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Two Hypotheses on the Exponential Class in the Class Of *O*-subexponential Infinitely Divisible Distributions

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Abstract
Two hypotheses on the class $\mathcal{L}(\gamma)$ in the class $\mathcal{OS} \cap \mathcal{ID}$ are discussed. Two weak hypotheses on the class $\mathcal{L}(\gamma)$ in the class $\mathcal{OS} \cap \mathcal{ID}$ are proved. A necessary and sufficient condition in order that, for every $t > 0$, the t-th convolution power of a distribution in the class $\mathcal{OS} \cap \mathcal{ID}$ belongs to the class $\mathcal{L}(\gamma)$ is given. Sufficient conditions are given for the validity of two hypotheses on the class $\mathcal{L}(\gamma)$.

Keywords Exponential class · O-subexponentiality · Infinite divisibility · Convolution roots

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1 Introduction and Results

In what follows, we denote by $\mathbb{R}$ the real line and by $\mathbb{R}_+$ the half line $[0, \infty)$. Denote by $\mathbb{N}$ the totality of positive integers and by $a\mathbb{N}$ the set $\{a, 2a, 3a, \dots\}$. The symbol $\delta_a(\mathrm{d}x)$ stands for the delta measure at $a \in \mathbb{R}$. Let η and ρ be probability distributions on $\mathbb{R}$. We denote by $\eta * \rho$ the convolution of η and ρ and by ρ^{n*} n-th convolution power of ρ with the understanding that $\rho^{0*}(\mathrm{d}x) = \delta_0(\mathrm{d}x)$. Denote by $\bar{\xi}(x)$ the tail $\xi((x, \infty))$ of a measure ξ on $\mathbb{R}$ for $x \in \mathbb{R}$. Let $\gamma \geq 0$. We define the γ-exponential moment $\widehat{\xi}(\gamma)$ as

$$\widehat{\xi}(\gamma) := \int_{-\infty}^{\infty} e^{\gamma x} \xi(\mathrm{d}x).$$

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If $\widehat{\xi}(\gamma) < \infty$, we define the Fourier–Laplace transform $\widehat{\xi}(\gamma + iz)$ for $z \in \mathbb{R}$ as

$$\widehat{\xi}(\gamma + iz) := \int_{-\infty}^{\infty} e^{(\gamma+iz)x} \xi(\mathrm{d}x).$$

An integral $\int_a^b g(x)\rho(\mathrm{d}x)$ means $\int_{a+}^{b+} g(x)\rho(\mathrm{d}x)$. For positive functions $f_1(x)$ and $g_1(x)$ on $[A, \infty)$ for some $A \in \mathbb{R}$, we define the relation $f_1(x) \sim g_1(x)$ by $\lim_{x\to\infty} f_1(x)/g_1(x) = 1$ and the relation $f_1(x) \asymp g_1(x)$ by

$$0 < \liminf_{x\to\infty} f_1(x)/g_1(x) \le \limsup_{x\to\infty} f_1(x)/g_1(x) < \infty.$$

Let $\gamma \ge 0$. A distribution ρ on $\mathbb{R}$ belongs to the class $\mathcal{L}(\gamma)$ if $\overline{\rho}(x) > 0$ for all $x > 0$ and, for every $a \in \mathbb{R}$,

$$\overline{\rho}(x + a) \sim e^{-\gamma a}\overline{\rho}(x).$$

A distribution ρ on $\mathbb{R}$ belongs to the class $\mathcal{S}(\gamma)$ if $\rho \in \mathcal{L}(\gamma)$, $\widehat{\rho}(\gamma) < \infty$, and

$$\overline{\rho^{2*}}(x) \sim 2\widehat{\rho}(\gamma)\overline{\rho}(x).$$

A distribution ρ on $\mathbb{R}$ belongs to the class $\mathcal{OL}$ if $\overline{\rho}(x) > 0$ for $x > 0$ and, for all $a \ge 0$,

$$\overline{\rho}(x - a) \asymp \overline{\rho}(x).$$

A distribution ρ on $\mathbb{R}$ belongs to the class $\mathcal{OS}$ if $\overline{\rho}(x) > 0$ for all $x > 0$ and

$$\overline{\rho^{2*}}(x) \asymp \overline{\rho}(x).$$

Note that the class $\mathcal{OS}$ is included in the class $\mathcal{OL}$. A distribution ρ on $\mathbb{R}$ belongs to the class $\mathcal{S}_\sharp$ if $\rho \in \mathcal{OS}$ and

$$\limsup_{A\to\infty} \limsup_{x\to\infty} \frac{\overline{\rho}(x - A)\bar{\rho}(A) + \int_A^{x-A} \bar{\rho}(x - u)\rho(\mathrm{d}u)}{\bar{\rho}(x)} = 0.$$

The class $\mathcal{S}_\sharp$ includes $\cup_{\gamma\ge0}\mathcal{S}(\gamma)$, and it is closed under convolution powers. A finite measure ξ satisfies the *Wiener condition* if $\widehat{\xi}(iz) \neq 0$ for every $z \in \mathbb{R}$. Denote by $\mathcal{W}$ the totality of finite measures on $\mathbb{R}$ satisfying the Wiener condition. We denote by $\mathcal{ID}$ the class of all infinitely divisible distributions on $\mathbb{R}$. For $\mu \in \mathcal{ID}$, denote by ν its Lévy measure. Under the assumption that $\bar{\nu}(c) > 0$ for every $c > 0$, define $\nu_1(\mathrm{d}x) := 1_{(1,\infty)}(x)\nu(\mathrm{d}x)/\bar{\nu}(1)$. Let $\mu \in \mathcal{ID}$. We define a compound Poisson distribution μ_1 with $c = \bar{\nu}(1)$ as

$$\mu_1(\mathrm{d}x) := e^{-c} \sum_{k=0}^{\infty} \frac{c^k}{k!} \nu_1^{k*}(\mathrm{d}x).$$

Denote by μ^{t*} the t-th convolution power of $\mu \in \mathcal{ID}$ for $t > 0$. Note that μ^{t*} is the distribution of X_t for a certain Lévy process $\{X_t\}$ on $\mathbb{R}$. Let $\gamma \geq 0$. Define $T(\mu, \gamma)$ as

$$T(\mu, \gamma) := \{t > 0 : \mu^{t*} \in \mathcal{L}(\gamma)\}.$$

Since the class $\mathcal{L}(\gamma)$ is closed under convolutions by Theorem 3 of Embrechts and Goldie [2], $T(\mu, \gamma)$ is empty or an additive semigroup in $(0, \infty)$. We see from Lemma 2.2 that for $\mu \in \mathcal{OS} \cap \mathcal{ID}$, there are positive integers n such that $\nu_1^{n*} \in \mathcal{OS}$. Let n_0 be the positive integer defined by (2.1). Note that we do not yet know an example of $\mu \in \mathcal{OS} \cap \mathcal{ID}$ such that $n_0 \geq 3$.

A class $\mathcal{C}$ of distributions is called *closed under convolution roots* if $\rho^{n*} \in \mathcal{C}$ for some $n \in \mathbb{N}$ implies $\rho \in \mathcal{C}$. We see from Shimura and Watanabe [11] that the class $\mathcal{OS}$ is not closed under convolution roots, but from Watanabe and Yamamuro [15] that the class $\mathcal{OS} \cap \mathcal{ID}$ is closed under convolution roots. Embrechts et al. [4] in the one-sided case and Watanabe [13] in the two-sided case proved that the class $\mathcal{S}(0)$ is closed under convolution roots, and Embrechts and Goldie [2] conjectured that the class $\mathcal{L}(\gamma)$ with $\gamma \geq 0$ is closed under convolution roots, but Shimura and Watanabe [12] showed that the class $\mathcal{L}(\gamma)$ with $\gamma \geq 0$ is not closed under convolution roots. Moreover, Watanabe and Yamamuro [16] proved that the class $\mathcal{S}_{ac}$ of all absolutely continuous distributions on $\mathbb{R}$ with subexponential densities is not closed under convolution roots. Embrechts and Goldie [3] conjectured that the class $\mathcal{S}(\gamma)$ with $\gamma > 0$ is closed under convolution roots. Watanabe [13] proved that $\mathcal{S}(\gamma) \cap \mathcal{ID}$ with $\gamma \geq 0$ is closed under convolution roots, but Watanabe [14] showed that the class $\mathcal{S}(\gamma)$ with $\gamma > 0$ is not closed under convolution roots. We add the following. Klüppelberg [5] showed that the class $\mathcal{OS}$ is closed under convolutions. The class $\mathcal{S}(\gamma)$ is closed under convolution powers for $\gamma \geq 0$, but Leslie [7], for $\gamma = 0$, and Klüppelberg and Villasenor [6], for $\gamma > 0$, proved that the class $\mathcal{S}(\gamma)$ is not closed under convolutions.

We consider the following two hypotheses on the class $\mathcal{L}(\gamma)$ in the class $\mathcal{OS} \cap \mathcal{ID}$:

Hypothesis I Let $\gamma \geq 0$. For every $\mu \in \mathcal{OS} \cap \mathcal{ID}$, if $\mu^{n*} \in \mathcal{L}(\gamma)$ for some $n \in \mathbb{N}$, then $\mu^{(n+1)*} \in \mathcal{L}(\gamma)$.

Hypothesis II Let $\gamma \geq 0$. For every $\mu \in \mathcal{OS} \cap \mathcal{ID}$, if $\mu^{n*} \in \mathcal{L}(\gamma)$ for some $n \in \mathbb{N}$, then $\mu \in \mathcal{L}(\gamma)$.

We also consider the weak version of the above hypotheses:

Hypothesis I′ Let $\gamma \geq 0$. For every $\mu \in \mathcal{OS} \cap \mathcal{ID}$, if $\mu^{n*}, \mu^{(n+1)*} \in \mathcal{L}(\gamma)$ for some $n \in \mathbb{N}$, then $\mu^{(n+2)*} \in \mathcal{L}(\gamma)$.

Hypothesis II′ Let $\gamma \geq 0$. For every $\mu \in \mathcal{OS} \cap \mathcal{ID}$, if $\mu^{n*}, \mu^{(n+1)*} \in \mathcal{L}(\gamma)$ for some $n \in \mathbb{N}$, then $\mu \in \mathcal{L}(\gamma)$.

Let $\gamma \geq 0$. Define

$$\mathcal{A}(\gamma) := \{\mu \in \mathcal{OS} \cap \mathcal{ID} : T(\mu, \gamma) = (0, \infty)\};$$
$$\mathcal{B}(\gamma) := \{\mu \in \mathcal{OS} \cap \mathcal{ID} : T(\mu, \gamma) = \emptyset\};$$

and

$$\mathcal{C}(\gamma) := \{\mu \in \mathcal{OS} \cap \mathcal{ID} : T(\mu, \gamma) = a_0\mathbb{N} \text{ with some } a_0 > 0\}.$$

Theorem 1.1 *Let $\gamma \geq 0$ and $\mu \in \mathcal{OS} \cap \mathcal{ID}$. We have the following:*

(i) *$\mathcal{OS} \cap \mathcal{ID} = \mathcal{A}(\gamma) \cup \mathcal{B}(\gamma) \cup \mathcal{C}(\gamma)$. Thus, Hypotheses I′ and II′ are true.*
(ii) *The relation $\mu \in \mathcal{A}(\gamma)$ holds if and only if, for all $a \geq 0$,*

$$\lim_{x\to\infty} \frac{e^{-\gamma a}\overline{\nu_1}(x-a) - \overline{\nu_1}(x)}{\overline{\nu_1^{n_0 *}}(x)} = 0. \tag{1.1}$$

If $\mu \in \mathcal{A}(\gamma)$, then $\nu_1^{n*} \notin \mathcal{L}(\gamma) \cap \mathcal{OS}$ for $1 \leq n \leq n_0 - 1$ and $\nu_1^{n*} \in \mathcal{L}(\gamma) \cap \mathcal{OS}$ for $n \geq n_0$.

Corollary 1.1 *Let $\gamma \geq 0$. Then the following are equivalent:*

(1) *Hypothesis I is true.*
(2) *Hypothesis II is true.*
(3) *$\mathcal{C}(\gamma)$ is empty.*
(4) *For every $\mu \in \mathcal{OS} \cap \mathcal{ID}$ it holds that, for every $2t \in T(\mu, \gamma)$ and for every $a \geq 0$,*

$$\limsup_{x\to\infty} \limsup_{\lambda\to\infty} \frac{\left|\int_x^{\lambda - x} (e^{-\gamma a}\overline{\mu_1^{t*}}(\lambda - a - u) - \overline{\mu_1^{t*}}(\lambda - u))\mu_1^{t*}(du)\right|}{\overline{\mu_1^{t*}}(\lambda)} = 0. \tag{1.2}$$

Remark 1.1 Let $\gamma = 0$. Then, $\mathcal{C}(0)$ is empty and Hypotheses I and II are true. The relation $\mu \in \mathcal{A}(0)$ holds if and only if

$$\lim_{x\to\infty} \frac{\nu_1((x, x+1])}{\overline{\nu_1^{n_0 *}}(x)} = 0.$$

If $\mu \in \mathcal{A}(0)$, then $\nu_1^{n*} \notin \mathcal{L}(0) \cap \mathcal{OS}$ for $1 \leq n \leq n_0 - 1$ and $\nu_1^{n*} \in \mathcal{L}(0) \cap \mathcal{OS}$ for $n \geq n_0$. Xu et al. showed in Theorem 2.2 of [18] an example of $\mu \in \mathcal{A}(0)$ with $n_0 = 2$.

For $\gamma > 0$, we cannot yet answer the question whether Hypotheses I and II are true. However, under some additional assumptions in terms of Lévy measure, we establish that $\mathcal{C}(\gamma)$ is empty.

Proposition 1.1 *Let $\gamma > 0$ and $\mu \in \mathcal{OS} \cap \mathcal{ID}$. Suppose that, for every $a \geq 0$,*

$$\liminf_{x\to\infty} e^{-\gamma a}\bar{\nu}_1(x-a)/\bar{\nu}_1(x) \geq 1. \tag{1.3}$$

Then, we have either $T(\mu, \gamma) = (0, \infty)$ or $\emptyset$.

Remark 1.2 Cui et al. [1] proved a result analogous to the above proposition under a stronger assumption. Xu et al. showed in Theorem 1.1 of [19] an example of the case where $T(\mu, \gamma) \neq \emptyset$ in the above proposition.

Proposition 1.2 *Let $\gamma > 0$ and $\mu \in \mathcal{OS} \cap \mathcal{ID}$. Suppose that $\nu_1^{2*} \in \mathcal{L}(\gamma)$ and the real part of $\widehat{\nu}_1(\gamma + iz)$ is not zero for every $z \in \mathbb{R}$. Then, either $T(\mu, \gamma) = (0, \infty)$ or $\emptyset$.*

Proposition 1.3 *Let $\gamma > 0$ and $\mu \in \mathcal{OS} \cap \mathcal{ID}$. Suppose that there exists $n_1 \in \mathbb{N}$ such that $\nu_1^{n_1 *} \in \mathcal{S}_\sharp$. Then, either $T(\mu, \gamma) = (0, \infty)$ or $\emptyset$. The equality $T(\mu, \gamma) = (0, \infty)$ holds if and only if $\nu_1 \in \mathcal{S}(\gamma)$.*

Remark 1.3 Watanabe made in Theorem 1.1 of [14] a distribution $\eta \in \mathcal{S}_\sharp$ such that $\eta^{n*} \in \mathcal{S}(\gamma)$ for every $n \geq 2$ but $\eta \notin \mathcal{S}(\gamma)$. Thus, taking this η as ν_1, then Proposition 1.3 holds with $T(\mu, \gamma) = \emptyset$.

2 Preliminaries

In this section, we give several basic results as preliminaries. Pakes [8] proved the following.

Lemma 2.1 (Lemmas 2.1 and 2.5 of [8]) *Let $\mu \in \mathcal{ID}$. Then we have $\mu \in \mathcal{L}(\gamma)$ if and only if $\mu_1 \in \mathcal{L}(\gamma)$.*

Watanabe and Yamamuro [15] proved the following.

Lemma 2.2 (Proposition 3.1 of [15]) *Suppose that $\mu \in \mathcal{ID}$. Then, we have $\mu \in \mathcal{OS}$ if and only if there is $n \in \mathbb{N}$ such that $\nu_1^{n*} \in \mathcal{OS}$ and $\overline{\mu_1^{t*}}(x) \asymp \overline{\nu_1^{n*}}(x)$ for any $t > 0$.*

For $\mu \in \mathcal{OS} \cap \mathcal{ID}$, define $n_0 \in \mathbb{N}$ as

$$n_0 := \min\{n \in \mathbb{N} : \nu_1^{n*} \in \mathcal{OS}\}. \tag{2.1}$$

Lemma 2.3 *Let $\mu \in \mathcal{OS} \cap \mathcal{ID}$.*

(i) *There exists $C(a) > 0$ such that, for all $a \geq 0$ and all $x > 0$,*

$$\overline{\nu_1^{n_0*}}(x - a) \leq C(a)\overline{\nu_1^{n_0*}}(x).$$

(ii) *There exists $K > 1$ such that, for all $n \in \mathbb{N}$ and all $x > 0$,*

$$\overline{\nu_1^{n*}}(x) \leq K^n \overline{\nu_1^{n_0*}}(x).$$

Proof Assertion (i) is clear since $\nu_1^{n_0*} \in \mathcal{OS} \subset \mathcal{OL}$. We see from Proposition 2.4 of Shimura and Watanabe [11] that there exists $K_1 > 1$ such that, for all $k \in \mathbb{N}$ and all $x > 0$,

$$\overline{\nu_1^{(kn_0)*}}(x) \leq K_1^k \overline{\nu_1^{n_0*}}(x).$$

Note that, for $m \leq n$,

$$\overline{\nu_1^{m*}}(x) \leq \overline{\nu_1^{n*}}(x).$$

Hence, we have, for $0 \le j \le n_0 - 1$ and for all $k \in \mathbb{N}$, with $K = K_1^{2/n_0} > 1$

$$\overline{\nu_1^{(kn_0+j)*}}(x) \le K^{(kn_0+j)}\overline{\nu_1^{n_0*}}(x).$$

This inequality holds for $k = 0$ too. Thus, assertion (ii) is true. □

Under the assumption that $\zeta \in \mathcal{OS} \subset \mathcal{OL}$, we define the following. Let

$$d^* := \limsup_{x\to\infty} \frac{\overline{\zeta^{2*}}(x)}{\bar{\zeta}(x)} < \infty.$$

Let Λ be the totality of increasing sequences $\{\lambda_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} \lambda_n = \infty$ such that, for every $x \in \mathbb{R}$, the following limit exists and is finite:

$$m(x; \{\lambda_n\}) := \lim_{n\to\infty} \frac{\bar{\zeta}(\lambda_n - x)}{\bar{\zeta}(\lambda_n)}. \tag{2.2}$$

Define, for each sequence $\{x_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} x_n = \infty$, $T_n(y)$ as

$$T_n(y) := \frac{\bar{\zeta}(x_n - y)}{\bar{\zeta}(x_n)}.$$

Since $\{T_n(y)\}_{n=1}^{\infty}$ is a sequence of increasing functions, uniformly bounded on every finite interval, by Helly's selection principle, there exists an increasing subsequence $\{\lambda_n\}$ of $\{x_n\}$ with $\lim_{n\to\infty} \lambda_n = \infty$ such that everywhere on $\mathbb{R}$ (2.2) holds. The limit function $m(x; \{\lambda_n\})$ is increasing and is finite. That is, $\{\lambda_n\} \in \Lambda$. It follows that, under the assumption that $\zeta \in \mathcal{OS}$, there exists an increasing subsequence $\{\lambda_n\} \in \Lambda$ of $\{x_n\}$ for each sequence $\{x_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} x_n = \infty$.

Lemma 2.4 *Suppose that $\zeta \in \mathcal{OS}$. Then, we have the following.*

(i) *If $\{\lambda_n\} \in \Lambda$, then $\{\lambda_n - a\} \in \Lambda$ for every $a \in \mathbb{R}$.*
(ii) *For $\{\lambda_n\} \in \Lambda$,*

$$\int_{-\infty}^{\infty} m(x; \{\lambda_n\})\zeta(dx) < \infty$$

and

$$\lim_{a\to\infty} m(a; \{\lambda_n\})\bar{\zeta}(a) = 0.$$

In particular, if $\zeta \in \mathcal{OS} \cap \mathcal{L}(\gamma)$, then $m(x; \{\lambda_n\}) = e^{\gamma x}$ and $\widehat{\zeta}(\gamma) < \infty$.

Proof We prove (i). Suppose that $\{\lambda_n\} \in \Lambda$. We have, for $x, a \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{\bar{\zeta}(\lambda_n - a - x)}{\bar{\zeta}(\lambda_n - a)} = \frac{m(x + a; \{\lambda_n\})}{m(a; \{\lambda_n\})}.$$

Thus, $\{\lambda_n - a\} \in \Lambda$. Next, we prove (ii). Let ρ be a distribution on $\mathbb{R}$. Note that, for $x > 2A$,

$$\overline{\rho^{2*}}(x) = 2\int_{-\infty}^{A+} \bar{\rho}(x-u)\rho(\mathrm{d}u) + \bar{\rho}(x-A)\bar{\rho}(A) + \int_{A}^{x-A} \bar{\rho}(x-u)\rho(\mathrm{d}u). \tag{2.3}$$

We see from (2.3) that, for $\{\lambda_n\} \in \Lambda$ and $s > 0$,

$$\begin{aligned} d^* &\ge \limsup_{n\to\infty} \frac{\overline{\zeta^{2*}}(\lambda_n)}{\bar{\zeta}(\lambda_n)} \\ &\ge 2\limsup_{n\to\infty} \int_{-\infty}^{s+} \frac{\bar{\zeta}(\lambda_n - x)}{\bar{\zeta}(\lambda_n)} \zeta(\mathrm{d}x) \\ &\ge 2\int_{-\infty}^{s+} m(x;\{\lambda_n\})\zeta(\mathrm{d}x). \end{aligned}$$

As $s \to \infty$, we have

$$\int_{-\infty}^{\infty} m(x;\{\lambda_n\})\zeta(\mathrm{d}x) < \infty.$$

Since $m(x;\{\lambda_n\})$ is increasing in x, we have

$$\begin{aligned} &\lim_{a\to\infty} m(a;\{\lambda_n\})\bar{\zeta}(a) \\ &\quad\le \lim_{a\to\infty} \int_{a+}^{\infty} m(x;\{\lambda_n\})\zeta(\mathrm{d}x) = 0. \end{aligned}$$

Hence, if $\zeta \in \mathcal{OS} \cap \mathcal{L}(\gamma)$, then $m(x;\{\lambda_n\}) = e^{\gamma x}$ and $\widehat{\zeta}(\gamma) < \infty$. Thus, we have proved the lemma. □

Pakes [8,9] asserted and Watanabe [13] finally proved the following.

Lemma 2.5 (Theorem 1.1 of [13]) *Let* $\gamma \ge 0$. *Then* $\mu \in \mathcal{ID} \cap \mathcal{S}(\gamma)$ *if and only if* $\nu_1 \in \mathcal{S}(\gamma)$.

Lemma 2.6 *Let* $\gamma \ge 0$. *Suppose that* $\rho \in \mathcal{S}_\sharp$.

(i) *If* $\bar{\eta}(x) \asymp \bar{\rho}(x)$, *then* $\eta \in \mathcal{S}_\sharp$.
(ii) $\rho \in \mathcal{S}(\gamma)$ *if and only if* $\rho \in \mathcal{L}(\gamma)$.

Proof Suppose that $\rho \in \mathcal{S}_\sharp$. We prove (i). If $\bar{\eta}(x) \asymp \bar{\rho}(x)$, then there is $C > 0$ such that $\bar{\eta}(x) \le C\bar{\rho}(x)$ for $x \in \mathbb{R}$. By using integration by parts in the second inequality, we obtain that

$$
\begin{aligned}
&\bar{\eta}(x-A)\bar{\eta}(A)+\int_A^{x-A}\bar{\eta}(x-u)\eta(\mathrm{d}u)\\
&\quad\le C^2\bar{\rho}(x-A)\bar{\rho}(A)+C\int_A^{x-A}\bar{\rho}(x-u)\eta(\mathrm{d}u)\\
&\quad\le 2C^2\bar{\rho}(x-A)\bar{\rho}(A)+C^2\int_A^{x-A}\bar{\rho}(x-u)\rho(\mathrm{d}u).
\end{aligned}
$$

Thus, we see that

$$
\limsup_{A\to\infty}\limsup_{x\to\infty}\frac{(\overline{\eta}(x-A)\bar{\eta}(A)+\int_A^{x-A}\bar{\eta}(x-u)\eta(\mathrm{d}u))}{\bar{\eta}(x)}=0.
$$

That is, $\eta\in\mathcal{S}_\sharp$. Next we prove (ii). If $\rho\in\mathcal{S}(\gamma)$, then clearly $\rho\in\mathcal{L}(\gamma)$. Note that, for $x>2A$, (2.3) holds. If $\rho\in\mathcal{S}_\sharp\cap\mathcal{L}(\gamma)$, then we have

$$
\begin{aligned}
&\lim_{x\to\infty}\frac{\overline{\rho^{2*}}(x)}{\bar{\rho}(x)}\\
&\quad=\lim_{A\to\infty}2\int_{-\infty}^{A+}\lim_{x\to\infty}\frac{\bar{\rho}(x-u)}{\bar{\rho}(x)}\rho(\mathrm{d}u)\\
&\quad=2\widehat{\rho}(\gamma)<\infty.
\end{aligned}
$$

Thus, we see that $\rho\in\mathcal{S}(\gamma)$. □

Watanabe [14] extended Wiener's approximation theorem in [17] as follows.

Lemma 2.7 (Lemma 2.6 of Watanabe [14]) *Let ξ be a finite measure on $\mathbb{R}$. The following are equivalent:*

(1) $\xi\in\mathcal{W}$.
(2) *If, for a bounded measurable function $g(x)$ on $\mathbb{R}$,*

$$
\int_{-\infty}^{\infty}g(x-t)\xi(dt)=0\quad\text{for a.e. } x\in\mathbb{R},
$$

then $g(x)=0$ for a.e. $x\in\mathbb{R}$.

3 Convolution Lemmas

In this section, we give important lemmas on convolutions.

Lemma 3.1 *Let $\gamma\ge 0$. Suppose that $\zeta\in\mathcal{OS}$. For $j=1,2$, let ρ_j be distributions on $\mathbb{R}_+$ satisfying*

$$
\bar{\rho}_j(x)\le C_j\bar{\zeta}(x)\quad\text{with some } C_j>0\text{ for all } x>0. \tag{3.1}
$$

Let $\{\lambda_n\}\in\Lambda$.

(i) *Let $\lambda_n>a+x$ and $x>0$. We have, for every $a\ge 0$,*

$$e^{-\gamma a}\overline{\rho_1 * \rho_2}(\lambda_n - a) - \overline{\rho_1 * \rho_2}(\lambda_n) =: \sum_{j=1}^{4} I_j, \tag{3.2}$$

where

$$\begin{aligned}
I_1 &:= -\int_{\lambda_n - a - x}^{\lambda_n - x} \overline{\rho_1}(\lambda_n - y)\rho_2(dy),\\
I_2 &:= \overline{\rho_1}(x)(e^{-\gamma a}\overline{\rho_2}(\lambda_n - a - x) - \overline{\rho_2}(\lambda_n - x)),\\
I_3 &:= \int_{0-}^{(\lambda_n - a - x)+} (e^{-\gamma a}\overline{\rho_1}(\lambda_n - a - y) - \overline{\rho_1}(\lambda_n - y))\rho_2(dy),
\end{aligned}$$

and

$$I_4 := \int_{0-}^{x+} (e^{-\gamma a}\overline{\rho_2}(\lambda_n - a - y) - \overline{\rho_2}(\lambda_n - y))\rho_1(dy).$$

(ii) *We have for* $j = 1, 2$

$$\limsup_{x\to\infty}\limsup_{n\to\infty} \frac{|I_j|}{\bar{\zeta}(\lambda_n)} = 0. \tag{3.3}$$

Proof By using integration by parts, we have

$$\begin{aligned}
&\overline{\rho_1 * \rho_2}(\lambda_n - a)\\
&\quad= \int_{0-}^{(\lambda_n - a - x)+} \overline{\rho_1}(\lambda_n - a - y)\rho_2(\mathrm{d}y)\\
&\qquad+ \int_{\lambda_n - a - x}^{\lambda_n - a} \overline{\rho_1}(\lambda_n - a - y)\rho_2(\mathrm{d}y) + \overline{\rho_2}(\lambda_n - a)\\
&\quad= \int_{0-}^{(\lambda_n - a - x)+} \overline{\rho_1}(\lambda_n - a - y)\rho_2(\mathrm{d}y) + \int_{0-}^{x+} \overline{\rho_2}(\lambda_n - a - y)\rho_1(\mathrm{d}y)\\
&\qquad+ \overline{\rho_1}(x)\overline{\rho_2}(\lambda_n - a - x),
\end{aligned}$$

and

$$\begin{aligned}
&\overline{\rho_1 * \rho_2}(\lambda_n)\\
&\quad= \int_{0-}^{(\lambda_n - a - x)+} \overline{\rho_1}(\lambda_n - y)\rho_2(\mathrm{d}y) + \int_{\lambda_n - a - x}^{\lambda_n - x} \overline{\rho_1}(\lambda_n - y)\rho_2(\mathrm{d}y)\\
&\qquad+ \int_{\lambda_n - x}^{\lambda_n} \overline{\rho_1}(\lambda_n - y)\rho_2(\mathrm{d}y) + \overline{\rho_2}(\lambda_n)\\
&\quad= \int_{0-}^{(\lambda_n - a - x)+} \overline{\rho_1}(\lambda_n - y)\rho_2(\mathrm{d}y) + \int_{\lambda_n - a - x}^{\lambda_n - x} \overline{\rho_1}(\lambda_n - y)\rho_2(\mathrm{d}y)\\
&\qquad+ \int_{0-}^{x+} \overline{\rho_2}(\lambda_n - y)\rho_1(\mathrm{d}y) + \overline{\rho_1}(x)\overline{\rho_2}(\lambda_n - x).
\end{aligned}$$

Thus, assertion (i) is valid. We have by Lemma 2.4 for $j = 1, 2$

$$\begin{aligned}
&\limsup_{x\to\infty}\limsup_{n\to\infty}\frac{|I_j|}{\bar{\zeta}(\lambda_n)}\\
&\quad\le \limsup_{x\to\infty}\overline{\rho_1}(x)\limsup_{n\to\infty}\frac{\overline{\rho_2}(\lambda_n-a-x)}{\bar{\zeta}(\lambda_n)}\\
&\quad\le C_1C_2\limsup_{x\to\infty}\bar{\zeta}(x)m(x;\{\lambda_n-a\})m(a;\{\lambda_n\})=0.
\end{aligned}$$

Lemma 3.2 *Let $\gamma \ge 0$. Suppose that $\zeta \in \mathcal{OS}$. For $j = 1, 2$, let ρ_j be distributions on $\mathbb{R}_+$ satisfying (3.1). Suppose further that for $j = 1, 2$ and every $a \ge 0$,*

$$\lim_{x\to\infty}\frac{e^{-\gamma a}\bar{\rho}_j(x-a)-\bar{\rho}_j(x)}{\bar{\zeta}(x)}=0.$$

Then, for every $a \ge 0$,

$$\lim_{x\to\infty}\frac{e^{-\gamma a}\overline{\rho_1*\rho_2}(x-a)-\overline{\rho_1*\rho_2}(x)}{\bar{\zeta}(x)}=0. \tag{3.4}$$

Proof. Let $\{\lambda_n\} \in \Lambda$. By the assumption for $j = 1$, there is $\epsilon(x) > 0$ such that $\epsilon(x) \to 0$ as $x \to \infty$ and

$$|e^{-\gamma a}\overline{\rho_1}(\lambda_n-a-y)-\overline{\rho_1}(\lambda_n-y)|\le\epsilon(x)\zeta(\lambda_n-y)$$

for $0 \le y \le \lambda_n - a - x$. Thus, we have

$$\begin{aligned}
&\limsup_{x\to\infty}\limsup_{n\to\infty}\frac{|I_3|}{\bar{\zeta}(\lambda_n)}\\
&\quad\le\limsup_{x\to\infty}\epsilon(x)\limsup_{n\to\infty}\frac{\int_{0-}^{(\lambda_n-a-x)+}\bar{\zeta}(\lambda_n-y)\rho_2(\mathrm{d}y)}{\bar{\zeta}(\lambda_n)}\\
&\quad\le\limsup_{x\to\infty}\epsilon(x)\limsup_{n\to\infty}\frac{\bar{\zeta}(\lambda_n)+\int_{a+x}^{\lambda_n}\overline{\rho_2}(\lambda_n-y)\zeta(\mathrm{d}y)}{\bar{\zeta}(\lambda_n)}\\
&\quad\le\limsup_{x\to\infty}\epsilon(x)\limsup_{n\to\infty}\frac{\bar{\zeta}(\lambda_n)+C_2\overline{\zeta^{2*}}(\lambda_n)}{\bar{\zeta}(\lambda_n)}=0.
\end{aligned} \tag{3.5}$$

As in the above argument, we have

$$\limsup_{x\to\infty}\limsup_{n\to\infty}\frac{|I_4|}{\bar{\zeta}(\lambda_n)}=0.$$

Thus, by (3.2) and (3.3) of Lemma 3.1, we have proved (3.4). □

Lemma 3.3 *Let $\gamma \ge 0$. Suppose that $\zeta \in \mathcal{OS}$. For $j = 1, 2$, let ρ_j be distributions on $\mathbb{R}_+$ satisfying (3.1). Suppose further that, for $j = 1, 2$, and for every $a \ge 0$,*

$$\liminf_{x\to\infty} \frac{e^{-\gamma a}\bar{\rho}_j(x-a)-\bar{\rho}_j(x)}{\bar{\zeta}(x)} \geq 0.$$

Then, we have, for every $a \geq 0$,

$$\liminf_{x\to\infty} \frac{e^{-\gamma a}\overline{\rho_1 * \rho_2}(x-a)-\overline{\rho_1 * \rho_2}(x)}{\bar{\zeta}(x)} \geq 0. \tag{3.6}$$

Proof Let $\{\lambda_n\} \in \Lambda$. Let $\epsilon > 0$ and $a \geq 0$ be arbitrary, and let $n \in \mathbb{N}$ and $x \in (0, \lambda_n - a)$ be sufficiently large such that

$$e^{-\gamma a}\overline{\rho_1}(\lambda_n - a - y) - \overline{\rho_1}(\lambda_n - y) \geq -\epsilon\bar{\zeta}(\lambda_n - y)$$

for $0 \leq y \leq \lambda_n - a - x$ and

$$e^{-\gamma a}\overline{\rho_2}(\lambda_n - a - y) - \overline{\rho_2}(\lambda_n - y) \geq -\epsilon\bar{\zeta}(\lambda_n - y)$$

for $0 \leq y \leq x$. By (3.2) and (3.3) of Lemma 3.1, we have only to prove that

$$\sum_{j=3}^{4} \liminf_{x\to\infty} \liminf_{n\to\infty} \frac{I_j}{\bar{\zeta}(\lambda_n)} \geq 0.$$

We have

$$\begin{aligned} I_3 &\geq -\epsilon \int_{0-}^{(\lambda_n-a-x)+} \bar{\zeta}(\lambda_n - y)\rho_2(\mathrm{d}y) \\ &\geq -\epsilon\left(\bar{\zeta}(\lambda_n) + \int_{a+x}^{\lambda_n} \overline{\rho_2}(\lambda_n - y)\zeta(\mathrm{d}y)\right) \\ &\geq -\epsilon\left(\bar{\zeta}(\lambda_n) + C_2\overline{\zeta^{2*}}(\lambda_n)\right), \end{aligned}$$

and

$$\begin{aligned} I_4 &\geq -\epsilon \int_{0-}^{x+} \bar{\zeta}(\lambda_n - y)\rho_1(\mathrm{d}y) \\ &\geq -\epsilon\left(\bar{\zeta}(\lambda_n) + \int_{\lambda_n - x}^{\lambda_n} \overline{\rho_1}(\lambda_n - y)\zeta(\mathrm{d}y)\right) \\ &\geq -\epsilon\left(\bar{\zeta}(\lambda_n) + C_1\overline{\zeta^{2*}}(\lambda_n)\right). \end{aligned}$$

Thus, we see that

$$\liminf_{n\to\infty} \frac{I_3}{\bar{\zeta}(\lambda_n)} \geq -\epsilon(1 + C_2 d^*),$$

and

$$\liminf_{n\to\infty} \frac{I_4}{\bar{\zeta}(\lambda_n)} \geq -\epsilon(1 + C_1 d^*).$$

Since $\epsilon > 0$ is arbitrary, we established, for $j = 3, 4$,

$$\liminf_{x\to\infty} \liminf_{n\to\infty} \frac{I_j}{\bar{\zeta}(\lambda_n)} \geq 0.$$

Thus, we have proved (3.6). □

Lemma 3.4 *Let $\gamma \geq 0$. Suppose that $\zeta \in \mathcal{OS} \cap \mathcal{L}(\gamma)$. For $j = 1, 2$, let ρ_j be distributions on $\mathbb{R}_+$ satisfying (3.1). Suppose further that, for every $a \geq 0$,*

$$\lim_{x\to\infty} \frac{e^{-\gamma a}\bar{\rho}_1(x-a) - \bar{\rho}_1(x)}{\bar{\zeta}(x)} = 0$$

and, for every $a \geq 0$,

$$\lim_{x\to\infty} \frac{e^{-\gamma a}\overline{\rho_1 * \rho_2}(x-a) - \overline{\rho_1 * \rho_2}(x)}{\bar{\zeta}(x)} = 0 \tag{3.7}$$

and that $e^{\gamma x}\rho_1(dx) \in \mathcal{W}$. Then, we have, for every $a \geq 0$,

$$\lim_{x\to\infty} \frac{e^{-\gamma a}\bar{\rho}_2(x-a) - \bar{\rho}_2(x)}{\bar{\zeta}(x)} = 0. \tag{3.8}$$

Proof Let Λ_2 be the totality of increasing sequences $\{\lambda_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} \lambda_n = \infty$ such that, for every $x \in \mathbb{R}$, the following limit exists and is finite:

$$m_2(x; \{\lambda_n\}) := \lim_{n\to\infty} \frac{\bar{\rho}_2(\lambda_n - x)}{\bar{\zeta}(\lambda_n)}.$$

We have $\Lambda_2 \subset \Lambda$. As for Λ, it follows that, under the assumption that $\zeta \in \mathcal{OS}$ and $\overline{\rho_2}(x) \leq C_2\bar{\zeta}(x)$, there exists an increasing subsequence $\{\lambda_n\} \in \Lambda_2$ of $\{x_n\}$ for each sequence $\{x_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} x_n = \infty$. Let $\{\lambda_n\} \in \Lambda_2$. Recall from Lemma 2.4 that $m(x; \{\lambda_n\}) = e^{\gamma x}$ and $\widehat{\zeta}(\gamma) < \infty$. As in the proof of Lemma 3.2, we have (3.5). We find that, for every $a \in \mathbb{R}$,

$$\begin{aligned} l(x) &:= \lim_{n\to\infty} \frac{I_4}{\bar{\zeta}(\lambda_n)} \\ &= \int_{0-}^{x+} (e^{-\gamma a} m_2(a+y; \{\lambda_n\}) - m_2(y; \{\lambda_n\}))\rho_1(\mathrm{d}y). \end{aligned}$$

Define $M_2(y;\{\lambda_n\}) := e^{-\gamma y} m_2(y;\{\lambda_n\})$. Then $M_2(y;\{\lambda_n\}) \le C_2$ on $\mathbb{R}$. Note that

$$l(x) = \int_{0-}^{x+} (M_2(a+y;\{\lambda_n\}) - M_2(y;\{\lambda_n\}))e^{\gamma y}\rho_1(\mathrm{d}y).$$

We see from (3.2), (3.3) of Lemma 3.1, (3.5), and (3.7) that, for every $a \in \mathbb{R}$,

$$\lim_{x\to\infty} l(x) = \int_{0-}^{\infty} (M_2(a+y;\{\lambda_n\}) - M_2(y;\{\lambda_n\}))e^{\gamma y}\rho_1(\mathrm{d}y) = 0.$$

Thus, we obtain that, for every $a, b \in \mathbb{R}$,

$$\int_{0-}^{\infty} (M_2(a+b+y;\{\lambda_n\}) - M_2(b+y;\{\lambda_n\}))e^{\gamma y}\rho_1(\mathrm{d}y) = 0.$$

Since $e^{\gamma y}\rho_1(\mathrm{d}y) \in \mathcal{W}$, we find from Lemma 2.7 that, for every $a \in \mathbb{R}$,

$$M_2(a+b;\{\lambda_n\}) = M_2(b;\{\lambda_n\}) \text{ for a.e. } b \in \mathbb{R}.$$

Since the function $m_2(x;\{\lambda_n\})$ is increasing, the functions $M_2(x+;\{\lambda_n\})$ and $M_2(x-;\{\lambda_n\})$ exist for all $x \in \mathbb{R}$. Taking $b_n = b_n(a) \downarrow 0$ and $b_n = b_n(a) \uparrow 0$, we have

$$M_2(a+;\{\lambda_n\}) = M_2(0+;\{\lambda_n\}) \text{ and } M_2(a-;\{\lambda_n\}) = M_2(0-;\{\lambda_n\}).$$

As $a \uparrow 0$ in the first equality, we see that

$$M_2(0-;\{\lambda_n\}) = M_2(0+;\{\lambda_n\})$$

and hence, for every $a \in \mathbb{R}$,

$$M_2(a;\{\lambda_n\}) = M_2(0;\{\lambda_n\}).$$

Since $\{\lambda_n\} \in \Lambda_2$ is arbitrary, we have (3.8). □

Lemma 3.5 *Let $\gamma \ge 0$. Suppose that $\zeta \in \mathcal{OS}$. For $j = 1, 2$, let ρ_j be distributions on $\mathbb{R}_+$ satisfying (3.1). Suppose further that, for $j = 1, 2$, and for every $a \ge 0$,*

$$\liminf_{x\to\infty} \frac{e^{-\gamma a}\bar{\rho}_j(x-a) - \bar{\rho}_j(x)}{\bar{\zeta}(x)} \ge 0.$$

If we have, for every $a \ge 0$,

$$\lim_{x\to\infty} \frac{e^{-\gamma a}\overline{\rho_1 * \rho_2}(x-a) - \overline{\rho_1 * \rho_2}(x)}{\bar{\zeta}(x)} = 0,$$

then, for $j = 1, 2$, *and for every* $a \geq 0$,

$$\lim_{x\to\infty} \frac{e^{-\gamma a}\bar{\rho}_j(x-a) - \bar{\rho}_j(x)}{\bar{\zeta}(x)} = 0.$$

Proof Suppose that, for some $a > 0$,

$$\limsup_{x\to\infty} \frac{e^{-\gamma a}\overline{\rho_2}(x-a) - \overline{\rho_2}(x)}{\bar{\zeta}(x)} > 0.$$

Then there is $\{\lambda_n\} \in \Lambda$ such that, for some $a > 0$,

$$\lim_{n\to\infty} \frac{e^{-\gamma a}\overline{\rho_2}(\lambda_n - a) - \overline{\rho_2}(\lambda_n)}{\bar{\zeta}(\lambda_n)} =: \delta_0 > 0.$$

So there is $\delta_1 > 0$ such that, for some $a > 0$,

$$\liminf_{n\to\infty} \frac{e^{-\gamma(a+\delta_1)}\overline{\rho_2}(\lambda_n - a) - \overline{\rho_2}(\lambda_n)}{\bar{\zeta}(\lambda_n)} =: \delta_2 > 0.$$

Take y_0 such that $x > y_0 > \delta_1$ and $\rho_1((y_0 - \delta_1, y_0]) > 0$. Let $\lambda_n' := \lambda_n + y_0$ and $a' := a + \delta_1$. Then we have

$$\begin{aligned} &\int_{y_0-\delta_1}^{y_0} (e^{-\gamma a'}\overline{\rho_2}(\lambda_n' - a' - y) - \overline{\rho_2}(\lambda_n' - y))\rho_1(\mathrm{d}y) \\ &\quad \geq \rho_1((y_0 - \delta_1, y_0])(e^{-\gamma a'}\overline{\rho_2}(\lambda_n - a) - \overline{\rho_2}(\lambda_n)). \end{aligned} \tag{3.9}$$

Let $\lambda_n' > a' + x$ and $x > 0$. Define J as

$$J := e^{-\gamma a'}\overline{\rho_1 * \rho_2}(\lambda_n' - a') - \overline{\rho_1 * \rho_2}(\lambda_n').$$

Then we have as in assertion (i) of Lemma 3.1

$$J = \sum_{j=1}^{4} I_j',$$

where

$$\begin{aligned} I_1' &:= -\int_{\lambda_n'-a'-x}^{\lambda_n'-x} \overline{\rho_1}(\lambda_n' - y)\rho_2(\mathrm{d}y), \\ I_2' &:= \overline{\rho_1}(x)(e^{-\gamma a'}\overline{\rho_2}(\lambda_n' - a' - x) - \overline{\rho_2}(\lambda_n' - x)), \\ I_3' &:= \int_{0-}^{(\lambda_n'-a'-x)+} (e^{-\gamma a'}\overline{\rho_1}(\lambda_n' - a' - y) - \overline{\rho_1}(\lambda_n' - y))\rho_2(\mathrm{d}y), \end{aligned}$$

and

$$I'_4 := \int_{0-}^{x+} (e^{-\gamma a'}\overline{\rho_2}(\lambda'_n - a' - y) - \overline{\rho_2}(\lambda'_n - y))\rho_1(\mathrm{d}y).$$

For $1 \le j \le 3$, let

$$J_j := I'_j,$$

and let

$$I'_4 = \sum_{j=4}^{6} J_j,$$

where

$$J_4 := \int_{0-}^{(y_0-\delta_1)+} (e^{-\gamma a'}\overline{\rho_2}(\lambda'_n - a' - y) - \overline{\rho_2}(\lambda'_n - y))\rho_1(\mathrm{d}y),$$
$$J_5 := \int_{y_0}^{x} (e^{-\gamma a'}\overline{\rho_2}(\lambda'_n - a' - y) - \overline{\rho_2}(\lambda'_n - y))\rho_1(\mathrm{d}y),$$

and

$$J_6 := \int_{y_0-\delta_1}^{y_0} (e^{-\gamma a'}\overline{\rho_2}(\lambda'_n - a' - y) - \overline{\rho_2}(\lambda'_n - y))\rho_1(\mathrm{d}y).$$

Then we have

$$J = \sum_{j=1}^{6} J_j.$$

As in the proof of Lemma 3.3, we see from the assumption and (3.9) that

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{J}{\bar{\zeta}(\lambda'_n)} \\
&\ge \sum_{j=1}^{6} \liminf_{x\to\infty} \liminf_{n\to\infty} \frac{J_j}{\bar{\zeta}(\lambda'_n)} \\
&\ge \liminf_{n\to\infty} \frac{J_6}{\bar{\zeta}(\lambda'_n)} \\
&\ge \liminf_{n\to\infty} \rho_1((y_0-\delta_1, y_0]) \frac{(e^{-\gamma a'}\overline{\rho_2}(\lambda_n - a) - \overline{\rho_2}(\lambda_n))}{\bar{\zeta}(\lambda'_n)} \\
&= \rho_1((y_0-\delta_1, y_0]) \frac{\delta_2}{m(-y_0; \{\lambda_n\})} > 0.
\end{aligned}$$

This is a contradiction. Thus, we have, for every $a \geq 0$,

$$\lim_{x\to\infty} \frac{e^{-\gamma a}\bar{\rho}_2(x-a) - \bar{\rho}_2(x)}{\bar{\zeta}(x)} = 0.$$

By the analogous argument, we have for every $a \geq 0$,

$$\lim_{x\to\infty} \frac{e^{-\gamma a}\bar{\rho}_1(x-a) - \bar{\rho}_1(x)}{\bar{\zeta}(x)} = 0.$$

Thus, we have proved the lemma. □

Lemma 3.6 *Let $\gamma \geq 0$. Let ρ be a distribution on $\mathbb{R}_+$. Suppose that $\rho \in \mathcal{OS}$ and, for every $a \geq 0$,*

$$\liminf_{x\to\infty} e^{-\gamma a}\bar{\rho}(x-a)/\bar{\rho}(x) \geq 1. \tag{3.10}$$

Then, for some positive integer $n \geq 2$, $\rho^{n*} \in \mathcal{L}(\gamma)$ implies that $\rho \in \mathcal{L}(\gamma)$.

Proof Let $\zeta := \rho$. Then we see from Lemma 3.3 that, for every $k \in \mathbb{N}$ and every $a \geq 0$,

$$\liminf_{x\to\infty} \frac{e^{-\gamma a}\overline{\rho^{k*}}(x-a) - \overline{\rho^{k*}}(x)}{\bar{\rho}(x)} \geq 0.$$

Thus, we find that $\rho_1 := \rho$ and $\rho_2 := \rho^{(n-1)*}$ satisfy the assumptions of Lemma 3.5. Hence, we have by Lemma 3.5, for every $a \geq 0$,

$$\lim_{x\to\infty} \frac{e^{-\gamma a}\bar{\rho}(x-a) - \bar{\rho}(x)}{\bar{\rho}(x)} = 0.$$

That is, $\rho \in \mathcal{L}(\gamma)$. □

Remark 3.1 For $\gamma = 0$, the assumption (3.10) necessarily holds, but for $\gamma > 0$, without the assumption (3.10) the lemma does not hold. For $\gamma > 0$, Watanabe [14] made a distribution $\eta \in \mathcal{OS}$ such that $\eta^{n*} \in \mathcal{L}(\gamma)$ for every $n \geq 2$ but $\eta \notin \mathcal{L}(\gamma)$.

4 Proof of Results

In this section, we prove the results stated in Sect. 1.

Lemma 4.1 *Let $\gamma \geq 0$ and $\mu \in \mathcal{OS} \cap \mathcal{ID}$. If, for every $a \geq 0$, (1.1) holds, then, for all $n \in \mathbb{N}$ and every $a \geq 0$,*

$$\lim_{x\to\infty} \frac{e^{-\gamma a}\overline{\nu_1^{n*}}(x-a) - \overline{\nu_1^{n*}}(x)}{\overline{\nu_1^{n_0*}}(x)} = 0, \tag{4.1}$$

and we have $T(\mu,\gamma)=(0,\infty)$.

Proof By induction, we see from Lemma 3.2 that if (1.1) holds for every $a\ge 0$, then, for all $n\in\mathbb{N}$ and every $a\ge 0$, we have (4.1). We have with $c:=\bar{\nu}(1)$, for $t>0$,

$$\mu_1^{t*}:=e^{-ct}\sum_{k=0}^{\infty}\frac{(ct)^k}{k!}\nu_1^{k*}.$$

Suppose that, for all $n\in\mathbb{N}$ and every $a\ge 0$, (4.1) holds. Let $\epsilon>0$ be arbitrary. By Lemma 2.3, we can choose sufficiently large $N\in\mathbb{N}$ such that, for $\epsilon>0$,

$$e^{-ct}\sum_{k=N+1}^{\infty}\frac{(ct)^k}{k!}\frac{|e^{-\gamma a}\overline{\nu_1^{k*}}(x-a)-\overline{\nu_1^{k*}}(x)|}{\overline{\nu_1^{n_0*}}(x)}<\epsilon.$$

We find from (4.1) that, for every $a\ge 0$,

$$\lim_{x\to\infty}e^{-ct}\sum_{k=1}^{N}\frac{(ct)^k}{k!}\frac{e^{-\gamma a}\overline{\nu_1^{k*}}(x-a)-\overline{\nu_1^{k*}}(x)}{\overline{\nu_1^{n_0*}}(x)}=0.$$

Thus, we see that, for every $a\ge 0$ and for every $t>0$,

$$\lim_{x\to\infty}\frac{e^{-\gamma a}\overline{\mu_1^{t*}}(x-a)-\overline{\mu_1^{t*}}(x)}{\overline{\nu_1^{n_0*}}(x)}=0.$$

Since $\overline{\mu_1^{t*}}(x)\asymp\overline{\nu_1^{n_0*}}(x)$ for every $t>0$, we have $T(\mu,\gamma)=(0,\infty)$. □

Lemma 4.2 *Let* $\gamma\ge 0$ *and* $\mu\in\mathcal{OS}\cap\mathcal{ID}$. *If* 0 *is a limit point of* $T(\mu,\gamma)$, *then, for every* $a\ge 0$, (1.1) *holds.*

Proof. Suppose that 0 is a limit point of $T(\mu,\gamma)$. Then, there exists a strictly decreasing sequence $\{t_n\}_{n=1}^{\infty}$ in $T(\mu,\gamma)$ converging to 0 as $n\to\infty$. We have with $c:=\bar{\nu}(1)$

$$\mu_1^{t_n*}:=e^{-ct_n}\sum_{k=0}^{\infty}\frac{(ct_n)^k}{k!}\nu_1^{k*}.$$

Since $\{t_n\}_{n=1}^{\infty}$ in $T(\mu,\gamma)$ and $\overline{\mu_1^{t_n*}}(x)\asymp\overline{\nu_1^{n_0*}}(x)$ from Lemma 2.2, we see that, for every $a\ge 0$,

$$\begin{aligned}&\lim_{x\to\infty}\frac{e^{-\gamma a}\overline{\mu_1^{t_n*}}(x-a)-\overline{\mu_1^{t_n*}}(x)}{\overline{\nu_1^{n_0*}}(x)}\\&=\lim_{x\to\infty}\frac{e^{-\gamma a}\overline{\mu_1^{t_n*}}(x-a)-\overline{\mu_1^{t_n*}}(x)}{\overline{\mu_1^{t_n*}}(x)}\frac{\overline{\mu_1^{t_n*}}(x)}{\overline{\nu_1^{n_0*}}(x)}=0.\end{aligned}$$

Thus, we obtain from Lemma 2.3 that, for every $a \geq 0$,

$$\limsup_{x\to\infty} \left|\frac{e^{-\gamma a}\overline{\nu_1}(x-a) - \overline{\nu_1}(x)}{\overline{\nu_1^{n_0*}}(x)}\right|$$
$$= \limsup_{n\to\infty}\limsup_{x\to\infty} \left|\frac{e^{ct_n}}{ct_n}\frac{e^{-\gamma a}\overline{\mu_1^{t_n*}}(x-a) - \overline{\mu_1^{t_n*}}(x)}{\overline{\nu_1^{n_0*}}(x)} - \frac{e^{-\gamma a}\overline{\nu_1}(x-a) - \overline{\nu_1}(x)}{\overline{\nu_1^{n_0*}}(x)}\right|$$
$$\leq \limsup_{n\to\infty}\limsup_{x\to\infty} \sum_{k=2}^{\infty}\frac{(ct_n)^{(k-1)}}{k!}\frac{e^{-\gamma a}\overline{\nu_1^{k*}}(x-a) + \overline{\nu_1^{k*}}(x)}{\overline{\nu_1^{n_0*}}(x)} = 0.$$

Thus, we have (1.1) for every $a \geq 0$. □

Lemma 4.3 *Let $\gamma \geq 0$ and $\mu \in \mathcal{OS} \cap \mathcal{ID}$. If $t_0, t_1 \in T(\mu,\gamma)$ with $t_1 > t_0$, then $t_1 - t_0 \in T(\mu,\gamma)$. If $T(\mu,\gamma)$ has a limit point, then $T(\mu,\gamma) = (0,\infty)$. If $T(\mu,\gamma)$ has the minimum $a_0 > 0$, then $T(\mu,\gamma) = a_0\mathbb{N}$.*

Proof Suppose that $t_0, t_1 \in T(\mu,\gamma)$ with $t_1 > t_0$. Let $\zeta := \rho_1 := \mu^{t_0*}$ and $\rho_2 := \mu^{(t_1-t_0)*}$. The distribution $e^{\gamma x}\rho_1(dx)/\widehat{\rho_1}(\gamma)$ is an exponentially tilted infinitely divisible distribution and hence itself is infinitely divisible, thus having a non-vanishing characteristic function. That is, $e^{\gamma x}\rho_1(dx) \in \mathcal{W}$. See (iii) of Theorem 25.17 of Sato [10]. Thus, we see from Lemma 3.4 that $\mu^{(t_1-t_0)*} \in \mathcal{L}(\gamma)$. Thus, if $T(\mu,\gamma)$ has a limit point, then 0 is a limit point of $T(\mu,\gamma)$, and hence, by Lemmas 4.1 and 4.2, $T(\mu,\gamma) = (0,\infty)$. If $T(\mu,\gamma)$ has the minimum $a_0 > 0$, then clearly $a_0\mathbb{N} \subset T(\mu,\gamma)$ and $T(\mu,\gamma)\backslash a_0\mathbb{N} = \emptyset$. □

Proof of Theorem 1.1 Assertion (i) is clear from Lemmas 4.1, 4.2, and 4.3. The first part of assertion (ii) is due to Lemmas 4.1 and 4.2. Suppose that $\mu \in \mathcal{A}(\gamma)$. If $n < n_0$, then $\nu_1^{n*} \notin \mathcal{OS}$ simply because of the definition of n_0. If $n \geq n_0$ and x is large, then $\overline{\nu_1^{n*}}(x) \geq \overline{\nu_1^{n_0*}}(x)$, and hence, (4.1) implies that $\nu_1^{n*} \in \mathcal{L}(\gamma)$. □

Proof of Corollary 1.1 Suppose that $\mathcal{C}(\gamma)$ is not empty. Then there is the minimum $a_0 > 0$ in $T(\mu,\gamma)$ for $\mu \in \mathcal{C}(\gamma)$. Since $a_0 > 0$ is a period of $T(\mu,\gamma)$, for $n = 2$, $\mu^{a_0*} = (\mu^{(a_0/n)*})^{n*} \in \mathcal{L}(\gamma)$ but $(\mu^{(a_0/n)*})^{(n+1)*} \notin \mathcal{L}(\gamma)$ and $\mu^{(a_0/n)*} \notin \mathcal{L}(\gamma)$. Thus, Hypotheses I and II are not true. Suppose that $\mathcal{C}(\gamma)$ is empty. Then, obviously, Hypotheses I and II are true. Thus, (1), (2), and (3) are equivalent. We prove the equivalence of (3) and (4). Suppose that $\mathcal{C}(\gamma)$ is empty. Then for every $\mu \in \mathcal{OS} \cap \mathcal{ID}$ it holds that, for every $2t \in T(\mu,\gamma)$, $\mu_1^{t*} \in \mathcal{L}(\gamma)$, and hence, for all $a \geq 0$, (1.2) holds. Conversely, suppose that $\mathcal{C}(\gamma)$ is not empty and, for $a_0 = 2t \in T(\mu,\gamma)$ with $\mu \in \mathcal{C}(\gamma)$ and for all $a \geq 0$, (1.2) holds. Letting $\rho_1 := \rho_2 := \mu_1^{t*}$, $\zeta := \mu_1^{2t*}$, define Λ_2 as in Lemma 3.4 and let $\{\lambda_n\} \in \Lambda_2 \subset \Lambda$. We have (3.3) by Lemma 3.1 for $j = 1, 2$. We have $I_3 + I_4 = 2I_4 + I_5$, where

$$I_5 := \int_x^{\lambda_n - a - x} (e^{-\gamma a}\overline{\rho_1}(\lambda_n - a - y) - \overline{\rho_1}(\lambda_n - y))\rho_2(dy),$$

We have by the assumption (1.2) for every $a \geq 0$

$$\limsup_{x\to\infty}\limsup_{n\to\infty}\frac{|I_5|}{\bar{\zeta}(\lambda_n)}=0.$$

Define $M_2(y;\{\lambda_n\}) := e^{-\gamma y} m_2(y;\{\lambda_n\})$. Thus, we find from (3.2), (3.3), and $2t \in T(\mu,\gamma)$ that, for every $a \geq 0$,

$$\begin{aligned}\lim_{x\to\infty}\lim_{n\to\infty}&\frac{I_4}{\bar{\zeta}(\lambda_n)}\\&=\int_{0-}^{\infty}(e^{-\gamma a}m_2(a+y;\{\lambda_n\})-m_2(y;\{\lambda_n\}))\rho_1(\mathrm{d}y)\\&=\int_{0-}^{\infty}(M_2(a+y;\{\lambda_n\})-M_2(y;\{\lambda_n\}))e^{\gamma y}\rho_1(\mathrm{d}y)=0.\end{aligned}$$

The distribution $e^{\gamma x}\rho_1(\mathrm{d}x)/\widehat{\rho_1}(\gamma)$ is an exponentially tilted infinitely divisible distribution and hence itself is infinitely divisible, thus having a non-vanishing characteristic function. That is,

$$e^{\gamma y}\rho_1(\mathrm{d}y)=e^{\gamma y}\mu_1^{t*}(\mathrm{d}y)\in\mathcal{W}.$$

As in the proof of Lemma 3.4, we have $\rho_2=\mu_1^{t*}\in\mathcal{L}(\gamma)$. This is a contradiction. Thus, (3) and (4) are equivalent. □

Proof of Remark 1.1 Let $\gamma = 0$. Then we see from Lemma 3.6 that Hypothesis II is true. Thus, $\mathcal{C}(0)$ is empty, and hence, Remark 1.1 follows from Theorem 1.1. □

Proof of Proposition 1.1 Let $\gamma > 0$ and $\mu \in \mathcal{OS}\cap\mathcal{ID}$. Suppose that (1.3) holds for every $a \geq 0$. Let $\zeta := \nu_1^{n_0*}$. Then, by induction, we see from (1.3) and Lemma 3.3 that, for every $n \in \mathbb{N}$ and every $a \geq 0$,

$$\liminf_{x\to\infty}\frac{e^{-\gamma a}\overline{\nu_1^{n*}}(x-a)-\overline{\nu_1^{n*}}(x)}{\overline{\nu_1^{n_0*}}(x)}\geq 0.$$

Let $\epsilon > 0$ be arbitrary. Thus, letting $N \in \mathbb{N}$ sufficiently large, we have, for every $t > 0$ and for every $a \geq 0$,

$$\begin{aligned}\liminf_{x\to\infty}&\frac{e^{-\gamma a}\overline{\mu_1^{t*}}(x-a)-\overline{\mu_1^{t*}}(x)}{\overline{\nu_1^{n_0*}}(x)}\\&=\liminf_{x\to\infty}e^{-ct}\sum_{k=1}^{N}\frac{(ct)^k}{k!}\frac{e^{-\gamma a}\overline{\nu_1^{k*}}(x-a)-\overline{\nu_1^{k*}}(x)}{\overline{\nu_1^{n_0*}}(x)}\\&\quad-\limsup_{x\to\infty}e^{-ct}\sum_{k=N+1}^{\infty}\frac{(ct)^k}{k!}\frac{e^{-\gamma a}\overline{\nu_1^{k*}}(x-a)+\overline{\nu_1^{k*}}(x)}{\overline{\nu_1^{n_0*}}(x)}\geq-\epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary and $\overline{\nu_1^{n0*}}(x) \asymp \overline{\mu_1^{(t/n)*}}(x)$ for every $n \in \mathbb{N}$, we obtain that $\rho := \mu_1^{(t/n)*}$ satisfies $\rho \in \mathcal{OS}$ and (3.10) holds. Hence, we find from Lemma 3.6 that if $t \in T(\mu, \gamma)$, then $t/n \in T(\mu, \gamma)$ for every $n \in \mathbb{N}$. Thus, by Lemmas 4.1 and 4.2, either $T(\mu, \gamma) = (0, \infty)$ or $\emptyset$. □

Proof of Proposition 1.2 Suppose that $\nu_1^{2*} \in \mathcal{L}(\gamma)$ and the real part of $\widehat{\nu_1}(\gamma + iz)$ is not 0 for every $z \in \mathbb{R}$. If $t \in T(\mu, \gamma)$, then

$$\mu_1^{t*} = e^{-ct} \sum_{k=0}^{\infty} \frac{(ct)^k}{k!} \nu_1^{k*} \in \mathcal{L}(\gamma) \cap \mathcal{OS}. \tag{4.2}$$

Define distributions η_1 and η_2 on $\mathbb{R}_+$ as

$$\eta_1 := (\cosh(ct))^{-1} \sum_{k=0}^{\infty} \frac{(ct)^{2k}}{(2k)!} \nu_1^{(2k)*}$$

and

$$\eta_2 := (\sinh(ct))^{-1} \sum_{k=0}^{\infty} \frac{(ct)^{2k+1}}{(2k+1)!} \nu_1^{(2k+1)*}.$$

We see from Proposition 3.1 of Shimura and Watanabe [11] that $\eta_j \in \mathcal{OS}$ and $\overline{\eta_j}(x) \asymp \overline{\nu_1^{n0*}}(x)$ for $j = 1, 2$. Let $\epsilon > 0$ be arbitrary. We obtain from Lemma 2.3 that there is a positive integer $N = N(a, \epsilon, t)$ such that

$$\limsup_{x\to\infty} (\cosh(ct))^{-1} \sum_{k=N+1}^{\infty} \frac{(ct)^{2k}}{(2k)!} \frac{e^{-\gamma a} \overline{\nu_1^{(2k)*}}(x-a) + \overline{\nu_1^{(2k)*}}(x)}{\overline{\nu_1^{n0*}}(x)} < \epsilon.$$

Since $\nu_1^{(2k)*} \in \mathcal{L}(\gamma)$ for every $k \geq 0$, we have, for every $a \geq 0$ and every $t > 0$,

$$\limsup_{x\to\infty} (\cosh(ct))^{-1} \sum_{k=0}^{N} \frac{(ct)^{2k}}{(2k)!} \frac{|e^{-\gamma a} \overline{\nu_1^{(2k)*}}(x-a) - \overline{\nu_1^{(2k)*}}(x)|}{\overline{\nu_1^{n0*}}(x)} = 0.$$

Thus, with some $C = C(t) > 0$ we have, for every $a \geq 0$ and every $t > 0$,

$$\begin{aligned}
&\limsup_{x\to\infty} \frac{|e^{-\gamma a}\overline{\eta_1}(x-a) - \overline{\eta_1}(x)|}{\overline{\eta_1}(x)} \\
&\leq \limsup_{x\to\infty} (\cosh(ct))^{-1} \sum_{k=0}^{N} \frac{(ct)^{2k}}{(2k)!} \frac{|e^{-\gamma a} \overline{\nu_1^{(2k)*}}(x-a) - \overline{\nu_1^{(2k)*}}(x)|}{C\overline{\nu_1^{n0*}}(x)} \\
&\quad + \limsup_{x\to\infty} (\cosh(ct))^{-1} \sum_{k=N+1}^{\infty} \frac{(ct)^{2k}}{(2k)!} \frac{e^{-\gamma a} \overline{\nu_1^{(2k)*}}(x-a) + \overline{\nu_1^{(2k)*}}(x)}{C\overline{\nu_1^{n0*}}(x)} \leq \epsilon/C.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have

$$\eta_1 \in \mathcal{L}(\gamma) \cap \mathcal{OS}. \tag{4.3}$$

Since

$$\sinh(ct)\eta_2 = e^{ct}\mu_1^{t*} - \cosh(ct)\eta_1,$$

we have by (4.2) and (4.3)

$$\eta_2 \in \mathcal{L}(\gamma) \cap \mathcal{OS}.$$

Let $\zeta := \rho_1 := \eta_2$ and $\rho_2 := \nu_1$. Then, by argument similar to the proof of (4.3),

$$\rho_1 * \rho_2 = (\sinh(ct))^{-1} \sum_{k=0}^{\infty} \frac{(ct)^{2k+1}}{(2k+1)!} \nu_1^{(2k+2)*} \in \mathcal{L}(\gamma) \cap \mathcal{OS}.$$

Since the real part of $\widehat{\nu}_1(\gamma + iz)$ is not 0 for every $z \in \mathbb{R}$,

$$2\sinh(ct)\widehat{\rho}_1(\gamma + iz) = \exp(ct\widehat{\nu}_1(\gamma + iz)) - \exp(-ct\widehat{\nu}_1(\gamma + iz)) \neq 0$$

for every $z \in \mathbb{R}$, that is, $e^{\gamma x}\rho_1(\mathrm{d}x) \in \mathcal{W}$. Thus, we see from Lemma 3.4 that

$$\lim_{x\to\infty} \frac{e^{-\gamma a}\bar{\nu}_1(x-a) - \bar{\nu}_1(x)}{\bar{\zeta}(x)} = 0.$$

Since $\bar{\zeta}(x) \asymp \overline{\nu_1^{n_0*}}(x)$, we see from Theorem 1.1 that $T(\mu, \gamma) = (0, \infty)$. Thus, we have proved the proposition. □

Proof of Proposition 1.3 Let $\gamma > 0$ and $\mu \in \mathcal{OS} \cap \mathcal{ID}$. Suppose that $\nu_1^{n_1*} \in \mathcal{S}_\sharp$. Since $\overline{\mu^{t*}}(x) \asymp \overline{\nu_1^{n_1*}}(x)$, we have $\mu^{t*} \in \mathcal{S}_\sharp$ for every $t > 0$. Thus, we see from Lemmas 2.5 and 2.6 that if $T(\mu, \gamma) \neq \emptyset$, then $\nu_1 \in \mathcal{S}(\gamma)$ and hence $T(\mu, \gamma) = (0, \infty)$. That is, either $T(\mu, \gamma) = (0, \infty)$ or $\emptyset$. Moreover, $T(\mu, \gamma) = (0, \infty)$ if and only if $\nu_1 \in \mathcal{S}(\gamma)$. □

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