Journal of Theoretical Probability (2020) 33:1196–1210
https://doi.org/10.1007/s10959-019-00894-0

Large Deviations for Scaled Sums of *p*-Adic-Valued Rotation-Symmetric Independent and Identically Distributed Random Variables

Kumi Yasuda[1]

Received: 3 December 2018 / Revised: 14 March 2019 / Published online: 22 March 2019

Abstract
The law of an appropriately scaled sum of p-adic-valued, independent, identically and rotation-symmetrically distributed random variables weakly converges to a semi-stable law, if the tail probabilities of the variables satisfy some assumption. If we consider a scaled sum of such random variables with a sufficiently much higher scaling order, it accumulates to the origin, and the mass of any set not including the origin gets small. The purpose of this article is to investigate the asymptotic order of the logarithm of the mass of such sets off the origin. The order is explicitly given under some assumptions on the tail probabilities of the random variables and the scaling order of their sum. It is also proved that the large deviation principle follows with a rate function being constant except at the origin, and the rate function is good only for the case its value is infinity off the origin.

Keywords p-adic field · Limit theorem · Large deviations · Scaled sum of independent and identically distributed

Mathematics Subject Classification (2010) 60B10 · 60F10

1 Introduction and Results

The p-adic field $\mathbf{Q}_p$ is identified with the set of formal series $x = \sum_{i=-m}^{\infty} \alpha_i p^i$ with integers m and $\alpha_i = 0, 1, \ldots, p-1$, equipped with the p-adic norm $\left|\sum_{i=-m}^{\infty} \alpha_i p^i\right|_p = p^m$ if $\alpha_{-m} \neq 0$, and $|0|_p = 0$. For a p-adic number a and an integer l, let $B\left(a, p^l\right) := \left\{x \in \mathbf{Q}_p \mid |x-a|_p \leq p^l\right\}$ be the ball of radius p^l centered at a. Let $B_l := B\left(0, p^l\right)$

✉ Kumi Yasuda
kumi.a6@keio.jp

1 Faculty of Business and Commerce, Keio University, 4-1-1 Hiyoshi, Kohoku-ku, Yokohama 223-8521, Japan

denote the ball centered at the origin, and $H_l := B_l \setminus B_{l-1}$ the sphere. By the ultra-metric property $|x + y|_p \le \max\{|x|_p, |y|_p\}$, all balls and spheres are compact and open. For the p-adic field and related fundamental subjects, we can refer to [4].

The p-adic field is a separable and complete metric space where a lot of standard methods of stochastic analysis are available. On the other hand, the ultra-metric property brings some unique phenomena different from the case of Euclidean spaces. The behavior of sums of p-adic-valued random variables is one of the interesting topics which has been studied since 1990s. Analysis based on p-adic-valued measures has been developed with reference to mathematical physics, and sums of random variables are discussed in this framework [2,3]. On the other hand, limit theorems on p-adic numbers in the context of real-valued measures have been established. A p-adic analog of the law of large numbers is covered in [11]. The p-adic central limit theorem is concerned in [5,9], and [8] gives estimates of convergence. As refinements of the central limit theorem, [10] derives a p-adic analog of the law of iterated logarithms, and this article proceeds to estimates for large deviations. Besides, related works on abstract probabilities are also remarkable [7].

Let $\xi_i (i = 1, 2, \dots)$ be independent identically distributed (I.I.D.) random variables on the p-adic field. We suppose its law is invariant by rotations around the origin; namely, $u\xi_i$ has the same law as ξ_i for any p-adic number u satisfying $|u|_p = 1$. We also suppose its tail probabilities satisfy

$$T_1(m) := P\left(|\xi_i|_p \ge p^m\right) = p^{-\alpha m} L(m),$$

for a constant positive number α and a function L on $\mathbf{Z}$ such that $\lim_{m\to\infty} \frac{L(m+1)}{L(m)} = 1$. Define a sequence $N(n) := \frac{p^{2\alpha}(p-1)}{p^{\alpha+1}-1} T_1(n)^{-1}$ for $n = 1, 2, \dots$, and let $[N(n)]$ be its integer part. Under these assumptions, the law of the scaled sum $p^n \sum_{i=1}^{[N(n)]} \xi_i$ converges to a rotation-symmetric α-semi-stable law as $n \to \infty$ [9].

Let $c_n (n = 1, 2, \dots)$ be a sequence of nonnegative integers. Concerning the growth rate of the scaled sum, $\limsup_{n\to\infty} \left| p^{n+c_n} \sum_{i=1}^{[N(n)]} \xi_i \right|_p = 0$ if c_n diverges faster than $\tilde{c}_n := \left[\frac{\log n}{\alpha \log p}\right]$, and $= +\infty$ if c_n is slower than $\tilde{c}_n$. At the critical order $c_n = \tilde{c}_n$, the result differs by the rate of convergence of $\frac{L(m+1)}{L(m)}$ to 1 [10].

In this article, we deal with general sequences $c_n (n = 1, 2, \dots)$ satisfying $\lim_{n\to\infty} c_n = +\infty$ and give estimates for large deviations of the law of the scaled sum. Let P_n be the law of the scaled sum $p^{n+c_n} \sum_{i=1}^{[N(n)]} \xi_i$ and put $\theta_- := \liminf_{n\to\infty} \frac{c_n}{n}$, $\theta_+ := \limsup_{n\to\infty} \frac{c_n}{n}$.

Proposition 1 *For any compact open set K in $\mathbf{Q}_p$ including the origin,*

$$\liminf_{n\to\infty} \frac{1}{n} \log P_n\left(K^c\right) = -\alpha\theta_+ \log p,$$
$$\limsup_{n\to\infty} \frac{1}{n} \log P_n\left(K^c\right) = -\alpha\theta_- \log p,$$

where K^c is the complement of K.

Proposition 2

i. For any closed set B *in* $\mathbf{Q}_p$ *not including the origin,*

$$\liminf_{n\to\infty} \frac{1}{n} \log P_n(B) \le -\alpha\theta_+ \log p,$$
$$\limsup_{n\to\infty} \frac{1}{n} \log P_n(B) \le -\alpha\theta_- \log p.$$

ii. For any open set A *in* $\mathbf{Q}_p$ *including the origin,*

$$\lim_{n\to\infty} \log P_n(A) = 0.$$

In order to determine the asymptotics of $P_n(B)$ for sets B off the origin, we require an additional assumption to the rate of convergence $\frac{L(m+1)}{L(m)} \to 1$. Define $\delta_n := \sup_{m\ge n} \left|1 - \frac{L(m+1)}{L(m)}\right|$ for $n \ge 1$.

Theorem 1 *Assume that either of the following two conditions is satisfied :*

i. $\theta_- = \theta_+ = +\infty$,
ii. $\limsup_{n\to\infty} c_n \log \frac{1+\delta_n}{1-\delta_n} < \alpha \log p$.

Then for any $a \in \mathbf{Q}_p$, $a \neq 0$, *and any integer* l *such that* $0 \notin B\left(a, p^l\right)$,

$$\liminf_{n\to\infty} \frac{1}{n} \log P_n\left(B\left(a, p^l\right)\right) = -\alpha\theta_+ \log p,$$
$$\limsup_{n\to\infty} \frac{1}{n} \log P_n\left(B\left(a, p^l\right)\right) = -\alpha\theta_- \log p.$$

By this result, we can discuss the large deviation principle of the sequence of probability measures P_n. For a general theory of the large deviation principle for a sequence of random variables or probability measures on a metric space, we can refer to [1].

Corollary 1 *Under the assumption of Theorem 1, the distributions* $P_n (n = 1, 2, \dots)$ *satisfy the large deviation principle if and only if the limit* $\theta := \lim_{n\to\infty} \frac{c_n}{n} \in [0, +\infty]$ *exists. If that is the case, the rate function is given by*

$$I(x) = \begin{cases} 0, & x = 0, \\ \alpha\theta \log p, & x \neq 0, \end{cases}$$

and I *is good only for the case* $\theta = +\infty$.

2 Tail Probabilities of the Sum of I.I.D.

For $n \geq 1$ and integers m, let $T_n(m) := P\left(\left|\sum_{i=1}^n \xi_i\right|_p \geq p^m\right)$ be the tail probabilities of the sum of ξ_i. As a preparation for a formula for $T_n(m)$, we shall derive an inductive relation of $U_n(m) := P\left(\left|\sum_{i=1}^n \xi_i\right|_p = p^m\right) = T_n(m) - T_n(m+1)$.

Lemma 1

$$\begin{aligned} U_n(m) &= U_1(m)(1 - T_{n-1}(m)) + U_{n-1}(m)(1 - T_1(m)) \\ &\quad + \frac{p-2}{p-1} U_{n-1}(m) U_1(m) + \sum_{k=m+1}^{\infty} p^{m-k} U_{n-1}(k) U_1(k). \end{aligned}$$

Proof By the ultra-metric property, p-adic numbers x and y satisfy $|x+y|_p = p^m$ if and only if one of the following exclusive events happens :

i. $|x|_p < p^m,\ |y|_p = p^m$,
ii. $|x|_p = p^m,\ |y|_p < p^m$,
iii. $|x|_p = |y|_p \geq p^m,\ |x+y|_p = p^m$.

Putting $x = \sum_{i=1}^{n-1} \xi_i$ and $y = \xi_n$, by the independence of ξ_i we have

$$\begin{aligned} U_n(m) &= P\left(\left|\sum_{i=1}^{n-1} \xi_i\right|_p < p^m,\ |\xi_n|_p = p^m\right) + P\left(\left|\sum_{i=1}^{n-1} \xi_i\right|_p = p^m,\ |\xi_n|_p < p^m\right) \\ &\quad + P\left(\left|\sum_{i=1}^{n-1} \xi_i\right|_p = |\xi_n|_p \geq p^m,\ \left|\sum_{i=1}^{n} \xi_i\right|_p = p^m\right) \\ &= U_1(m) \sum_{k=-\infty}^{m-1} U_{n-1}(k) + U_{n-1}(m) \sum_{k=-\infty}^{m-1} U_1(k) \\ &\quad + \sum_{l=0}^{\infty} P\left(\left|\sum_{i=1}^{n-1} \xi_i\right|_p = |\xi_n|_p = p^{m+l},\ \left|\sum_{i=1}^{n} \xi_i\right|_p = p^m\right). \end{aligned} \tag{1}$$

For the case $l = 0$ in the last sum, the sphere H_m consists of $p-1$ disjoint balls $B\left(\alpha_{-m} p^{-m}, p^{m-1}\right)$ $(\alpha_{-m} = 1, 2, \ldots, p-1)$ of radius p^{m-1}. We can see that the event $|x|_p = |y|_p = |x+y|_p = p^m$ happens if and only if $x \in B\left(\alpha_{-m} p^{-m}, p^{m-1}\right)$ and $-y \in B\left(\alpha'_{-m} p^{-m}, p^{m-1}\right)$ for some $\alpha_{-m} \neq \alpha'_{-m}$. Since the balls $B\left(\alpha_{-m} p^{-m}, p^{m-1}\right)$ are mapped to each other by rotations around the origin, and the law of ξ_i is invariant by the rotations,

$$P\left(\sum_{i=1}^{n-1} \xi_i \in B\left(\alpha_{-m} p^{-m}, p^{m-1}\right)\right) = \frac{U_{n-1}(m)}{p-1}$$

and

$$P\left(-\xi_n \in B\left(\alpha'_{-m} p^{-m}, p^{m-1}\right)\right) = \frac{U_1(m)}{p-1}$$

hold for all α_{-m} and α'_{-m}. Therefore,

$$\begin{aligned}
&P\left(\left|\sum_{i=1}^{n-1} \xi_i\right|_p = |\xi_n|_p = \left|\sum_{i=1}^{n} \xi_i\right|_p = p^m\right) \\
&= \sum_{\alpha_{-m}} \sum_{\alpha'_{-m} \neq \alpha_{-m}} P\left(\sum_{i=1}^{n-1} \xi_i \in B\left(\alpha_{-m} p^{-m}, p^{m-1}\right)\right) \\
&\quad \times P\left(-\xi_n \in B\left(\alpha'_{-m} p^{-m}, p^{m-1}\right)\right) \\
&= (p-1)(p-2)\frac{U_{n-1}(m)}{p-1}\frac{U_1(m)}{p-1} \\
&= \frac{p-2}{p-1} U_{n-1}(m) U_1(m).
\end{aligned}$$

As for $l \geq 1$, the sphere H_{m+l} consists of $p^l(p-1)$ disjoint balls $B\left(\sum_{i=-m-l}^{-m} \alpha_i p^i, p^{m-1}\right)$ ($\alpha_{-m-l} = 1, 2, \ldots, p-1$, and $\alpha_{-m-l+1}, \ldots, \alpha_{-m} = 0, 1, \ldots, p-1$) of radius p^{m-1}. The event $|x|_p = |y|_p = p^{m+l}$, $|x+y|_p = p^m$ happens if and only if $x \in B\left(\sum_{i=-m-l}^{-m} \alpha_i p^i, p^{m-1}\right)$ and $-y \in B\left(\sum_{i=-m-l}^{-m-1} \alpha_i p^i + \alpha'_{-m} p^{-m}, p^{m-1}\right)$ for some $\alpha_{-m-l}, \ldots, \alpha_{-m}$, and $\alpha'_{-m} \neq \alpha_{-m}$. Hence, we have

$$\begin{aligned}
&P\left(\left|\sum_{i=1}^{n-1} \xi_i\right| = |\xi_n|_p = p^{m+l}, \left|\sum_{i=1}^{n} \xi_i\right| = p^m\right) \\
&= \sum_{\alpha_{-m-l},\ldots,\alpha_{-m}} \sum_{\alpha'_{-m} \neq \alpha_{-m}} P\left(\sum_{i=1}^{n-1} \xi_i \in B\left(\sum_{i=-m-l}^{-m} \alpha_i p^i, p^{m-1}\right)\right) \\
&\quad \times P\left(\xi_n \in B\left(\sum_{i=-m-l}^{-m-1} \alpha_i p^i + \alpha'_{-m} p^{-m}, p^{m-1}\right)\right) \\
&= p^l(p-1) \cdot (p-1)\frac{U_{n-1}(m+l)}{p^l(p-1)}\frac{U_1(m+l)}{p^l(p-1)} \\
&= p^{-l} U_{n-1}(m+l) U_1(m+l).
\end{aligned}$$

Consequently, (1) leads to

$$U_n(m) = U_1(m) \sum_{k=-\infty}^{m-1} U_{n-1}(k) + U_{n-1}(m) \sum_{k=-\infty}^{m-1} U_1(k)$$

$$+\frac{p-2}{p-1}U_{n-1}(m)U_1(m)+\sum_{l=1}^{\infty}p^{-l}U_{n-1}(m+l)U_1(m+l)$$
$$=U_1(m)(1-T_{n-1}(m))+U_{n-1}(m)(1-T_1(m))$$
$$+\frac{p-2}{p-1}U_{n-1}(m)U_1(m)+\sum_{k=m+1}^{\infty}p^{m-k}U_{n-1}(k)U_1(k).$$

□

Proposition 3

$$T_n(m)=1-\left(1-p^{-1}\right)\sum_{k=0}^{\infty}p^{-k}\left(1-\frac{T_1(m+k)-p^{-1}T_1(m+k+1)}{1-p^{-1}}\right)^n.$$

Proof Let us put $V_n(m) := T_n(m)-p^{-1}T_n(m+1)$. Since $T_n(m)=\sum_{l=m}^{\infty}U_n(l)$, Lemma 1 gives

$$V_n(m)$$
$$=\sum_{l=m}^{\infty}\Bigg\{U_1(l)(1-T_{n-1}(l))+U_{n-1}(l)(1-T_1(l))$$
$$+\frac{p-2}{p-1}U_{n-1}(l)U_1(l)+\sum_{k=l+1}^{\infty}p^{l-k}U_{n-1}(k)U_1(k)\Bigg\}$$
$$-p^{-1}\sum_{l=m+1}^{\infty}\Bigg\{U_1(l)(1-T_{n-1}(l))+U_{n-1}(l)(1-T_1(l))$$
$$+\frac{p-2}{p-1}U_{n-1}(l)U_1(l)+\sum_{k=l+1}^{\infty}p^{l-k}U_{n-1}(k)U_1(k)\Bigg\}$$
$$=U_1(m)(1-T_{n-1}(m))+U_{n-1}(m)(1-T_1(m))+\frac{p-2}{p-1}U_{n-1}(m)U_1(m)$$
$$+\left(1-p^{-1}\right)\sum_{l=m+1}^{\infty}\{U_1(l)(1-T_{n-1}(l))+U_{n-1}(l)(1-T_1(l))$$
$$+U_{n-1}(l)U_1(l)\}$$
$$=(T_1(m)-T_1(m+1))(1-T_{n-1}(m))$$
$$+(T_{n-1}(m)-T_{n-1}(m+1))(1-T_1(m))$$
$$+\frac{p-2}{p-1}(T_{n-1}(m)-T_{n-1}(m+1))(T_1(m)-T_1(m+1))$$
$$+\left(1-p^{-1}\right)\sum_{l=m+1}^{\infty}\{(T_1(l)-T_1(l+1))(1-T_{n-1}(l))$$
$$+(T_{n-1}(l)-T_{n-1}(l+1))(1-T_1(l))$$

$$
\begin{aligned}
&\quad + (T_{n-1}(l) - T_{n-1}(l+1))(T_1(l) - T_1(l+1))\} \\
&= T_1(m) - p^{-1}T_1(m+1) + T_{n-1}(m) - p^{-1}T_{n-1}(m+1) \\
&\quad - \frac{1}{1-p^{-1}}\left(T_1(m) - p^{-1}T_1(m+1)\right)\left(T_{n-1}(m) - p^{-1}T_{n-1}(m+1)\right) \\
&= V_1(m) + V_{n-1}(m) - \frac{1}{1-p^{-1}} V_1(m)V_{n-1}(m). \qquad (2)
\end{aligned}
$$

By this equation, we can derive inductively that

$$
V_n(m) = \left(1 - p^{-1}\right)\left(1 - \left(1 - \frac{V_1(m)}{1-p^{-1}}\right)^n\right). \qquad (3)
$$

Indeed, this is trivial for $n = 1$. Provided it is true for $n = n_0 - 1$, then (2) yields

$$
\begin{aligned}
V_n(m) &= V_1(m) + \left(1 - p^{-1}\right)\left(1 - \left(1 - \frac{V_1(m)}{1-p^{-1}}\right)^{n_0-1}\right)\left(1 - \frac{V_1(m)}{1-p^{-1}}\right) \\
&= \left(1 - p^{-1}\right)\left(1 - \left(1 - \frac{V_1(m)}{1-p^{-1}}\right)^{n_0}\right).
\end{aligned}
$$

By (3), taking the sum of $p^{-k}V_n(m+k) = p^{-k}T_n(m+k) - p^{-k-1}T_n(m+k+1)$ for $k = 0, 1, \ldots$, we obtain

$$
\begin{aligned}
T_n(m) &= \sum_{k=0}^{\infty} p^{-k}\left(1 - p^{-1}\right)\left(1 - \left(1 - \frac{V_1(m+k)}{1-p^{-1}}\right)^n\right) \\
&= 1 - \left(1 - p^{-1}\right)\sum_{k=0}^{\infty} p^{-k}\left(1 - \frac{T_1(m+k) - p^{-1}T_1(m+k+1)}{1-p^{-1}}\right)^n.
\end{aligned}
$$

□

Remark 1 If we import a result of [9], Proposition 3 can be derived more concisely by using Fourier transform. Let φ be the character on the p-adic field defined by

$$
\varphi\left(\sum_{i=-m}^{\infty} \alpha_i p^i\right) := \begin{cases} \exp\left(2\pi\sqrt{-1}\sum_{i=-m}^{-1} \alpha_i p^i\right), & \text{if } m \geq 1, \\ 1, & \text{if } m \leq 0, \end{cases}
$$

then the characteristic function of a probability measure μ on $\mathbf{Q}_p$ is defined by

$$
\hat{\mu}(y) := \int_{\mathbf{Q}_p} \varphi(xy)\mu(\mathrm{d}x), \quad y \in \mathbf{Q}_p.
$$

We can see the Fourier transform of the indicator function $\mathbf{1}_{B_l}$ of the ball B_l is given by

$$\mathcal{F}\mathbf{1}_{B_l}(x) := \int_{\mathbf{Q}_p} \mathbf{1}_{B_l}(y)\varphi(xy)\mathrm{d}y = p^l \mathbf{1}_{B_{-l}}(x),$$

(see, e.g., Chapter XIV of [6]), where $\int \cdot \mathrm{d}y$ denotes the integration with respect to Haar measure of $\mathbf{Q}_p$ normalized so that $\int \mathbf{1}_{B_l}(y)\mathrm{d}y = p^l$. Let μ be the law of ξ_i, then we have

$$\begin{aligned} 1 - T_n(m) &= \int_{\mathbf{Q}_p} \mathbf{1}_{B_{m-1}}(x)\mu^{*n}(\mathrm{d}x) \\ &= \int_{\mathbf{Q}_p} p^{m-1}\mathcal{F}\mathbf{1}_{B_{-(m-1)}}(x)\mu^{*n}(\mathrm{d}x) \\ &= p^{m-1}\int_{\mathbf{Q}_p} \mathbf{1}_{B_{-(m-1)}}(y)\left(\int_{\mathbf{Q}_p} \varphi(xy)\mu^{*n}(\mathrm{d}x)\right)\mathrm{d}y \\ &= p^{m-1}\int_{\mathbf{Q}_p} \mathbf{1}_{B_{-(m-1)}}(y)\hat{\mu}(y)^n \mathrm{d}y \\ &= p^{m-1}\sum_{k=m-1}^{\infty}\int_{|y|_p=p^{-k}} \hat{\mu}(y)^n\mathrm{d}y. \end{aligned}$$

The characteristic function of μ is calculated in Lemma 3 of [9] as

$$\hat{\mu}(y) = 1 - (p-1)^{-1}(pT_1(k+1) - T_1(k+2)), \quad \text{if } |y|_p = p^{-k},$$

and then Proposition 3 follows immediately.

3 Proofs

For proofs of Propositions 1, 2, and Theorem 1, the following estimate is crucial.

Lemma 2 *There exist positive constants C_1 and C_2 independent of $n \geq 1$ and $l \in \mathbf{Z}$ such that, for every fixed integer l,*

$$C_1\left(p^{-\alpha}(1-\delta_n)\right)^{c_n+l} \leq P_n\left(B_l^c\right) \leq C_2\left(p^{-\alpha}(1+\delta_n)\right)^{c_n+l}$$

holds for sufficiently large n. Furthermore, the both constants C_1 and C_2 can be taken arbitrarily close to $C := C(\alpha) := \frac{p^\alpha(p-1)}{p^{\alpha+1}-1}$, and accordingly,

$$\limsup_{n\to\infty}\frac{P_n\left(B_l^c\right)}{\left(p^{-\alpha}(1+\delta_n)\right)^{c_n+l}} \leq C \leq \liminf_{n\to\infty}\frac{P_n\left(B_l^c\right)}{\left(p^{-\alpha}(1-\delta_n)\right)^{c_n+l}}$$

holds for any integer l.

Proof Fix an integer l and take any $\varepsilon > 0$. By Proposition 3, we have

$$\begin{aligned} P_n\left(B_l^c\right) &= P\left(\left|p^{n+c_n}\sum_{i=1}^{[N(n)]}\xi_i\right|_p \geq p^{l+1}\right) \\ &= T_{[N(n)]}(n+c_n+l+1) \\ &= 1-\left(1-p^{-1}\right)\sum_{k=0}^{\infty}p^{-k}(1-v(n+c_n+l+k))^{[N(n)]}, \end{aligned}$$

where

$$\begin{aligned} v(m) &:= \left(1-p^{-1}\right)^{-1}\left(T_1(m+1)-p^{-1}T_1(m+2)\right) \\ &= \left(1-p^{-1}\right)^{-1}\left(1-p^{-\alpha-1}\frac{L(m+2)}{L(m+1)}\right)T_1(m+1). \end{aligned}$$

Since $v(m)$ tends to 0 as $m \to \infty$,

$$e^{-(1+\varepsilon)} < (1-v(n+c_n+l+k))^{\frac{1}{v(n+c_n+l+k)}} < e^{-(1-\varepsilon)}$$

holds for sufficiently large n and any $k \geq 0$, and then

$$\begin{aligned} &1-\left(1-p^{-1}\right)\sum_{k=0}^{\infty}p^{-k}\left(e^{-(1-\varepsilon)}\right)^{v(n+c_n+l+k)[N(n)]} \\ &< P_n\left(B_l^c\right) \\ &< 1-\left(1-p^{-1}\right)\sum_{k=0}^{\infty}p^{-k}\left(e^{-(1+\varepsilon)}\right)^{v(n+c_n+l+k)[N(n)]}, \end{aligned}$$

namely,

$$\begin{aligned} &\left(1-p^{-1}\right)\sum_{k=0}^{\infty}p^{-k}\left(1-e^{-(1-\varepsilon)v(n+c_n+l+k)[N(n)]}\right) \\ &< P_n\left(B_l^c\right) \\ &< \left(1-p^{-1}\right)\sum_{k=0}^{\infty}p^{-k}\left(1-e^{-(1+\varepsilon)v(n+c_n+l+k)[N(n)]}\right). \end{aligned} \tag{4}$$

We have

$$\begin{aligned}
&\left(p^{-\alpha}(1-\delta_n)\right)^{c_n+l+k+1}\\
&\quad\le \frac{T_1(n+c_n+l+k+1)}{T_1(n)} = p^{-\alpha(c_n+l+k+1)}\prod_{i=0}^{c_n+l+k}\frac{L(n+i+1)}{L(n+i)}\\
&\quad\le \left(p^{-\alpha}(1+\delta_n)\right)^{c_n+l+k+1},
\end{aligned}$$

and

$$1-\varepsilon < 1-\delta_{n+c_n+l} \le \frac{L(n+c_n+l+k+2)}{L(n+c_n+l+k+1)} \le 1+\delta_{n+c_n+l} < 1+\varepsilon,$$

for all $k \ge 0$, if n is large enough so that $\delta_{n+c_n+l} < \varepsilon$. We also have

$$(1-\varepsilon)N(n) \le [N(n)] \le N(n) \tag{5}$$

for large n, since $N(n) = \frac{p^{2\alpha}(p-1)}{p^{\alpha+1}-1}T_1(n)^{-1} \to \infty$ as $n \to \infty$. Applying these inequalities to $v(n+c_n+l+k)[N(n)] = \left(1-p^{-1}\right)^{-1}\left(1-p^{-\alpha-1}\frac{L(n+c_n+l+k+2)}{L(n+c_n+l+k+1)}\right)T_1(n+c_n+l+k+1)\left[\frac{p^{2\alpha}(p-1)}{p^{\alpha+1}-1}T_1(n)^{-1}\right]$, we see that

$$\begin{aligned}
&(1-\varepsilon)\left(1-p^{-1}\right)^{-1}\left(1-p^{-\alpha-1}(1+\varepsilon)\right)\frac{p^{2\alpha}(p-1)}{p^{\alpha+1}-1}\left(p^{-\alpha}(1-\delta_n)\right)^{c_n+l+k+1}\\
&\quad< v(n+c_n+l+k+1)[N(n)]\\
&\quad< \left(1-p^{-1}\right)^{-1}\left(1-p^{-\alpha-1}(1-\varepsilon)\right)\frac{p^{2\alpha}(p-1)}{p^{\alpha+1}-1}\left(p^{-\alpha}(1+\delta_n)\right)^{c_n+l+k+1}.
\end{aligned} \tag{6}$$

In particular, take n large enough so that $\delta_n < \varepsilon \wedge (p^\alpha - 1)$, then the assumption $c_n \to +\infty$ implies that the right-hand side goes to 0 as $n \to \infty$ uniformly for $k \ge 0$. Since $\lim_{t\to 0}\frac{1-e^{-t}}{t} = 1$, we have

$$\begin{aligned}
1-e^{-(1-\varepsilon)v(n+c_n+l+k+1)[N(n)]} &\ge (1-\varepsilon)^2 v(n+c_n+l+k+1)[N(n)],\\
1-e^{-(1+\varepsilon)v(n+c_n+l+k+1)[N(n)]} &\le (1+\varepsilon)^2 v(n+c_n+l+k+1)[N(n)]
\end{aligned}$$

for large n, and therefore, inequalities (4) lead to

$$\begin{aligned}
&\left(1-p^{-1}\right)\sum_{k=0}^{\infty}p^{-k}(1-\varepsilon)^2 v(n+c_n+l+k+1)[N(n)]\\
&\quad< P_n\left(B_l^c\right)\\
&\quad< \left(1-p^{-1}\right)\sum_{k=0}^{\infty}p^{-k}(1+\varepsilon)^2 v(n+c_n+l+k+1)[N(n)].
\end{aligned}$$

Applying (6) to the above, we obtain

$$
\begin{aligned}
&(1-\varepsilon)^3\left(1-p^{-\alpha-1}(1+\varepsilon)\right)\frac{p^{2\alpha}(p-1)}{p^{\alpha+1}-1}\left(p^{-\alpha}(1-\delta_n)\right)^{c_n+l+1}\\
&\quad\times\frac{1}{1-p^{-\alpha-1}(1-\delta_n)}\\
&\quad< P_n\left(B_l^c\right)\\
&\quad< (1+\varepsilon)^2\left(1-p^{-\alpha-1}(1-\varepsilon)\right)\frac{p^{2\alpha}(p-1)}{p^{\alpha+1}-1}\left(p^{-\alpha}(1+\delta_n)\right)^{c_n+l+1}\\
&\quad\times\frac{1}{1-p^{-\alpha-1}(1+\delta_n)}.
\end{aligned}
$$

Put $C_1=(1-\varepsilon)^4\frac{1-p^{-\alpha-1}(1+\varepsilon)}{1-p^{-\alpha-1}(1-\varepsilon)}\frac{p^{\alpha}(p-1)}{p^{\alpha+1}-1}$ and $C_2=(1+\varepsilon)^3\frac{1-p^{-\alpha-1}(1-\varepsilon)}{1-p^{-\alpha-1}(1+\varepsilon)}\frac{p^{\alpha}(p-1)}{p^{\alpha+1}-1}$, then since $\delta_n<\varepsilon$ we obtain

$$
C_1\left(p^{-\alpha}(1-\delta_n)\right)^{c_n+l}\le P_n\left(B_l^c\right)\le C_2\left(p^{-\alpha}(1+\delta_n)\right)^{c_n+l}
$$

for sufficiently large n.

We can see C_1 and C_2 both approach C as $\varepsilon\to 0$; then, the second assertion is clear. □

Now let us give proofs to Propositions 1, 2, and Theorem 1, using the estimates of Lemma 2.

Proof *(Proposition 1)* Since K is assumed to be compact open and $0\in K$, there exist integers $l_1\ge l_2$ such that $B_{l_2}\subset K\subset B_{l_1}$. Then, Lemma 2 implies

$$
C_1\left(p^{-\alpha}(1-\delta_n)\right)^{c_n+l_1}\le P_n\left(K^c\right)\le C_2\left(p^{-\alpha}(1+\delta_n)\right)^{c_n+l_2}.
$$

Take their logarithm and divide by n, then we have

$$
\begin{aligned}
&\frac{c_n+l_1}{n}(-\alpha\log p+\log(1-\delta_n))+\frac{\log C_1}{n}\\
&\le\frac{1}{n}\log P_n\left(K^c\right)\\
&\le\frac{c_n+l_2}{n}(-\alpha\log p+\log(1+\delta_n))+\frac{\log C_2}{n}.
\end{aligned}
$$

Since $\delta_n\to 0$ as $n\to\infty$, taking lim sup and lim inf of each side, the assertion is proved. □

Proof *(Proposition 2)* (i) Since the complement B^c is an open set including the origin, we can take an integer l such that $B_l \subset B^c$. Apply Proposition 1 to $K = B_l$, then

$$\liminf_{n\to\infty} \frac{1}{n} \log P_n(B) \le \liminf_{n\to\infty} \frac{1}{n} \log P_n\left(B_l^c\right) = -\alpha\theta_+ \log p,$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \log P_n(B) \le \limsup_{n\to\infty} \frac{1}{n} \log P_n\left(B_l^c\right) = -\alpha\theta_- \log p.$$

(ii) We can take an integer l such that $B_l \subset A$, and Lemma 2 implies

$$\log\left(1 - C_2\left(p^{-\alpha}(1+\delta_n)\right)^{c_n+l}\right) \le \log P_n(B_l) \le \log P_n(A) \le 0.$$

By the assumption $\lim_{n\to\infty} c_n = +\infty$, we have $C_2\left(p^{-\alpha}(1+\delta_n)\right)^{c_n+l} \to 0$ as $n \to \infty$, and hence, the left-hand side tends to 0. □

Proof *(Theorem 1)* For the case (i), our claim is trivial by Proposition 2 (i). Let us assume (ii) and put $|a|_p = p^k$. The sphere H_k is the disjoint union of $p^{k-l-1}(p-1)$ balls of radius p^l. Since each of these balls is mapped to the ball $B\left(a, p^l\right)$ by a rotation around the origin, and the law of ξ_i is invariant by the rotation, it follows that

$$P_n\left(B\left(a, p^l\right)\right) = \frac{P_n(H_k)}{p^{k-l-1}(p-1)} = \frac{P_n\left(B_{k-1}^c\right) - P_n\left(B_k^c\right)}{p^{k-l-1}(p-1)}. \tag{7}$$

Applying Lemma 2 to $l = k-1$ and $l = k$, we obtain

$$\begin{aligned} &C_1\left(p^{-\alpha}(1-\delta_n)\right)^{c_n+k-1} - C_2\left(p^{-\alpha}(1+\delta_n)\right)^{c_n+k} \\ &\quad \le P_n\left(B_{k-1}^c\right) - P_n\left(B_k^c\right) \\ &\quad \le C_2\left(p^{-\alpha}(1+\delta_n)\right)^{c_n+k-1} - C_1\left(p^{-\alpha}(1-\delta_n)\right)^{c_n+k}. \end{aligned} \tag{8}$$

Here let us verify the left-hand side of this inequality is positive for large n. By the assumption $\limsup_{n\to\infty} c_n \log \frac{1+\delta_n}{1-\delta_n} < \alpha \log p$, we have $p^\alpha \left(\frac{1-\delta_n}{1+\delta_n}\right)^{c_n} > 1$ for sufficiently large n. Since $\frac{C_1}{C_2}$ can be arbitrarily close to 1 and $\delta_n \to 0$ as $n \to \infty$, the ratio

$$\frac{C_1\left(p^{-\alpha}(1-\delta_n)\right)^{c_n+k-1}}{C_2\left(p^{-\alpha}(1+\delta_n)\right)^{c_n+k}} = \frac{C_1}{C_2} \frac{(1-\delta_n)^{k-1}}{(1+\delta_n)^k} \cdot p^\alpha \left(\frac{1-\delta_n}{1+\delta_n}\right)^{c_n}$$

is greater than 1 for sufficiently large n. Therefore, the left-hand side of (8) is positive, and then we can estimate the logarithm of (7) as

$$\log\left(C_1\left(p^{-\alpha}(1-\delta_n)\right)^{c_n+k-1} - C_2\left(p^{-\alpha}(1+\delta_n)\right)^{c_n+k}\right)$$

$$
\begin{aligned}
&-(k-l-1)\log p - \log(p-1) \\
&\le \log P_n\left(B\left(a, p^l\right)\right) \\
&\le \log\left(C_2\left(p^{-\alpha}(1+\delta_n)\right)^{c_n+k-1} - C_1\left(p^{-\alpha}(1-\delta_n)\right)^{c_n+k}\right) \\
&-(k-l-1)\log p - \log(p-1).
\end{aligned}
$$

Divide the each side by n, then we proceed to

$$
\begin{aligned}
&\frac{c_n+k-1}{n}(-\alpha\log p + \log(1-\delta_n)) \\
&\quad +\frac{1}{n}\left(\log\left(C_1 - C_2 p^{-\alpha}(1-\delta_n)\left(\frac{1+\delta_n}{1-\delta_n}\right)^{c_n+k}\right)\right. \\
&\quad \left. -(k-l-1)\log p - \log(p-1)\right) \\
&\le \frac{1}{n}\log P_n\left(B\left(a, p^l\right)\right) \\
&\le \frac{c_n+k-1}{n}(-\alpha\log p + \log(1+\delta_n)) \\
&\quad +\frac{1}{n}\left(\log\left(C_2 - C_1 p^{-\alpha}(1+\delta_n)\left(\frac{1-\delta_n}{1+\delta_n}\right)^{c_n+k}\right)\right. \\
&\quad \left. -(k-l-1)\log p - \log(p-1)\right).
\end{aligned}
$$

Since $\delta_n \to 0$ and $\limsup_{n\to\infty}\left(\frac{1+\delta_n}{1-\delta_n}\right)^{c_n} < p^{\alpha}$, the second terms of the left- and the right-hand side go to 0 as $n \to \infty$. Hence, taking lim inf and lim sup of the both sides, our assertion follows. □

Proof *(Corollary 1)* If we suppose P_n $(n = 1, 2, \dots)$ satisfy the large deviation principle with some rate function I, then Theorem 1 requires

$$
\alpha\theta_+ \log p \le \inf_{x\in B(a,p^l)} I(x) \le \alpha\theta_- \log p;
$$

therefore, $\theta_+ = \theta_-$. Thus for the large deviation principle, it is necessary that $\lim_{n\to\infty}\frac{c_n}{n}$ exists.

Conversely, suppose that $\theta := \lim_{n\to\infty}\frac{c_n}{n}$ exists, and let $I(x) = \alpha\theta\log p$ for $x \ne 0$ and $I(0) = 0$. For the large deviation principle, what we have to show are

$$
\liminf_{n\to\infty}\frac{1}{n}\log P_n(A) \ge -\inf_{x\in A} I(x) \tag{9}
$$

for any open set A in $\mathbf{Q}_p$, and

$$\limsup_{n\to\infty} \frac{1}{n} \log P_n(B) \leq -\inf_{x\in B} I(x) \tag{10}$$

for any closed set B.

For the empty set ϕ, (9) and (10) are trivial. If an open set A includes the origin, then by Proposition 2 (ii),

$$\liminf_{n\to\infty} \frac{1}{n} \log P_n(A) = 0 = -I(0) = -\inf_{x\in A} I(x).$$

Suppose next $A \neq \phi$ is an open set not including the origin. Take $a \in \mathbf{Q}_p$ and an integer l such that $B\left(a, p^l\right) \subset A$, then by Theorem 1,

$$\liminf_{n\to\infty} \frac{1}{n} \log P_n(A) \geq \liminf_{n\to\infty} \frac{1}{n} \log P_n\left(B\left(a, p^l\right)\right) = -\alpha\theta \log p = -\inf_{x\in A} I(x).$$

Let B be a closed set including the origin, then trivially we have

$$\limsup_{n\to\infty} \frac{1}{n} \log P_n(B) \leq 0 = -I(0) = -\inf_{x\in B} I(x).$$

In case a closed set $B \neq \phi$ does not include the origin, we can take an integer l such that $B_l \subset B^c$, and then Proposition 1 implies

$$\limsup_{n\to\infty} \frac{1}{n} \log P_n(B) \leq \limsup_{n\to\infty} \frac{1}{n} \log P_n\left(B_l^c\right) = -\alpha\theta \log p = -\inf_{x\in B} I(x).$$

Therefore, the large deviation principle holds with the rate function I.

In case $\theta = +\infty$, it holds that $\{x \in \mathbf{Q}_p \mid I(x) \leq c\} = \{0\}$ for any $c \geq 0$, and in case $\theta < +\infty$, we have $\{x \in \mathbf{Q}_p \mid I(x) \leq c\} = \mathbf{Q}_p$ for $c \geq \alpha\theta \log p$. Therefore, I is good if and only if $\theta = +\infty$. □

References

1. Feng, J., Kurtz, T.G.: Large Deviations for Stochastic Processes. Mathematical Surveys and Monographs, vol. 131. American Mathematical Society, Providence (2006)
2. Khrennikov, A.: p-adic analogues of the law of large numbers and the central limit theorem. Indag. Math. **8**(1), 61–77 (1997)
3. Khrennikov, A.: Limit behaviour of sums of independent random variables with respect to the uniform p-adic distribution. Stat. Probab. Lett. **51**, 269–276 (2001)
4. Koblitz, N.: p-Adic Numbers, p-Adic Analysis, and Zeta-Functions. Graduate Texts in Mathematics, vol. 58. Springer, Cham (1984)
5. Kochubei, A.N.: Limit theorems for sums of p-adic random variables. Expo. Math. **16**, 425–439 (1998)
6. Lang, S.: Algebraic Number Theory. Graduate Texts in Mathematics, vol. 110, 2nd edn. Springer, Cham (1994)
7. Maximov, V.M.: Multidimensional and abstract probability. Proc. Steklov Inst. Math. **287**, 174–201 (2014)

8. Mikhailov, A.E.: Estimates of convergence rates to stable distributions on $\mathbf{Q}_p$. J. Math. Sci. **206**, 207–211 (2015)
9. Yasuda, K.: On infinitely divisible distributions on locally compact abelian groups. J. Theor. Probab. **13**, 635–657 (2000)
10. Yasuda, K.: On the asymptotic growth rate of the sum of p-adic-valued independent identically distributed random variables. Arab. J. Math. **3**, 449–460 (2014)
11. Zelenov, E.I.: The p-adic law of large numbers. Izv. Math. **80**, 489–499 (2016)