal{D}(A_e)$ and $g = A_e f$ if and only if

(i) $f \in \mathcal{B}_b(\mathbb{R}^d)$ and g is a measurable function such that $\|R_\lambda(|g|)\|_\infty < \infty$ for some (all) $\lambda > 0$,
(ii) $f = R_\lambda(\lambda f - g)$ for all $\lambda > 0$.

The mapping $g = A_e f$ is defined up to a set of potential zero, i.e. up to a set $B \in \mathcal{B}(\mathbb{R}^d)$ which satisfies $\mathbb{E}^x \int_{(0,\infty)} \mathbb{1}_B(X_t)\,\mathrm{d}t = 0$ for all $x \in \mathbb{R}^d$. We will often choose a representative with a certain property; for instance if we write "$A_e f$ is continuous", this means that there exists a continuous function g such that (i),(ii) hold. In abuse of notation, we set

$$\|A_e f\|_\infty := \inf\{c > 0; |A_e f| \leq c \text{ up to a set of potential zero}\}.$$

Clearly, the extended infinitesimal generator $(A_e, \mathcal{D}(A_e))$ extends the (strong) infinitesimal generator $(A, \mathcal{D}(A))$. The following result is essentially due to Airault and Föllmer [1] and shows the connection to the Favard space of order 1 (cf. (6)).

Theorem 2.1 *Let $(X_t)_{t\geq 0}$ be a Feller process with semigroup $(P_t)_{t\geq 0}$ and extended generator $(A_e, \mathcal{D}(A_e))$. The associated Favard space F_1 of order 1 satisfies*

$$F_1 = \{f \in \mathcal{D}(A_e); \|A_e f\|_\infty < \infty\}.$$

If $f \in F_1$ then

$$\sup_{t\in(0,1)} \frac{1}{t}\|P_t f - f\|_\infty = \|A_e f\|_\infty \tag{10}$$

and, moreover, Dynkin's formula

$$\mathbb{E}^x f(X_\tau) - f(x) = \mathbb{E}^x\left(\int_0^\tau A_e f(X_s)\,\mathrm{d}s\right) \tag{11}$$

holds for any $x \in \mathbb{R}^d$ *and any stopping time* τ *such that* $\mathbb{E}^x \tau < \infty$.

The next corollary shows how the Favard space can be defined in terms of the stopped process $X_{t \wedge \tau_r^x}$. Since we will frequently use stopping techniques, it plays an important role in our proofs .

Corollary 2.2 *Let* $(X_t)_{t \geq 0}$ *be a Feller process with semigroup* $(P_t)_{t \geq 0}$*, extended generator* $(A_e, \mathcal{D}(A_e))$ *and symbol* q*. Denote by*

$$\tau_r^x := \inf\{t > 0; |X_t - x| > r\}$$

the exit time of $(X_t)_{t \geq 0}$ *from the closed ball* $\overline{B(x,r)}$*. If* q *has bounded coefficients, then the following statements are equivalent for any* $f \in \mathcal{B}_b(\mathbb{R}^d)$*:*

(i) $f \in F_1$*, i.e.* $f \in \mathcal{D}(A_e)$ *and* $\sup_{t \in (0,1)} t^{-1} \|P_t f - f\|_\infty = \|A_e f\|_\infty < \infty$*;*
(ii) There exists $r > 0$ *such that*

$$K_r(f) := \sup_{t \in (0,1)} \frac{1}{t} \sup_{x \in \mathbb{R}^d} |\mathbb{E}^x f(X_{t \wedge \tau_r^x}) - f(x)| < \infty.$$

If one (hence both) of the conditions is satisfied, then

$$A_e f(x) = \lim_{t \to 0} \frac{\mathbb{E}^x f(X_{t \wedge \tau_r^x}) - f(x)}{t} \tag{12}$$

up to a set of potential zero for any $r > 0$*. In particular,* $\|A_e f\|_\infty \leq K_r(f)$ *for* $r > 0$*.*

For the proof of Theorem 2.1 and Corollary 2.2 and some further remarks, we refer to Appendix A.

3 Main Results

Let $(X_t)_{t \geq 0}$ be a Feller process with semigroup $(P_t)_{t \geq 0}$. Throughout this section,

$$F_1^X := F_1 := \left\{ f \in \mathcal{B}_b(\mathbb{R}^d); \sup_{t \in (0,1)} \left\| \frac{P_t f - f}{t} \right\|_\infty < \infty \right\}$$

is the Favard space of order 1 associated with $(X_t)_{t \geq 0}$. By Theorem 2.1, we have

$$F_1 = \{f \in \mathcal{D}(A_e); \|A_e f\|_\infty < \infty\},$$

where A_e denotes the extended infinitesimal generator. The results which we present in this section will be proved in Sect. 5.

Our first result, Proposition 3.1, shows how regularity estimates for the semigroup $(P_t)_{t \geq 0}$ can be used to obtain Schauder estimates of the form

$$\|f\|_{\mathcal{C}_b^\kappa(\mathbb{R}^d)} \leq C(\|f\|_\infty + \|A_e f\|_\infty), \quad f \in F_1.$$

Proposition 3.1 *Let $(X_t)_{t\geq 0}$ be a Feller process with semigroup $(P_t)_{t\geq 0}$, extended generator $(A_e, \mathcal{D}(A_e))$ and Favard space F_1. If there exist constants $M > 0$, $T > 0$, $\kappa \geq 0$ and $\beta \in (0,1)$ such that*

$$\|P_t u\|_{\mathcal{C}_b^\kappa(\mathbb{R}^d)} \leq M t^{-\beta} \|u\|_\infty \tag{13}$$

for all $u \in \mathcal{B}_b(\mathbb{R}^d)$ and $t \in (0, T]$, then

$$F_1 \subseteq \mathcal{C}_b^\kappa(\mathbb{R}^d)$$

and

$$\|f\|_{\mathcal{C}_b^\kappa(\mathbb{R}^d)} \leq C(\|f\|_\infty + \|A_e f\|_\infty), \quad f \in F_1,$$

for some constant $C = C(T, M, \kappa, \beta)$.

Since the domain $\mathcal{D}(A)$ of the (strong) infinitesimal generator of $(X_t)_{t\geq 0}$ is contained in F_1, Proposition 3.1 gives, in particular, $\mathcal{D}(A) \subseteq \mathcal{C}_b^\kappa(\mathbb{R}^d)$.

Proposition 3.1 is a useful tool, but it does not, in general, give optimal regularity results. Since Feller processes are inhomogeneous in space, the regularity of $f \in F_1$ will, in general, depend on the space variable x, e.g.

$$|\Delta_h^2 f(x)| = |f(x+2h) - 2f(x+h) + f(x)| \leq C|h|^{\kappa(x)}, \quad |h| \leq 1, \tag{14}$$

and therefore it is much more natural to use Hölder–Zygmund spaces of variable order to describe the regularity; this is also indicated by the results obtained in [30].

Our second result, Theorem 3.2, shows how Hölder estimates for Feller semigroups can be used to establish local Hölder estimates (14). Before stating the result, let us explain the idea. Let $(X_t)_{t\geq 0}$ be a Feller process with symbol q and Favard space F_1^X, and fix $x \in \mathbb{R}^d$. Let $(Y_t)_{t\geq 0}$ be another Feller process which has the same behaviour as $(X_t)_{t\geq 0}$ in a neighbourhood of x, in the sense that its symbol p satisfies

$$p(z,\xi) = q(z,\xi), \quad z \in B(x,\delta),\ \xi \in \mathbb{R}^d \tag{15}$$

for some $\delta > 0$. The aim is to choose $(Y_t)_{t\geq 0}$ in such a way that its semigroup $(T_t)_{t\geq 0}$ satisfies a "good" regularity estimate

$$\|T_t u\|_{\mathcal{C}_b^\kappa(\mathbb{R}^d)} \leq M t^{-\beta} \|u\|_\infty, \quad u \in \mathcal{B}_b(\mathbb{R}^d);$$

here "good" means that κ is large. Because of (15), it is intuitively clear that

$$|\mathbb{E}^z f(X_t) - f(z)| \approx |\mathbb{E}^z f(Y_t) - f(z)| \quad \text{for } z \text{ close to } x \text{ and "small" } t. \tag{16}$$

(We will use stopping to specify what "small" means; see Lemma 5.2.) If χ is a truncation function such that $\mathbb{1}_{B(x,\varepsilon)} \leq \chi \leq \mathbb{1}_{B(x,2\varepsilon)}$ for small $\varepsilon > 0$, then it is, because of (16), natural to expect that for any $f \in F_1^X$ the truncated mapping $g := f \cdot \chi$ is in the Favard space F_1^Y associated with $(Y_t)_{t \geq 0}$, i. e.

$$\sup_{t \in (0,1)} \sup_{z \in \mathbb{R}^d} t^{-1} |\mathbb{E}^z (f \cdot \chi)(Y_t) - (f \cdot \chi)(z)| < \infty.$$

Since, by Proposition 3.1, $g \in F_1^Y \subseteq \mathcal{C}_b^\kappa(\mathbb{R}^d)$, and $g = f$ in a neighbourhood of x, this entails that $f(\cdot)$ is κ-Hölder continuous in a neighbourhood of x. Since $\kappa = \kappa(x)$ depends on the point $x \in \mathbb{R}^d$, which we fixed at the beginning, this localizing procedure allows us to obtain local Hölder estimates (14) for f.

Theorem 3.2 *Let $(X_t)_{t \geq 0}$ be a Feller process with extended generator $(A_e, \mathcal{D}(A_e))$ and Favard space F_1^X such that*

$$A_e f(z) = -q(z, D) f(z), \quad f \in C_c^\infty(\mathbb{R}^d),\ z \in \mathbb{R}^d,$$

for a continuous negative definite symbol q (cf. (7)). Let $x \in \mathbb{R}^d$ and $\delta \in (0,1)$ be such that there exists a Feller process $(Y_t^{(x)})_{t \geq 0}$ with the following properties:

(C1) *The infinitesimal generator $(L^{(x)}, \mathcal{D}(L^{(x)}))$ of $(Y_t^{(x)})_{t \geq 0}$ restricted to $C_c^\infty(\mathbb{R}^d)$ is a pseudo-differential operator with negative definite symbol $p^{(x)}$,*

$$p^{(x)}(z, \xi) = -i b^{(x)}(z) \cdot \xi + \int_{y \neq 0} \left(1 - e^{iy \cdot \xi} + iy \cdot \xi \mathbb{1}_{(0,1)}(|y|)\right) \nu^{(x)}(z, dy), \quad z, \xi \in \mathbb{R}^d;$$

$p^{(x)}$ has bounded coefficients, and

$$p^{(x)}(z, \xi) = q(z, \xi) \quad \text{for all } \xi \in \mathbb{R}^d,\ |z - x| \leq 4\delta. \tag{17}$$

(C2) *The $(L^{(x)}, C_c^\infty(\mathbb{R}^d))$-martingale problem is well-posed.*

(C3) *There exist constants $M(x) > 0$, $\kappa(x) \in [0, 2]$ and $\beta(x) \in (0, 1)$ such that the semigroup $(T_t^{(x)})_{t \geq 0}$ associated with $(Y_t^{(x)})_{t \geq 0}$ satisfies*

$$\|T_t^{(x)} u\|_{\mathcal{C}_b^{\kappa(x)}(\mathbb{R}^d)} \leq M(x) t^{-\beta(x)} \|u\|_\infty$$

for all $u \in \mathcal{B}_b(\mathbb{R}^d)$, $t \in (0, 1)$.

If $f \in F_1^X$ and $\varrho(x) \in [0, 1]$ are such that

$$\|f\|_{\mathcal{C}_b^{\varrho(x)}(\overline{B(x, 4\delta)})} < \infty \quad \text{and} \quad \sup_{|z - x| \leq 4\delta} \int_{|y| \leq 1} |y|^{1 + \varrho(x)} \nu^{(x)}(z, dy) < \infty, \tag{18}$$

then

$$|\Delta_h^2 f(x)| \leq C |h|^{\kappa(x)} \left(\|f\|_\infty + \|A_e f\|_\infty + \|f\|_{\mathcal{C}_b^{\varrho(x)}(\overline{B(x, 4\delta)})}\right) \tag{19}$$

for all $|h| \leq \delta/2$. *The finite constant* $C > 0$ *depends continuously on* $M(x) \in [0, \infty)$, $\beta(x) \in [0, 1)$ *and* $K(x) \in [0, \infty)$ *with*

$$K(x) := \sup_{z \in \mathbb{R}^d} \left(|b^{(x)}(z)| + \int_{y \neq 0} \min\{1, |y|^2\} \, \nu^{(x)}(z, dy) \right) + \sup_{|z-x| \leq 4\delta} \int_{y \neq 0} \min\{|y|^{\varrho(x)+1}, 1\} \, \nu^{(x)}(z, dy).$$

Remark 3.3 (i) The assumption $f \in C_b^{\varrho(x)}(\overline{B(x, 4\delta)})$ is an a priori estimate on the regularity of f. If the semigroup $(P_t)_{t \geq 0}$ of $(X_t)_{t \geq 0}$ satisfies a regularity estimate of form (13), then such an a priori estimate can be obtained from Proposition 3.1. Note that, by (18), there is a trade-off between the required a priori regularity of f and the roughness of the measures $\nu^{(x)}(z, dy)$, $z \in \overline{B(x, 4\delta)}$. If the measures $\nu^{(x)}(z, dy)$ only have a weak singularity at $y = 0$, in the sense that

$$\sup_{|z-x| \leq 4\delta} \int_{|y| \leq 1} |y| \, \nu^{(x)}(z, dy) < \infty,$$

then we can choose $\varrho(x) = 0$, i. e. it suffices that f is continuous. In contrast, if (at least) one of the measures has a strong singularity at $y = 0$, then we need a higher regularity of f (in a neighbourhood of x).

(ii) It is not very restrictive to assume that $(Y_t^{(x)})_{t \geq 0}$ has bounded coefficients since $(Y_t^{(x)})_{t \geq 0}$ is only supposed to mimic the behaviour of $(X_t)_{t \geq 0}$ in a neighbourhood of x (cf. (17)). We are, essentially, free to choose the behaviour of the process far away from x. In dimension $d = 1$, it is, for instance, a natural idea to consider

$$p^{(x)}(z, \xi) := \begin{cases} q(x - 4\delta, \xi), & z \leq x - 4\delta, \\ q(z, \xi), & |z - x| < 4\delta, \\ q(x + 4\delta, \xi), & z \geq x + 4\delta; \end{cases}$$

note that $p^{(x)}$ has bounded coefficients even if q has unbounded coefficients.

(iii) Condition (C2) is automatically satisfied if $C_c^\infty(\mathbb{R}^d)$ is a core for the infinitesimal generator of $(Y_t^{(x)})_{t \geq 0}$; see e.g. [20, Proposition 3.9.3] or [22, Theorem 1.38].

(iv) It is possible to extend Theorem 3.2 to Feller processes with a non-vanishing diffusion part. The idea of the proof is similar, but we need to impose stronger assumptions on the regularity on f, e.g. that $f|_{B(x,4\delta)}$ is differentiable.

As a direct consequence of Theorem 3.2, we obtain the following corollary.

Corollary 3.4 *Let* $(X_t)_{t \geq 0}$ *be a Feller process with extended generator* $(A_e, \mathcal{D}(A_e))$ *and symbol* q. *If there exist* $U \subseteq \mathbb{R}^d$ *open,* $\delta > 0$ *and* $\varrho : U \to [0, 1]$ *such that for any* $x \in U$ *the assumptions of Theorem* 3.2 *hold, then the Favard space of order 1 satisfies*

$$\mathcal{C}^{\varrho(\cdot)}(U) \cap F_1 \subseteq \mathcal{C}^{\kappa(\cdot)}(U).$$

If additionally

$$\sup_{x\in U}(M(x)+K(x))<\infty \quad \textit{and} \quad \sup_{x\in U}\beta(x)<1, \tag{20}$$

then $\mathcal{C}_b^{\varrho(\cdot)}(U)\cap F_1\subseteq \mathcal{C}_b^{\kappa(\cdot)}(U)$ *and there exists a constant* $C>0$ *such that*

$$\|f\|_{\mathcal{C}_b^{\kappa(\cdot)}(U)}\leq C\left(\|f\|_\infty+\|A_ef\|_\infty+\|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(U)}\right) \quad \textit{for all } f\in \mathcal{C}_b^{\varrho(\cdot)}(U)\cap F_1; \tag{21}$$

in particular, the domain $\mathcal{D}(A)$ *of the (strong) infinitesimal generator* A *satisfies* $\mathcal{C}_b^{\varrho(\cdot)}(U)\cap\mathcal{D}(A)\subseteq \mathcal{C}_b^{\kappa(\cdot)}(U)$ *and* (21) *holds for any* $f\in \mathcal{C}_b^{\varrho(\cdot)}(U)\cap\mathcal{D}(A)$.

In many examples (see e.g. Sect. 4), it is possible to choose the mapping ϱ in such a way that $F_1\subseteq \mathcal{C}_b^{\varrho(\cdot)}(U)$; in this case, Corollary 3.4 shows that $F_1\subseteq \mathcal{C}^{\kappa(\cdot)}(U)$ (resp. $F_1\subseteq \mathcal{C}_b^{\kappa(\cdot)}(U)$) and the Schauder estimate (21) holds for any function $f\in F_1$. In our applications, we will even have $\|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(U)}\leq c(\|f\|_\infty+\|A_ef\|_\infty)$, and therefore (21) becomes

$$\|f\|_{\mathcal{C}_b^{\kappa(\cdot)}(U)}\leq C'\left(\|f\|_\infty+\|A_ef\|_\infty\right) \quad \text{for all } f\in F_1.$$

In Sect. 4, we will apply Corollary 3.4 to isotropic stable-like processes, i. e. Feller processes with symbol of the form $q(x,\xi)=|\xi|^{\alpha(x)}$. The study of the domain $\mathcal{D}(A)$ of the infinitesimal generator A is particularly interesting since A is an operator of variable order. We will show that any function $f\in\mathcal{D}(A)$ satisfies the Hölder estimate of variable order

$$|\Delta_h^2 f(x)|\leq C_\varepsilon |h|^{\alpha(x)-\varepsilon}(\|f\|_\infty+\|Af\|_\infty), \quad |h|\leq 1,\ x\in\mathbb{R}^d,$$

for $\varepsilon>0$ (cf. Theorem 4.1) for the precise statement.

Our final result in this section is concerned with Schauder estimates for solutions to the equation $A_ef=g$ for Hölder continuous mappings g. To establish such Schauder estimates, we need additional assumptions on the regularity of the symbol and improved regularity estimates for the semigroup of the "localizing" Feller process $(Y_t^{(x)})_{t\geq 0}$ in Theorem 3.2.

Theorem 3.5 *Let* $(X_t)_{t\geq 0}$ *be a Feller process with extended generator* $(A_e,\mathcal{D}(A_e))$ *and Favard space* F_1^X *such that*

$$A_ef(z)=-q(z,D)f(z), \quad f\in C_c^\infty(\mathbb{R}^d),\ z\in\mathbb{R}^d,$$

for a continuous negative definite symbol q. *Assume that there exists* $\delta\in(0,1)$ *such that for any* $x\in\mathbb{R}^d$ *there exists a Feller process* $(Y_t^{(x)})_{t\geq 0}$ *with symbol*

$$p^{(x)}(z,\xi) = -ib^{(x)}(z)\cdot\xi + \int_{y\neq 0}\left(1 - e^{iy\cdot\xi} + iy\cdot\xi\mathbb{1}_{(0,1)}(|y|)\right)\nu^{(x)}(z,dy), \quad (22)$$

satisfying (C1)-(C3) in Theorem 3.2. Assume additionally that the following conditions hold for absolute constants $C_1, C_2 > 0$:

(S1) *For any $x, z \in \mathbb{R}^d$, there exists $\alpha^{(x)}(z) \in (0,2)$ such that*

$$\nu^{(x)}(z,dy) \leq C_1|y|^{-d-\alpha^{(x)}(z)}\,dy \quad \text{on } B(0,1)$$

and $0 < \inf_{x,z\in\mathbb{R}^d}\alpha^{(x)}(z) \leq \sup_{x,z\in\mathbb{R}^d}\alpha^{(x)}(z) < 2$.

(S2) *There exists $\theta \in (0,1]$ such that*

$$|b^{(x)}(z) - b^{(x)}(z+h)| \leq C_2|h|^\theta, \quad x,z,h \in \mathbb{R}^d, \quad (23)$$

and the following statement holds for every $r \in (0,1)$ and every $x, z \in \mathbb{R}^d$: If $u : \mathbb{R}^d \to \mathbb{R}$ is a measurable mapping such that

$$|u(y)| \leq c_u \min\{|y|^{\alpha^{(x)}(z)+r}, 1\}, \quad y \in \mathbb{R}^d,$$

for some $c_u > 0$, then there exist $C_{3,r} > 0$ and $H_r > 0$ (not depending on u, x, z) such that

$$\left|\int u(y)\,\nu^{(x)}(z,dy) - \int u(y)\,\nu^{(x)}(z+h,dy)\right| \leq C_{3,r}c_u|h|^\theta \quad (24)$$

for all $|h| \leq H_r$.

(S3) *There exists $\Lambda > 0$ such that the semigroup $(T_t^{(x)})_{t\geq 0}$ of the Feller process $(Y_t^{(x)})_{t\geq 0}$ satisfies*

$$\|T_t^{(x)}u\|_{\mathcal{C}_b^{\lambda+\kappa(x)}(\mathbb{R}^d)} \leq M(x)t^{-\beta(x)}\|u\|_{\mathcal{C}_b^{\lambda}(\mathbb{R}^d)}, \quad u \in \mathcal{C}_b^\lambda(\mathbb{R}^d),\ t \in (0,1), \quad (25)$$

for any $x \in \mathbb{R}^d$ and $\lambda \in [0,\Lambda]$; here $M(x)$, $\kappa(x)$ and $\beta(x)$ denote the constants from (C3).

(S4) *The mapping $\kappa : \mathbb{R}^d \to (0,\infty)$ is uniformly continuous and bounded away from zero, i.e. $\kappa_0 := \inf_{x\in\mathbb{R}^d}\kappa(x) > 0$.*

(S5) $\sup_{x\in\mathbb{R}^d} M(x) < \infty$, $\sup_{x\in\mathbb{R}^d}\beta(x) < 1$, *and*

$$\sup_{x,z\in\mathbb{R}^d}\left(|b^{(x)}(z)| + \int_{|y|\geq 1}\nu^{(x)}(z,dy)\right) < \infty.$$

Let $\varrho : \mathbb{R}^d \to [0,2]$ be a uniformly continuous function satisfying

$$\sigma := \inf_{x\in\mathbb{R}^d}\inf_{|z-x|\leq 4\delta}\left(1 + \varrho(x) - \alpha^{(x)}(z)\right) > 0. \quad (26)$$

If $f \in F_1^X$ *is such that* $f \in \mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)$ *and*

$$A_e f = g \in \mathcal{C}_b^{\lambda}(\mathbb{R}^d)$$

for some $\lambda \in [0, \Lambda]$, *then* $f \in \mathcal{C}_b^{(\kappa(\cdot)+\min\{\theta,\lambda,\sigma\})-}(\mathbb{R}^d)$, *i. e.*

$$f \in \bigcap_{\varepsilon \in (0,\kappa_0)} \mathcal{C}_b^{\kappa(\cdot)+\min\{\theta,\lambda,\sigma\}-\varepsilon}(\mathbb{R}^d). \tag{27}$$

Moreover, the Schauder estimate

$$\|f\|_{\mathcal{C}_b^{\kappa(\cdot)+\min\{\theta,\lambda,\sigma\}-\varepsilon}(\mathbb{R}^d)} \leq C_\varepsilon \left(\|A_e f\|_{\mathcal{C}_b^{\lambda}(\mathbb{R}^d)} + \|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)} \right) \tag{28}$$

holds for any $\varepsilon \in (0, \kappa_0)$ *and some finite constant* C_ε *which does not depend on* f, g.

Remark 3.6 (i) In our examples in Sect. 4, we will be able to choose ϱ in such a way that $\alpha^{(x)}(z) - \varrho(z)$ is arbitrarily small for $x \in \mathbb{R}^d$ and $z \in \overline{B(x, 4\delta)}$, and therefore the constant σ in (26) will be close to 1. Noting that $\theta \leq 1$, it follows that we can discard σ in (27) and (28) i. e. we get

$$f \in \mathcal{C}_b^{\kappa(\cdot)+\min\{\theta,\lambda\}-\varepsilon}(\mathbb{R}^d), \quad \varepsilon \in (0, \kappa_0). \tag{29}$$

We would like to point out that it is, in general, *not* possible to improve this estimate and to obtain that $f \in \mathcal{C}_b^{\kappa(\cdot)+\lambda-\varepsilon}(\mathbb{R}^d)$, $\varepsilon \in (0, \kappa_0)$. To see this, consider a Feller process $(X_t)_{t\geq 0}$ with symbol $q(x, \xi) = ib(x)\xi$, $x, \xi \in \mathbb{R}$, for a mapping $b \in C_b(\mathbb{R}^d)$ with $\inf_x b(x) > 0$. If we define

$$f(x) := \int_0^x \frac{1}{b(y)}\, dy, \quad x \in \mathbb{R}^d,$$

then $A_e f = b f' = 1$ is smooth. However, the regularity of f clearly depends on the regularity of b,

$$\text{regularity of } f \approx 1 + \text{regularity of } b,$$

which means that f is *less* regular than $A_e f$.

(ii) It suffices to check (25) for $\lambda = \Lambda$; for $\lambda \in (0, \Lambda)$, the inequality then follows from the interpolation theorem (see e.g. [52, Section 1.3.3] or [39, Theorem 1.6]) and the fact that $\mathcal{C}_b^{\gamma}(\mathbb{R}^d)$ can be written as a real interpolation space (see [52, Theorem 2.7.2.1] for details).

(iii) (24) is an assumption on the regularity of $z \mapsto \nu^{(x)}(z, dy)$. If $\nu^{(x)}(z, dy)$ has a density, say $m^{(x)}(z, y)$, with respect to Lebesgue measure, then a sufficient condition for (24) is

$$\int_{y\neq 0} \min\{1, |y|^{\alpha^{(x)}(z)+r}\} |m^{(x)}(z, y) - m^{(x)}(z + h, y)|\, dy \leq C_{3,r}|h|^\theta.$$

(iv) Condition (S1) is not strictly necessary for the proof of Theorem 3.5; essentially we need suitable upper bounds for

$$\int_{|y|\leq r} |y|^{\gamma}\, \nu^{(x)}(z, \mathrm{d}y) \quad \text{and} \quad \int_{r<|y|\leq R} |y|^{\gamma}\, \nu^{(x)}(z, \mathrm{d}y),$$

where $0 < r < R < 1$, $x, z \in \mathbb{R}^d$ and $\gamma \in (0, 3)$.

(v) In (S2), we assume that $\theta \leq 1$; this assumption can be relaxed. To this end, we have to replace in (23) and (24) the differences of first order,

$$|b^{(x)}(z) - b^{(x)}(z+h)| \quad \text{and} \quad \left| \int u(y)\, \nu^{(x)}(z, \mathrm{d}y) - \int u(y)\, \nu^{(x)}(z+h, \mathrm{d}y) \right|,$$

by iterated differences of higher order (cf. (5)). This makes the proof more technical, but the idea of the proof stays the same.

The proofs of the results stated in this section will be presented in Sect. 5.

4 Applications

In this section, we apply the results from the previous section to various classes of Feller processes. We will study processes of variable order (Theorem 4.1 and Corollary 4.3), random time changes of Lévy processes (Proposition 4.5) and solutions to Lévy-driven SDEs (Proposition 4.7). Our aim is to illustrate the range of applications, and therefore, we do not strive for the greatest generality of the examples; we will, however, point the reader to possible extensions of the results which we present. We remind the reader of the notation

$$\mathcal{C}_b^{\alpha(\cdot)+}(\mathbb{R}^d) := \bigcup_{\varepsilon>0} \mathcal{C}_b^{\alpha(\cdot)+\varepsilon}(\mathbb{R}^d) \qquad \mathcal{C}_b^{\alpha(\cdot)-}(\mathbb{R}^d) := \bigcap_{\varepsilon>0} \mathcal{C}_b^{\max\{\alpha(\cdot)-\varepsilon,0\}}(\mathbb{R}^d)$$

introduced in Sect. 2.

The first part of this section is devoted to isotropic stable-like processes, i. e. Feller processes $(X_t)_{t\geq 0}$ with symbol of the form $q(x, \xi) = |\xi|^{\alpha(x)}$; they appeared first in papers by Bass [3]. A sufficient condition for the existence of such a Feller process is that $\alpha : \mathbb{R}^d \to (0, 2]$ is Hölder continuous and bounded from below (cf. [22, Theorem 5.2]). If $\alpha(\mathbb{R}^d) \subseteq (0, 2)$, then the infinitesimal generator A of $(X_t)_{t\geq 0}$ satisfies

$$Af(x) = c_{d,\alpha(x)} \int_{y\neq 0} \big(f(x+y) - f(x) - y \cdot \nabla f(x) \mathbb{1}_{(0,1)}(|y|)\big) \frac{1}{|y|^{d+\alpha(x)}}\, \mathrm{d}y,$$

for all $f \in C_c^{\infty}(\mathbb{R}^d)$, which means that A is a fractional Laplacian of variable order, i.e. $A = -(-\Delta)^{\alpha(\cdot)/2}$. This makes A—and hence the stable-like process $(X_t)_{t\geq 0}$—an interesting object of study. To our knowledge, there are no Schauder estimates for the Poisson equation $Af = g$ available in the existing literature. Using the results from the previous section, we are able to derive Schauder estimates for functions f in the Favard space F_1 (and, hence in particular, for $f \in \mathcal{D}(A)$) (cf. Theorem 4.1), as well as Schauder estimates for solutions to $Af = g$ (cf. Corollary 4.3).

Theorem 4.1 *Let $(X_t)_{t\geq 0}$ be a Feller process with symbol $q(x,\xi) = |\xi|^{\alpha(x)}$ for a Hölder continuous function $\alpha : \mathbb{R}^d \to (0,2)$ such that*

$$0 < \alpha_L := \inf_{x\in\mathbb{R}^d} \alpha(x) \leq \sup_{x\in\mathbb{R}^d} \alpha(x) < 2.$$

The associated Favard space F_1 of order 1 (cf. (6)) satisfies

$$F_1 \subseteq \mathcal{C}_b^{\alpha(\cdot)-}(\mathbb{R}^d).$$

For any $\varepsilon \in (0,\alpha_L)$, there exists a finite constant $C = C(\varepsilon,\alpha)$ such that

$$\|f\|_{\mathcal{C}_b^{\alpha(\cdot)-\varepsilon}(\mathbb{R}^d)} \leq C(\|f\|_\infty + \|A_e f\|_\infty), \quad f \in F_1, \tag{30}$$

where A_e denotes the extended generator of $(X_t)_{t\geq 0}$. In particular, (30) *holds for any f in the domain $\mathcal{D}(A)$ of the (strong) generator of $(X_t)_{t\geq 0}$, and $\mathcal{D}(A) \subseteq \mathcal{C}_b^{\alpha(\cdot)-}(\mathbb{R}^d)$.*

Remark 4.2 (i) Theorem 4.1 allows us to obtain information on the regularity of the transition density $p(t,x,y)$ of $(X_t)_{t\geq 0}$. Since $p(t,\cdot,y) \in \mathcal{D}(A)$ for each $t > 0$ and $y \in \mathbb{R}^d$ (cf. [22, Corollary 3.6]), it follows from Theorem 4.1 that $p(t,\cdot,y) \in \mathcal{C}_b^{\alpha(\cdot)-}(\mathbb{R}^d)$; in particular, $x \mapsto p(t,x,y)$ is differentiable at $x \in \{\alpha > 1\}$. Moreover, $(\partial_t - A_x)p(t,x,y) = 0$ entails by [22, Theorem 3.8] that

$$\|p(t,\cdot,y)\|_{\mathcal{C}_b^{\alpha(\cdot)-\varepsilon}(\mathbb{R}^d)} \leq Ct^{-1-d/\alpha_L}, \quad t \in (0,T), \ y \in \mathbb{R}^d,$$

for a finite constant $C = C(\varepsilon,\alpha,T)$. Some related results on the regularity of the transition density were recently obtained in [10].

(ii) Theorem 4.1 gives a necessary condition for a function $f \in C_\infty(\mathbb{R}^d)$ to be in the domain $\mathcal{D}(A)$ of the infinitesimal generator; sufficient conditions were established in [30, Example 5.5]. Combining both results, it should be possible to show that $\mathcal{D}(A)$ is an algebra, i.e. $f,g \in \mathcal{D}(A)$ implies $f\cdot g \in \mathcal{D}(A)$, and that

$$A(f\cdot g) = fAg + gAf + \Gamma(f,g), \quad f,g \in \mathcal{D}(A),$$

see [27, Proof of Theorem 4.3(iii)] for the idea of the proof; here

$$\Gamma(f,g)(x) := c_{d,\alpha(x)} \int_{y\neq 0} (f(x+y)-f(x))\,(g(x+y)-g(x))\,\frac{1}{|y|^{d+\alpha(x)}}\,\mathrm{d}y$$

is the so-called carré du champ operator (cf. [8,12]) and $\nu(x, \mathrm{d}y) = c_{d,\alpha(x)} |y|^{-d-\alpha(x)}\,\mathrm{d}y$ is the family of Lévy measures associated with the symbol $|\xi|^{\alpha(x)}$ via the Lévy–Khintchine representation.

(iii) Theorem 4.1 can be generalized to a larger class of "stable-like" Feller processes, e.g. relativistic stable-like processes and tempered stable-like processes (cf. [22, Section 5.1] or [25, Example 4.7]) for the existence of such processes. In order to apply the results from Sect. 3, we need two key ingredients: general existence results—which ensure the existence of a "nice" Feller process $(Y_t)_{t\geq 0}$ whose symbol is "truncated" in a suitable way (cf. Step 1 in the proof of Theorem 4.1)—and certain heat kernel estimates needed to establish Hölder estimates for the semigroup; in [22], both ingredients were established for a wide class of stable-like processes.

As a corollary of Theorem 4.1 and Theorem 3.5, we will establish the following Schauder estimates for the elliptic equation $Af = g$ associated with the infinitesimal generator A of the isotropic stable-like process.

Corollary 4.3 *Let $(X_t)_{t\geq 0}$ be a Feller process with infinitesimal generator $(A, \mathcal{D}(A))$ and symbol $q(x,\xi) = |\xi|^{\alpha(x)}$ for a mapping $\alpha : \mathbb{R}^d \to (0,2)$ which satisfies*

$$0 < \alpha_L := \inf_{x\in\mathbb{R}^d} \leq \sup_{x\in\mathbb{R}^d} \alpha(x) < 2 \tag{31}$$

and $\alpha \in C_b^\gamma(\mathbb{R}^d)$ for some $\gamma \in (0,1)$. If $f \in \mathcal{D}(A)$ is such that

$$Af = g \in \mathcal{C}_b^\lambda(\mathbb{R}^d)$$

for some $\lambda > 0$, then $f \in \mathcal{C}_b^{(\alpha(\cdot)+\min\{\lambda,\gamma\})-}(\mathbb{R}^d)$. For any $\varepsilon \in (0, \alpha_L)$, there exists a constant $C_\varepsilon > 0$ (not depending on f, g) such that

$$\|f\|_{\mathcal{C}_b^{\alpha(\cdot)+\min\{\lambda,\gamma\}-\varepsilon}(\mathbb{R}^d)} \leq C_\varepsilon \left(\|Af\|_{\mathcal{C}_b^{\min\{\lambda,\gamma\}}(\mathbb{R}^d)} + \|f\|_\infty \right). \tag{32}$$

It is possible to extend Corollary 4.3 to a larger class of "stable-like" processes (see also Remark 4.2(ii)). Let us give some remarks on the assumption that $\alpha \in C_b^\gamma(\mathbb{R}^d)$ for $\gamma \in (0,1)$.

Remark 4.4 (i) Let $\alpha : \mathbb{R}^d \to (0,2)$ be Lipschitz continuous function satisfying (31). Since $\alpha \in \mathcal{C}_b^{1-\varepsilon}(\mathbb{R}^d)$ for every $\varepsilon \in (0,1)$, the Schauder estimate (32) holds with $\gamma = 1 - \varepsilon/2$ and $\varepsilon \rightsquigarrow \varepsilon/2$, and this entails that (32) holds with $\gamma = 1$. This means that Corollary 4.3 remains valid for Lipschitz continuous functions (with $\gamma = 1$ in (32)).

(ii) If $\alpha \in C_b^\gamma(\mathbb{R}^d)$ for $\gamma > 1$, we can apply Corollary 4.3 with $\gamma = 1$, but this gives a weaker regularity estimate for f than we would expect; this is because we lose some information on the regularity of α. The reason why we have to restrict ourselves to $\gamma \in (0,1)$ is that two tools which we need for the proof (Theorem 3.5 and Proposition 6.2) are only available for $\gamma \in (0,1)$. However, we believe that both results are valid for $\gamma > 0$, and hence that that the assumption $\gamma \in (0, 1)$ in Corollary 4.3 can be dropped.

Since the proofs of Theorem 4.1 and Corollary 4.3 are quite technical, we defer them to Sect. 6. The idea is to apply Theorem 3.2 and Theorem 3.5. As "localizing" process $(Y_t^{(x)})_{t\geq 0}$, we will use a Feller process with symbol

$$p^{(x)}(z, \xi) := |\xi|^{\alpha^{(x)}(z)}, \quad z, \xi \in \mathbb{R}^d,$$

where

$$\alpha^{(x)}(z) := (\alpha(x) - \varepsilon) \vee \alpha(z) \wedge (\alpha(x) + \varepsilon), \quad z \in \mathbb{R}^d,$$

for fixed $x \in \mathbb{R}^d$ and small $\varepsilon > 0$. In order to apply the results from the previous section, we need suitable regularity estimates for the semigroup $(P_t)_{t\geq 0}$ associated with an isotropic stable-like process $(Y_t)_{t\geq 0}$. We will study the regularity of $x \mapsto P_t u(x)$ using the parametrix construction of (the transition density of) $(Y_t)_{t\geq 0}$ in [22]; the results are of independent interest, we refer the reader to Sect. 6.1.

Next we study Feller processes with symbols of the form $q(x, \xi) = m(x)|\xi|^\alpha$. They can be constructed as random time changes of isotropic α-stable Lévy processes (see e.g. [6, Section 4.1] and [26] for further details). This class of Feller processes includes, in particular, solutions to SDEs

$$\mathrm{d}X_t = \sigma(X_{t-})\,\mathrm{d}L_t, \quad X_0 = x,$$

driven by a one-dimensional isotropic α-stable Lévy process $(L_t)_{t\geq 0}$, $\alpha \in (0, 2]$; for instance if $\sigma > 0$ is continuous and at most of linear growth, then there exists a unique weak solution to the SDE, and the solution is a Feller process with symbol $q(x, \xi) = |\sigma(x)|^\alpha |\xi|^\alpha$ (cf. [23, Example 5.4]).

Proposition 4.5 *Let $(X_t)_{t\geq 0}$ be a Feller process with symbol $q(x, \xi) = m(x)|\xi|^\alpha$ for $\alpha \in (0, 2)$ and a Hölder continuous function $m : \mathbb{R}^d \to (0, \infty)$ such that*

$$0 < \inf_{x\in\mathbb{R}^d} m(x) \leq \sup_{x\in\mathbb{R}^d} m(x) < \infty.$$

(i) The infinitesimal generator $(A, \mathcal{D}(A))$ and the Favard space F_1 of order 1 satisfy

$$\mathcal{C}_\infty^{\alpha+}(\mathbb{R}^d) \subseteq \mathcal{D}(A) \subseteq F_1 \subseteq \mathcal{C}_b^{\alpha-}(\mathbb{R}^d),$$

where

$$\mathcal{C}_\infty^{\alpha+}(\mathbb{R}^d) := \mathcal{C}_b^{\alpha+}(\mathbb{R}^d) \cap C_\infty^{\lfloor\alpha\rfloor}(\mathbb{R}^d) = \begin{cases} \mathcal{C}_b^{\alpha}(\mathbb{R}^d) \cap C_\infty(\mathbb{R}^d), & \alpha \in (0,1), \\ \mathcal{C}_b^{\alpha}(\mathbb{R}^d) \cap C_\infty^1(\mathbb{R}^d), & \alpha \in [1,2). \end{cases} \tag{33}$$

For any $\kappa \in (0, \alpha)$ *there exists a finite constant* $C_1 > 0$ *such that*

$$\|f\|_{\mathcal{C}_b^\kappa(\mathbb{R}^d)} \leq C_1(\|f\|_\infty + \|A_e f\|_\infty) \quad \text{for all } f \in F_1; \tag{34}$$

here A_e *denotes the extended infinitesimal generator.*

(ii) Let $\theta \in (0, 1]$ *be such that* $m \in C_b^\theta(\mathbb{R}^d)$*. If* $f \in \mathcal{D}(A)$ *is such that*

$$Af = g \in \mathcal{C}_b^\lambda(\mathbb{R}^d)$$

for some $\lambda > 0$*, then* $f \in \mathcal{C}_b^{(\alpha+\min\{\lambda,\theta\})-}(\mathbb{R}^d)$ *and for any* $\kappa \in (0, \alpha)$ *there exists a constant* $C_2 > 0$ *(not depending on* f*,* Af*) such that*

$$\|f\|_{\mathcal{C}_b^{\kappa+\min\{\lambda,\theta\}}(\mathbb{R}^d)} \leq C_2 \left(\|f\|_\infty + \|Af\|_{\mathcal{C}_b^{\min\{\lambda,\theta\}}(\mathbb{R}^d)} \right).$$

Proof It follows from [22, Theorem 3.3] that there exists a unique Feller process $(X_t)_{t\geq 0}$ with symbol $q(x, \xi) = m(x)|\xi|^\alpha$, $x, \xi \in \mathbb{R}^d$. As in the proof of Proposition 6.1 and Proposition 6.2, it follows from the parametrix construction of the transition density p in [22] that the semigroup $(P_t)_{t\geq 0}$ satisfies

$$\|P_t u\|_{\mathcal{C}_b^\kappa(\mathbb{R}^d)} \leq c_{1,\kappa} t^{-\kappa/\alpha} \|u\|_\infty, \quad u \in \mathcal{B}_b(\mathbb{R}^d),\ t \in (0, 1),$$

and

$$\|P_t u\|_{\mathcal{C}_b^{\kappa+\lambda}(\mathbb{R}^d)} \leq c_{2,\kappa} t^{-\kappa/\alpha} \|u\|_{\mathcal{C}_b^\lambda(\mathbb{R}^d)}, \quad u \in \mathcal{C}_b^\lambda(\mathbb{R}^d),\ t \in (0, 1),$$

for any $\kappa \in (0, \alpha)$ and $\lambda \in [0, \theta]$; for the particular case $\alpha \in (0, 1]$ the first inequality follows from [37]. Applying Proposition 3.1, we get (34); in particular $F_1 \subseteq \mathcal{C}_b^{\alpha-}(\mathbb{R}^d)$. The inclusion $\mathcal{C}_\infty^{\alpha+}(\mathbb{R}^d) \subseteq \mathcal{D}(A)$ is a direct consequence of [30, Example 5.4]. The Schauder estimate in (ii) follows Theorem 3.5 applied with $Y_t^{(x)} := X_t$ for all $x \in \mathbb{R}^d$ (using the regularity estimates for $(P_t)_{t\geq 0}$ from above). □

Remark 4.6 (Possible extensions of Proposition 4.5)

(i) Proposition 4.5 can be extended to symbols $q(x, \xi) = m(x)\psi(\xi)$ for "nice" continuous negative definite functions ψ, e.g. the characteristic exponent of a relativistic stable or tempered stable Lévy process (cf. [22, Table 5.2] for further examples).

(ii) The family of Lévy kernels associated with the Feller process $(X_t)_{t\geq 0}$ is of the form $\nu(x, dy) = m(x)|y|^{-d-\alpha}\, dy$. More generally, it is possible to consider Feller processes with Lévy kernels $\nu(x, dy) = m(x, y)\, \nu(dy)$, for instance [5,37,50] establish existence results as well as Hölder estimates under suitable assumptions on m and ν (in particular, $x \mapsto m(x, y)$ needs to satisfy some Hölder condition). Combining the results with Proposition 3.1, we can obtain Schauder estimates for functions in the domain of the infinitesimal generator of $(X_t)_{t\geq 0}$. Let us mention that for $\nu(x, y) = m(x, y)|y|^{-d-\alpha}\, dy$ Schauder estimates were studied in [4].

We close this section with some results on solutions to Lévy-driven SDEs.

Proposition 4.7 *Let $(L_t)_{t\geq 0}$ be a 1-dimensional isotropic α-stable Lévy process for some $\alpha \in (0,2)$. Consider the SDE*

$$\mathrm{d}X_t = b(X_{t-})\,\mathrm{d}t + \sigma(X_{t-})\,\mathrm{d}L_t, \quad X_0 = x, \tag{35}$$

for a bounded β-Hölder continuous mapping $b : \mathbb{R} \to \mathbb{R}$ and a bounded Lipschitz continuous mapping $\sigma : \mathbb{R} \to (0,\infty)$. If

$$\beta + \alpha > 1 \quad \text{and} \quad \sigma_L := \inf_{x\in\mathbb{R}} \sigma(x) > 0, \tag{36}$$

then there exists a unique weak solution $(X_t)_{t\geq 0}$ to (35), and it gives rise to a Feller process with infinitesimal generator $(A, \mathcal{D}(A))$. The associated Favard space F_1 of order 1 satisfies

$$\mathcal{D}(A) \subseteq F_1 \subseteq \bigcap_{k\in\mathbb{N}} \mathcal{C}_b^{\min\{1,\alpha-1/k\}}(\mathbb{R}),$$

and there exists for any $k \in \mathbb{N}$ a finite constant $C > 0$ such that

$$\|f\|_{\mathcal{C}_b^{\min\{\alpha-1/k,1\}}(\mathbb{R})} \leq C(\|f\|_\infty + \|A_e f\|_\infty) \quad \text{for all } f \in F_1, \tag{37}$$

where A_e denotes the extended generator. In particular, (37) holds for any $f \in \mathcal{D}(A)$ with $A_e f = Af$.

Proof It follows from (36) that SDE (35) has a unique weak solution $(X_t)_{t\geq 0}$ for any $x \in \mathbb{R}$ (cf. [33]). By [49] (see also [24]), $(X_t)_{t\geq 0}$ is a Feller process. Moreover, [36] shows that for any $\kappa < \alpha$ there exists a constant $c > 0$ such that the semigroup $(P_t)_{t\geq 0}$ satisfies

$$\|P_t u\|_{\mathcal{C}_b^{\kappa\wedge 1}(\mathbb{R})} \leq c\|u\|_\infty t^{-\kappa/\alpha}$$

for all $t \in (0,1)$ and $u \in \mathcal{B}_b(\mathbb{R})$. Applying Proposition 3.1 proves the assertion. □

Before giving some remarks on possible extensions of Proposition 4.7, let us mention that sufficient conditions for a function f to be in the domain $\mathcal{D}(A)$ were studied in [30]. For instance, if the SDE has no drift part, i.e. $b = 0$, then it follows from Proposition 4.7 and [30, Example 5.6] that

$$\mathcal{C}_\infty^{\alpha+}(\mathbb{R}) \subseteq \mathcal{D}(A) \subseteq \mathcal{C}_b^{\alpha-}(\mathbb{R}) \quad \text{if } \alpha \in (0,1] \tag{38}$$

and

$$\mathcal{C}_\infty^{\alpha+}(\mathbb{R}) \subseteq \mathcal{D}(A) \subseteq \mathcal{C}_b^{1}(\mathbb{R}) \quad \text{if } \alpha \in (1,2); \tag{39}$$

see (33) for the definition of $\mathcal{C}_\infty^{\alpha+}(\mathbb{R})$. Intuitively, one would expect that (38) holds for $\alpha \in (0,2)$. If we knew that the semigroup $(P_t)_{t\geq 0}$ of the solution to (35) satisfies

$$\|P_t u\|_{\mathcal{C}_b^\kappa(\mathbb{R})} \leq c t^{-\kappa/\alpha} \|u\|_\infty, \quad u \in \mathcal{B}_b(\mathbb{R}),\ t \in (0,1),\ \kappa \in (0,\alpha) \tag{40}$$

for some constant $c = c(\kappa) > 0$, this would immediately follow from Proposition 3.1. We could not find (40) in the literature, but we strongly believe that the parametrix construction of the transition density in [33] can be used to establish such an estimate; this is also indicated by the proof of Theorem 4.1 (see in particular the proof of Proposition 6.1). In fact, we believe that the parametrix construction in [33] entails estimates of the form

$$\|P_t u\|_{\mathcal{C}_b^{\kappa+\min\{\lambda,\beta\}}(\mathbb{R})} \leq c t^{-\kappa/\alpha} \|u\|_{\mathcal{C}_b^{\min\{\lambda,\beta\}}(\mathbb{R})}, \qquad u \in \mathcal{C}_b^\lambda(\mathbb{R}),\ t \in (0,1),$$

for $\kappa \in (0,\alpha)$, $\lambda > 0$ (recall that β is the Hölder exponent of the drift b), which would then allow us to establish Schauder estimates to the equation $Af = g$ for $g \in \mathcal{C}_b^\lambda(\mathbb{R})$ using Theorem 3.5.

Remark 4.8 (Possible extensions of Proposition 4.7)

(i) The gradient estimates in [36] were obtained under more general conditions, and (the proof of) Proposition 4.7 extends naturally to this more general framework. Firstly, Proposition 4.7 can be extended to higher dimensions; the assumption $\sigma_L > 0$ in (36) is then replaced by the assumption that σ is uniformly non-degenerate in the sense that

$$\begin{aligned} M^{-1}|\xi| &\leq \inf_{x\in\mathbb{R}^d} \min\{|\sigma(x)\xi|, |\sigma(x)^{-1}\xi|\} \\ &\leq \sup_{x\in\mathbb{R}^d} \max\{|\sigma(x)\xi|, |\sigma(x)^{-1}\xi|\} \leq M|\xi| \end{aligned}$$

for some absolute constant $M > 0$ which does not depend on $\xi \in \mathbb{R}^d$. Secondly, Proposition 4.7 holds for a larger class of driving Lévy processes; it suffices to assume that the Lévy measure ν satisfies $\nu(dz) \geq c|z|^{-d-\alpha}\mathbb{1}_{\{|z|\leq\eta\}}$ for some $c, \eta > 0$ and that SDE (35) has a unique weak solution. Under the stronger balance condition $\beta + \alpha/2 > 1$ this is automatically satisfied for a large class of Lévy processes, e.g. if $(L_t)_{t\geq 0}$ is an relativistic stable or a tempered stable Lévy process (cf. [11]).

(ii) Recently, Kulczycki et al. [32] established Hölder estimates for the semigroup associated with the solution to the SDE

$$dX_t = \sigma(X_{t-})\, dL_t$$

driven by a d-dimensional Lévy process $(L_t)_{t\geq 0}$, $d \geq 2$, whose components are independent α-stable Lévy processes, $\alpha \in (0,1)$, under the assumption that the coefficient $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is bounded, Lipschitz continuous and satisfies $\inf_x \det(\sigma(x)) > 0$. Combining the estimates with Proposition 3.1, we find that the assertion of Proposition 4.7 remains valid in this framework, i.e. the Favard space F_1 associated with the unique solution $(X_t)_{t\geq 0}$ satisfies $F_1 \subseteq \mathcal{C}_b^{\alpha-}(\mathbb{R}^d)$ and

$$\|f\|_{\mathcal{C}_b^{\alpha-1/k}(\mathbb{R}^d)} \leq C_k(\|f\|_\infty + \|A_e f\|_\infty), \quad f \in F_1.$$

(iii) Using coupling methods, Luo and Wang [41, Section 5.1] and Liang et. al [38] recently studied the regularity of semigroups associated with solutions to SDEs with additive noise

$$\mathrm{d}X_t = b(X_{t-})\,\mathrm{d}t + \mathrm{d}L_t$$

for a large class of driving Lévy processes $(L_t)_{t\geq 0}$. The results from [38,41] and Sect. 3 can be used to obtain Schauder estimates for functions in the domain of the infinitesimal generator of $(X_t)_{t\geq 0}$.

5 Proofs of Results from Sect. 3

For the proof of Proposition 3.1, we use the following lemma which shows how Hölder estimates for a Feller semigroup translate to regularity properties of the λ-potential operator

$$R_\lambda u := \int_{(0,\infty)} \mathrm{e}^{-\lambda t} P_t u \,\mathrm{d}t, \quad u \in \mathcal{B}_b(\mathbb{R}^d),\ \lambda > 0.$$

Lemma 5.1 *Let $(X_t)_{t\geq 0}$ be a Feller process with semigroup $(P_t)_{t\geq 0}$ and λ-potential operators $(R_\lambda)_{\lambda>0}$.*

(i) If there exist $T > 0$, $M \geq 0$, $\kappa \geq 0$ and $\beta \geq 0$ such that

$$\|P_t u\|_{\mathcal{C}_b^\kappa(\mathbb{R}^d)} \leq M t^{-\beta} \|u\|_\infty$$

for all $t \in (0, T)$ and $u \in \mathcal{B}_b(\mathbb{R}^d)$, then

$$\|P_t u\|_{\mathcal{C}_b^\kappa(\mathbb{R}^d)} \leq M \mathrm{e}^{mt} t^{-\beta} \|u\|_\infty \tag{41}$$

for all $t > 0$ and $u \in \mathcal{B}_b(\mathbb{R}^d)$, where $m := \log(2)\beta/T$.

(ii) If $u \in \mathcal{B}_b(\mathbb{R}^d)$ is such that (41) holds for some $\beta \in [0, 1)$, then $R_\lambda u \in \mathcal{C}_b^\kappa(\mathbb{R}^d)$ for any $\lambda > m$ and

$$\|R_\lambda u\|_{\mathcal{C}_b^\kappa(\mathbb{R}^d)} \leq \|u\|_\infty \left(\frac{1}{\lambda - m} + \frac{1}{1-\beta}\right)(M+1).$$

Proof (i) By the contraction property of $(P_t)_{t\geq 0}$, $\|P_t u\|_{\mathcal{C}_b^\kappa(\mathbb{R}^d)} \leq \|P_{t/2}u\|_{\mathcal{C}_b^\kappa(\mathbb{R}^d)}$ for all $t \geq 0$, and so

$$\|P_t u\|_{\mathcal{C}_b^\kappa(\mathbb{R}^d)} \leq M \left(\frac{t}{2}\right)^{-\beta} = M 2^\beta t^{-\beta} \quad \text{for all } t \in (0, 2T).$$

Iterating the procedure, it follows easily that (41) holds.

(ii) Let $u \in \mathcal{B}_b(\mathbb{R}^d)$ be such that (41) holds for some $\beta < 1$. If we choose $K > \kappa$, then (41) gives that the iterated difference operator Δ_h^K (cf. (5)) satisfies

$$|\Delta_h^K P_t u(x)| \leq M \mathrm{e}^{mt} t^{-\beta} \|u\|_\infty |h|^\kappa$$

for any $x \in \mathbb{R}^d$ and $|h| \leq 1$. Since, by the linearity of the integral,

$$\Delta_h^K R_\lambda u(x) = \int_{(0,\infty)} \mathrm{e}^{-\lambda t} \Delta_h^K P_t u(x)\, \mathrm{d}t,$$

we find that

$$|\Delta_h^K R_\lambda u(x)| \leq M |h|^\kappa \|u\|_\infty \int_{(0,\infty)} \mathrm{e}^{-t(\lambda - m)} t^{-\beta}\, \mathrm{d}t.$$

On the other hand, we have $\|R_\lambda u\|_\infty \leq \lambda^{-1} \|u\|_\infty$, and therefore we get for all $\lambda > m$

$$\|R_\lambda u\|_{\mathcal{C}_b^\kappa(\mathbb{R}^d)} \leq \lambda^{-1} \|u\|_\infty + M \|u\|_\infty \left(\int_0^1 t^{-\beta}\, \mathrm{d}t + \int_1^\infty \mathrm{e}^{-t(\lambda - m)}\, \mathrm{d}t \right),$$

which proves the assertion. □

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1 By Lemma 5.1(i), (41) holds with $m := \log(2)\beta/T$ for any $u \in \mathcal{B}_b(\mathbb{R}^d)$. If we set $\lambda := 2m$ and $u := \lambda f - A_e f$ for $f \in F_1$, then $f = R_\lambda u$. Applying Lemma 5.1(ii), we find that

$$\|f\|_{\mathcal{C}_b^\kappa(\mathbb{R}^d)} = \|R_\lambda u\|_{\mathcal{C}_b^\kappa(\mathbb{R}^d)} \leq K \|u\|_\infty \leq \lambda K \|f\|_\infty + K \|A_e f\|_\infty$$

for $K := 2m^{-1} + (1-\beta)^{-1}$. □

For the proof of Theorem 3.2, we need two auxiliary results.

Lemma 5.2 *Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be Feller processes with infinitesimal generator $(A, \mathcal{D}(A))$ and $(L, \mathcal{D}(L))$, respectively, such that*

$$Af(z) = -q(z, D) f(z) \quad \textit{and} \quad Lf(z) = -p(z, D) f(z) \quad \textit{for all } f \in C_c^\infty(\mathbb{R}^d),\ z \in \mathbb{R}^d$$

(cf. (7)) and assume that the $(A, C_c^\infty(\mathbb{R}^d))$-martingale problem is well-posed. Let $U \subseteq \mathbb{R}^d$ be an open set such that

$$p(z,\xi) = q(z,\xi) \quad \textit{for all } z \in U,\ \xi \in \mathbb{R}^d.$$

If $x \in U$ and $r > 0$ are such that $\overline{B(x,r)} \subseteq U$, then for the stopping times

$$\tau^X := \inf\{t > 0; |X_t - x| > r\} \quad \tau^Y := \inf\{t > 0; |Y_t - x| > r\} \tag{42}$$

the random variables $X_{t\wedge\tau^X}$ and $Y_{t\wedge\tau^Y}$ are equal in distribution with respect to $\mathbb{P}^x$ for any $t \geq 0$.[1]

Proof Set

$$\sigma^X := \inf\{t > 0; X_t \notin U \text{ or } X_{t-} \notin U\}, \quad \sigma^Y := \inf\{t > 0; Y_t \notin U \text{ or } Y_{t-} \notin U\}.$$

It follows from the well-posedness of the $(A, C_c^\infty(\mathbb{R}^d))$-martingale problem that the local martingale problem for U is well-posed (cf. [15, Theorem 4.6.1] or [18] for details). On the other hand, Dynkin's formula shows that both $(X_{t\wedge\sigma^X})_{t\geq 0}$ and $(Y_{t\wedge\sigma^Y})_{t\geq 0}$ are solutions to the local martingale problem, and therefore $(X_{t\wedge\sigma^X})_{t\geq 0}$ equals in distribution $(Y_{t\wedge\sigma^Y})_{t\geq 0}$ with respect to $\mathbb{P}^x$ for any $x \in U$. If $x \in U$ and $r > 0$ are such that $\overline{B(x,r)} \subseteq U$, then it follows from the definition of τ^X and τ^Y that $\tau^X \leq \sigma^X$ and $\tau^Y \leq \sigma_U^Y$; in particular,

$$X_{t\wedge\tau^X} = X_{t\wedge\tau^X\wedge\sigma^X} \quad \text{and} \quad Y_{t\wedge\tau^Y} = Y_{t\wedge\tau^Y\wedge\sigma^Y}.$$

Approximating τ^X and τ^Y from above by sequences of discrete-valued stopping times, we conclude from $(X_{t\wedge\sigma^X})_{t\geq 0} \overset{d}{=} (Y_{t\wedge\sigma^Y})_{t\geq 0}$ that $X_{t\wedge\tau^X} \overset{d}{=} Y_{t\wedge\tau^Y}$. □

Lemma 5.3 *Let $(Y_t)_{t\geq 0}$ be a Feller process with infinitesimal generator $(A, \mathcal{D}(A))$ and symbol*

$$p(x,\xi) = -ib(x)\cdot\xi + \int_{y\neq 0} \left(1 - e^{iy\cdot\xi} + iy\cdot\xi \mathbb{1}_{(0,1)}(|y|)\right) \nu(x, dy), \quad x, \xi \in \mathbb{R}^d.$$

If $\alpha > 1$ and $U \in \mathcal{B}(\mathbb{R}^d)$ are such that

$$\sup_{z\in U}\left(|b(z)| + \int_{y\neq 0} \min\{1, |y|^\alpha\}\, \nu(z, dy)\right) < \infty,$$

[1] Here and below we are a bit sloppy in our notation. The Feller processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ each come with a family of probability measures, i.e. their semigroups are of the form $\int f(X_t)\, \mathbb{P}^x(dy)$ and $\int f(Y_t)\, \tilde{\mathbb{P}}^x(dy)$, respectively, for families of probability measures $(\mathbb{P}^x)_{x\in\mathbb{R}^d}$ and $(\tilde{\mathbb{P}}^x)_{x\in\mathbb{R}^d}$. To keep the notation simple, we will not distinguish these two families. Formally written, the assertion of Lemma 3.5 reads $\mathbb{P}^x(X_{t\wedge\tau^X} \in \cdot) = \tilde{\mathbb{P}}^x(Y_{t\wedge\tau^Y} \in \cdot)$.

then there exists an absolute constant $c > 0$ *such that the stopped process* $(Y_{t\wedge\tau_U})_{t\geq 0}$, *where*

$$\tau_U := \inf\{t \geq 0;\, Y_t \notin U\},$$

satisfies

$$\mathbb{E}^x(|Y_{t\wedge\tau_U} - x|^\alpha \wedge 1) \leq ct \sup_{z\in U}\left(|b(z)| + \int_{y\neq 0} \min\{1, |y|^\alpha\}\, \nu(z, \mathrm{d}y)\right) \tag{43}$$

for all $x \in U$, $t \geq 0$.

Note that (43) implies, by Jensen's inequality, that the moment estimate

$$\mathbb{E}^x(|Y_{t\wedge\tau_U} - x|^\beta \wedge 1) \leq c' t^{\beta/\alpha} \sup_{z\in U}\left(|b(z)| + \int_{y\neq 0} \min\{1, |y|^\alpha\}\, \nu(z, \mathrm{d}y)\right)^{\beta/\alpha} \tag{44}$$

holds for any $\beta \in [0, \alpha]$, $x \in U$ and $t \geq 0$. If $(Y_t)_{t\geq 0}$ has a compensated drift, in the sense that $b(z) = \int_{|y|<1} y\, \nu(z, \mathrm{d}y)$ for all $z \in U$, then Lemma 5.3 holds also for $\alpha \in (0, 1]$. Let us mention that estimates for fractional moments of Feller processes were studied in [21]; it is, however, not immediate how Lemma 5.3 can be derived from the results in [21].

Proof of Lemma 5.3 Let $(f_k)_{k\in\mathbb{N}} \subseteq \mathcal{C}_b^\alpha(\mathbb{R}^d) \cap C_c(\mathbb{R}^d)$ be a sequence such that $f_k \geq 0$, $f_k(z) = \min\{1, |z|^\alpha\}$ for $|z| \leq k$ and $M := \sup_k \|f_k\|_{\mathcal{C}_b^\alpha} < \infty$. Pick $\chi \in C_c^\infty(\mathbb{R}^d)$, $\chi \geq 0$, such that $\int_{\mathbb{R}^d} \chi(x)\,\mathrm{d}x = 1$ and set $\chi_\varepsilon(z) := \varepsilon^{-1}\chi(\varepsilon^{-1}z)$. If we define for fixed $x \in U$

$$f_{k,\varepsilon}(z) := (f_k(\cdot - x) * \chi_\varepsilon)(z) := \int_{\mathbb{R}^d} f_k(z - x - y)\chi_\varepsilon(y)\,\mathrm{d}y, \quad z \in \mathbb{R}^d,$$

then $f_{k,\varepsilon} \to f_k(\cdot - x)$ uniformly as $\varepsilon \to 0$ and $\|f_{k,\varepsilon}\|_{\mathcal{C}_b^\alpha(\mathbb{R}^d)} \leq M$ for all $k \in \mathbb{N}$. As $f_{k,\varepsilon} \in C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A)$, an application of Dynkin's formula shows that

$$\mathbb{E}^x f_{k,\varepsilon}(Y_{t\wedge\tau_U}) - f_{k,\varepsilon}(x) = \mathbb{E}^x\left(\int_{(0,t\wedge\tau_U)} Af_{k,\varepsilon}(Y_s)\,\mathrm{d}s\right)$$

for all $t \geq 0$. Since $\alpha > 1$, there exists an absolute constant $C > 0$ such that

$$|\nabla f_{k,\varepsilon}(z)| \leq C\|f_{k,\varepsilon}\|_{\mathcal{C}_b^\alpha(\mathbb{R}^d)} \leq CM$$

and

$$\left|f_{k,\varepsilon}(z + y) - f_{k,\varepsilon}(z) - y\cdot\nabla f_{k,\varepsilon}(z)\mathbb{1}_{(0,1)}(|y|)\right| \leq C\|f_{k,\varepsilon}\|_{\mathcal{C}_b^\alpha(\mathbb{R}^d)} \min\{1, |y|^\alpha\}$$

for all $z \in \mathbb{R}^d$. This implies

$$\begin{aligned}|Af_{k,\varepsilon}(z)| &\le |b(z)|\,|\nabla f_{k,\varepsilon}(z)| \\ &\quad + \int_{y\neq 0} \left| f_{k,\varepsilon}(z+y) - f_{k,\varepsilon}(z) - y\cdot \nabla f_{k,\varepsilon}(z)\mathbb{1}_{(0,1)}(|y|)\right| \nu(z,\mathrm{d}y) \\ &\le CM\left(|b(z)| + \int_{y\neq 0} \min\{1,|y|^\alpha\}\,\nu(z,\mathrm{d}y)\right)\end{aligned}$$

for any $z \in U$. Hence,

$$\mathbb{E}^x f_{k,\varepsilon}(Y_{t\wedge\tau_U}) \le f_{k,\varepsilon}(x) + 2CMt \sup_{z\in U}\left(|b(z)| + \int_{y\neq 0}\min\{1,|y|^\alpha\}\,\nu(z,\mathrm{d}y)\right)$$

for $x \in U$. Applying Fatou's lemma twice, we conclude that

$$\begin{aligned}\mathbb{E}^x \min\{1,|Y_{t\wedge\tau_U} - x|^\alpha\} &\le \liminf_{k\to\infty}\liminf_{\varepsilon\to 0} \mathbb{E}^x f_{k,\varepsilon}(Y_{t\wedge\tau_U}) \\ &\le 2CMt\sup_{z\in U}\left(|b(z)| + \int_{y\neq 0}\min\{1,|y|^\alpha\}\,\nu(z,\mathrm{d}y)\right).\end{aligned}$$

□

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2 Since $x \in \mathbb{R}^d$ is fixed throughout this proof, we will omit the superscript x in the notation which we used in the statement of Theorem 3.2, e.g. we will write $(Y_t)_{t\ge 0}$ instead of $(Y_t^{(x)})_{t\ge 0}$, L instead of $L^{(x)}$ etc.

Denote by $(L_e, \mathcal{D}(L_e))$ the extended generator of $(Y_t)_{t\ge 0}$, and fix a truncation function $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $\mathbb{1}_{\overline{B(x,\delta)}} \le \chi \le \mathbb{1}_{\overline{B(x,2\delta)}}$ and $\|\chi\|_{C_b^2(\mathbb{R}^d)} \le 10\delta^{-2}$. To prove the assertion, it suffices by (C3) and Proposition 3.1 to show that $v := f\cdot\chi$ is in $\mathcal{D}(L_e)$ and

$$\|L_e v\|_\infty \le C\left(\|A_e f\|_\infty + \|f\|_\infty + \|f\|_{\mathcal{C}_b^{\varrho(x)}(\overline{B(x,4\delta)})}\right) \tag{45}$$

for a suitable constant $C > 0$. The first- and main- step is to estimate

$$\sup_{t\in(0,1)} \frac{1}{t} \sup_{z\in\mathbb{R}^d} |\mathbb{E}^z v(Y_{t\wedge\tau_\delta^z}) - v(z)| \tag{46}$$

for the stopping time

$$\tau_\delta^z := \inf\{t > 0;\, |Y_t - z| > \delta\}.$$

We consider separately the cases $z \in \overline{B(x,3\delta)}$ and $z \in \mathbb{R}^d\backslash\overline{B(x,3\delta)}$. For fixed $z \in \mathbb{R}^d\backslash B(x,3\delta)$ it follows from $\operatorname{supp}\chi \subseteq \overline{B(x,2\delta)}$ that $v = 0$ on $\overline{B(z,\delta)}$, and so

$$v(Y_{t\wedge\tau_\delta^z}(\omega)) - v(z) = 0 \quad \text{for all } \omega \in \{\tau_\delta^z > t\}.$$

Hence,

$$|\mathbb{E}^z v(Y_{t\wedge\tau_\delta^z}) - v(z)| \leq 2\|v\|_\infty \mathbb{P}^z(\tau_\delta^z \leq t).$$

Applying the maximal inequality (9) for Feller processes, we find that there exists an absolute constant $c_1 > 0$ such that

$$|\mathbb{E}^z v(Y_{t\wedge\tau_\delta^z}) - v(z)| \leq c_1 t \|f\|_\infty \sup_{y\in\mathbb{R}^d} \sup_{|\xi|\leq\delta^{-1}} |p(y,\xi)|$$

for all $z \in \mathbb{R}^d \backslash B(x, 3\delta)$; the right-hand side is finite since p has, by assumption, bounded coefficients.

For $z \in B(x, 3\delta)$ we write

$$|\mathbb{E}^z v(Y_{t\wedge\tau_\delta^z}) - v(z)| \leq I_1 + I_2 + I_3$$

for

$$\begin{aligned} I_1 &:= |\chi(z)\mathbb{E}^z(f(Y_{t\wedge\tau_\delta^z}) - f(z))|, \\ I_2 &:= |f(z)\mathbb{E}^z(\chi(Y_{t\wedge\tau_\delta^z}) - \chi(z))|, \\ I_3 &:= \left|\mathbb{E}^z\left[\big(f(Y_{t\wedge\tau_\delta^z}) - f(z)\big)\big(\chi(Y_{t\wedge\tau_\delta^z}) - \chi(z)\big)\right]\right|. \end{aligned}$$

We estimate the terms separately. By (17) and (C2), it follows from Lemma 5.2 that

$$\mathbb{E}^z f(X_{t\wedge\tau_\delta^z(X)}) = \mathbb{E}^z f(Y_{t\wedge\tau_\delta^z}) \quad \text{for all } t \geq 0,$$

where $\tau_\delta^z(X)$ is the exit time of $(X_t)_{t\geq0}$ from $\overline{B(z,\delta)}$. As $0 \leq \chi \leq 1$ we thus find

$$I_1 \leq |\mathbb{E}^z(f(X_{t\wedge\tau_\delta^z(X)}) - f(z))|.$$

Since $f \in F_1^X$, an application of Dynkin's formula (11) shows that

$$I_1 \leq \|A_e f\|_\infty \mathbb{E}^z(t \wedge \tau_\delta^z(X)) \leq \|A_e f\|_\infty t.$$

We turn to I_2. As $\chi \in C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(L)$ we find from the (classical) Dynkin formula that

$$\begin{aligned} |I_2| \leq \|f\|_\infty |\mathbb{E}^z(\chi(Y_{t\wedge\tau_\delta^z}) - \chi(z))| &= \|f\|_\infty \left|\mathbb{E}^z\left(\int_{(0,t\wedge\tau_\delta^z)} L\chi(Y_s)\,\mathrm{d}s\right)\right| \\ &\leq t\|f\|_\infty \sup_{|z-x|\leq 4\delta} |L\chi(z)|. \end{aligned}$$

A straightforward application of Taylor's formula shows that

$$|L\chi(z)| \leq 2\|\chi\|_{C_b^2(\mathbb{R}^d)} \left(|b(z)| + \int_{y\neq 0} \min\{1, |y|^2\} \, \nu(z, \mathrm{d}y) \right).$$

Since $0 \leq \varrho(x) \leq 1$ and χ is chosen such that $\|\chi\|_{C_b^2(\mathbb{R}^d)} \leq 10\delta^{-2}$, we thus get

$$I_2 \leq 20\delta^{-2} t \|f\|_\infty \sup_{|z-x|\leq 4\delta} \left(|b(z)| + \int_{y\neq 0} \min\{1, |y|^{1+\varrho(x)}\} \, \nu(z, \mathrm{d}y) \right).$$

It remains to estimate I_3. Because of the assumptions on the Hölder regularity of f on $\overline{B(x, 4\delta)}$, we have

$$I_3 \leq 16\delta^{-2} (\|f\|_{\mathcal{C}_b^{\varrho(x)}(\overline{B(x,4\delta)})} + \|f\|_\infty) \|\chi\|_{C_b^1(\mathbb{R}^d)} \mathbb{E}^z (|Y_{t\wedge\tau_\delta^z} - z|^{1+\varrho(x)} \wedge 1).$$

It follows from Lemma 5.3 that there exists an absolute constant $c_2 > 0$ such that I_3 is bounded above by

$$c_2 \delta^{-4} t \left(\|f\|_{\mathcal{C}_b^{\varrho(x)}(\overline{B(x,4\delta)})} + \|f\|_\infty \right) \sup_{|z-x|\leq 4\delta} \left(|b(z)| + \int_{y\neq 0} \min\{|y|^{\varrho(x)+1}, 1\} \, \nu(z, \mathrm{d}y) \right).$$

Combining the estimates and applying Corollary 2.2, we find that $v = \chi \cdot f \in \mathcal{D}(L_e)$ and

$$\|L_e v\|_\infty \leq C' \left(\|A_\varepsilon f\|_\infty + \|f\|_\infty + \|f\|_{\mathcal{C}_b^{\varrho(x)}(\overline{B(x,4\delta)})} \right),$$

where

$$\begin{aligned} C' := {} & c_3 \sup_{z\in\mathbb{R}^d} \sup_{|\xi|\leq\delta^{-1}} |p(z,\xi)| \\ & + c_3 \delta^{-4} \sup_{|z-x|\leq 4\delta} \left(|b(z)| + \int_{y\neq 0} \min\{|y|^{1+\varrho(x)}, 1\} \, \nu(z, \mathrm{d}y) \right) \end{aligned}$$

for some absolute constant $c_3 > 0$. Since there exists an absolute constant $c_4 > 0$ such that

$$\sup_{z\in\mathbb{R}^d} \sup_{|\xi|\leq\delta^{-1}} |p(z,\delta)| \leq c_4 \sup_{z\in\mathbb{R}^d} \left(|b(z)| + \int_{y\neq 0} \min\{1, |y|^2\} \, \nu(z, \mathrm{d}y) \right) \delta^{-2}$$

for $\delta \in (0, 1)$ (cf. [49, Lemma 6.2] and [6, Theorem 2.31]), we obtain, in particular, that

$$\|L_e v\|_\infty \leq C'' \left(\|A_\varepsilon f\|_\infty + \|f\|_\infty + \|f\|_{\mathcal{C}_b^{\varrho(x)}(\overline{B(x,4\delta)})} \right)$$

for

$$C'' := c_5\delta^{-4} \sup_{z\in\mathbb{R}^d}\left(|b(z)| + \int_{y\neq 0} \min\{1,|y|^2\}\,\nu(z,\mathrm{d}y)\right) + c_5\delta^{-4} \sup_{|z-x|\leq 4\delta} \int_{|y|\leq 1} \min\{|y|^{1+\varrho(x)},1\}\,\nu(z,\mathrm{d}y).$$

This finishes the proof of (45). The continuous dependence of the constant $C > 0$ in (19) on the parameters $\beta(x) \in [0,1)$, $M(x) \in [0,\infty)$, $K(x) \in [0,\infty)$ follows from the fact that each of the constants in this proof depends continuously on these parameters; see also Lemma 5.1. □

The remaining part of this section is devoted to the proof of Theorem 3.5. We need the following auxiliary result.

Lemma 5.4 *Let $(Y_t)_{t\geq 0}$ be a Feller process with infinitesimal generator $(L,\mathcal{D}(L))$, symbol p and characteristics $(b(x), Q(x), \nu(x,\mathrm{d}y))$. For $x \in \mathbb{R}^d$ and $r > 0$ denote by*

$$\tau_r^x = \inf\{t > 0;\, |Y_t - x| > r\}$$

the exit time from the closed ball $\overline{B(x,r)}$. For any fixed $x \in \mathbb{R}^d$ and $r > 0$, the family of measures

$$\mu_t(x,B) := \frac{1}{t}\mathbb{P}^x(Y_{t\wedge\tau_r^x} - x \in B), \quad t > 0,\; B \in \mathcal{B}(\mathbb{R}^d\backslash\{0\}),$$

converges vaguely to $\nu(x,\mathrm{d}y)$, i. e.

$$\lim_{t\to 0}\frac{1}{t}\mathbb{E}^x f(Y_{t\wedge\tau_r^x} - x) = \int_{y\neq 0} f(y)\,\nu(x,\mathrm{d}y) \quad \textit{for all } f \in C_c(\mathbb{R}^d\backslash\{0\}).$$

The main ingredient for the proof of Lemma 5.4 is [30, Theorem 4.2], which states that the family of measures $p_t(x,B) := t^{-1}\mathbb{P}^x(Y_t - x \in B)$, $t > 0$, converges vaguely to $\nu(x,\mathrm{d}y)$ as $t \to 0$.

Proof of Lemma 5.4 By the Portmanteau theorem, it suffices to show that

$$\limsup_{t\to 0} \mu_t(x,K) \leq \nu(x,K) \tag{47}$$

for any compact set $K \subseteq \mathbb{R}^d\backslash\{0\}$. For given $K \subseteq \mathbb{R}^d\backslash\{0\}$ compact, there exists by Urysohn's lemma a sequence $(\chi_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^d)$ and a constant $\delta > 0$ such that $\operatorname{supp}\chi_n \subseteq B(0,\delta)^c$ for all $n \in \mathbb{N}$ and $\mathbb{1}_K = \inf_{n\in\mathbb{N}}\chi_n$. It follows from [30, Theorem 4.2] that

$$\lim_{t\to 0}\frac{\mathbb{E}^x\chi_n(Y_t - x)}{t} = \int_{y\neq 0}\chi_n(y)\,\nu(x,\mathrm{d}y)$$

for all $n \in \mathbb{N}$. On the other hand, an application of Dynkin's formula yields that

$$|\mathbb{E}^x \chi_n(Y_{t\wedge\tau_r^x} - x) - \mathbb{E}^x \chi_n(Y_t - x)| \leq \|L\chi_n\|_\infty \mathbb{E}^x(t - \min\{t, \tau_r^x\}) \\ \leq t\|L\chi_n\|_\infty \mathbb{P}^x(\tau_r^x \leq t).$$

Since $(Y_t)_{t\geq 0}$ has right-continuous sample paths, we have $\mathbb{P}^x(\tau_r^x \leq t) \to 0$ as $t \to 0$, and therefore we obtain that

$$\lim_{t\to 0} \frac{\mathbb{E}^x \chi_n(Y_{t\wedge\tau_r^x} - x)}{t} = \int_{y\neq 0} \chi_n(y)\, \nu(x, dy).$$

Hence,

$$\limsup_{t\to 0} \mu_t(x, K) \leq \limsup_{t\to 0} \frac{1}{t}\mathbb{E}^x \chi_n(Y_{t\wedge\tau_r^x} - x) = \int_{y\neq 0} \chi_n(y)\, \nu(x, dy).$$

As $\mathbb{1}_K = \inf_{n\in\mathbb{N}} \chi_n$, the monotone convergence theorem gives (47). □

Proof of Theorem 3.5 For fixed $x \in \mathbb{R}^d$ let $(Y_t^{(x)})_{t\geq 0}$ be the Feller process from Theorem 3.5. Let $\chi_0 \in C_c^\infty(\mathbb{R}^d)$ be a truncation function such that $\mathbb{1}_{\overline{B(0,\delta)}} \leq \chi_0 \leq \mathbb{1}_{\overline{B(0,2\delta)}}$, and set $\chi^{(x)}(z) := \chi_0(z - x)$, $z \in \mathbb{R}^d$. Since $x \in \mathbb{R}^d$ is fixed throughout Step 1–3 of this proof, we will often omit the superscript x in our notation, i.e. we will write $(Y_t)_{t\geq 0}$ instead of $(Y_t^{(x)})_{t\geq 0}$, $\chi(z)$ instead of $\chi^{(x)}(z)$, etc.

Step 1 Show that $v := \chi \cdot f$ is in the domain $\mathcal{D}(L_e)$ of the extended generator of $(Y_t)_{t\geq 0}$ and determine $L_e(v)$.

First of all, we note that $(X_t)_{t\geq 0}$, $(Y_t)_{t\geq 0}$ and f satisfy the assumptions of Theorem 3.2. Since we have seen in the proof of Theorem 3.2 that $v = \chi \cdot f$ is in the Favard space F_1^Y of order 1 associated with $(Y_t)_{t\geq 0}$, it follows that $v \in \mathcal{D}(L_e)$ and $\|L_e(v)\|_\infty < \infty$. Applying Corollary 2.2, we find that

$$L_e v(z) = \lim_{t\to 0} \frac{\mathbb{E}^z v(Y_{t\wedge\tau_\delta^z}) - v(z)}{t}$$

(up to a set of potential zero), where

$$\tau_\delta^z := \inf\{t > 0; |Y_t - z| > \delta\}.$$

On the other hand, the proof of Theorem 3.2 shows that

$$\frac{\mathbb{E}^z v(Y_{t\wedge\tau_\delta^z}) - v(z)}{t} = I_1(t) + I_2(t) + I_3(t),$$

where

$$I_1(t) := t^{-1} f(z)(\mathbb{E}^z \chi(Y_{t\wedge\tau_\delta^z}) - \chi(z)),$$
$$I_2(t) := t^{-1} \chi(z)(\mathbb{E}^z f(X_{t\wedge\tau_\delta^z(X)}) - f(z)),$$
$$I_3(t) := t^{-1} \mathbb{E}^z\big[(f(Y_{t\wedge\tau_\delta^z}) - f(z))(\chi(Y_{t\wedge\tau_\delta^z}) - \chi(z))\big];$$

here $\tau_\delta^z(X)$ denotes the exit time of $(X_t)_{t\geq 0}$ from $\overline{B(z,\delta)}$. Since $\chi \in C_c^\infty(\mathbb{R}^d)$ is in the domain of the (strong) infinitesimal generator L of $(Y_t)_{t\geq 0}$ and f is the Favard space F_1^X associated with $(X_t)_{t\geq 0}$, another application of Corollary 2.2 shows that

$$\lim_{t\to 0} I_1(t) = f(z) L\chi(z) \quad \text{and} \quad \lim_{t\to 0} I_2(t) = \chi(z) A_e f(z)$$

for all $z \in \mathbb{R}^d$. We claim that

$$\lim_{t\to 0} I_3(t) = \Gamma(f,\chi)(z) := \int_{y\neq 0} (f(z+y) - f(z))(\chi(z+y) - \chi(z))\, \nu(z, dy) \tag{48}$$

for all $z \in \mathbb{R}^d$, where $\nu(z, dy) = \nu^{(x)}(z, dy)$ denotes the family of Lévy measures associated with $(Y_t)_{t\geq 0} = (Y_t^{(x)})_{t\geq 0}$ (cf. (22)). Once we have shown this, it follows that

$$L_e v = fL\chi + \chi A_e f + \Gamma(f,\chi). \tag{49}$$

To prove (48), we fix a function $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that $\mathbb{1}_{B(0,1)} \leq \varphi \leq \mathbb{1}_{B(0,2)}$ and set $\varphi_\varepsilon(y) := \varphi(\varepsilon^{-1} y)$ for $\varepsilon > 0$, $y \in \mathbb{R}^d$. Since $y \mapsto (1 - \varphi_\varepsilon(y))$ is zero in a neighbourhood of 0, we find from Lemma 5.4 that

$$\frac{\mathbb{E}^z\big[(1 - \varphi_\varepsilon(Y_{t\wedge\tau_\delta^z} - z))(f(Y_{t\wedge\tau_\delta^z}) - f(z))(\chi(Y_{t\wedge\tau_\delta^z}) - \chi(z))\big]}{t}$$
$$\xrightarrow{t\to 0} \int_{y\neq 0} (1 - \varphi_\varepsilon(y))(f(y+z) - f(z))(\chi(z+y) - \chi(z))\, \nu(z, dy).$$

If $z \in \mathbb{R}^d \backslash B(x, 3\delta)$, then $\chi = 0$ on $B(z,\delta)$, and therefore the integrand on the right hand side equals zero for $|y| < \delta$. By dominated convergence, the right-hand side converges to $\Gamma(f,\chi)(z)$, defined in (48), as $\varepsilon \to 0$. For $z \in B(x, 3\delta)$ we note that $\chi \in C_b^1(\mathbb{R}^d)$ and $f \in \mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)$ for ϱ satisfying (26); it now follows from (S1) and dominated convergence that the right-hand side converges to $\Gamma(f,\chi)(z)$ as $\varepsilon \to 0$. To prove (48), it remains to show that

$$J(\varepsilon, t, z) := \Big|\mathbb{E}^z\big[\varphi_\varepsilon(Y_{t\wedge\tau_\delta^z} - z)(f(Y_{t\wedge\tau_\delta^z}) - f(z))(\chi(Y_{t\wedge\tau_\delta^z}) - \chi(z))\big]\Big|$$

satisfies

$$\limsup_{\varepsilon\to 0}\limsup_{t\to 0}\frac{1}{t}J(\varepsilon,t,z)=0 \quad \text{for all } z\in\mathbb{R}^d.$$

By (26) and (S1), there exists some constant $\gamma>0$ such that

$$1+\min\{\varrho(z),1\}\geq \alpha(z)+2\gamma \quad \text{for all } z\in\overline{B(x,3\delta)}. \tag{50}$$

Indeed: On $\{\varrho\geq 1\}$ this inequality holds since α is bounded away from 2 (cf. (S1)), and on $\{\varrho<1\}$ this is a direct consequence of (26). Now fix some $z\in\overline{B(x,3\delta)}$. As $\operatorname{supp}\varphi_\varepsilon\subseteq\overline{B(0,2\varepsilon)}$, it follows from $f\in\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)$ and $\chi\in C_b^1(\mathbb{R}^d)$ that

$$J(\varepsilon,t,z)\leq c_1\varepsilon^\gamma\|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)}\|\chi\|_{C_b^1(\mathbb{R}^d)}\mathbb{E}^z\min\{|Y_{t\wedge\tau_\delta^z}-z|^{\alpha(z)+\gamma},1\}$$

with γ from (50) and some constant $c_1>0$ (not depending on f,x,z). An application of Lemma 5.3 now yields

$$J(\varepsilon,t,z)\leq c_2\varepsilon^\gamma t\sup_{|z-x|\leq 4\delta}\left(|b(z)|+\int_{y\neq 0}\min\{|y|^{\alpha(z)+\gamma},1\}\nu(z,dy)\right),$$

which is finite because of (S1) and (S5). Hence,

$$\limsup_{t\to 0}\limsup_{\varepsilon\to 0}\frac{1}{t}J(\varepsilon,t,z)=0 \quad \text{for all } |z-x|\leq 3\delta.$$

If $z\in\mathbb{R}^d\backslash B(x,3\delta)$, then it follows from $\chi|_{B(z,\delta)}=0$ and $\operatorname{supp}\varphi\subseteq B(0,2\varepsilon)$ that

$$J(\varepsilon,t,z)\leq 4\varepsilon\|f\|_\infty\|\chi\|_{C_b^1(\mathbb{R}^d)}\mathbb{P}^z(\tau_\delta^z\leq t).$$

Applying the maximal inequality (9) for Feller processes, we conclude that

$$\limsup_{\varepsilon\to 0}\limsup_{t\to 0}t^{-1}J(\varepsilon,t,z)=0 \quad \text{for all } z\in\mathbb{R}^d\backslash B(x,3\delta).$$

Step 2 If $\varrho:\mathbb{R}^d\to[0,2]$ is a uniformly continuous function satisfying (26) and $\varrho_0:=\inf_z\varrho(z)>0$, then

$$\begin{aligned}&f\in F_1^X\cap\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d),\ A_e f=g\in\mathcal{C}_b^\lambda(\mathbb{R}^d)\\ &\quad\Longrightarrow \forall\varepsilon>0:\ L_e(f\chi)\in\mathcal{C}_b^{(\varrho_0\wedge\lambda\wedge\theta\wedge\sigma)-\varepsilon}(\mathbb{R}^d)\end{aligned}$$

for any $\lambda\in[0,\Lambda]$ where $\chi=\chi^{(x)}$ is the truncation function chosen at the beginning of the proof; see (S2), (S3) and (26) for the definition of θ, Λ and σ.

Indeed: We know from Step 1 that

$$L_e(f\chi) = fL\chi + \chi A_e f + \Gamma(f,\chi) =: I_1 + I_2 + I_3.$$

As $\theta \leq 1$ we have $\varrho_0 \wedge \lambda \wedge \theta \wedge \sigma \leq 1$, and therefore it suffices to estimate

$$\sup_{z\in\mathbb{R}^d} |I_k(z)| + \sup_{z,h\in\mathbb{R}^d} |I_k(z+h) - I_k(z)|$$

for $k = 1, 2, 3$.

Estimate of $I_1 = fL\chi$: First we estimate the Hölder norm of $L\chi$. As $\chi \in C_c^\infty(\mathbb{R}^d)$ a straight-forward application of Taylor's formula shows that

$$\|L\chi\|_\infty \leq 2\|\chi\|_{C_b^2(\mathbb{R}^d)} \sup_{z\in\mathbb{R}^d} \left(|b(z)| + \int_{y\neq 0} \min\{1,|y|^2\}\, \nu(z,\mathrm{d}y) \right).$$

If we set $D_y\chi(z) := \chi(z+y) - \chi(z) - \chi'(z)y\mathbb{1}_{(0,1)}(|y|)$, then

$$\begin{aligned}
&|L\chi(z) - L\chi(z+h)| \\
&\quad \leq |b(z)|\,|\nabla\chi(z+h) - \nabla\chi(z)| + |b(z+h) - b(z)|\,|\nabla\chi(z+h)| \\
&\qquad + \int_{y\neq 0} |D_y\chi(z+h) - D_y\chi(z)|\,\nu(z,\mathrm{d}y) \\
&\qquad + \left| \int_{y\neq 0} D_y\chi(z+h)\,(\nu(z+h,\mathrm{d}y) - \nu(z,\mathrm{d}y)) \right|
\end{aligned}$$

for all $z, h \in \mathbb{R}^d$. To estimate the first two terms on the right-hand side we use the Hölder continuity of b (cf. (S2)) and the fact that $\chi \in C_b^2(\mathbb{R}^d)$. For the third term, we use

$$|D_y\chi(z+h) - D_y\chi(z)| \leq \|\chi\|_{C_b^3(\mathbb{R}^d)} |h| \min\{|y|^2, 1\};$$

cf. [4, Theorem 5.1] for details, and noting that

$$|D_y\chi(z+h)| \leq 2\|\chi\|_{C_b^2(\mathbb{R}^d)} \min\{1,|y|^2\}$$

we can estimate the fourth term for small h by applying (S2). Hence,

$$\begin{aligned}
|L\chi(z) - L\chi(z+h)| \leq |h|\|\chi\|_{C_b^3(\mathbb{R}^d)} \left(|b(z)| + \int_{y\neq 0} \min\{1,|y|^2\}\,\nu(z,\mathrm{d}y) \right) \\
+2C|h|^\theta \|\chi\|_{C_b^2(\mathbb{R}^d)}
\end{aligned}$$

for small $h > 0$. Hence,

$$\|L\chi\|_{\mathcal{C}_b^\theta(\mathbb{R}^d)} \leq c_1 \|\chi\|_{C_b^3(\mathbb{R}^d)} \sup_{z\in\mathbb{R}^d} \left(1 + |b(z)| + \int_{y\neq 0} \min\{1,|y|^2\}\,\nu(z,\mathrm{d}y) \right)$$

for some absolute constant $c_1 > 0$. Since $f \in \mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d) \subseteq \mathcal{C}_b^{\varrho_0}(\mathbb{R}^d)$, this entails that

$$\begin{aligned}&\|fL\chi\|_{\mathcal{C}_b^{\theta\wedge\varrho_0}(\mathbb{R}^d)}\\ &\quad\leq c_1'\|f\|_{\mathcal{C}_b^{\varrho_0}(\mathbb{R}^d)}\|\chi\|_{C_b^3(\mathbb{R}^d)}\sup_{z\in\mathbb{R}^d}\left(1+|b(z)|+\int_{y\neq 0}\min\{1,|y|^2\}\,\nu(z,\mathrm{d}y)\right).\end{aligned}$$

Estimate of $I_2 = \chi A_e f$: By assumption, $A_e f = g \in \mathcal{C}_b^{\lambda}(\mathbb{R}^d)$ and $\chi \in C_c^\infty(\mathbb{R}^d)$. Thus,

$$\|\chi A_e f\|_{\mathcal{C}_b^\lambda} \leq 2\|\chi\|_{C_b^\lambda}\|A_e f\|_{\mathcal{C}_b^\lambda} < \infty.$$

Estimate of $I_3 = \Gamma(f,\chi)$: As $f \in C_b^{\varrho(\cdot)}(\mathbb{R}^d)$ and $\chi \in C_b^1(\mathbb{R}^d)$, it follows from the definition of $\Gamma(f,\chi)$ (cf. (48)) that

$$\begin{aligned}&|\Gamma(f,\chi)(z)|\\ &\quad\leq 4\|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)}\|\chi\|_{C_b^1(\mathbb{R}^d)}\int_{y\neq 0}\min\{|y|^{1+\min\{1,\varrho(z)\}}\wedge 1,1\}\,\nu(z,\mathrm{d}y) < \infty\end{aligned}$$

for all $|z-x| \leq 3\delta$. If $z \in \mathbb{R}^d\backslash B(x,3\delta)$, then $\Delta_y\chi(z) = 0$ for all $|y| \leq \delta$, and so

$$|\Gamma(f,\chi)(z)| \leq 4\|f\|_\infty\int_{|y|>\delta/2}\nu(z,\mathrm{d}y)$$

for all $z \in \mathbb{R}^d\backslash B(x,3\delta)$. Combining both estimates and using (26), (S1) and (S5), we get

$$\|\Gamma(f,\chi)\|_\infty \leq c_2\|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)}$$

for some constant $c_2 > 0$ not depending on x, z and f. To study the regularity of $\Gamma(f,\chi)$ we consider separately the cases $\|\varrho\|_\infty \leq 1$ and $\|\varrho\|_\infty > 1$. We start with the case $\|\varrho\|_\infty \leq 1$; see the end of this step for the other case. To estimate $\Delta_h\Gamma(f,\chi)$, we note that

$$|\Delta_h\Gamma(f,\chi)(z)| = |\Gamma(f,\chi)(z+h) - \Gamma(f,\chi)(z)| \leq J_1 + J_2 + J_3, \tag{51}$$

where

$$\begin{aligned}J_1(z) &:= \int_{y\neq 0}|\Delta_y f(z+h) - \Delta_y f(z)|\,|\Delta_y\chi(z+h)|\,\nu(z,\mathrm{d}y),\\ J_2(z) &:= \int_{y\neq 0}|\Delta_y f(z)|\,|\Delta_y\chi(z+h) - \Delta_y\chi(z)|\,\nu(z,\mathrm{d}y),\\ J_3(z) &:= \left|\int_{y\neq 0}\Delta_y f(z+h)\Delta_y\chi(z+h)(\nu(z,\mathrm{d}y) - \nu(z+h,\mathrm{d}y))\right|.\end{aligned}$$

We estimate the terms separately, and start with J_1. Fix $\varepsilon \in (0, \min\{\varrho_0, \sigma\}/2)$ (cf. (26) for the definition of σ). Since ϱ is uniformly continuous, there exists $r \in (0,1)$ such that

$$|\varrho(z) - \varrho(z+h)| \leq \varepsilon \quad \text{for all } z \in \mathbb{R}^d,\ |h| \leq r.$$

For $|h| \leq r$ and $|y| \leq r$ it then follows from $f \in \mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)$ that

$$\begin{aligned} |\Delta_y f(z+h) - \Delta_y f(z)| &\leq 2\|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)} \min\{|y|^{\varrho(z)\wedge\varrho(z+h)}, |h|^{\varrho(y+z)\wedge\varrho(z)}\} \\ &\leq 2\|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)} \min\{|y|^{\varrho(z)-\varepsilon}, |h|^{\varrho(z)-\varepsilon}\}. \end{aligned}$$

(Here we use $\|\varrho\|_\infty \leq 1$; otherwise we would need to replace $\varrho(z)$ by $\varrho(z) \wedge 1$ etc.) On the other hand, we also have

$$|\Delta_y f(z+h) - \Delta_y f(z)| \leq 2\|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)} |h|^{\varrho_0} \tag{52}$$

for all $y \in \mathbb{R}^d$. Combining both estimates yields

$$\begin{aligned} J_1(z) &\leq 2\|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)} \|\chi\|_{C_b^1(\mathbb{R}^d)} \\ &\quad \cdot \left(\int_{|y|\leq r} \min\{|y|^{\varrho(z)-\varepsilon}, |h|^{\varrho(z)-\varepsilon}\}|y|\, \nu(z, dy) + |h|^{\varrho_0} \int_{|y|>r} \nu(z, dy) \right) \end{aligned}$$

for $|h| \leq r$. It is now not difficult to see from (S1) and (S5) that there exists a constant $c_3 > 0$ (not depending on x, z, f) such that

$$J_1(z) \leq c_3 \|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)} (|h|^{\varrho_0} + |h|^{\varrho(z)+1-\alpha(z)-\varepsilon}) \quad \text{for all } |h| \leq r, z \in B(x, 3\delta).$$

By the very definition of σ (cf. (26)), this implies that

$$\sup_{z\in B(x,3\delta)} J_1(z) \leq c_3 \|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)} |h|^{\min\{\varrho_0,\sigma\}-\varepsilon} \quad \text{for all } |h| \leq r.$$

If $z \in \mathbb{R}^d \backslash B(x, 3\delta)$, then $\Delta_y \chi(z+h) = 0$ for $|h| \leq \delta/2$ and $|y| \leq \delta/2$. Using (52), we get

$$J_1(z) \leq 2|h|^{\varrho_0} \|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)} \int_{|y|\geq\delta/2} \nu(z, dy) \quad \text{for all } |h| \leq \delta/2.$$

Invoking once more (S1) and (S5), we obtain that

$$\sup_{z\in\mathbb{R}^d\backslash B(x,3\delta)} J_1(z) \leq c_4 |h|^{\varrho_0} \|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)}, \quad |h| \leq \delta/2,$$

for some constant c_4 not depending on x, z and f. In summary, we have shown that

$$\sup_{z\in\mathbb{R}^d} J_1(z) \le c_5 |h|^{\min\{\varrho_0,\sigma\}-\varepsilon} \|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)}.$$

To estimate J_2, consider again separately the cases $z \in B(x, 3\delta)$ and $z \in \mathbb{R}^d \backslash B(x, 3\delta)$. If $z \in \mathbb{R}^d \backslash B(x, 3\delta)$, then $\Delta_y \chi(z+h) = 0 = \Delta_y \chi(z)$ for all $|y| \le \delta/2$ and $|h| \le \delta/2$. Since we also have

$$|\Delta_y \chi(z+h) - \Delta_y \chi(z)| \le 2\|\chi\|_{C_b^2(\mathbb{R}^d)} \min\{|y|, |h|\}, \tag{53}$$

we find that

$$J_2(z) \le 4\|f\|_\infty \|\chi\|_{C_b^2(\mathbb{R}^d)} |h| \int_{|y|\ge\delta/2} \nu(z, dy)$$

for $|h| \le \delta/2$. Because of (S1) and (S5), this gives the existence of a constant $c_6 > 0$ (not depending on f, x and z) such that

$$\sup_{z\in\mathbb{R}^d\backslash B(x,3\delta)} J_2(z) \le c_6 \|f\|_\infty |h|.$$

For $z \in B(x, 3\delta)$, we combine

$$|\Delta_y f(z)| \le 2\|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)} \min\{|y|^{\varrho(z)}, 1\}$$

with (53) to get

$$J_2(z) \le 4\|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)} \|\chi\|_{C_b^2(\mathbb{R}^d)} \int_{y\neq 0} \min\{|y|^{\varrho(z)}, 1\} \min\{|y|, |h|\}\, \nu(z, dy),$$

which implies, by (S1), (S5) and (26), that

$$\sup_{z\in B(x,3\delta)} J_2 \le c_7 \|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)} |h|^{\sigma\wedge 1}.$$

We conclude that

$$\sup_{z\in\mathbb{R}^d} J_2(z) \le c_8 |h|^{\sigma\wedge 1} \|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)}.$$

It remains to estimate J_3. By the uniform continuity of ϱ there exists $r \in (0, 1)$ such that $|\Delta_h \varrho(z)| \le \sigma/2$ for all $|h| \le r$. Since $f \in \mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)$ we have

$$|\Delta_y f(z+h) \Delta_y \chi(z+h)| \le 4\|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)} \|\chi\|_{C_b^1(\mathbb{R}^d)} \min\{|y|^{\varrho(z+h)+1}, 1\},$$

and thus, by (26) and our choice of $r \in (0, 1)$,

$$\begin{aligned}|\Delta_y f(z+h)\Delta_y \chi(z+h)| &\leq 4\|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)} \|\chi\|_{C_b^1(\mathbb{R}^d)} \min\{|y|^{\varrho(z)+1-\sigma/2}, 1\} \\ &\leq 4\|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)} \|\chi\|_{C_b^1(\mathbb{R}^d)} \min\{|y|^{\sigma/2+\alpha(z)}, 1\}\end{aligned}$$

for all $|z-x| \leq 3\delta$ and $|h| \leq r$. On the other hand, if $z \in \mathbb{R}^d \backslash \overline{B(x, 3\delta)}$, then $\chi = 0$ on $\overline{B(z,\delta)}$ and so

$$|\Delta_y f(z+h)\Delta_y \chi(z+h)| = 0 \quad \text{for all } |h| \leq \delta/2,\ |y| \leq \delta/2.$$

Consequently, there exists a constant $c_9 = c_9(\delta, r) > 0$ such that

$$|\Delta_y f(z+h)\Delta_y \chi(z+h)| \leq c_9 \|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)} \|\chi\|_{C_b^1(\mathbb{R}^d)} \min\{|y|^{\sigma+\alpha(z)}, 1\}$$

for all $z \in \mathbb{R}^d$, $y \in \mathbb{R}^d$ and $|h| \leq \min\{r, \delta\}/2$. Applying (S2), we thus find

$$\sup_{z \in \mathbb{R}^d} J_3(z) \leq c_{10}|h|^\theta \|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)}.$$

Combining the above estimates, we conclude that

$$\|\Gamma(f,\chi)\|_{\mathcal{C}_b^{\varrho_0 \wedge \theta \wedge \sigma - \varepsilon}(\mathbb{R}^d)} \leq c_{11} \|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)},$$

provided that $\|\varrho\|_\infty \leq 1$. In the other case, i.e. if ϱ takes values strictly larger than one, then we need to consider second differences $\Delta_h^2 \Gamma(f,\chi)(z)$ in order to capture the full information on the regularity of f. The calculations are very similar to the above ones but quite lengthy (it is necessary to consider nine terms separately) and so we do not present the details here.

Conclusion of Step 2 For any small $\varepsilon > 0$ there exists a finite constant $K_{1,\varepsilon} > 0$ such that

$$\|L_e(f\chi)\|_{\mathcal{C}_b^{\min\{\varrho_0,\lambda,\theta,\sigma\}-\varepsilon}(\mathbb{R}^d)} \leq K_{1,\varepsilon}\left(\|A_e f\|_{\mathcal{C}_b^{\lambda}(\mathbb{R}^d)} + \|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)}\right). \tag{54}$$

The constant $K_{1,\varepsilon}$ does not depend on x, z and f.

Step 3 If $u \in \mathcal{D}(L_e)$ is such that $u \in \mathcal{C}_b^\lambda(\mathbb{R}^d)$ and $L_e u \in \mathcal{C}_b^\lambda(\mathbb{R}^d)$ for some $\lambda \leq \Lambda$ (cf. (S3)), then

$$\|u\|_{\mathcal{C}_b^{\kappa(x)+\lambda}(\mathbb{R}^d)} \leq K_2(\|u\|_{\mathcal{C}_b^\lambda(\mathbb{R}^d)} + \|L_e u\|_{\mathcal{C}_b^\lambda(\mathbb{R}^d)})$$

for some constant $K_2 > 0$ which does not depend on x, z and f. (Recall that $L_e = L_{\mathrm{e}}^{(x)}$ is the extended generator of the Feller process $(Y_t)_{t\geq 0} = (Y_t^{(x)})_{t \geq 0}$; this explains the x-dependence of the regularity on the left-hand side of the inequality.)

Indeed The μ-potential operators $(R_\mu)_{\mu>0}$ associated with $(Y_t)_{t\geq 0} = (Y_t^{(x)})_{t\geq 0}$ satisfies

$$\|R_\mu v\|_{\mathcal{C}_b^{\kappa(x)+\lambda}(\mathbb{R}^d)} \leq K\|v\|_{\mathcal{C}_b^{\lambda}(\mathbb{R}^d)}, \quad v \in \mathcal{C}_b^{\lambda}(\mathbb{R}^d),\ \lambda \leq \Lambda \tag{55}$$

for μ sufficiently large and some constant $K = K(\mu) > 0$. This is a direct consequence of (S3) and Lemma 5.1. Now if $u \in \mathcal{D}(L_e)$ is such that $u \in \mathcal{C}_b^{\lambda}(\mathbb{R}^d)$ and $L_e u \in \mathcal{C}_b^{\lambda}(\mathbb{R}^d)$, then we have $u = R_\mu v$ for $v := \mu u - L_e u \in \mathcal{C}_b^{\lambda}(\mathbb{R}^d)$. Applying (55) proves the desired estimate.

Conclusion of the proof Let $f \in \mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d) \cap F_1^X$ for ϱ satisfying (26) be such that $A_e f \in \mathcal{C}_b^{\lambda}(\mathbb{R}^d)$ for some $\lambda \leq \Lambda$. Without loss of generality, we may assume that $\varrho_0 := \inf_x \varrho(x) > 0$. *Indeed*: It follows from Corollary 3.4 that $f \in \mathcal{C}_b^{\kappa(\cdot)-\varepsilon}(\mathbb{R}^d)$ for $\varepsilon := \kappa_0/2 := \inf_x \kappa(x)/2 > 0$, and therefore we may replace ϱ by

$$\tilde{\varrho}(z) := \max\{\varrho(z), \kappa(z) - \varepsilon\},$$

which is clearly bounded away from zero and satisfies the assumptions of Theorem 3.5.

For fixed $x \in \mathbb{R}^d$, denote by $\chi = \chi^{(x)}$ the truncation function chosen at the beginning of the proof, and fix $\varepsilon \in (0, \min\{\varrho_0, \kappa_0\}/2)$. It follows from Step 2 and Step 3 that there exists a constant $c_1 > 0$ such that

$$\|f\chi^{(x)}\|_{\mathcal{C}_b^{\kappa(x)+\min\{\varrho_0,\sigma,\theta,\lambda\}-\varepsilon}(\mathbb{R}^d)} \leq c_1 \left(\|A_e f\|_{\mathcal{C}_b^{\lambda}(\mathbb{R}^d)} + \|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)}\right)$$

for all $x \in \mathbb{R}^d$. As $\chi^{(x)} = 1$ on $B(x, \delta)$, we obtain that

$$\|f\|_{\mathcal{C}_b^{\kappa(\cdot)+\min\{\varrho_0,\sigma,\theta,\lambda\}-\varepsilon}(\mathbb{R}^d)} \leq c_1' \left(\|A_e f\|_{\mathcal{C}_b^{\lambda}(\mathbb{R}^d)} + \|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)}\right).$$

Since, by assumption, $f \in \mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)$, this implies $f \in \mathcal{C}_b^{\varrho^1(\cdot)}(\mathbb{R}^d)$ for

$$\varrho^1(x) := \max\{\varrho(x), \kappa(x) - \varepsilon + \min\{\varrho_0, \sigma, \theta, \lambda\}\}, \quad x \in \mathbb{R}^d,$$

and we have

$$\|f\|_{\mathcal{C}_b^{\varrho^1(\cdot)}(\mathbb{R}^d)} \leq (c_1' + 1) \left(\|A_e f\|_{\mathcal{C}_b^{\lambda}(\mathbb{R}^d)} + \|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)}\right).$$

As ϱ^1 satisfies (26) (with ϱ replaced by ϱ^1), we each apply Step 2 with ϱ replaced by ϱ^1 to obtain

$$\|f\chi^{(x)}\|_{\mathcal{C}_b^{\kappa(x)+\min\{\varrho_0^1,\sigma,\theta,\lambda\}-\varepsilon}(\mathbb{R}^d)} \leq c_2 \left(\|A_e f\|_{\mathcal{C}_b^{\lambda}(\mathbb{R}^d)} + \|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)}\right),$$

where $\varrho_0^1 := \inf_{x\in\mathbb{R}^d} \varrho^1(x)$. Repeating the argument, i.e. using that $\chi^{(x)} = 1$ on $B(x, \delta)$, we obtain $f \in \mathcal{C}_b^{\varrho^2(\cdot)}(\mathbb{R}^d)$ for

$$\varrho^2(x) := \max\{\varrho(x), \kappa(x) - \varepsilon + \min\{\varrho_0^1, \sigma, \theta, \lambda\}\}$$

and

$$\|f\|_{\mathcal{C}_b^{\varrho^2(\cdot)}(\mathbb{R}^d)} \leq c_2' \left(\|A_e f\|_{\mathcal{C}_b^{\lambda}(\mathbb{R}^d)} + \|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)} \right).$$

We proceed iteratively, i.e. we set

$$\varrho^n(x) := \max\{\varrho(x), \kappa(x) - \varepsilon + \min\{\varrho_0^{n-1}, \sigma, \theta, \lambda\}\}, n \geq 2,$$

where $\varrho_0^{n-1} := \inf_x \varrho^{n-1}(x)$. By Steps 2 and 3, we then have

$$\|f\|_{\mathcal{C}_b^{\varrho^n(\cdot)}(\mathbb{R}^d)} \leq c_n \left(\|A_e f\|_{\mathcal{C}_b^{\lambda}(\mathbb{R}^d)} + \|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)} \right) \tag{56}$$

for some constant $c_n > 0$. Since $\kappa_0 = \inf_x \kappa(x) > 0$ and $\varepsilon < \kappa_0/2$, it is not difficult to see that we can choose $n \in \mathbb{N}$ sufficiently large such that $\varrho_0^n \geq \min\{\sigma, \theta, \lambda\}$, and so

$$\varrho^{n+1}(x) \geq \kappa(x) - \varepsilon + \min\{\sigma, \theta, \lambda\}.$$

Using (56) (with n replaced by $n+1$), we conclude that

$$\|f\|_{\mathcal{C}_b^{\kappa(\cdot)+\min\{\sigma,\theta,\lambda\}-\varepsilon}(\mathbb{R}^d)} \leq c_{n+1} \left(\|A_e f\|_{\mathcal{C}_b^{\lambda}(\mathbb{R}^d)} + \|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)} \right).$$

□

6 Proof of Schauder Estimates for Isotropic Stable-Like Processes

In this section we present the proof of the Schauder estimates for isotropic stable-like processes which we stated in Theorem 4.1 and Corollary 4.3. Throughout this section, $(X_t)_{t\geq 0}$ is an isotropic stable-like process, i. e. a Feller process with symbol of the form $q(x,\xi) = |\xi|^{\alpha(x)}$, $x, \xi \in \mathbb{R}^d$, for a mapping $\alpha : \mathbb{R}^d \to (0, 2]$. We remind the reader that such a Feller process exists if α is Hölder continuous and bounded away from zero.

We will apply the results from Sect. 3 to establish the Schauder estimates. To this end, we need regularity estimates for the semigroup $(P_t)_{t\geq 0}$ associated with $(X_t)_{t\geq 0}$. The results, which we obtain, are of independent interest and we present them in Sect. 6.1. Once we have established another auxiliary statement in Sect. 6.2, we will present the proof of Theorem 4.1 and Corollary 4.3 in Sect. 6.3.

6.1 Regularity Estimates for the Semigroup of Stable-Like Processes

Let $(P_t)_{t\geq 0}$ be the semigroup of an isotropic stable-like process $(X_t)_{t\geq 0}$ with symbol $q(x,\xi) = |\xi|^{\alpha(x)}$. In this subsection, we study the regularity of the mapping $x \mapsto P_t u(x)$. We will see that there are several parameters which influence the regularity of $P_t u$:

- the regularity of $x \mapsto u(x)$,
- the regularity of $x \mapsto \alpha(x)$,
- $\alpha_L := \inf_{x\in\mathbb{R}^d} \alpha(x)$;

the larger these quantities are, the higher the regularity of $P_t u$. The regularity estimates we present rely on the parametrix construction of (the transition density of) $(X_t)_{t\geq 0}$ in [22]. We mention that there are other approaches to obtain regularity estimates for the semigroup. Using coupling methods, Luo and Wang [40] showed that for any $\kappa \in (0, \alpha_L)$ there exists $c > 0$ such that

$$\|P_t u\|_{\mathcal{C}_b^{\kappa\wedge 1}(\mathbb{R}^d)} \leq c\|u\|_\infty t^{-(\kappa\wedge 1)/\alpha_L} \quad \text{for all } u \in \mathcal{B}_b(\mathbb{R}^d),\ t \in (0, T].$$

For $\alpha_L > 1$, this estimate is not good enough for our purpose; we need a higher regularity of $P_t u$.

Proposition 6.1 *Let $(X_t)_{t\geq 0}$ be a Feller process with symbol $q(x, \xi) = |\xi|^{\alpha(x)}$, $x, \xi \in \mathbb{R}^d$, for a mapping $\alpha : \mathbb{R}^d \to (0, 2)$ bounded away from zero, i.e. $\alpha_L := \inf_{x\in\mathbb{R}^d} \alpha(x) > 0$, and γ-Hölder continuous for $\gamma \in (0, 1)$. For any $T > 0$ and $\kappa \in (0, \alpha_L)$ there exists a constant $C > 0$ such that the semigroup $(P_t)_{t\geq 0}$ satisfies*

$$\|P_t u\|_{\mathcal{C}_b^{\kappa}(\mathbb{R}^d)} \leq C\|u\|_\infty t^{-\kappa/\alpha_L} \quad \textit{for all } u \in \mathcal{B}_b(\mathbb{R}^d),\ t \in (0, T]. \tag{57}$$

In particular, $(P_t)_{t\geq 0}$ has the strong Feller property. The constant $C > 0$ depends continuously on $\alpha_L \in (0, 2)$, $\alpha_L - \kappa \in (0, \alpha_L)$, $\|\alpha\|_{\mathcal{C}_b^{\gamma}(\mathbb{R}^d)} \in [0, \infty)$ and $T \in [0, \infty)$.

For the proof of Proposition 6.1, we use a representation for the transition density p which was obtained in [22] using a parametrix construction; see also [25]. For $\varrho \in (0, 2)$, denote by $p^{\varrho}(t, x)$ the transition density of an isotropic ϱ-stable Lévy process and set

$$p_0(t, x, y) := p^{\alpha(y)}(t, x - y), \quad t > 0,\ x, y \in \mathbb{R}^d.$$

The transition density p of $(X_t)_{t\geq 0}$ has the representation

$$p(t, x, y) = p_0(t, x, y) + (p_0 \circledast \Phi)(t, x, y), \quad t > 0,\ x, y \in \mathbb{R}^d, \tag{58}$$

where $\circledast$ is the time-space convolution and Φ is a suitable function satisfying

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |\Phi(t, x, y)|\, dy \leq C_1 t^{-1+\lambda}, \quad t \in (0, T), \tag{59}$$

for some constant $\lambda > 0$ and $C_1 = C_1(T) > 0$. For further details, we refer the reader to "Appendix B" where we collect the material from [22] which we need in this article.

Proof of Proposition 6.1 Fix $T > 0$, $u \in \mathcal{B}_b(\mathbb{R}^d)$ and $\kappa \in (0, \alpha_L)$. By contractivity $\|P_t u\|_\infty \leq \|u\|_\infty$, it suffices to show that the iterated differences of order 2 (cf. (5)) satisfy

$$\sup_{x\in\mathbb{R}^d} |\Delta_h^2 P_t u(x)| \leq C t^{-\kappa/\alpha_L} \|u\|_\infty \quad \text{for all } t \in (0, T],\ |h| \leq 1.$$

By (58),

$$|\Delta_h^2 P_t u(x)| \leq |\Delta_h^2 P_t^{(0)} u(x)| + |\Delta_h^2 P_t^{(1)} u(x)|$$

for any $x, h \in \mathbb{R}^d$ and $t \in (0, T]$, where

$$P_t^{(0)} u(z) := \int_{\mathbb{R}^d} u(y) p_0(t, z, y) \, dy \quad \text{and} \quad P_t^{(1)} u(z) := \int_{\mathbb{R}^d} u(y) (p_0 \circledast \Phi)(t, z, y) \, dy.$$

We estimate the terms separately; we start with $P^{(0)}$. The transition density $p^{\varrho}(t, x)$ of an isotropic ϱ-stable Lévy process is twice differentiable, and there exists a constant $c_1 > 0$ such that

$$\begin{aligned} |p^{\varrho}(t, x)| &\leq c_1 S(x, \varrho, t), \\ |\partial_{x_i} p^{\varrho}(t, x)| &\leq c_1 t^{-1/\varrho} S(x, \varrho, t), \\ |\partial_{x_i} \partial_{x_j} p^{\varrho}(t, x)| &\leq c_1 t^{-2/\varrho} S(x, \varrho, t), \end{aligned} \tag{60}$$

where

$$S(x, \varrho, t) := \min \left\{ t^{-d/\alpha}, \frac{t}{|x|^{d+\alpha}} \right\}, \tag{61}$$

and $\varrho \in [\alpha_L, \|\alpha\|_\infty]$, $t \in (0, T)$, $x \in \mathbb{R}^d$ and $i, j \in \{1, \ldots, d\}$ (cf. Lemma B.1). For the parametrix $p_0(t, x, y) = p^{\alpha(y)}(t, x - y)$ this implies, by Taylor's formula, that there exists is $c_2 > 0$ such that

$$\begin{aligned} &|p_0(t, x + 2h, y) - 2p_0(t, x + h, y) + p_0(t, x, y)| \\ &\quad \leq c_2 t^{-2/\alpha(y)} |h|^2 S(\eta(x, h) - y, \alpha(y), t), \quad x, h \in \mathbb{R}^d \end{aligned}$$

for some intermediate value $\eta(x, h) \in B(x, 2h)$. As $t \leq T$, we find that

$$\begin{aligned} &|p_0(t, x + 2h, y) - 2p_0(t, x + h, y) + p_0(t, x, y)| \\ &\quad \leq c_3 t^{-2/\alpha_L} |h|^2 S(\eta(x, h) - y, \alpha(y), t), \quad x, h \in \mathbb{R}^d \end{aligned}$$

for a suitable constant $c_3 = c_3(T, \alpha_L, \|\alpha\|_\infty)$. On the other hand, (60) gives

$$\begin{aligned} &|p_0(t, x + 2h, y) - 2p_0(t, x + h, y) + p_0(t, x, y)| \\ &\quad \leq c_1 (S(x + 2h - y, \alpha(y), t) + 2S(x + h - y, \alpha(y), t) + S(x - y, \alpha(y), t)). \end{aligned}$$

Combining both estimates, we obtain that there exists a constant $c_4 = c_4(T, \alpha_L, \|\alpha\|_\infty)$ such that

$$|p_0(t, x + 2h, y) - 2p_0(t, x + h, y) + p_0(t, x, y)| \leq c_4 |h|^\kappa t^{-\kappa/\alpha_L} U(t, x, y, h) \tag{62}$$

for

$$U(t,x,y,h) := S(\eta(x,h)-y,\alpha(y),t) + S(x+h-y,\alpha(y),t) + S(x-h-y,\alpha(y),t) + S(x-y,\alpha(y),t);$$

cf. Lemma C.1 with $r := t^{1/\alpha_L}$. Hence,

$$\begin{aligned} &|P_t^{(0)}u(x+2h) - 2P_t^{(0)}u(x+h) + P_t^{(0)}u(x)| \\ &\leq c_4\|u\|_\infty t^{-\kappa/\alpha_L}|h|^\kappa \int_{\mathbb{R}^d} U(t,x,y,h)\,\mathrm{d}y \end{aligned}$$

for any $x,h \in \mathbb{R}^d$ and $t \in (0,T)$. Since

$$c_T := \sup_{t\in(0,T)} \sup_{z\in\mathbb{R}^d} \int_{\mathbb{R}^d} S(z-y,\alpha(y),t)\,\mathrm{d}y < \infty, \tag{63}$$

(cf. Appendix B), we have

$$\sup_{t\in(0,T)} \sup_{z\in\mathbb{R}^d} \int_{\mathbb{R}^d} U(t,z,y,h)\,\mathrm{d}y \leq 4c_T < \infty, \tag{64}$$

and so we conclude that

$$|P_t^{(0)}u(x+2h) - 2P_t^{(0)}u(x+h) + P_t^{(0)}u(x)| \leq 4c_4c_T\|u\|_\infty t^{-\kappa/\alpha_L}|h|^\kappa.$$

It remains to establish the Hölder estimate for $P_t^{(1)}$. By (62),

$$\begin{aligned} &|(p_0 \circledast \Phi)(t,x+2h,y) - 2(p_0 \circledast \Phi)(t,x+h,y) + (p_0 \circledast \Phi)(t,x,y)| \\ &\leq c_4|h|^\kappa \int_0^t \int_{\mathbb{R}^d} (t-s)^{-\kappa/\alpha_L} U(t-s,x,z,h)|\Phi(s,z,y)|\,\mathrm{d}z\,\mathrm{d}s. \end{aligned}$$

Integrating with respect to $y \in \mathbb{R}^d$, it follows from (59) and (64) that

$$\begin{aligned} &|P_t^{(1)}u(x+2h) - 2P_t^{(1)}u(x+h) + P_t^{(1)}u(x)| \\ &\leq c_6|h|^\kappa\|u\|_\infty \int_0^t (t-s)^{-\kappa/\alpha_L} s^{-1+\lambda}\,\mathrm{d}s \\ &\leq c_7|h|^\kappa t^{-\kappa/\alpha_L}\|u\|_\infty \end{aligned}$$

for suitable constants c_6 and c_7. Combining the estimates, (57) holds for some finite constant $C > 0$. The continous dependence of C on the parameters $\alpha_L - \kappa \in (0,\alpha_L)$, $\alpha_L \in (0,2)$, $\|\alpha\|_{C_b^\gamma} > 0$ and $T > 0$ follows from the fact that each of the constants in this proof depends continuously on these parameters. □

In Proposition 6.1, we studied the regularity of $x \mapsto P_t u(x)$ for measurable functions u. The next result is concerned with the regularity of $P_t u(\cdot)$ for Hölder continuous functions u. It is natural to expect that $P_t u$ "inherits" some regularity from u.

Proposition 6.2 *Let $(X_t)_{t\geq 0}$ be a Feller process with symbol $q(x,\xi) = |\xi|^{\alpha(x)}$, $x, \xi \in \mathbb{R}^d$, for a mapping $\alpha : \mathbb{R}^d \to (0,2)$ such that $\alpha_L := \inf_{x\in\mathbb{R}^d} \alpha(x)$ is strictly positive and $\alpha \in C_b^\gamma(\mathbb{R}^d)$ for some $\gamma \in (0,1)$ satisfying*

$$\gamma > \gamma_0 := \|\alpha\|_\infty - \alpha_L.$$

For any $T > 0$, $\kappa \in (0, \alpha_L)$ and $\varepsilon \in (\gamma_0, \min\{\gamma, \alpha_L\})$, there exists a constant $C > 0$ such that the semigroup $(P_t)_{t\geq 0}$ of $(X_t)_{t\geq 0}$ satisfies

$$\|P_t u\|_{\mathcal{C}_b^{\kappa+\min\{\delta,\gamma\}-\varepsilon}(\mathbb{R}^d)} \leq C(1 + |\log t|)t^{-\kappa/\alpha_L} \|u\|_{\mathcal{C}_b^{\min\{\delta,\gamma\}}(\mathbb{R}^d)}, \quad u \in \mathcal{C}_b^\delta(\mathbb{R}^d), \tag{65}$$

for all $\delta > 0$ and $t \in (0, T]$. The constant $C > 0$ depends continuously on $\alpha_L \in (0,2)$, $\kappa - \alpha_L \in (0,2)$, $(\varepsilon - \|\alpha\|_\infty)/\alpha_L \in (1,\infty)$, $\|\alpha\|_{C_b^\gamma(\mathbb{R}^d)} \in [0,\infty)$ and $T \in [0,\infty)$.

For the proof of the Schauder estimates, Corollary 4.3, we will apply Proposition 6.2 for an isotropic stable-like process $(X_t)_{t\geq 0}$ with symbol $q(x,\xi) = |\xi|^{\alpha(x)}$ for a "truncated" function α of the form

$$\alpha(x) := (\varrho(x_0) - \delta) \vee \varrho(x) \wedge (\varrho(x_0) + \delta), \quad x \in \mathbb{R}^d,$$

where $x_0 \in \mathbb{R}^d$ is fixed and $\delta > 0$ is a constant which we can choose as small as we like; in particular $\gamma_0 := \|\alpha\|_\infty - \alpha_L \leq 2\delta$ is small, and so the assumptions $\varepsilon > \gamma_0$ and $\gamma > \gamma_0$ in Proposition 6.2 are not a restriction. Let us mention that both assumptions, i.e. $\varepsilon > \gamma_0$ and $\gamma > \gamma_0$, come into play when estimating one particular term in the proof of Proposition 6.2; see (76); a more careful analysis of this term would probably allow us to relax these two conditions.

Proof of Proposition 6.2 Fix $\varepsilon \in (\gamma_0, \gamma \wedge \alpha_L)$, $\kappa \in (0, \alpha_L)$ and $T > 0$. First of all, we note that it clearly suffices to show (65) for $u \in \mathcal{C}_b^\delta(\mathbb{R}^d)$ with $\delta \leq \gamma \leq 1$. Throughout the first part of this proof, we will assume that

$$\kappa \leq 1. \tag{66}$$

Under (66), the assertion follows if we can show that

$$|\Delta_h^2 P_t u(x)| \leq C\|u\|_{\mathcal{C}_b^\delta(\mathbb{R}^d)}(1 + |\log(t)|)t^{-\kappa/\alpha_L}|h|^{\kappa+\delta-\varepsilon},$$

for all $x \in \mathbb{R}^d$, $|h| \leq 1$ and $t \in (0, T]$, where Δ_h^2 denotes as usual the iterated difference operator (cf. (5)). For the proof of this inequality, we use again the parametrix construction of the transition density p of $(X_t)_{t\geq 0}$,

$$p(t,x,y) = p_0(t,x,y) + (p_0 \circledast \Phi)(t,x,y), \quad t > 0, \text{ut}\, x, y \in \mathbb{R}^d, \tag{67}$$

where

$$p_0(t,x,y) = p^{\alpha(y)}(t, x-y), \quad t>0,\ x,y \in \mathbb{R}^d, \tag{68}$$

see Appendix B for details. Since

$$\begin{aligned}\Delta_h P_t u(x) &= \int_{\mathbb{R}^d} \Delta_h u(y) p(t,x,y)\,\mathrm{d}y \\ &\quad - \int_{\mathbb{R}^d} (u(y+h)p(t,x,y) - u(y)p(t,x+h,y))\,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} \Delta_h u(y) p(t,x,y)\,\mathrm{d}y \\ &\quad - \int_{\mathbb{R}^d} u(y+h)(p(t,x,y) - p(t,x+h,y+h))\,\mathrm{d}y,\end{aligned}$$

we find that $\Delta_h^2 P_t f(x) = J_1 - J_2$, where

$$\begin{aligned}J_1 &:= \int_{\mathbb{R}^d} \Delta_h u(y)\,(p(t,x+h,y) - p(t,x,y))\,\mathrm{d}y, \\ J_2 &:= \int_{\mathbb{R}^d} u(y+h)\Big(p(t,x+h,y) - p(t,x+2h,y+h) \\ &\qquad\qquad - p(t,x,y) + p(t,x+h,y+h)\Big)\,\mathrm{d}y.\end{aligned}$$

We estimate the terms separately. For fixed $h \in \mathbb{R}^d$, $|h| \leq 1$, define an auxiliary function v by $v(y) := \Delta_h u(y)$. Proposition 6.1 gives

$$|J_1| \leq |h|^\kappa \|P_t v\|_{\mathcal{C}_b^\kappa(\mathbb{R}^d)} \leq C_1 |h|^\kappa \|v\|_\infty t^{-\kappa/\alpha_L}, \quad t \in (0,T],$$

and so, by the definition of v and the Hölder continuity of u,

$$|J_1| \leq C_1 |h|^{\kappa+\delta} \|u\|_{\mathcal{C}_b^\delta(\mathbb{R}^d)} t^{-\kappa/\alpha_L}, \quad t \in (0,T].$$

It remains to establish the corresponding estimate for J_2, and to this end we use representation (67) for the transition density p.

Step 1 There exists a constant $c_1 > 0$ such that

$$\begin{aligned}q(t,x,y) := p_0(t,x+h,y) - p_0(t,x+2h,y+h) - p_0(t,x,y) \\ + p_0(t,x+h,y+h)\end{aligned} \tag{69}$$

satisfies

$$\int_{\mathbb{R}^d} |q(t,x,y)|\,\mathrm{d}y \leq c_1 |h|^{\kappa+\gamma}(1+|\log(t)|) t^{-\kappa/\alpha_L} \quad \text{for all } x,h \in \mathbb{R}^d,\ t \in (0,T].$$

Indeed: If we denote by p^ϱ the transition density of the d-dimensional isotropic ϱ-stable Lévy process, $\varrho \in (0,2)$, then there is a constant $c_2 > 0$ such that

$$\int_{\mathbb{R}^d} \left| \frac{\partial}{\partial \varrho} p^\varrho(t,x) \right| \, \mathrm{d}x \le c_2(1+|\log(t)|)$$
$$\int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x_j} \frac{\partial}{\partial \varrho} p^\varrho(t,x) \right| \, \mathrm{d}x \le c_2(1+|\log(t)|) t^{-1/\alpha_L} \tag{70}$$

for all $t \in (0,T]$, $j \in \{1,\ldots,d\}$ and $\varrho \in [\alpha_L, \|\alpha\|_\infty] \subseteq (0,2]$ (cf. Lemma B.1). To shorten the notation, we fix $x, h \in \mathbb{R}^d$ and $t \in (0,T]$, and write $q(y)$ for the function defined in (69). By definition of p_0 (cf. (68)), we have

$$\begin{aligned} |q(y)| = \Big| & p^{\alpha(y)}(t, x+h-y) - p^{\alpha(y+h)}(t, x+h-y) \\ & - p^{\alpha(y)}(t, x-y) + p^{\alpha(y+h)}(t, x-y) \Big|, \end{aligned}$$

and so, by the fundamental theorem of calculus and the mean-value theorem,

$$\begin{aligned} |q(y)| &= \left| \int_{\alpha(y)}^{\alpha(y+h)} \big(\partial_\varrho p^\varrho(t, x+h-y) - \partial_\varrho p^\varrho(t, x-y) \big) \, \mathrm{d}\varrho \right| \\ &\le |h| \int_{\alpha(y)}^{\alpha(y+h)} \left| \nabla_x \partial_\varrho p^\varrho(t, \eta_\varrho(x,h) - y) \right| \, \mathrm{d}\varrho \end{aligned} \tag{71}$$

for some intermediate value $\eta_\varrho(x,h) \in B(x,h)$. Integrating with respect to y and using (70), we obtain that

$$\int_{\mathbb{R}^d} |q(y)| \, \mathrm{d}y \le c_3(1+|\log(t)|) t^{-1/\alpha_L} |h| \sup_{z \in \mathbb{R}^d} \int_{\alpha(z)}^{\alpha(z+h)} \mathrm{d}\varrho \tag{72}$$
$$\le c_3 \|\alpha\|_{\mathcal{C}_b^\gamma(\mathbb{R}^d)} (1+|\log(t)|) t^{-1/\alpha_L} |h|^{1+\gamma}. \tag{73}$$

On the other hand, it follows from (71) and the Hölder continuity of α that

$$\int_{\mathbb{R}^d} |q(y)| \, \mathrm{d}y \le |h|^\gamma \|\alpha\|_{\mathcal{C}_b^\gamma(\mathbb{R}^d)} \sup_{\varrho \in [\alpha_L, \|\alpha\|_\infty]} \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\partial_\varrho p^\varrho(t, \eta - y)| \, \mathrm{d}y.$$

Hence, by (70),

$$\int_{\mathbb{R}^d} |q(y)| \, \mathrm{d}y \le c_4 |h|^\gamma (1+|\log(t)|). \tag{74}$$

Combining (73) and (74), we find that

$$\int_{\mathbb{R}^d} |q(y)| \, \mathrm{d}y \le c_5 |h|^{\kappa+\gamma} (1+|\log(t)|) t^{-\kappa/\alpha_L}, \quad \kappa \in [0, \alpha_L];$$

the reasoning is very similar to the proof of Lemma C.1. Alternatively, we can use an interpolation theorem.

Step 2 There exists a constant $c > 0$ such that

$$|J_2| \leq c|h|^{\kappa+\delta-\varepsilon} \|u\|_{\mathcal{C}_b^\delta(\mathbb{R}^d)} (1+|\log(t)|) t^{-\kappa/\alpha_L} \quad \text{for all } t \in (0, T],\ |h| \leq 1,\ x \in \mathbb{R}^d;$$

recall that $\varepsilon \in (\gamma_0, \alpha_L \wedge \gamma)$ has been fixed at the beginning of the proof.

Indeed: Because of decomposition (67), we have $J_2 = J_{2,1} + J_{2,2}$ for

$$\begin{aligned} J_{2,1} &:= \int_{\mathbb{R}^d} u(y+h) q(t, x, y)\, \mathrm{d}y, \\ J_{2,2} &:= \int_{\mathbb{R}^d} u(y+h) \left((p_0 \circledast \Phi)(t, x+h, y) - (p_0 \circledast \Phi)(t, x+2h, y+h) \right) \mathrm{d}y \\ &\quad + \int_{\mathbb{R}^d} u(y+h) \left((p_0 \circledast \Phi)(t, x+h, y+h) - (p_0 \circledast \Phi)(t, x, y) \right) \mathrm{d}y, \end{aligned}$$

with q defined in (69). It follows from Step 1 that

$$|J_{2,1}| \leq c_1 \|u\|_{\mathcal{C}_b^\delta(\mathbb{R}^d)} (1 + |\log(t)|) t^{-\kappa/\alpha_L} |h|^{\kappa+\delta}, \quad t \in (0, T].$$

It remains to estimate $J_{2,2}$. By the definition of the time-space convolution,

$$\begin{aligned} &(p_0 \circledast \Phi)(t, x+h, y) - (p_0 \circledast \Phi)(t, x+2h, y+h) - (p_0 \circledast \Phi)(t, x, y) \\ &\quad + (p_0 \circledast \Phi)(t, x+h, y+h) \\ &= \int_0^t \int_{\mathbb{R}^d} (p_0(t-s, x+h, z) - p_0(t-s, x, z)) \Phi(s, z, y)\, \mathrm{d}z\, \mathrm{d}s \\ &\quad - \int_0^t \int_{\mathbb{R}^d} (p_0(t-s, x+2h, z) - p_0(t-s, x+h, z)) \Phi(s, z, y+h)\, \mathrm{d}z\, \mathrm{d}s \\ &= \int_0^t \int_{\mathbb{R}^d} q(t-s, x, z) \Phi(s, z, y)\, \mathrm{d}z\, \mathrm{d}s \\ &\quad - \int_0^t \int (p_0(t-s, x+2h, z+h) - p_0(t-s, x+h, z+h)) \\ &\qquad\qquad (\Phi(s, z+h, y+h) - \Phi(s, z, y))\, \mathrm{d}z\, \mathrm{d}s \\ &=: H_1(t, y) - H_2(t, y). \end{aligned}$$

Integrating with respect to y and applying Tonelli's theorem,

$$\begin{aligned} &\left| \int_{\mathbb{R}^d} u(y+h) H_1(t, y)\, \mathrm{d}y \right| \\ &\quad \leq \|u\|_\infty \int_0^t \left(\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\Phi(s, \eta, y)|\, \mathrm{d}y \right) \left(\int_{\mathbb{R}^d} |q(t-s, x, z)|\, \mathrm{d}z \right) \mathrm{d}s. \end{aligned}$$

Thus, by (59) and Step 1,

$$\left|\int_{\mathbb{R}^d} u(y+h)H_1(t,y)\,\mathrm{d}y\right| \\ \leq c_6|h|^{\kappa+\gamma}\|u\|_\infty \int_0^t s^{-1+\lambda_1}(1+|\log(t-s)|)(t-s)^{-\kappa/\alpha_L}\,\mathrm{d}s \tag{75}$$

for a suitable constant $c_6 > 0$ and $\lambda_1 > 0$. It remains to estimate H_2. We claim that there exist constants $c_7 > 0$ and $\lambda_2 > 0$ such that

$$\sup_{z\in\mathbb{R}^d} \int_{\mathbb{R}^d} |\Phi(t,z+h,y+h) - \Phi(t,z,y)|\,\mathrm{d}y \leq c_7|h|^{\gamma-\varepsilon}t^{-1+\lambda_2} \tag{76}$$

for all $t \in (0,T]$ and $|h| \leq 1$; here $\varepsilon \in (\gamma_0, \alpha_L \wedge \gamma)$ is as above. We postpone the proof of (76) to the end of this subsection (see Lemma 6.3). Using (76) and the fact that

$$\int_{\mathbb{R}^d} |p_0(t-s,x+2h,z+h) - p_0(t-s,x+h,z+h)|\,\mathrm{d}z \leq c_8|t-s|^{-\kappa/\alpha_L}|h|^\kappa$$

for some constant $c_8 > 0$ (which follows by a similar reasoning to that in the first part of the proof of Proposition 6.1), we obtain

$$\left|\int_{\mathbb{R}^d} u(y+h)H_2(t,y)\,\mathrm{d}y\right| \leq c_7c_8\|u\|_\infty|h|^{\gamma+\kappa-\varepsilon}\int_0^t s^{-1+\lambda_2}(t-s)^{-\kappa/\alpha_L}\,\mathrm{d}s.$$

Combining this estimate with (75) gives

$$|J_{2,2}| \leq (c_6+c_7c_8)\|u\|_\infty|h|^{\gamma+\kappa-\varepsilon}\int_0^t s^{-1+\lambda}(t-s)^{-\kappa/\alpha_L}(1+|\log(t-s)|)\,\mathrm{d}s.$$

Hence,

$$|J_{2,2}| \leq c_9\|u\|_\infty|h|^{\gamma+\kappa-\varepsilon}t^{-\kappa/\alpha_L}\int_0^1 r^{-1+\lambda}(1-r)^{-\kappa/\alpha_L}(1+|\log(1-r)|)\,dr$$

for all $t \in (0,T]$ where $\lambda := \min\{\lambda_1,\lambda_2\}$. This finishes the proof of Step 2 and hence of Proposition 6.2 for the case $\kappa \leq 1$. If $\kappa > 1$, we need to estimate the iterated differences of third order $\Delta_h^3 P_t u(x)$; the calculations then become more technical and lengthy, but the idea of the proof does not change. We refer the reader to the arXiv version [28] of this paper for full details. □

Lemma 6.3 *Let $(X_t)_{t\geq 0}$ be a Feller process with symbol $q(x,\xi) = |\xi|^{\alpha(x)}$ satisfying the assumptions of Proposition 6.2, and denote by*

$$p(t,x,y) = p_0(t,x,y) + (p_0 \circledast \Phi)(t,x,y)$$

the parametrix representation of the transition density p of $(X_t)_{t\geq 0}$ (cf. Appendix C). For any $T>0$ and any $\varepsilon \in (\gamma_0, \gamma \wedge \alpha_L)$, there exist finite constants $C>0$ and $\lambda>0$ such that

$$\int_{\mathbb{R}^d} |\Phi(t,x+h,y+h)-\Phi(t,x,y)|\,\mathrm{d}y \leq C|h|^{\gamma-\varepsilon}t^{-1+\lambda}$$

for all $x \in \mathbb{R}^d$, $|h|\leq 1$ and $t \in (0,T]$. The constant $C>0$ depends continuously on $\alpha_L \in (0,2)$, $\kappa-\alpha_L \in (0,2)$, $(\varepsilon-\|\alpha\|_\infty)/\alpha_L \in (1,\infty)$, $\|\alpha\|_{C_b^\gamma(\mathbb{R}^d)} \in [0,\infty)$ and $T \in [0,\infty)$. The constant $\lambda>0$ depends continuously on $(\varepsilon-\|\alpha\|_\infty)/\alpha_L \in (1,\infty)$ and $(\gamma-\|\alpha\|_\infty)/\alpha_L \in (1,\infty)$.

Proof Fix $\varepsilon \in (\gamma_0, \alpha_L \wedge \gamma)$. To keep the calculations as simple as possible, we consider $T:=1$. To prove the assertion, we will use that

$$\Phi(t,x,y)=\sum_{i=1}^\infty F^{\circledast i}(t,x,y), \quad t>0,\ x,y \in \mathbb{R}^d, \tag{77}$$

where $F^{\circledast i} := F \circledast F^{\circledast (i-1)}$ denotes the ith convolution power of

$$F(t,x,y) := (2\pi)^{-d} \int_{\mathbb{R}^d} \left(|\xi|^{\alpha(y)}-|\xi|^{\alpha(x)}\right) \mathrm{e}^{i\xi\cdot(y-x)}\mathrm{e}^{-t|\xi|^{\alpha(y)}}\,\mathrm{d}\xi,$$

cf. Appendix C.

Step 1 There exist constants $C>0$ and $\lambda>0$ such that

$$\int_{\mathbb{R}^d} |F(t,x+h,y+h)-F(t,x,y)|\,\mathrm{d}y \leq C|h|^{\gamma-\varepsilon}t^{-1+\lambda} \tag{78}$$

for all $x \in \mathbb{R}^d$, $|h|\leq 1$, $t \in (0,1)$.

Indeed: For fixed $|h|\leq 1$, we write

$$F(t,x+h,y+h)-F(t,x,y)=(2\pi)^{-d}\left(D_1(t,x,y)+D_2(t,x,y)\right)$$

where

$$\begin{aligned} D_1(t,x,y) &:= \int_{\mathbb{R}^d} \left(\left(|\xi|^{\alpha(y+h)}-|\xi|^{\alpha(y)}\right)-\left(|\xi|^{\alpha(x+h)}-|\xi|^{\alpha(x)}\right)\right) \\ &\qquad \mathrm{e}^{i\xi\cdot(y-x)}\mathrm{e}^{-t|\xi|^{\alpha(y)}}\,\mathrm{d}\xi, \\ D_2(t,x,y) &:= \int_{\mathbb{R}^d} \left(|\xi|^{\alpha(y)}-|\xi|^{\alpha(x)}\right)\mathrm{e}^{i\xi\cdot(y-x)}\left(\mathrm{e}^{-t|\xi|^{\alpha(y+h)}}-\mathrm{e}^{-t|\xi|^{\alpha(y)}}\right)\mathrm{d}\xi. \end{aligned}$$

We estimate the terms separately. As $\alpha \in \mathcal{C}_b^\gamma(\mathbb{R}^d)$, it follows that $x \mapsto r^{\alpha(x)} \in \mathcal{C}_b^\gamma(\mathbb{R}^d)$ for any fixed $r \geq 0$ and

$$\|r^{\alpha(\cdot)}\|_{\mathcal{C}_b^\gamma(\mathbb{R}^d)} \leq \left(\|\alpha\|_{\mathcal{C}_b^\gamma(\mathbb{R}^d)} |\log(r)| + 1\right) \max\{r^{\alpha_L}, r^{\|\alpha\|_\infty}\}.$$

By Lemma C.2, there exists a constant $c_1 > 0$ such that

$$\begin{aligned}
&\left|\left(r^{\alpha(y+h)} - r^{\alpha(y)}\right) - \left(r^{\alpha(x+h)} - r^{\alpha(x)}\right)\right| \\
&\quad \leq c_1 |h|^{\gamma-\varepsilon} |x-y|^\varepsilon \|r^{\alpha(\cdot)}\|_{\mathcal{C}_b^\gamma(\mathbb{R}^d)} \\
&\quad \leq c_1' |h|^{\gamma-\varepsilon} |x-y|^\varepsilon (|\log(r)| + 1) \max\{r^{\alpha_L}, r^{\|\alpha\|_\infty}\},
\end{aligned}$$

for all $r \geq 0$, $x, y \in \mathbb{R}^d$ and $|h| \leq 1$. By [22, (proof of) Theorem 4.7], this implies that there is a constant $c_2 > 0$ such that

$$\begin{aligned}
|D_1(t,x,y)| &\leq c_2 |h|^{\gamma-\varepsilon} |x-y|^\varepsilon \\
&\quad \cdot \left\{ (1+|\log(t)|) t^{-(d+\|\alpha\|_\infty)/\alpha_L} \wedge \frac{1+|\log(|x-y|)|}{\min\{|x-y|^{d+\alpha_L}, |x-y|^{d+\|\alpha\|_\infty}\}} \right\},
\end{aligned}$$

for all $x, y \in \mathbb{R}^d$, $t \in (0,1)$ and $|h| \leq 1$. Splitting up the domain of integration into three parts

$$\{y \in \mathbb{R}^d; |x-y| < t^{1/\alpha_L}\} \quad \{y \in \mathbb{R}^d; t^{1/\alpha_L} \leq |x-y| \leq 1\} \quad \{y \in \mathbb{R}^d; |x-y| > 1\}$$

we find that $\int_{\mathbb{R}^d} |D_1(t,x,y)| \, dy$ is bounded by

$$\begin{aligned}
&c_2 |h|^{\gamma-\varepsilon} \Bigg((1+|\log(t)|) t^{-(d+|\alpha\|_\infty-\varepsilon)/\alpha_L} \int_{|z|<t^{1/\alpha_L}} dz \\
&\qquad + \int_{t^{1/\alpha_L} \leq |z| \leq 1} \frac{1+|\log(|z|)|}{|z|^{d+\|\alpha\|_\infty-\varepsilon}} \, dz + \int_{|z|>1} \frac{1+|\log(|z|)|}{|z|^{d+\alpha_L-\varepsilon}} \, dz \Bigg) \\
&\leq c_2' |h|^{\gamma-\varepsilon} (1+|\log(t)|) t^{-(|\alpha\|_\infty-\varepsilon)/\alpha_L}.
\end{aligned}$$

As $\varepsilon > \gamma_0 = \|\alpha\|_\infty - \alpha_L$, there exists $\lambda_1 > 0$ such that

$$\int_{\mathbb{R}^d} |D_1(t,x,y)| \, dy \leq c_3 t^{-1+\lambda_1} |h|^{\gamma-\varepsilon}, \quad t \in (0,1), \; x \in \mathbb{R}^d.$$

To estimate the second term, note that

$$D_2(t,x,y) = -t \int_{\alpha(y)}^{\alpha(y+h)} \int_{\alpha(x)}^{\alpha(y)} \int_{\mathbb{R}^d} (\log(|\xi|))^2 |\xi|^u e^{i\xi\cdot(y-x)} e^{-t|\xi|^\varrho} \, d\xi \, du \, d\varrho.$$

From [22, Theorem 4.7] and the Hölder continuity of α, there exists a constant $c_4 > 0$ such that

$$|D_2(t,x,y)| \leq c_4 t|h|^\gamma |x-y|^\gamma \cdot \left\{(1+|\log(t)|^2)t^{-(d+\|\alpha\|_\infty)/\alpha_L} \wedge \frac{1+|\log(|x-y|)|^2}{\min\{|x-y|^{d+\alpha_L}, \|x-y|^{d+\|\alpha\|_\infty}\}}\right\}.$$

Now we can proceed exactly as in the first part of this step to conclude that

$$\int_{\mathbb{R}^d} |D_2(t,x,y)|\,\mathrm{d}y \leq c_5|h|^\gamma(1+|\log(t)|^2)t^{-(\|\alpha\|_\infty-\gamma)/\alpha_L} \leq c_5'|h|^\gamma t^{-1+\lambda_2}$$

for all $x \in \mathbb{R}^d$, $|h| \leq 1$ and $t \in (0,1)$ and suitable constants $c_5, c_5', \lambda_2 > 0$; for the second estimate, we used that $\gamma > \gamma_0 = \|\alpha\|_\infty - \alpha_L$.

Step 2 For any $\varepsilon \in (\gamma_0, \min\{\gamma, \alpha_L\})$ there exist constants $C > 0$ and $\lambda > 0$ such that

$$\int_{\mathbb{R}^d} |F^{\circledast i}(t,x+h,y+h) - F^{\circledast i}(t,x,y)|\,\mathrm{d}y \leq 2^i C^i \frac{\Gamma(\lambda)^i}{\Gamma(i\lambda)} t^{-1+i\lambda}|h|^{\gamma-\varepsilon} \tag{79}$$

for all $i \in \mathbb{N}$, $x \in \mathbb{R}^d$, $|h| \leq 1$ and $t \in (0,1)$.

Indeed Fix $\epsilon \in (\gamma_0, \min\{\gamma, \alpha\})$. There exist constants $C > 0$ and $\lambda > 0$ such that

$$\int_{\mathbb{R}^d} |F^{\circledast i}(t,x,y)|\,\mathrm{d}y \leq C^i \frac{\Gamma(\lambda)^i}{\Gamma(i\lambda)} t^{-1+i\lambda} \tag{80}$$

for all $x \in \mathbb{R}^d$, $i \geq 1$ and $t \in (0,1)$ (cf. Appendix C). Without loss of generality, we may assume that $C > 0$ and $\lambda > 0$ are such that (78) holds (otherwise increase $C > 0$ and decrease $\lambda > 0$). We claim that (79) holds for this choice of $C > 0$ and $\lambda > 0$, and prove this by induction. For $i = 1$ the estimate is a direct consequence of (78). Now assume that (79) holds for some $i \geq 1$. By the definition of the time-space convolution,

$$\begin{aligned}
&(F \circledast F^{\circledast i})(t,x+h,y+h) \\
&= \int_0^t \int_{\mathbb{R}^d} F(t-s,x+h,z)F^{\circledast i}(s,z,y+h)\,\mathrm{d}z\,\mathrm{d}s \\
&= \int_0^t \int_{\mathbb{R}^d} F(t-s,x+h,z+h)F^{\circledast i}(s,z+h,y+h)\,\mathrm{d}z\,\mathrm{d}s,
\end{aligned}$$

so

$$|(F \circledast F^{\circledast i})(t,x+h,y+h) - (F \circledast F^{\circledast i})(t,x,y)| \leq I_1(t,x,y) + I_2(t,x,y)$$

for

$$I_1(t,x,y) := \int_0^t \int_{\mathbb{R}^d} \Big|(F(t-s,x+h,z+h) \\ - F(t-s,x,z))F^{\circledast i}(s,z+h,y+h)\Big| \, \mathrm{d}z\, \mathrm{d}s,$$

$$I_2(t,x,y) := \int_0^t \int_{\mathbb{R}^d} \Big|(F^{\circledast i}(s,z+h,y+h) - F^{\circledast i}(s,z,y))F(t-s,x,z)\Big| \, \mathrm{d}z\, \mathrm{d}s.$$

From first (80) and then (78),

$$\int_{\mathbb{R}^d} |I_1(t,x,y)| \, \mathrm{d}y \leq C^{i+1} \frac{\Gamma(\lambda)^i}{\Gamma(i\lambda)} |h|^{\gamma-\varepsilon} \int_0^t (t-s)^{-1+\lambda} s^{-1+i\lambda} \, \mathrm{d}s,$$

for all $x \in \mathbb{R}^d$, $|h| \leq 1$ and $t \in (0,1)$. To estimate the second term, we use (80) with $i = 1$ and our induction hypothesis to find that

$$\int_{\mathbb{R}^d} |I_2(t,x,y)| \, \mathrm{d}y \leq 2^i C^{i+1} \frac{\Gamma(\lambda)^i}{\Gamma(i\lambda)} |h|^{\gamma-\varepsilon} \int_0^t (t-s)^{-1+\lambda} s^{-1+i\lambda} \, \mathrm{d}s$$

for all $x \in \mathbb{R}^d$, $|h| \leq 1$ and $t \in (0,1)$. Combining these, $F^{\circledast(i+1)} = F \circledast F^{\circledast i}$ satisfies

$$\int_{\mathbb{R}^d} |F^{\circledast(i+1)}(t,x+h,y+h) - F^{\circledast(i+1)}(t,x,y)| \, \mathrm{d}y \\ \leq (2C)^{i+1} \frac{\Gamma(\lambda)^i}{\Gamma(i\lambda)} |h|^{\gamma-\varepsilon} \int_0^t (t-s)^{-1+\lambda} s^{-1+i\lambda} \, \mathrm{d}s.$$

By a change of variables $s \rightsquigarrow tr$ and Euler's formula for the Beta function, $B(u,v) = \Gamma(u)\Gamma(v)/\Gamma(u+v)$,

$$\int_0^t (t-s)^{-1+\lambda} s^{-1+i\lambda} \, \mathrm{d}s = t^{-1+(i+1)\lambda} B(\lambda, i\lambda) = t^{-1+(i+1)\lambda} \frac{\Gamma(i)\Gamma(i\lambda)}{\Gamma((i+1)\lambda)}.$$

Plugging this identity in the previous estimate shows that (79) holds for $i+1$, and this finishes the proof of Step 2.

Conclusion of the proof Fix $\varepsilon \in (\gamma_0, \gamma \wedge \alpha_L)$. Since, by (77),

$$|\Phi(t,x+h,y+h) - \Phi(t,x,y)| \leq \sum_{i=1}^{\infty} |F^{\circledast i}(t,x+h,y+h) - F^{\circledast i}(t,x,y)|,$$

the monotone convergence theorem gives

$$\int_{\mathbb{R}^d} |\Phi(t,x+h,y+h) - \Phi(t,x,y)|\,dy \\ \leq \sum_{i=1}^{\infty} \int_{\mathbb{R}^d} |F^{\circledast i}(t,x+h,y+h) - F^{\circledast i}(t,x,y)|\,dy.$$

So, by Step 2,

$$\int_{\mathbb{R}^d} |\Phi(t,x+h,y+h) - \Phi(t,x,y)|\,dy \leq |h|^{\gamma-\varepsilon} t^{-1+\lambda} \sum_{i\geq 1} 2^i C^i \frac{\Gamma(\lambda)^i}{\Gamma(i\lambda)},$$

for all $x \in \mathbb{R}^d$, $|h| \leq 1$ and $t \in (0,1)$ and suitable constants $C > 0$ and $\lambda > 0$ (not depending on x, h, t). It is not difficult to see that the series on the right-hand side converges, and consequently, we have proved the desired estimate. □

6.2 Auxiliary Result for the Proof of Theorem 4.1

Let $(X_t)_{t\geq 0}$ be an isotropic stable-like process with symbol $q(x,\xi) = |\xi|^{\alpha(x)}$ for a Hölder continuous mapping $\alpha : \mathbb{R}^d \to (0,2)$ with $\alpha_L := \inf_x \alpha(x) > 0$. By Proposition 6.1 and Proposition 3.1, any function f in the Favard space F_1 associated with $(X_t)_{t\geq 0}$ satisfies the a priori estimate

$$\|f\|_{\mathcal{C}_b^\kappa(\mathbb{R}^d)} \leq c(\|A_e f\|_\infty + \|f\|_\infty) \tag{81}$$

for $\kappa \in (0,\alpha_L)$; in particular, $F_1 \subseteq \mathcal{C}_b^{\alpha_L -}(\mathbb{R}^d)$. For the proof of Theorem 4.1, we need the following auxiliary result, which will give us an improved a priori estimate once we have shown that $f \in F_1$ is sufficiently regular on $\{x \in \mathbb{R}^d; \alpha(x) \leq 1\}$.

Lemma 6.4 *Let $(X_t)_{t\geq 0}$ be a Feller process with extended infinitesimal generator $(A_e, \mathcal{D}(A_e))$, Favard space F_1 and symbol $q(x,\xi) = |\xi|^{\alpha(x)}$ for a Hölder continuous mapping $\alpha : \mathbb{R}^d \to (0,2)$ such that*

$$0 < \alpha_L := \inf_{x\in\mathbb{R}^d} \alpha(x) \leq \sup_{x\in\mathbb{R}^d} \alpha(x) < 2.$$

Let $f \in F_1$ be such that for any $\varepsilon \in (0,\alpha_L)$ there exists a constant $M(\varepsilon) > 0$ such that

$$|\Delta_h f(x)| = |f(x+h) - f(x)| \leq M(\varepsilon)|h|^{\alpha(x)-\varepsilon}, \quad |h| \leq 1, \tag{82}$$

for any $x \in \{\alpha \leq 1\}$. Then there exists for any $\theta \in (0,1)$ a constant $C = C(\alpha,\theta)$ such that

$$|\Delta_h^2 f(x)| \leq C|h|^{1-\theta}(\|A_e f\|_\infty + \|f\|_\infty + M(\theta/12)), \quad |h| \leq 1,$$

for any $x \in \{\alpha \geq 1\}$.

Proof The idea of the proof is similar to the proof of Theorem 3.2. For fixed $0 < \theta < \min\{\alpha_L, 1/4\}$, define $\tilde{\alpha}(x) := \max\{1 - 3\theta, \alpha(x)\}$. By [22, Theorem 5.2], there exists a Feller process $(Y_t)_{t\geq 0}$ with symbol $p(x, \xi) := |\xi|^{\tilde{\alpha}(x)}$, and the $(L, C_c^\infty(\mathbb{R}^d))$-martingale problem for the generator L of $(Y_t)_{t\geq 0}$ is well-posed. Since α is Hölder continuous, there exists $\delta > 0$ such that

$$|x - z| \leq 2\delta \implies |\alpha(x) - \alpha(z)| \leq \theta. \tag{83}$$

As usual, we denote by

$$\tau_\delta^x := \inf\{t > 0; |Y_t - x| > \delta\}$$

the exit time from the closed ball $\overline{B(x, \delta)}$. Pick $\kappa \in C_b^\infty(\mathbb{R}^d)$, $0 \leq \kappa \leq 1$, such that $\kappa(x) = 0$ for any $x \in \{\alpha \leq 1-2\theta\}$ and $\kappa(x) = 1$ for $x \in \{\alpha \geq 1-\theta\}$; see Lemma D.1 for the existence of such a mapping.

Step 1 We show that for any $f \in F_1$ the product $v := f \cdot \kappa$ is in the domain $\mathcal{D}(L_e)$ of the extended generator of $(Y_t)_{t\geq 0}$; we will use a similar reasoning as in the proof of Theorem 3.2, i. e. we will estimate

$$\frac{1}{t} \sup_{x\in\mathbb{R}^d} |\mathbb{E}^x v(Y_{t\wedge\tau_\delta^x}) - v(x)|.$$

Clearly,

$$|\mathbb{E}^x v(Y_{t\wedge\tau_\delta^x}) - v(x)| \leq I_1(x) + I_2(x) + I_3(x),$$

where

$$\begin{aligned}
I_1(x) &:= |\kappa(x)\mathbb{E}^x(f(Y_{t\wedge\tau_\delta^x}) - f(x))|, \\
I_2(x) &:= |f(x)\mathbb{E}^x(\kappa(Y_{t\wedge\tau_\delta^x}) - \kappa(x))|, \\
I_3(x) &:= \left|\mathbb{E}^x\big((f(Y_{t\wedge\tau_\delta^x}) - f(x))(\kappa(Y_{t\wedge\tau_\delta^x}) - \kappa(x))\big)\right|.
\end{aligned}$$

We estimate the terms separately; we start with I_1. If $x \in \{\alpha \geq 1-2\theta\}$, then it follows from (83) that $B(x, 2\delta) \subseteq \{\alpha \geq 1 - 3\theta\}$ and therefore

$$q(z, \xi) = |\xi|^{\alpha(z)} = |\xi|^{\tilde{\alpha}(z)} = p(z, \xi) \quad \text{for all } z \in B(x, 2\delta),\ \xi \in \mathbb{R}^d. \tag{84}$$

Applying Lemma 5.2,

$$I_1(x) = |\kappa(x)\mathbb{E}^x(f(X_{t\wedge\tau_\delta^x(X)}) - f(x))|,$$

with $\tau_\delta^x(X)$ the exit time of $(X_t)_{t\geq 0}$ from $\overline{B(x,\delta)}$. As $f \in F_1$, Dynkin's formula (11) gives

$$I_1(x) \leq t\|A_e f\|_\infty.$$

If $x \in \{\alpha < 1-2\theta\}$, then $\kappa(x) = 0$ by the very definition of κ, and so $I_1(x) = 0$. Hence,

$$\sup_{x\in\mathbb{R}^d} I_1(x) \leq t\|A_e f\|_\infty.$$

For I_2, we note that $\kappa \in C_b^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(L)$, and so the (classical) Dynkin formula gives

$$\sup_{x\in\mathbb{R}^d} I_2(x) \leq t\|f\|_\infty \|L\kappa\|_\infty.$$

To estimate I_3, we consider two cases separately. If $x \in \{\alpha \leq 1\}$, then from our assumption on the regularity of f, cf. (82), and the Lipschitz continuity of κ,

$$\begin{aligned}
&|f(Y_{t\wedge\tau_\delta^x}) - f(x)| \cdot |\kappa(Y_{t\wedge\tau_\delta^x}) - \kappa(x)| \\
&\quad \leq 4(\|f\|_\infty + M(\theta/3))\|\kappa\|_{C_b^1(\mathbb{R}^d)} \min\{|Y_{t\wedge\tau_\delta^x} - x|^{\alpha(x)-\theta/3+1}, 1\}.
\end{aligned}$$

By Lemma 5.3, there exists a constant $c_2 = c_2(\alpha_L, \|\alpha\|_\infty) > 0$ such that

$$\begin{aligned}
I_3(x) &\leq c_2(\|f\|_\infty + M(\theta/3))\|\kappa\|_{C_b^1(\mathbb{R}^d)} \\
&\quad \cdot \sup_{|z-x|\leq\delta} \int_{y\neq 0} \min\{1, |y|^{\alpha(x)-\theta/3+1}\} \frac{1}{|y|^{d+\tilde{\alpha}(z)}}\, dy.
\end{aligned} \tag{*}$$

For $x \in \mathbb{R}^d$ with $\alpha(x) \leq 1-2\theta$ we note that it follows from the definition of $\tilde{\alpha}$ that $\tilde{\alpha}(z) \geq 1-3\theta$ for all $z \in \mathbb{R}^d$, and so

$$\begin{aligned}
&\sup_{x\in\{\alpha\leq 1-2\theta\}} I_3(x) \\
&\quad \leq c_2(\|f\|_\infty + M(\theta/3)) \left(\int_{|y|\leq 1} |y|^{-d+2\theta/3}\, dy + \int_{|y|>1} |y|^{-d-1+3\theta}\, dy \right) < \infty.
\end{aligned}$$

If $1-2\theta \leq \alpha(x) \leq 1$, then $\alpha(z) = \tilde{\alpha}(z)$ for all $|z-x| \leq \delta$; using (83), we find from (*) that

$$\begin{aligned}
&\sup_{x\in\{1-2\theta\leq\alpha\leq 1\}} I_3(x) \\
&\quad \leq c_2(\|f\|_\infty + M(\theta/3)) \left(\int_{|y|\leq 1} |y|^{-d+1-4\theta/3}\, dy + \int_{|y|>1} |y|^{-d-\alpha_L}\, dy \right) < \infty.
\end{aligned}$$

Finally, if $x \in \{\alpha > 1\}$, then $\overline{B(x,\delta)} \subseteq \{\alpha \geq 1-\theta\}$, and therefore $\kappa(z) = 1$ for any $|z-x| \leq \delta$; hence,

$$|f(Y_{t\wedge\tau_\delta^x}) - f(x)| \cdot |\kappa(Y_{t\wedge\tau_\delta^x}) - \kappa(x)| \leq 2\|f\|_\infty \mathbb{1}_{\{\tau_\delta^x \leq t\}},$$

which implies

$$I_3(x) \leq 2\|f\|_\infty \mathbb{P}^x(\tau_\delta^x \leq t).$$

By the maximal inequality (9),

$$I_3(x) \leq c_3\|f\|_\infty t \sup_{|z-x|\leq\delta} \sup_{|\xi|\leq\delta^{-1}} |p(z,\xi)|,$$

for some absolute constant $c_3 > 0$. As $|p(z,\xi)| \leq |\xi|^2$ for all $\xi \in \mathbb{R}^d$, this shows that

$$\sup_{x\in\{\alpha>1\}} I_3(x) \leq c_3\|f\|_\infty t\delta^{-2}.$$

Combining the estimates,

$$\sup_{t>0} \frac{1}{t} \sup_{x\in\mathbb{R}^d} |\mathbb{E}^x v(Y_{t\wedge\tau_\delta^x}) - v(x)| \leq c_4(\|f\|_\infty + \|A_e f\|_\infty + M(\theta/3))$$

for some constant $c_4 = c_4(\theta, \delta, \alpha_L, \|\alpha\|_\infty, \|L\kappa\|_\infty)$.

Step 2 Applying Corollary 2.2 we find that $v = f \cdot \kappa$ is in the Favard space F_1^Y of order 1 associated with $(Y_t)_{t\geq 0}$ and

$$\|L_e(f\cdot\kappa)\|_\infty \leq c_5(\|f\|_\infty + \|A_e f\|_\infty + M(\theta/3)).$$

Since Proposition 6.1 shows that the semigroup $(T_t)_{t\geq 0}$ associated with $(Y_t)_{t\geq 0}$ satisfies the Hölder estimate

$$\|T_t u\|_{\mathcal{C}_b^{1-4\theta}(\mathbb{R}^d)} \leq c_6\|u\|_\infty t^{-(1-4\theta)/(1-3\theta)}, \quad t \in (0,1],\ u \in \mathcal{B}_b(\mathbb{R}^d),$$

for $c_6 = c_6(\alpha,\theta) > 0$, Proposition 3.1 gives

$$\|f\cdot\kappa\|_{\mathcal{C}_b^{1-4\theta}(\mathbb{R}^d)} \leq c_7(\|f\|_\infty + \|A_e f\|_\infty + M(\theta/3))$$

for some constant $c_7 > 0$ which does not depend on f. Finally, we note that for any $x \in \{\alpha \geq 1\}$ we have $\kappa(z) = 1$ for $z \in \overline{B(x,\delta)}$, and so for all $|h| \leq \delta/2$

$$\begin{aligned}
&|f(x+2h) - 2f(x+h) + f(x)| \\
&\quad = |\kappa(x+2h)f(x+2h) - 2\kappa(x+h)f(x+h) + \kappa(x) + f(x)| \\
&\quad \leq c_7|h|^{1-4\theta}(\|f\|_\infty + \|A_e f\|_\infty + M(\theta/3)).
\end{aligned}$$

□

6.3 Proof of Theorem 4.1 and Corollary 4.3

Proof of Theorem 4.1 Fix $\varepsilon \in (0, \alpha_L)$. Since α is Hölder continuous, there exists $\delta > 0$ such that

$$|\alpha(x) - \alpha(y)| \leq \frac{\varepsilon}{2} \quad \text{for all } |x - y| \leq 4\delta. \tag{*}$$

Moreover, as $\|\alpha\|_\infty < 2$, we can choose $\theta \in (0, \alpha_L)$ such that $\alpha(x) < 2 - \theta$ for all $x \in \mathbb{R}^d$; without loss of generality, we may assume that $\varepsilon \leq \theta$. We divide the proof in two steps. First, we will establish the Hölder regularity of functions $f \in F_1$ at points $x \in \mathbb{R}^d$ such that $\alpha(x) \leq 1 + \alpha_L - \theta$. In the second part, we will consider the remaining points.

Step 1 There exists a constant $C_1 > 0$ such that

$$|\Delta_h^2 f(x)| \leq C_1 |h|^{\alpha(x)-\varepsilon} (\|A_e f\|_\infty + \|f\|_\infty) \tag{85}$$

for all $f \in F_1, \ |h| \leq \delta, \ x \in \{\alpha \leq \alpha_L + 1 - \theta\}$.

Indeed: Fix $x \in \mathbb{R}^d$ such that $\alpha(x) \leq \alpha_L + 1 - \theta$, and define

$$\alpha^x(z) := \max\{\alpha(z), \alpha(x) - \varepsilon/2\}, \quad z \in \mathbb{R}^d.$$

It is not difficult to see that $\|\alpha^x\|_{\mathcal{C}_b^\gamma(\mathbb{R}^d)} \leq \|\alpha\|_{\mathcal{C}_b^\gamma(\mathbb{R}^d)}$ and, moreover,

$$\alpha_L^x := \inf_{z \in \mathbb{R}^d} \alpha^x(z) \geq \alpha(x) - \frac{\varepsilon}{2} > 0.$$

It follows from [22, Theorem 5.2] that there exists a Feller process with symbol $p(z, \xi) := |\xi|^{\alpha^x(z)}$ and that the $(L, C_c^\infty(\mathbb{R}^d))$-martingale problem for the generator L of $(Y_t)_{t \geq 0}$ is well-posed. Note that, by (*), $\alpha^x(z) = \alpha(z)$ for $|z - x| \leq 4\delta$, and so

$$q(z, \xi) = |\xi|^{\alpha(z)} = |\xi|^{\alpha^x(z)} = p^{(x)}(z, \xi) \quad \text{for all } \xi \in \mathbb{R}^d, \ |z - x| \leq 4\delta.$$

Moreover, an application of Lemma 6.4 shows that there exists a constant $c_1 = c_1(\varepsilon, \alpha)$ such that the semigroup $(T_t)_{t \geq 0}$ associated with $(Y_t)_{t \geq 0}$ satisfies

$$\|T_t u\|_{\mathcal{C}_b^{\alpha(x)-\varepsilon}(\mathbb{R}^d)} \leq c_1 \|u\|_\infty t^{-(\alpha(x)-\varepsilon)/(\alpha(x)-\varepsilon/2)} \tag{86}$$

for any $u \in \mathcal{B}_b(\mathbb{R}^d)$ and $t \in (0, 1]$. So the conditions (C1)-(C3) in Theorem 3.2 are satisfied. By (81), it follows from Theorem 3.2 (with $\varrho(x) := \alpha_L - \theta/4$) that there exists a constant $c_2 = c_2(\varepsilon, \alpha)$ such that

$$|\Delta_h^2 f(x)| \leq c_2 K(x) |h|^{\alpha(x)-\varepsilon} (\|A_e f\|_\infty + \|f\|_\infty), \quad f \in F_1, \ |h| \leq \delta,$$

where

$$K(x) := \sup_{z\in\mathbb{R}^d} \int_{y\neq 0} \min\{1,|y|^2\} \frac{1}{|y|^{d+\alpha^x(z)}} \, dy \\ + \sup_{|z-x|\leq 4\delta} \int_{y\neq 0} \min\{1,|y|^{1+\alpha_L-\theta/4}\} \frac{1}{|y|^{d+\alpha^x(z)}} \, dy;$$

if we can show that $K := \sup_{x\in\{\alpha\leq\alpha_L+1-\theta\}} K(x) < \infty$ this gives (85). To this end, we note that $\varepsilon \leq \theta$ and (*) imply

$$\alpha^x(z) = \alpha(z) \leq \alpha(x) + \frac{\varepsilon}{2} \leq (\alpha_L + 1 - \theta) + \frac{\theta}{2} = \alpha_L + 1 - \frac{\theta}{2}$$

for all $|z-x| \leq 4\delta$, and so

$$K \leq \sup_{\beta\in[\alpha_L,\|\alpha\|_\infty]} \int_{y\neq 0} \min\{1,|y|^2\} \frac{1}{|y|^{d+\beta}} \, dy \\ + \sup_{\beta\in[\alpha_L,\alpha_L+1-\theta/2]} \int_{y\neq 0} \min\{1,|y|^{1+\alpha_L-\theta/4}\} \frac{1}{|y|^{d+\beta}} \, dy < \infty.$$

Step 2 There exists $C_2 > 0$ such that

$$|\Delta_h^2 f(x)| \leq C_2 |h|^{\alpha(x)-\varepsilon} (\|A_e f\|_\infty + \|f\|_\infty)$$

for all $f \in F_1,\ |h| \leq \delta,\ x \in \{\alpha \geq \alpha_L + 1 - \theta\}$.

Indeed: It follows from Lemma 6.4 and Step 1 that there exists a constant $c_3 > 0$ such that

$$|\Delta_h^2 f(x)| \leq c_3 |h|^{1-\theta/2} (\|A_e f\|_\infty + \|f\|_\infty), \quad |h| \leq 1, \tag{87}$$

for any $f \in F_1$ and $x \in \{\alpha \geq 1\}$. Thanks to this improved a priori estimate for $f \in F_1$, we can use a very similar reasoning to that in the first part of the proof to deduce the desired estimate. If we set $\alpha^x(z) := \max\{\alpha(z), \alpha(x) - \varepsilon/2\}$ for fixed $x \in \{\alpha \geq 1+\alpha_L-\theta\}$, then it follows exactly as in Step 1 that the Feller process $(Y_t)_{t\geq 0}$ with symbol $p(z,\xi) := |\xi|^{\alpha^x(z)}$ satisfies (C1)-(C3) in Theorem 3.2; in particular, (86) holds for the associated semigroup $(T_t)_{t\geq 0}$. By (87), we may apply Theorem 3.2 with $\varrho(x) := 1 - \theta/2$ to obtain

$$|\Delta_h^2 f(x)| \leq c_4 K(x) |h|^{\alpha(x)-\varepsilon} (\|A_e f\|_\infty + \|f\|_\infty), \quad f \in F_1,$$

for some constant c_4 (not depending on f and x) and

$$K(x) := \sup_{z\in\mathbb{R}^d} \int_{y\neq 0} \min\{1,|y|^2\} \frac{1}{|y|^{d+\alpha^x(z)}}\,\mathrm{d}y + \sup_{|z-x|\leq 4\delta} \int_{y\neq 0} \min\{1,|y|^{2-\theta/2}\} \frac{1}{|y|^{d+\alpha^x(z)}}\,\mathrm{d}y.$$

By our choice of θ, we have $\alpha_L \leq \alpha^x(z) \leq \|\alpha\|_\infty < 2-\theta$, and so

$$\sup_{x\in\{\alpha\geq 1+\alpha_L-\theta\}} K(x) \leq 2 \sup_{\beta\in[\alpha_L,\|\alpha\|_\infty]} \int_{y\neq 0} \min\{1,|y|^2\} \frac{1}{|y|^{d+\beta}}\,\mathrm{d}y + \int_{|y|\leq 1} |y|^{-d+\theta/2}\,\mathrm{d}y < \infty.$$

□

Proof of Corollary 4.3 We are going to apply Theorem 3.5 to prove the assertion. To this end, we first need to construct for each $x \in \mathbb{R}^d$ a Feller process $(Y_t^{(x)})_{t\geq 0}$ which satisfies (C1)-(C3) from Theorem 3.2, as well as (S1)-(S5) from Theorem 3.5. Recall that $\alpha_L = \inf_x \alpha(x) > 0$ and that $\gamma \in (0,1)$ is the Hölder exponent of α.

Fix $\varepsilon \in (0, \alpha_L \wedge \gamma)$ and $x \in \mathbb{R}^d$. Since α is Hölder continuous, there exists $\delta > 0$ such that

$$|\alpha(z+y) - \alpha(z)| \leq \frac{\varepsilon}{4} \quad \text{for all } z \in \mathbb{R}^d,\ |h| \leq \delta. \tag{*}$$

If we define

$$\alpha^x(z) := (\alpha(x) - \varepsilon/4) \vee \alpha(z) \wedge (\alpha(x) + \varepsilon/4), \quad z \in \mathbb{R}^d,$$

then it follows from [22, Theorem 5.2] that there exists a Feller process $(Y_t^{(x)})_{t\geq 0}$ with symbol $p^{(x)}(z,\xi) := |\xi|^{\alpha^x(z)}$ such that the martingale problem for its generator is well-posed. Moreover, by our choice of δ,

$$q(z,\xi) = |\xi|^{\alpha(z)} = |\xi|^{\alpha^x(z)} = p^{(x)}(z,\xi) \quad \text{for all } \xi \in \mathbb{R}^d,\ |z-x| \leq 4\delta,$$

and so (C1) and (C2) from Theorem 3.2 hold. By Proposition 6.1 and Proposition 6.2, the semigroup $(T_t^{(x)})_{t\geq 0}$ associated with $(Y_t^{(x)})_{t\geq 0}$ satisfies

$$\|T_t^{(x)} u\|_{\mathcal{C}_b^{\kappa(x)}(\mathbb{R}^d)} \leq c_1 \|u\|_\infty t^{-\beta(x)}, \quad u \in \mathcal{B}_b(\mathbb{R}^d),\ t \in (0,1),$$

and

$$\|T_t^{(x)} u\|_{\mathcal{C}_b^{\kappa(x)+\lambda}(\mathbb{R}^d)} \leq c_1 \|u\|_{\mathcal{C}_b^{\lambda}(\mathbb{R}^d)} t^{-\beta(x)}, \quad u \in \mathcal{C}_b^{\lambda}(\mathbb{R}^d),\ t \in (0,1),$$

for any $\lambda \leq \Lambda := \gamma$, where $c_1 > 0$ is some constant (not depending on u, t, x) and

$$\kappa(x) := \alpha(x) - \varepsilon, \quad \beta(x) := \frac{\alpha(x) - 2\varepsilon}{\alpha(x) - \varepsilon/4}.$$

Consequently, we have established (C3) and (S3). Since κ is clearly uniformly continuous and bounded away from zero, we get immediately that (S4) holds. Moreover, as α is bounded away from zero and from two, it follows easily that (S1) and (S5) hold with $\alpha^{(x)}(z) := \alpha^x(z)$. Finally, we note that the Hölder condition (S2) on the symbol $p^{(x)}$ is a consequence of the Hölder continuity of α; see Lemma 6.5 for details.

We are ready to apply Theorem 3.5. Let $f \in \mathcal{D}(A)$ be such that $Af = g \in \mathcal{C}_b^\lambda(\mathbb{R}^d)$ for some $\lambda > 0$. Without loss of generality, we may assume that $\lambda \leq \gamma$. Since $(X_t)_{t\geq 0}$ satisfies the assumptions of Theorem 4.1, it follows that $f \in \mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)$ for $\varrho(x) := \alpha(x) - \varepsilon/4$ and, moreover,

$$\|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)} \leq C_\varepsilon(\|Af\|_\infty + \|f\|_\infty). \tag{88}$$

Furthermore, by our choice of δ (cf. (*)), we find that

$$\sigma := \inf_{x\in\mathbb{R}^d} \inf_{|z-x|\leq 4\delta} (1 + \varrho(x) - \alpha^x(z))$$

satisfies $\sigma \geq 1 - \varepsilon/4$. Applying Theorem 3.5, we conclude that

$$f \in \mathcal{C}_b^{\kappa(\cdot)+\min\{\gamma,\lambda,1-\varepsilon/4\}-\varepsilon/4}(\mathbb{R}^d) \subseteq \mathcal{C}_b^{\alpha(\cdot)+\min\{\gamma,\lambda\}-2\varepsilon}(\mathbb{R}^d)$$

and

$$\begin{aligned}\|f\|_{\mathcal{C}_b^{\alpha(\cdot)+\min\{\gamma,\lambda\}-2\varepsilon}(\mathbb{R}^d)} &\leq C'_\varepsilon(\|Af\|_{\mathcal{C}_b^\lambda(\mathbb{R}^d)} + \|f\|_{\mathcal{C}_b^{\varrho(\cdot)}(\mathbb{R}^d)}) \\ &\leq C''_\varepsilon(\|Af\|_{\mathcal{C}_b^\lambda(\mathbb{R}^d)} + \|f\|_\infty),\end{aligned}$$

where we used (88) for the last inequality. □

Lemma 6.5 *For fixed* $\alpha \in (0, 2)$, *denote by* ν_α *the Lévy measure of the isotropic* α*-stable Lévy process, i. e.*

$$|\xi|^\alpha = \int_{y\neq 0} (1 - \cos(y \cdot \xi))\, \nu_\alpha(\mathrm{d}y), \quad \xi \in \mathbb{R}^d. \tag{89}$$

Let $\beta : \mathbb{R}^d \to (0, 2)$ *be such that* $\beta \in C_b^\gamma(\mathbb{R}^d)$ *for some* $\gamma \in (0, 1]$ *and*

$$0 < \beta_L := \inf_{z\in\mathbb{R}^d} \beta(z) \leq \sup_{z\in\mathbb{R}^d} \beta(z) < 2.$$

If $u : \mathbb{R}^d \to \mathbb{R}$ *is a measurable mapping such that*

$$|u(y)| \leq M \min\{|y|^{\beta(z)+r}, 1\}, \quad y \in \mathbb{R}^d, \tag{90}$$

for some $z \in \mathbb{R}^d$, $r > 0$ and $M > 0$, then there exist constants $K > 0$ and $H > 0$ (not depending on u or z) such that

$$\left| \int u(y)\, \nu_{\beta(z)}(\mathrm{d}y) - \int u(y)\, \nu_{\beta(z+h)}(\mathrm{d}y) \right| \leq MK|h|^\gamma \quad \textit{for all } |h| \leq H.$$

Proof It is well known that $\nu_\alpha(\mathrm{d}y) = c(\alpha)|y|^{-d-\alpha}$ with $c(\alpha)$ a normalizing constant such that (89) holds. Noting that, by the rotational invariance of $\xi \mapsto |\xi|^\alpha$,

$$|\xi|^\alpha = c(\alpha) \int_{y \neq 0} (1 - \cos(y_1|\xi|)) \frac{1}{|y|^{d+\alpha}}\, \mathrm{d}y = |\xi|^\alpha c(\alpha) \int_{y \neq 0} (1 - \cos(y_1)) \frac{1}{|y|^{d+\alpha}}\, \mathrm{d}y$$

for all $\xi \in \mathbb{R}^d$, we find that $c(\alpha) = 1/h(\alpha)$, where

$$h(\alpha) := \int_{y \neq 0} (1 - \cos(y_1)) \frac{1}{|y|^{d+\alpha}}\, \mathrm{d}y.$$

From

$$\begin{aligned} \left| \frac{1}{r^{d+\alpha}} - \frac{1}{r^{d+\tilde{\alpha}}} \right| &= \frac{1}{r^{2d+\alpha+\beta}} |r^{d+\tilde{\alpha}} - r^{d+\alpha}| \\ &\leq |\log(r)| r^{-d} \max\{r^{-\alpha}, r^{-\tilde{\alpha}}\} |\alpha - \tilde{\alpha}|, \quad r > 0, \end{aligned} \tag{91}$$

and $\alpha, \tilde{\alpha} \in I := [\beta_L, \|\beta\|_\infty] \subseteq (0, 2)$, it follows that

$$|h(\alpha) - h(\tilde{\alpha})| \leq C_1 |\alpha - \tilde{\alpha}|, \quad \alpha, \tilde{\alpha} \in I$$

for some constant $C_1 > 0$. As $\inf_{\alpha \in I} h(\alpha) > 0$, this implies that $c(\alpha) = 1/h(\alpha)$ satisfies

$$|c(\alpha) - c(\tilde{\alpha})| \leq C_2 |\alpha - \tilde{\alpha}|, \quad \alpha, \tilde{\alpha} \in I, \tag{92}$$

for some constant $C_2 > 0$.

Now let $u : \mathbb{R}^d \to \mathbb{R}$ be a measurable mapping such that (90) holds for some $z \in \mathbb{R}^d$, $M > 0$ and $r > 0$. Since $\nu_\alpha(\mathrm{d}y) = c(\alpha)|y|^{-d-\alpha}\, \mathrm{d}y$,

$$\left| \int u(y)\, \nu_{\beta(z)}(\mathrm{d}y) - \int u(y)\, \nu_{\beta(z+h)}(\mathrm{d}y) \right| \leq I_1 + I_2,$$

where

$$\begin{aligned} I_1 &:= |c(\beta(z)) - c(\beta(z+h))| \int_{y \in \mathbb{R}^d} |u(y)| \frac{1}{|y|^{d+\beta(z)}}\, \mathrm{d}y, \\ I_2 &:= c(\beta(z+h)) \int_{y \neq 0} |u(y)| \left| \frac{1}{|y|^{d+\beta(z)}} - \frac{1}{|y|^{d+\beta(z+h)}} \right| \mathrm{d}y. \end{aligned}$$

By the first part of the proof (cf. (92)) and by (90),

$$I_1 \le C_2 M|\beta(z)-\beta(z+h)| \int_{y\in\mathbb{R}^d} \min\{|y|^{\beta(z)+r},1\} \frac{1}{|y|^{d+\beta(z)}}\, dy,$$

and so

$$I_1 \le C_2 M|h|^\gamma \|\beta\|_{C_b^\gamma(\mathbb{R}^d)} \sup_{\alpha\in I} \int_{y\neq 0} \min\{|y|^{\alpha+r},1\}|y|^{d-\alpha}\, dy =: C_3 M|h|^\gamma,$$

for all $h\in\mathbb{R}^d$. To estimate I_2, we choose $H>0$ such that

$$|\beta(x)-\beta(x+h)| \le \frac{\min\{r,\beta_L\}}{2} \quad \text{for all } x\in\mathbb{R}^d, |h|\le H.$$

By (90) and (91),

$$\begin{aligned} I_2 &\le M|\beta(z)-\beta(z+h)| \sup_{\alpha\in I} c(\alpha) \\ &\quad \cdot \int_{y\neq 0} \min\{|y|^{\beta(z)+r},1\}|\log(|y|)| \frac{\max\{|y|^{-\beta(z)},|y|^{-\beta(z+h)}\}}{|y|^d}\, dy, \end{aligned}$$

for all $|h|\le H$. By our choice of H,

$$\frac{\beta(z)}{2} \le \beta(z)-\frac{\beta_L}{2} \le \beta(z+h) \le \beta(z)+\frac{r}{2} \quad \text{for all } |h|\le H,$$

and so

$$\begin{aligned} I_2 &\le M|\beta(z)-\beta(z+h)| \sup_{\alpha\in I} c(\alpha) \\ &\quad \cdot \left(\int_{|y|\le 1} |y|^{-d+r/2}|\log(|y|)|\, dy + \int_{|y|>1} |y|^{-d-\beta(z)/2}\log(|y|)\, dy \right) \\ &\le C_4 M|h|^\gamma, \end{aligned}$$

for all $|h|\le H$ and

$$\begin{aligned} C_4 &:= \|\beta\|_{C_b^\gamma(\mathbb{R}^d)} \sup_{\alpha\in I} c(\alpha) \\ &\quad \cdot \left(\int_{|y|\le 1} |y|^{-d+r/2}|\log(|y|)|\, dy + \sup_{\alpha\in I} \int_{|y|>1} |y|^{-d-\alpha/2}\log(|y|)\, dy \right) < \infty. \end{aligned}$$

□

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Appendix A: Extended Generator

In this section, we collect some material on the extended generator of a Feller process; in particular, we present the proofs of Theorem 2.1 and Corollary 2.2. The extended infinitesimal generator was originally introduced by Kunita [34] and was studied quite intensively in the 1980s, e.g. by Airault and Föllmer [1], Bouleau [7], Hirsch [17], Meyer [44] and Mokobodzki [45]. Recall the following definition (cf. Sect. 2).

Definition A.1 Let $(X_t)_{t\geq 0}$ be a Feller process with μ-potential operators $(R_\lambda)_{\lambda>0}$. A function f is in the domain $\mathcal{D}(A_e)$ of the extended generator and $g = A_e f$ if

(i) $f \in \mathcal{B}_b(\mathbb{R}^d)$ and g is a measurable function such that $\|R_\lambda(|g|)\|_\infty < \infty$ for some (all) $\lambda > 0$,

(ii) $f = R_\lambda(\lambda f - g)$ for all $\lambda > 0$.

Condition A.1(ii) may be replaced by

(ii') $M_t := f(X_t) - f(X_0) - \int_0^t g(X_s)\,\mathrm{d}s$, $t \geq 0$, is a local $\mathbb{P}^x$-martingale for any $x \in \mathbb{R}^d$;

cf. Meyer [44] or Bouleau [7]. Moreover, it was shown in [1] that the extended generator can also be defined in terms of pointwise limits

$$\lim_{t\to 0} t^{-1}(\mathbb{E}^x f(X_t) - f(x)), \tag{93}$$

see also Corollary A.4. The domain $\mathcal{D}(A_e)$ is, in general, quite large; an indication is that it is possible to show, under relatively weak assumptions (e.g. $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A_e)$) that $\mathcal{D}(A_e)$ is closed under multiplication (cf. [44, pp. 144] or [8, Theorem 4.3.6]). There is a close connection between the extended generator and the carré du champ operator (cf. [8, Section 4.3] or [12]). The following statement is essentially due to Airault and Föllmer [1].

Theorem A.2 *Let $(X_t)_{t\geq 0}$ be a Feller process with semigroup $(P_t)_{t\geq 0}$ and extended generator $(A_e, \mathcal{D}(A_e))$. The associated Favard space F_1 of order 1 (cf. (6)) satisfies*

$$F_1 = \{f \in \mathcal{D}(A_e);\ \|A_e f\|_\infty < \infty\}.$$

If $f \in F_1$, then

$$K(f) := \sup_{t\in(0,1)} \frac{1}{t}\|P_t f - f\|_\infty = \|A_e f\|_\infty,$$

and, moreover, Dynkin's formula

$$\mathbb{E}^x f(X_\tau) - f(x) = \mathbb{E}^x \left(\int_0^\tau A_e f(X_s)\,\mathrm{d}s \right) \tag{94}$$

holds for any $x \in \mathbb{R}^d$ and any stopping time τ such that $\mathbb{E}^x \tau < \infty$.

Proof Denote by $(R_\lambda)_{\lambda>0}$ the λ-potential operators of $(X_t)_{t\geq 0}$, and set

$$\mathcal{D} := \{f \in \mathcal{B}_b(\mathbb{R}^d);\ \|A_e f\|_\infty < \infty\}.$$

First we prove $F_1 \subseteq \mathcal{D}$. Let $f \in F_1$. Airault and Föllmer [1, p. 320–322] showed that the limit $g(x) = \lim_{t\to 0} t^{-1}(P_t f(x) - f(x))$ exists outside a set of potential zero, and that

$$M_t := f(X_t) - f(X_0) - \int_0^t g(X_s)\,\mathrm{d}s, \quad t \geq 0,$$

is a $\mathbb{P}^x$-martingale for any $x \in \mathbb{R}^d$; we set $g = 0$ on the set of potential zero where the limit does not exist. Clearly, $\|g\|_\infty \leq K(f) < \infty$, and so it is obvious that $R_\lambda(|g|)$ is bounded for any $\lambda > 0$. It remains to check A.1(ii). Since the martingale $(M_t)_{t\geq 0}$ has constant expectation, we have $P_t f = f + \int_0^t P_s g\,\mathrm{d}s$, and thus

$$\begin{aligned} \lambda \int_{(0,\infty)} \mathrm{e}^{-\lambda t} P_t f(x)\,\mathrm{d}t &= \lambda \int_{(0,\infty)} \mathrm{e}^{-\lambda t} \left(f(x) + \int_0^t P_s g(x)\,\mathrm{d}s \right) \mathrm{d}t \\ &= f(x) - \int_{(0,\infty)} \left(\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{-\lambda t} \right) \left(\int_0^t P_s g(x)\,\mathrm{d}s \right) \mathrm{d}t. \end{aligned}$$

Integrating by parts,

$$\lambda \int_{(0,\infty)} \mathrm{e}^{-\lambda t} P_t f(x)\,\mathrm{d}t = f(x) + \int_{(0,\infty)} \mathrm{e}^{-\lambda t} P_t g(x)\,\mathrm{d}t,$$

i.e. $\lambda R_\lambda f = f + R_\lambda g$. This proves $f \in \mathcal{D}(A_e)$, $A_e f = g$ and $\|A_e f\|_\infty \leq K(f)$.

If $f \in \mathcal{D}$, then the local martingale

$$M_t = f(X_t) - f(X_0) - \int_0^t A_e f(X_s)\,\mathrm{d}s$$

satisfies

$$\mathbb{E}^x(M_{t\wedge\tau}^2) \leq (2\|f\|_\infty + \|A_e f\|_\infty)^2 (1+t), \quad t \geq 0,\ x \in \mathbb{R}^d,$$

for any stopping time τ. By Doob's maximal inequality, $\sup_{s\leq t} |M_s|$ is square-integrable, and hence $(M_t)_{t\geq 0}$ is a martingale. In particular, $\mathbb{E}^x(M_t) = \mathbb{E}^x(M_0)$,

i. e.

$$\mathbb{E}^x f(X_t) - f(x) = \mathbb{E}^x \left(\int_0^t A_e f(X_s)\, ds \right),$$

and so $K(f) \leq \|A_e f\|_\infty < \infty$ and $f \in F_1$. Finally, we note that Dynkin's formula (94) was shown in [1, Corollary 5.11] for any function $f \in \mathcal{B}_b(\mathbb{R}^d)$ satisfying $K(f) < \infty$. □

Remark A.3 (i) Airault and Föllmer [1] show Dynkin's formula (94), more generally, for Markov processes (not necessarily having the Feller property). If $(X_t)_{t\geq 0}$ is a time-homogeneous Markov process with semigroup $(P_t)_{t\geq 0}$ and Favard space F_1, then Dynkin's formula (94) holds for all $f \in F_1$, where $A_e f$ is *defined* by

$$A_e f(x) = \lim_{t\to 0} \frac{P_t f(x) - f(x)}{t}, \quad f \in F_1,\ x \in \mathbb{R}^d;$$

this limit exists up to a set of potential zero (cf. [1]).

(ii) The *weak infinitesimal generator* $\tilde{A}$ in the sense of Dynkin [13] is the linear operator $\tilde{A} : \mathcal{D}(\tilde{A}) \to \mathcal{B}_b(\mathbb{R}^d)$,

$$\mathcal{D}(\tilde{A}) := \left\{ f \in F_1; \exists g \in \mathcal{B}_b(\mathbb{R}^d)\ \forall x \in \mathbb{R}^d\ :\ g(x) = \lim_{t\to 0} \frac{P_t f(x) - f(x)}{t} \right\},$$

$$\tilde{A} f(x) := \lim_{t\to 0} \frac{P_t f(x) - f(x)}{t}.$$

By (the proof of) Theorem A.2, the extended generator $(A_e, \mathcal{D}(A_e))$ is an extension of the weak generator $(\tilde{A}, \mathcal{D}(\tilde{A}))$. In view of the previous remark, this is not only true for Feller processes but also for general Markov processes.

Corollary A.4 *Let $(X_t)_{t\geq 0}$ be a Feller process with semigroup $(P_t)_{t\geq 0}$, extended generator $(A_e, \mathcal{D}(A_e))$ and symbol q. Denote by*

$$\tau_r^x := \inf\{t > 0; |X_t - x| > r\}$$

the exit time of $(X_t)_{t\geq 0}$ from the closed ball $\overline{B(x,r)}$. If the symbol q has bounded coefficients, then the following statements are equivalent for any $f \in \mathcal{B}_b(\mathbb{R}^d)$.

(i) $f \in F_1$, i.e. $f \in \mathcal{D}(A_e)$ and $\sup_{t\in(0,1)} t^{-1}\|P_t f - f\|_\infty = \|A_e f\|_\infty < \infty$.

(ii) There exists $r > 0$ such that

$$K_r^{(1)}(f) := \sup_{t\in(0,1)} \sup_{x\in\mathbb{R}^d} \frac{1}{\mathbb{E}^x(t \wedge \tau_r^x)} |\mathbb{E}^x f(X_{t\wedge\tau_r^x}) - f(x)| < \infty.$$

(iii) There exists $r > 0$ such that

$$K_r^{(2)}(f) := \sup_{t\in(0,1)} \frac{1}{t} \sup_{x\in\mathbb{R}^d} |\mathbb{E}^x f(X_{t\wedge\tau_r^x}) - f(x)| < \infty.$$

If one (hence all) of the conditions is satisfied, then

$$A_e f(x) = \lim_{t\to 0} \frac{\mathbb{E}^x f(X_{t\wedge\tau_r^x}) - f(x)}{t} = \lim_{t\to 0} \frac{\mathbb{E}^x f(X_{t\wedge\tau_r^x}) - f(x)}{\mathbb{E}^x(t\wedge\tau_r^x)}, \tag{95}$$

up to a set of potential zero for any $r \in (0,\infty]$. *In particular,* $\|A_e f\|_\infty \leq K_r^{(i)}(f)$ *for* $i \in \{1,2\}$ *and* $r \in (0,\infty]$.

The proof of Corollary A.4 shows that the implications (i) $\implies$ (ii), (i) $\implies$ (iii) and (i) $\implies$ (95) remain valid if the symbol q has unbounded coefficients.

Proof of Corollary A.4 (i) $\implies$ (ii): If $f \in F_1$, then it follows from Dynkin's formula (94) that

$$K_r^{(1)}(f) \leq \|A_e f\|_\infty < \infty \quad \text{for all } r > 0.$$

(ii) $\implies$ (iii): This is obvious because $\mathbb{E}^x(t\wedge\tau_r^x) \leq t$.

(iii) $\implies$ (i): Fix $t \in (0,1)$. Clearly,

$$|\mathbb{E}^x f(X_t) - f(x)| \leq |\mathbb{E}^x f(X_{t\wedge\tau_r^x}) - f(x)| + |\mathbb{E}^x(f(X_{t\wedge\tau_r^x}) - f(X_t))|.$$

By assumption, the first term on the right-hand side is bounded by $K_r^{(2)}(f)t$. For the second term, we note that

$$|\mathbb{E}^x(f(X_{t\wedge\tau_r^x}) - f(X_t))| \leq 2\|f\|_\infty \mathbb{P}^x(\tau_r^x \leq t).$$

The maximal inequality (9) for Feller processes shows that there exists an absolute constant $c > 0$ such that

$$\begin{aligned} |\mathbb{E}^x(f(X_{t\wedge\tau_r^x}) - f(X_t))| &\leq 2ct\|f\|_\infty \sup_{|y-x|\leq r} \sup_{|\xi|\leq r^{-1}} |q(y,\xi)| \\ &\leq 2ct\|f\|_\infty \sup_{y\in\mathbb{R}^d} \sup_{|\xi|\leq r^{-1}} |q(y,\xi)|; \end{aligned}$$

note that the right-hand side is finite because q has bounded coefficients. Combining both estimates gives (i).

Proof of (95): For $r=\infty$, this follows from [1]; see the proof of Theorem A.2. Fix $r \in (0,\infty)$. By Dynkin's formula (94), we find

$$\begin{aligned} \left| \frac{\mathbb{E}^x f(X_t) - f(x)}{t} - \frac{\mathbb{E}^x f(X_{t\wedge\tau_r^x}) - f(x)}{t} \right| &\leq \frac{1}{t}\|A_e f\|_\infty \mathbb{E}^x(t - \min\{\tau_r^x, t\}) \\ &\leq \|A_e f\|_\infty \mathbb{P}^x(\tau_r^x \leq t). \end{aligned}$$

The right-continuity of the sample paths of $(X_t)_{t\geq 0}$ gives $\mathbb{P}^x(\tau_r^x \leq t) \to 0$ as $t \to 0$, and so

$$\lim_{t\to 0} \frac{\mathbb{E}^x f(X_{t\wedge\tau_r^x}) - f(x)}{t} = \lim_{t\to 0} \frac{\mathbb{E}^x f(X_t) - f(x)}{t}.$$

Since the right-hand side equals $A_e f(x)$ up to a set of potential zero (see the proof of Theorem A.2), this proves the first equality in (95). Similarly, it follows from Dynkin's formula that

$$\begin{aligned}&\left|\frac{\mathbb{E}^x f(X_{t\wedge\tau_r^x})-f(x)}{t}-\frac{\mathbb{E}^x f(X_{t\wedge\tau_r^x})-f(x)}{\mathbb{E}^x(t\wedge\tau_r^x)}\right|\\&\quad\leq \|A_e f\|_\infty \mathbb{E}^x(\tau_r^x\wedge t)\left|\frac{1}{t}-\frac{1}{\mathbb{E}^x(t\wedge\tau_r^x)}\right|\\&\quad\leq \|A_e f\|_\infty \mathbb{P}^x(\tau_r^x\leq t).\end{aligned}$$

As $\mathbb{P}^x(\tau_r^x \leq t) \to 0$ we find that the right-hand side converges to 0 as $t \to 0$, and this proves the second equality in (95). □

Appendix B: Parametrix Construction of the Transition Density

Let $(X_t)_{t\geq 0}$ be a Feller process with symbol $q(x,\xi) = |\xi|^{\alpha(x)}$ for a Hölder continuous mapping $\alpha : \mathbb{R}^d \to (0,2)$ with $\alpha_L := \inf_x \alpha(x) > 0$. For the proof of Proposition 6.1, the parametrix construction of the transition density of $(X_t)_{\geq 0}$ from [22] plays a crucial role (see also [25]). In this section, we collect some results from [22] needed for our proofs. Throughout, $p^\varrho(t,x)$ denotes the transition density of an isotropic ϱ-stable Lévy process, $\varrho \in (0,2]$,

$$p^\varrho(t,x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{ix\cdot\xi}\mathrm{e}^{-t|\xi|^\varrho}\,\mathrm{d}\xi, \quad x\in\mathbb{R}^d,\ t>0, \tag{96}$$

and $\circledast$ is the time-space convolution, i. e.

$$(f\circledast g)(t,x,y) := \int_0^t\int_{\mathbb{R}^d} f(t-s,x,z)g(s,z,y)\,\mathrm{d}z\,\mathrm{d}s, \quad t>0,\ x,y\in\mathbb{R}^d.$$

By [22, Theorem 5.2, Theorem 4.25], the transition density p of $(X_t)_{t\geq 0}$ has the representation

$$p(t,x,y) = p_0(t,x,y) + (p_0\circledast\Phi)(t,x,y), \quad t>0,\ x,y\in\mathbb{R}^d, \tag{97}$$

where p_0 is the zero-order approximation of p, defined by,

$$p_0(t,x,y) := p^{\alpha(y)}(t,x-y), \quad t>0,\ x,y\in\mathbb{R}^d, \tag{98}$$

and Φ is a suitable function; see (99) for the precise definition. There exists for any $T>0$ a constant $C_1>0$ such that

$$|p_0(t,x,y)| \leq C_1 S(x-y,\alpha(y),t), \quad t\in(0,T),\ x,y\in\mathbb{R}^d,$$

where

$$S(x,\alpha,t) := \min\left\{t^{-d/\alpha},\ \frac{t}{|x|^{d+\alpha}}\right\};$$

cf. [22, Section 4.1]. A straightforward computation yields

$$\forall 0<a<b\le 2:\ \sup_{t\in(0,T)}\sup_{z\in\mathbb{R}^d}\sup_{\varrho\in[a,b]}\int_{\mathbb{R}^d} S(z-y,\varrho,t)\,\mathrm{d}y<\infty;$$

cf. [22, Lemma 4.16] for details. The function Φ in (97) has the representation

$$\Phi(t,x,y)=\sum_{i=1}^{\infty} F^{\circledast i}(t,x,y),\quad t>0,\ x,y\in\mathbb{R}^d, \tag{99}$$

where $F^{\circledast i} := F \circledast F^{\circledast(i-1)}$ denotes the ith convolution power of

$$F(t,x,y):=(2\pi)^{-d}\int_{\mathbb{R}^d}\left(|\xi|^{\alpha(y)}-|\xi|^{\alpha(x)}\right)\mathrm{e}^{i\xi\cdot(y-x)}\mathrm{e}^{-t|\xi|^{\alpha(y)}}\,\mathrm{d}\xi,\quad t>0,\ x,y\in\mathbb{R}^d.$$

It is possible to show that

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|\Phi(t,x,y)|\,\mathrm{d}y\le C_2 t^{-1+\lambda},\quad t\in(0,T),$$

for some constant $\lambda>0$ and $C_2=C_2(T)>0$ (cf. [22, Theorem 4.25(iii), Lemma A.8]). Moreover, by [22, Lemma 4.21 and 4.24], there exist constants $C_3=C_3(T)>0$ and $\lambda>0$ such that

$$\int_{\mathbb{R}^d}|F^{\circledast i}(t,x,y)|\,\mathrm{d}y\le C_3^i\frac{\Gamma(\lambda)^i}{\Gamma(i\lambda)}t^{-1+i\lambda},\quad x\in\mathbb{R}^d,\ t\in(0,T).$$

Because of representation (98), the following estimates are a useful tool to derive estimates for the transition density p.

Lemma B.1 *Let $I=[a,b]\subset(0,2)$. For all $T>0$ and $k\in\mathbb{N}_0$, there exists a constant $C>0$ such that the following estimates hold for any $\varrho\in[a,b]$, $x\in\mathbb{R}^d$, $t\in(0,T)$, and any multiindex $\beta\in\mathbb{N}_0^d$ with $|\beta|=k$:*

$$|\partial_x^\beta p^\varrho(t,x)|\le Ct^{-|\beta|/\varrho}S(x,\varrho,t), \tag{100}$$

$$\int_{\mathbb{R}^d}\left|\frac{\partial^\beta}{\partial x^\beta}\frac{\partial}{\partial\varrho}p^\varrho(t,x)\right|\,\mathrm{d}x\le C(1+|\log(t)|)t^{-|\beta|/\varrho}. \tag{101}$$

Proof We only prove (101); for the pointwise estimate (100), see [22, Theorem 4.12]. Denote by $p^\varrho=p^{\varrho,d}$ the transition density of the d-dimensional isotropic ϱ-stable

Lévy process, $\varrho \in (0,2)$. It follows from the Fourier representation (96) of p^ϱ that $\varrho \mapsto p^{\varrho,d}(t,x)$ and $x \mapsto p^{\varrho,d}(t,x)$ are infinitely often differentiable, and

$$\partial_\varrho \partial_x^\beta p^{\varrho,d}(t,x) = -\frac{t}{(2\pi)^d} \int_{\mathbb{R}^d} (i\xi)^\beta \mathrm{e}^{ix\cdot\xi} \mathrm{e}^{-t|\xi|^\varrho} |\xi|^\varrho \log(|\xi|)\, \mathrm{d}\xi,$$

for all $\varrho \in [a,b]$, $x \in \mathbb{R}^d$, $t > 0$ and $\beta \in \mathbb{N}_0^d$. In particular,

$$\frac{\partial}{\partial\varrho} p^{\varrho,d}(t,x) = -t\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{ix\cdot\xi} \mathrm{e}^{-t|\xi|^\varrho} |\xi|^\varrho \log(|\xi|)\, \mathrm{d}\xi, \quad t > 0,\ x \in \mathbb{R}^d, \tag{102}$$

and, by [22, Theorem 4.7], there exists a constant $c_2 > 0$ such that

$$\left|\frac{\partial}{\partial\varrho} p^{\varrho,d}(t,x)\right| \leq c_2 \min\left\{(1+|\log(t)|)t^{-d/\varrho}, \frac{t}{|x|^{d+\varrho}}(1+|\log(|x|)|)\right\}, \tag{103}$$

for all $t \in (0,T]$, $x \in \mathbb{R}^d$ and $\varrho \in [a,b] \subseteq (0,2)$. By (102), $\partial_\varrho p^{\varrho,d}$ is the Fourier transform of a rotationally invariant function, and so it follows from the dimension-walk formula for the Fourier transform that

$$\frac{\partial}{\partial x_j}\frac{\partial}{\partial\varrho} p^{\varrho,d}(t,x) = -2\pi x_j \frac{\partial}{\partial\varrho} p^{\varrho,d+2}(t,x),$$

for $j = 1,\dots,d$, $t > 0$, $x \in \mathbb{R}^d$ and $\varrho \in (0,2)$; the dimension-walk formula goes back to Matheron [42, pp. 31–37] (see also [43]), and has been subsequently "rediscovered" by several authors (see the article [29] and the references therein). Using (103) for dimension $d+2$, there is a constant $c_3 > 0$ such that

$$\int_{\mathbb{R}^d} \left|\frac{\partial}{\partial x_j}\frac{\partial}{\partial\varrho} p^{\varrho,d}(t,x)\right| \mathrm{d}x \leq c_3(1+|\log(t)|)t^{-1/\alpha_L}, \tag{104}$$

for all $t \in (0,T]$, $j \in \{1,\dots,d\}$ and $\varrho \in [a,b] \subseteq (0,2)$. By iteration, we get (101). □

Appendix C: Inequalities for Hölder Continuous Functions

We present two inequalities for Hölder continuous functions which we used in Sect. 6.

Lemma C.1 *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function. If $x \in \mathbb{R}^d$ and $M_1, M_2 > 0$ are such that*

$$|\Delta_h^2 f(x)| \leq M_1|h|^2 \quad \textit{and} \quad |\Delta_h^2 f(x)| \leq M_2$$

for all $h \in \mathbb{R}^d$, then

$$|\Delta_h^2 f(x)| \leq |h|^\kappa \max\{M_1 r^{2-\kappa}, M_2 r^{-\kappa}\}$$

for any $r > 0$, $h \in \mathbb{R}^d$ and $\kappa \in [0, 2]$.

Proof Fix $\kappa \in [0, 2]$ and $r > 0$. If $h \in \mathbb{R}^d$ is such that $|h| > r$, then

$$|\Delta_h^2 f(x)| \leq M_2 \leq M_2 \frac{|h|^\kappa}{r^\kappa}.$$

If $|h| \leq r$ then

$$|\Delta_h^2 f(x)| \leq M_1 |h|^2 \leq M_1 |h|^\kappa r^{2-\kappa}.$$

□

Lemma C.2 *Let $f \in \mathcal{C}_b^\gamma(\mathbb{R}^d)$ for some $\gamma \in (0, 1)$. There exists a constant $C = C(\gamma) > 0$ such that*

$$|\Delta_h f(x) - \Delta_h f(y)| \leq C \|f\|_{\mathcal{C}_b^\gamma(\mathbb{R}^d)} |x - y|^\alpha |h|^{\gamma-\alpha} \tag{105}$$

for all $\alpha \in [0, \gamma]$ and $x, y, h \in \mathbb{R}^d$.

If $f : \mathbb{R}^d \to \mathbb{R}$ is Lipschitz continuous and bounded, then (105) holds for $\gamma = 1$; the norm $\|f\|_{C_b^\gamma(\mathbb{R}^d)}$ needs to be replaced by the sum of the supremum norm and the Lipschitz constant of f.

Proof By definition of the Hölder–Zygmund space $\mathcal{C}_b^\gamma(\mathbb{R}^d)$,

$$|f(x+h) - f(x)| \leq \|f\|_{\mathcal{C}_b^\gamma(\mathbb{R}^d)} |h|^\gamma \mathbb{1}_{\{|h| \leq 1\}} + 2\|f\|_\infty \mathbb{1}_{\{|h|>1\}} \leq 2\|f\|_{\mathcal{C}_b^\gamma(\mathbb{R}^d)} |h|^\gamma,$$

for any $x, h \in \mathbb{R}^d$. Hence,

$$\begin{aligned} |\Delta_h f(x) - \Delta_h f(y)| &\leq |f(x+h) - f(x)| + |f(y+h) - f(y)| \\ &\leq 4\|f\|_{\mathcal{C}_b^\gamma(\mathbb{R}^d)} |h|^\gamma, \end{aligned} \tag{106}$$

and

$$\begin{aligned} |\Delta_h f(x) - \Delta_h f(y)| &\leq |f(x) - f(y)| + |f(x+h) - f(y+h)| \\ &\leq 4\|f\|_{\mathcal{C}_b^\gamma(\mathbb{R}^d)} |x - y|^\gamma, \end{aligned} \tag{107}$$

for all $x, y, h \in \mathbb{R}^d$, i.e. (105) holds for $\alpha = 0$ and $\alpha = \gamma$. Next we show that (105) holds for $\alpha = \gamma/2$, for which we use interpolation theory. Let $f = u + v$ for $u \in C_b(\mathbb{R}^d)$ and $v \in C_b^2(\mathbb{R}^d)$. Clearly,

$$|\Delta_h u(x) - \Delta_h u(y)| \leq 4\|u\|_\infty,$$

and, by the gradient theorem,

$$\begin{aligned}|\Delta_h v(x) - \Delta_h v(y)| &= \left| h \int_0^1 (\nabla v(x+rh) - \nabla v(y+rh))\, dr \right| \\ &\leq |h|\,|x-y|\,\|v\|_{C_b^2(\mathbb{R}^d)},\end{aligned}$$

for all $x, y, h \in \mathbb{R}^d$. Hence,

$$|\Delta_h f(x) - \Delta_h f(y)| \leq 4\|u\|_\infty + |h|\,|x-y|\,\|v\|_{C_b^2(\mathbb{R}^d)}, \quad x, y, h \in \mathbb{R}^d.$$

Since $\mathcal{C}_b^\gamma(\mathbb{R}^d)$ is the real interpolation space[2] $(C_b(\mathbb{R}^d), C_b^2(\mathbb{R}^d))_{\gamma/2,\infty}$ (cf. [52, Section 2.7.2]), this implies that there exists a constant $C > 0$ such that

$$|\Delta_h f(x) - \Delta_h f(y)| \leq C|h|^{\gamma/2}|x-y|^{\gamma/2}\|f\|_{\mathcal{C}_b^\gamma(\mathbb{R}^d)}, \tag{108}$$

which shows (105) for $\alpha = \gamma/2$. Now let $\alpha \in (0, \gamma/2)$. For $|h| \leq |x-y|$, (106) gives

$$|\Delta_h f(x) - \Delta_h f(y)| \leq 4\|f\|_{\mathcal{C}_b^\gamma(\mathbb{R}^d)}|h|^\gamma \leq 4\|f\|_{\mathcal{C}_b^\gamma(\mathbb{R}^d)}|h|^\alpha|x-y|^{\gamma-\alpha}.$$

If $|h| > |x-y|$, then (108) gives

$$\begin{aligned}|\Delta_h f(x) - \Delta_h f(y)| &\leq C\|f\|_{\mathcal{C}_b^\gamma(\mathbb{R}^d)}|x-y|^{\gamma/2}|h|^{\gamma/2} \\ &\leq C\|f\|_{\mathcal{C}_b^\gamma(\mathbb{R}^d)}|x-y|^\alpha|h|^{\gamma/2+(\gamma/2-\alpha)},\end{aligned}$$

where we used $\alpha < \gamma/2$ for the second estimate. For $\alpha \in (\gamma/2, \gamma)$, a very similar reasoning shows that (105) follows from (107) and (108). □

Appendix D: A Separation Theorem for Closed Subsets

In Sect. 6, we used the following result on the smooth separation of closed subsets of $\mathbb{R}^d$.

Lemma D.1 *Let $F, G \subseteq \mathbb{R}^d$ be closed sets. If*

$$d(F, G) = \inf\{|x-y|; x \in F, y \in G\} > 0, \tag{109}$$

then there exists a function $f \in C_b^\infty(\mathbb{R}^d)$, $0 \leq f \leq 1$, such that

$$f^{-1}(\{0\}) = F \quad and \quad f^{-1}(\{1\}) = G. \tag{110}$$

[2] More precisely, the norm on the interpolation space $(C_b(\mathbb{R}^d), C_b^2(\mathbb{R}^d))_{\gamma/2,\infty}$ is equivalent to the norm on $\mathcal{C}_b^\gamma(\mathbb{R}^d)$.

It is well known (see e.g. [35]) that for closed sets $F, G \subseteq \mathbb{R}^d$ satisfying (109), there exists $f \in C^\infty(\mathbb{R}^d)$, $0 \leq f \leq 1$, satisfying (110); however, we could not find a reference for the fact that (109) implies boundedness of the derivatives of f. It is not difficult to see that boundedness of the derivatives fails, in general, to hold if $d(F, G) = 0$; consider for instance $F := \mathbb{R} \times (-\infty, 0]$ and $G := \{(x, y); \, y \geq e^x\}$.

Proof of Lemma D.1 As $d(F, G) > 0$, we can choose $\varepsilon > 0$ such that the sets

$$F_\varepsilon := F + \overline{B(0,\varepsilon)}, \quad G_\varepsilon := G + \overline{B(0,\varepsilon)}$$

are disjoint. It is known (see e.g. [35, Problem 2–14]) that there exists $h \in C^\infty(\mathbb{R}^d)$, $0 \leq h \leq 1$, such that $h^{-1}(\{0\}) = F_\varepsilon$ and $h^{-1}(\{1\}) = G_\varepsilon$. Pick $\varphi \in C_c^\infty(\mathbb{R}^d)$, $\varphi \geq 0$, such that $\operatorname{supp} \varphi = \overline{B(0,\varepsilon)}$ and $\int_{\mathbb{R}^d} \varphi(y) \, dy = 1$, and set

$$f(x) := (h * \varphi)(x) = \int_{\mathbb{R}^d} h(y)\varphi(x - y) \, dy, \quad x \in \mathbb{R}^d.$$

Since f is the convolution of a bounded continuous function with a smooth function with compact support, it follows that f is smooth and its derivatives are given by

$$\partial_x^\alpha f(x) = \int_{\mathbb{R}^d} h(y)\partial_x^\alpha \varphi(x - y) \, dy, \quad x \in \mathbb{R}^d,$$

for any multi-index $\alpha \in \mathbb{N}_0^d$ (see e.g. [48]). In particular, $\|\partial^\alpha f\|_\infty \leq \|\partial^\alpha \varphi\|_{L^1} < \infty$, and so $f \in C_b^\infty(\mathbb{R}^d)$. Moreover, as $\operatorname{supp} \varphi \subseteq \overline{B(0,\varepsilon)}$, it is obvious that $f(x) = 0$ for any $x \in F$ and $f(x) = 1$ for $x \in G$. It remains to check that $0 < f(x) < 1$ for any $x \in (F \cup G)^c$.

Case 1: $x \in \mathbb{R}^d \backslash (F_\varepsilon \cup G_\varepsilon)$. Then $0 < h(x) < 1$, and so we can choose $r \in (0, \varepsilon)$ such that

$$0 < \inf_{|y-x| \leq r} h(y) \leq \sup_{|y-x| \leq r} h(y) < 1.$$

Since $\operatorname{supp} \varphi = \overline{B(0,\varepsilon)} \supseteq \overline{B(0,r)}$, this implies

$$\begin{aligned} f(x) &\leq \int_{\mathbb{R}^d \backslash \overline{B(x,r)}} \varphi(x - y) \, dy + \sup_{|y-x| \leq r} h(y) \int_{\overline{B(x,r)}} \varphi(x - y) \, dy \\ &< \int_{\mathbb{R}^d} \varphi(x - y) \, dy = 1. \end{aligned}$$

A very similar estimate shows $f(x) > 0$.

Case 2: $x \in F_\varepsilon \backslash F$. We have $\overline{B(x,\varepsilon)} \cap F^c \neq \emptyset$, and so there exist $y \in \mathbb{R}^d$ and $r > 0$ such that

$$\overline{B(y,r)} \subseteq F^c \cap \overline{B(x,\varepsilon)}.$$

In particular,

$$0 < \inf_{z \in \overline{B(y,r)}} h(z) \leq \sup_{z \in \overline{B(y,r)}} h(z) < 1.$$

As $\operatorname{supp} \varphi = \overline{B(0, \varepsilon)}$, it follows much as in the first case that $0 < f(x) < 1$.

Case 3: $x \in G_\varepsilon \backslash G$. Analogous to Case 2. □

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