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Higher-Order Derivative of Intersection Local Time for Two Independent Fractional Brownian Motions

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Abstract In this article, we obtain sharp conditions for the existence of the high-order derivatives (k-th order) of intersection local time $\widehat{\alpha}^{(k)}(0)$ of two independent d-dimensional fractional Brownian motions $B_t^{H_1}$ and $\widetilde{B}_s^{H_2}$ of Hurst parameters H_1 and H_2, respectively. We also study their exponential integrability.

Keywords Fractional Brownian motion · Intersection local time · k-th derivative of intersection local time · Exponential integrability

Mathematics Subject Classification (2010) 60G22 · 60J55

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1 Introduction and Main Result

Intersection local time or self-intersection local time when the two processes are the same are important subjects in probability theory and their derivatives have received much attention recently, see, e.g., [9,11–13]. Jung and Markowsky [6,7] discussed Tanaka formula and occupation time formula for derivative self-intersection local time of fractional Brownian motions. On the other hand, several authors paid attention to the renormalized self-intersection local time of fractional Brownian motions, see, e.g., Hu et al. [3,4].

Motivated by Jung and Markowsky [6] and Hu [2], higher-order derivative of intersection local time for two independent fractional Brownian motions is studied in this paper.

To state our main result we let $B^{H_1} = \{B_t^{H_1}, t \geq 0\}$ and $\widetilde{B}^{H_2} = \{\widetilde{B}_t^{H_2}, t \geq 0\}$ be two independent d-dimensional fractional Brownian motions of Hurst parameters $H_1, H_2 \in (0, 1)$, respectively. This means that B^{H_1} and $\widetilde{B}^{H_2}$ are independent centered Gaussian processes with covariance

$$\mathbb{E}\left[B_s^{H_1} B_t^{H_1}\right] = \frac{1}{2}\left(s^{2H_1} + t^{2H_1} - |\, s - t\,|^{2H_1}\right)$$

(similar identity for $\tilde{B}$). In this paper we concern with the derivatives of intersection local time of B^{H_1} and $\widetilde{B}^{H_2}$, defined by

$$\hat{\alpha}^{(k)}(x) := \frac{\partial^k}{\partial x_1^{k_1} \ldots \partial x_d^{k_d}} \int_0^T \int_0^T \delta\left(B_t^{H_1} - \widetilde{B}_s^{H_2} + x\right) \mathrm{d}t \mathrm{d}s,$$

where $k = (k_1, \ldots, k_d)$ is a multi-index with all k_i being nonnegative integers and δ is the Dirac delta function of d-variables. In particular, we consider exclusively the case when $x = 0$ in this work. Namely, we are studying

$$\hat{\alpha}^{(k)}(0) := \int_0^T \int_0^T \delta^{(k)}\left(B_t^{H_1} - \widetilde{B}_s^{H_2}\right) \mathrm{d}t \mathrm{d}s, \tag{1.1}$$

where $\delta^{(k)}(x) = \frac{\partial^k}{\partial x_1^{k_1} \ldots \partial x_d^{k_d}} \delta(x)$ is k-th order partial derivative of the Dirac delta function. Since $\delta(x) = 0$ when $x \neq 0$ the intersection local time $\hat{\alpha}(0)$ (when $k = 0$) measures the frequency that processes B^{H_1} and $\widetilde{B}^{H_2}$ intersect each other.

Since the Dirac delta function δ is a generalized function, we need to give a meaning to $\hat{\alpha}^{(k)}(0)$. To this end, we approximate the Dirac delta function δ by

$$f_\varepsilon(x) := \frac{1}{(2\pi\varepsilon)^{\frac{d}{2}}} \mathrm{e}^{-\frac{|x|^2}{2\varepsilon}} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{ipx} \mathrm{e}^{-\frac{\varepsilon |p|^2}{2}} \mathrm{d}p, \tag{1.2}$$

where and throughout this paper, we use $px = \sum_{j=1}^d p_j x_j$ and $|p|^2 = \sum_{j=1}^d p_j^2$. Thus, we approximate $\delta^{(k)}$ by

$$f_\varepsilon^{(k)}(x) := \frac{\partial^k}{\partial x_1^{k_1} \dots \partial x_d^{k_d}} f_\varepsilon(x) = \frac{i^k}{(2\pi)^d} \int_{\mathbb{R}^d} p_1^{k_1} \dots p_d^{k_d} e^{ipx} e^{-\frac{\varepsilon|p|^2}{2}} dp. \tag{1.3}$$

We say that $\hat{\alpha}^{(k)}(0)$ exists (in L^2) if

$$\hat{\alpha}_\varepsilon^{(k)}(0) := \int_0^T \int_0^T f_\varepsilon^{(k)} \left(B_t^{H_1} - \widetilde{B}_s^{H_2} \right) dt ds \tag{1.4}$$

converges to a random variable (denoted by $\hat{\alpha}^{(k)}(0)$) in L^2 when $\varepsilon \downarrow 0$.

Here is the main result of this work.

Theorem 1 *Let B^{H_1} and $\widetilde{B}^{H_2}$ be two independent d-dimensional fractional Brownian motions of Hurst parameter H_1 and H_2, respectively.*

(i) *Assume $k = (k_1, \dots, k_d)$ is an index of nonnegative integers (meaning that $k_1, \dots, k_d$ are nonnegative integers) satisfying*

$$\frac{H_1 H_2}{H_1 + H_2}(|k| + d) < 1, \tag{1.5}$$

where $|k| = k_1 + \dots + k_d$. Then, the k-th order derivative intersection local time $\hat{\alpha}^{(k)}(0)$ exists in $L^p(\Omega)$ for any $p \in [1, \infty)$.

(ii) *Assume condition (1.5) is satisfied. There is a strictly positive constant $C_{d,k,T} \in (0, \infty)$ such that*

$$\mathbb{E}\left[\exp\left\{ C_{d,k,T} \left| \widehat{\alpha}^{(k)}(0) \right|^\beta \right\} \right] < \infty,$$

where $\beta = \frac{H_1 + H_2}{2dH_1H_2}$.

(iii) *If $\hat{\alpha}^{(k)}(0) \in L^1(\Omega)$, where $k = (0, \dots, 0, k_i, 0, \dots, 0)$ with k_i being even integer, then condition (1.5) must be satisfied.*

Remark 1 (i) When $k = 0$, we have that $\widehat{\alpha}^{(0)}(0)$ is in L^p for any $p \in [1, \infty)$ if $\frac{H_1 H_2}{H_1 + H_2} d < 1$. In the special case $H_1 = H_2 = H$, this condition becomes $Hd < 2$, which is the condition obtained in Nualart et al. [8].

(ii) When $H_1 = H_2 = \frac{1}{2}$, we have the exponential integrability exponent $\beta = 2/d$, which implies an earlier result [2, Theorem 9.4].

(iii) Part (iii) of the theorem states that the inequality (1.5) is also a necessary condition for the existence of $\hat{\alpha}^{(k)}(0)$. This is the first time for such a statement.

2 Proof of the Theorem

Proof of Parts (i) and (ii). This section is devoted to the proof of the theorem. We shall first find a good bound for $\mathbb{E}\left|\widehat{\alpha}^{(k)}(0)\right|^n$ which gives a proof for (i) and (ii) simultaneously. We introduce the following notations.

$$p_j = (p_{1j}, \ldots, p_{dj}), \quad p_j^k = \left(p_{1j}^{k_1}, \ldots, p_{dj}^{k_d}\right), \quad j = 1, 2, \ldots, n;$$
$$p = (p_1, \ldots, p_n), \qquad dp = \prod_{i=1}^{d}\prod_{j=1}^{n} dp_{ij}.$$

We also denote $s = (s_1, \ldots, s_n)$, $t = (t_1, \ldots, t_n)$, $ds = ds_1 \ldots ds_n$ and $dt = dt_1 \ldots dt_n$.

Fix an integer $n \geq 1$. Denote $T_n = \{0 < t, s < T\}^n$. We have

$$\begin{aligned}
\mathbb{E}\left[\left|\widehat{\alpha}_\varepsilon^{(k)}(0)\right|^n\right] &\leq \frac{1}{(2\pi)^{nd}}\int_{T_n}\int_{\mathbb{R}^{nd}}\left|\mathbb{E}\left[\exp\left\{ip_1\left(B_{s_1}^{H_1} - \widetilde{B}_{t_1}^{H_2}\right) + \cdots\right.\right.\right. \\
&\quad \left.\left.\left. + ip_n\left(B_{s_n}^{H_1} - \widetilde{B}_{t_n}^{H_2}\right)\right\}\right]\right| \exp\left\{-\frac{\varepsilon}{2}\sum_{j=1}^{n} \mid p_j \mid^2\right\}\prod_{j=1}^{n} \mid p_j^k \mid \mathrm{d}p\mathrm{d}t\mathrm{d}s \\
&= \frac{1}{(2\pi)^{nd}}\int_{T_n}\int_{\mathbb{R}^{nd}} \exp\left\{-\frac{1}{2}\mathbb{E}\left[\sum_{j=1}^{n} p_j\left(B_{s_j}^{H_1} - \widetilde{B}_{t_j}^{H_2}\right)\right]^2\right\} \\
&\quad \times \exp\left\{-\frac{\varepsilon}{2}\sum_{j=1}^{n} \mid p_j \mid^2\right\}\prod_{j=1}^{n} \mid p_j^k \mid \mathrm{d}p\mathrm{d}t\mathrm{d}s \\
&\leq \frac{1}{(2\pi)^{nd}}\int_{T_n}\int_{\mathbb{R}^{nd}}\prod_{i=1}^{d}\left(\prod_{j=1}^{n} \mid p_{ij}^{k_i} \mid\right)\exp\left\{-\frac{1}{2}\mathbb{E}\left[p_{i1}B_{s_1}^{H_1,i} + \cdots\right.\right. \\
&\quad \left.\left. + p_{in}B_{s_n}^{H_1,i}\right]^2 - \frac{1}{2}\mathbb{E}\left[p_{i1}B_{t_1}^{H_2,i} + \cdots + p_{in}B_{t_n}^{H_2,i}\right]^2\right\}\mathrm{d}p\mathrm{d}t\mathrm{d}s.
\end{aligned}$$

The expectations in the above exponent can be computed by

$$\mathbb{E}\left[p_{i1}B_{s_1}^{H_1,i} + \cdots + p_{in}B_{s_n}^{H_1,i}\right]^2 = (p_{i1}, \ldots, p_{in})Q_1(p_{i1}, \ldots, p_{in})^{\mathrm{T}},$$
$$\mathbb{E}\left[p_{i1}\tilde{B}_{s_1}^{H_2,i} + \cdots + p_{in}\tilde{B}_{s_n}^{H_2,i}\right]^2 = (p_{i1}, \ldots, p_{in})Q_2(p_{i1}, \ldots, p_{in})^{\mathrm{T}},$$

where

$$Q_1 = \mathbb{E}\left(B_j^{H_1,i}B_k^{H_1,i}\right)_{1\leq j,k\leq n} \quad \text{and} \quad Q_2 = \mathbb{E}\left(\tilde{B}_j^{H_2,i}\tilde{B}_k^{H_2,i}\right)_{1\leq j,k\leq n}$$

denote, respectively, covariance matrices of n-dimensional random vectors $(B_{s_1}^{H_1,i}, \ldots, B_{s_n}^{H_1,i})$ and that of $(\widetilde{B}_{t_1}^{H_2,i}, \ldots, \widetilde{B}_{t_n}^{H_2,i})$. Thus, we have

$$\mathbb{E}\left[\left|\widehat{\alpha}_\varepsilon^{(k)}(0)\right|^n\right] \leq \frac{1}{(2\pi)^{nd}}\int_{T_n}\prod_{i=1}^{d} I_i(t, s)\mathrm{d}t\mathrm{d}s, \tag{2.1}$$

where

$$I_i(t,s) := \int_{\mathbb{R}^n} \mid x^{k_i} \mid \exp\left\{-\frac{1}{2}x^T(Q_1+Q_2)x\right\} \mathrm{d}x.$$

Here we recall $x = (x_1, \ldots, x_n)$ and $x_i^k = x_1^{k_i} \ldots x_n^{k_i}$. For each fixed i let us compute integral $I_i(t,s)$ first. Denote $B = Q_1 + Q_2$. Then B is a strictly positive definite matrix, and hence $\sqrt{B}$ exists. Making substitution $\xi = \sqrt{B}x$. Then

$$I_i(t,s) = \int_{\mathbb{R}^n} \prod_{j=1}^n \mid (B^{-\frac{1}{2}}\xi)_j \mid^{k_i} \exp\left\{-\frac{1}{2} \mid \xi \mid^2\right\} \det(B)^{-\frac{1}{2}} \mathrm{d}\xi.$$

To obtain a nice bound for the above integral, let us first diagonalize B:

$$B = Q\Lambda Q^{-1},$$

where $\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$ is a strictly positive diagonal matrix with $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_d$ and $Q = (q_{ij})_{1\le i,j\le d}$ is an orthogonal matrix. Hence, we have $\det(B) = \lambda_1 \ldots \lambda_d$. Denote

$$\eta = \left(\eta_1, \eta_2, \ldots, \eta_n\right)^{\mathrm{T}} = Q^{-1}\xi,$$

Hence,

$$\begin{aligned} B^{-\frac{1}{2}}\xi &= Q\Lambda^{-1/2}Q^{-1}\xi = Q\Lambda^{-1/2}\eta \\ &= Q\begin{pmatrix} \lambda_1^{-\frac{1}{2}}\eta_1 \\ \lambda_2^{-\frac{1}{2}}\eta_2 \\ \vdots \\ \lambda_n^{-\frac{1}{2}}\eta_n \end{pmatrix} = \begin{pmatrix} q_{1,1} & q_{1,2} & \cdots & q_{1,n} \\ q_{2,1} & q_{2,2} & \cdots & q_{2,n} \\ \vdots & \vdots & \cdots & \vdots \\ q_{n,1} & q_{n,2} & \cdots & q_{n,n} \end{pmatrix} \begin{pmatrix} \lambda_1^{-\frac{1}{2}}\eta_1 \\ \lambda_2^{-\frac{1}{2}}\eta_2 \\ \vdots \\ \lambda_n^{-\frac{1}{2}}\eta_n. \end{pmatrix}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mid (B^{-\frac{1}{2}}\xi)_j \mid &= \left|\sum_{k=1}^n q_{jk}\lambda_k^{-\frac{1}{2}}\eta_k\right| \le \lambda_1^{-\frac{1}{2}} \sum_{k=1}^n \mid q_{jk}\eta_k \mid \\ &\le \lambda_1^{-\frac{1}{2}} \left(\sum_{k=1}^n q_{jk}^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^n \eta_k^2\right)^{\frac{1}{2}} \le \lambda_1^{-\frac{1}{2}} \mid \eta \mid_2 = \lambda_1^{-\frac{1}{2}} \mid \xi \mid_2 . \end{aligned}$$

Since both Q_1 and Q_2 are positive definite, we see that

$$\lambda_1 \ge \lambda_1(Q_1), \qquad \text{and} \qquad \lambda_1 \ge \lambda_1(Q_2),$$

where $\lambda_1(Q_i)$ is the smallest eigenvalue of Q_i, $i = 1, 2$. This means that

$$\lambda_1 \ge \lambda_1(Q_1)^\rho \lambda_1(Q_2)^{1-\rho} \quad \text{for any} \quad \rho \in [0, 1].$$

This implies

$$| (B^{-\frac{1}{2}}\xi)_j | \leq \lambda_1(Q_1)^{-\frac{1}{2}\rho}\lambda_1(Q_2)^{-\frac{1}{2}(1-\rho)} | \xi |_2 .$$

Consequently, we have

$$I_i(t,s) = \det(B)^{-\frac{1}{2}}\lambda_1(Q_1)^{-\frac{1}{2}\rho k_i}\lambda_1(Q_2)^{-\frac{1}{2}(1-\rho)k_i} \int_{\mathbb{R}^n} | \xi |_2^{k_i} \exp\left\{-\frac{1}{2} | \xi |^2\right\} \mathrm{d}\xi, \tag{2.2}$$

for any $\rho \in [0, 1]$.

Now we are going to find a lower bound for $\lambda_1(Q_1)$ ($\lambda_1(Q_2)$ can be dealt with the same way. We only need to replace s by t). Without loss of generality we can assume $0 \leq s_1 < s_2 < \cdots < s_n \leq T$. From the definition of Q_1 we have for any vector $u = (u_1, \ldots, u_d)^T$,

$$\begin{aligned} u^T Q_1 u &= \mathrm{Var}\left(u_1 B_{s_1}^{H_1} + u_2 B_{s_2}^{H_1} + \cdots + u_n B_{s_n}^{H_1}\right) \\ &= \mathrm{Var}\Big((u_1 + \cdots + u_n) B_{s_1}^{H_1} + (u_2 + \cdots + u_n)\left(B_{s_2}^{H_1} - B_{s_1}^{H_1}\right) \\ &\quad + \cdots + (u_{n-1} + u_n)\left(B_{s_{n-1}}^{H_1} - B_{s_{n-2}}^{H_1}\right) + u_n\left(B_{s_n}^{H_1} - B_{s_{n-1}}^{H_1}\right)\Big) \end{aligned}$$

Now we use Proposition 1 in "Appendix" to conclude

$$\begin{aligned} u^T Q_1 u &\geq c^n \Big((u_1 + \cdots + u_n)^2 s_1^{2H_1} + (u_2 + \cdots + u_n)^2 (s_2 - s_1)^{2H_1} \\ &\quad + \cdots + (u_{n-1} + u_n)^2 (s_{n-1} - s_{n-2})^{2H_1} + u_n^2 (s_n - s_{n-1})^{2H_1}\Big) \\ &\geq c^n \min\left\{s_1^{2H_1}, (s_2 - s_1)^{2H_1}, \ldots, (s_n - s_{n-1})^{2H_1}\right\} \\ &\quad \cdot \left[(u_1 + \cdots + u_n)^2 + (u_2 + \cdots + u_n)^2 + \cdots + (u_{n-1} + u_n)^2 + u_n^2\right]. \end{aligned}$$

Consider the function

$$\begin{aligned} f(u_1, \ldots, u_n) &= (u_1 + \cdots + u_n)^2 + (u_2 + \cdots + u_n)^2 + \cdots + (u_{n-1} + u_n)^2 + u_n^2 \\ &= (u_1, \ldots, u_n) G (u_1, \ldots, u_n)^{\mathrm{T}}, \end{aligned}$$

where

$$G = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

It is easy to see that the matrix $G^T G$ has a minimum eigenvalue independent of n. Thus, this function f attains its minimum value $f_{\min}$ independent of n on the sphere $u_1^2 + \cdots + u_n^2 = 1$. It is also easy to see that $f_{\min} > 0$.

As a consequence we have

$$
\begin{aligned}
\lambda_1(Q_1) &= \inf_{|u|=1} u^T Q_1 u \\
&\geq c^n \min\left\{s_1^{2H_1}, (s_2-s_1)^{2H_1}, \ldots, (s_n-s_{n-1})^{2H_1}\right\} \inf_{|u|=1} f(u_1,\ldots,u_n) \\
&\geq c^n f_{\min} \min\left\{s_1^{2H_1}, (s_2-s_1)^{2H_1}, \ldots, (s_n-s_{n-1})^{2H_1}\right\} \\
&\geq K c^n \min\left\{s_1^{2H_1}, (s_2-s_1)^{2H_1}, \ldots, (s_n-s_{n-1})^{2H_1}\right\}. \qquad (2.3)
\end{aligned}
$$

In a similar way we have

$$
\lambda_1(Q_2) \geq K c^n \min\left\{t_1^{2H_2}, (t_2-t_1)^{2H_2}, \ldots, (t_n-t_{n-1})^{2H_2}\right\}. \qquad (2.4)
$$

The integral in (2.2) can be bounded as

$$
\begin{aligned}
I_2 &:= \int_{\mathbb{R}^n} \mid \xi \mid^{k_i} \exp\left\{-\frac{1}{2} \mid \xi \mid^2\right\} \mathrm{d}\xi \\
&\leq n^{\frac{k_i}{2}} \int_{\mathbb{R}^{nd}} \max_{1\leq j\leq n} \mid \xi_j \mid^{k_i} \exp\left\{-\frac{1}{2} \mid \xi \mid^2\right\} \mathrm{d}\xi \\
&\leq n^{\frac{k_i}{2}} \int_{\mathbb{R}^n} \sum_{j=1}^n \mid \xi_j \mid^{k_j} \exp\left\{-\frac{1}{2} \mid \xi \mid^2\right\} \mathrm{d}\xi \\
&\leq n^{\frac{k_i}{2}+1} \int_{\mathbb{R}^n} \mid \xi_1 \mid^{k_i} \exp\left\{-\frac{1}{2} \mid \xi \mid^2\right\} \mathrm{d}\xi \\
&\leq n^{\frac{k_i}{2}+1} C^n \leq C^n. \qquad (2.5)
\end{aligned}
$$

Substitute (2.3)-(2.5) into (2.2) we obtain

$$
\begin{aligned}
I_i(t,s) \leq C^n \det(B)^{-\frac{1}{2}} & \min_{j=1,\ldots,n} (s_j - s_{j-1})^{-\rho H_1 k_i} \\
& \min_{j=1,\ldots,n} (t_j - t_{j-1})^{-(1-\rho) H_2 k_i}
\end{aligned}
$$

for possibly a different constant C, independent of n.

Next we obtain a lower bound for $\det(B)$. According to [2, Lemma 9.4]

$$
\det(Q_1 + Q_2) \geq \det(Q_1)^\gamma \det(Q_2)^{1-\gamma},
$$

for any two symmetric positive definite matrices Q_1 and Q_2 and for any $\gamma \in [0, 1]$. Now it is well known that (see also the usages in [2–4]).

$$
\det(Q_1) \geq C^n s_1^{2H_1} (s_2 - s_1)^{2H_1} \cdots (s_n - s_{n-1})^{2H_1}.
$$

and

$$\det(Q_2) \geq C^n t_1^{2H_2}(t_2 - t_1)^{2H_2} \cdots (t_n - t_{n-1})^{2H_2}.$$

As a consequence, we have

$$I_i(t,s) \leq C^n \min_{j=1,\ldots,n}(s_j - s_{j-1})^{-\rho H_1 k_i} \min_{j=1,\ldots,n}(t_j - t_{j-1})^{-(1-\rho)H_2 k_i} \left[s_1(s_2 - s_1)\ldots(s_n - s_{n-1})\right]^{-\gamma H_1} \left[t_1(t_2 - t_1)\ldots(t_n - t_{n-1})\right]^{-(1-\gamma)H_2}$$

Thus,

$$\begin{aligned}\mathbb{E}\left[\left|\widehat{\alpha}_\varepsilon^{(k)}(0)\right|^n\right] &\leq (n!)^2 C^n \int_{\Delta_n^2} \min_{j=1,\ldots,n}(s_j - s_{j-1})^{-\rho H_1|k|} \\ &\quad \min_{j=1,\ldots,n}(t_j - t_{j-1})^{-(1-\rho)H_2|k|}\left[s_1(s_2 - s_1)\ldots(s_n - s_{n-1})\right]^{-\gamma H_1 d} \\ &\quad \left[t_1(t_2 - t_1)\ldots(t_n - t_{n-1})\right]^{-(1-\gamma)H_2 d}\,\mathrm{d}t\mathrm{d}s \\ &\leq (n!)^2 C^n \sum_{i,j=1}^n \int_{\Delta_n^2}(s_i - s_{i-1})^{-\rho H_1|k|} \\ &\quad (t_j - t_{j-1})^{-(1-\rho)H_2|k|}\left[s_1(s_2 - s_1)\ldots(s_n - s_{n-1})\right]^{-\gamma H_1 d} \\ &\quad \left[t_1(t_2 - t_1)\ldots(t_n - t_{n-1})\right]^{-(1-\gamma)H_2 d}\,\mathrm{d}t\mathrm{d}s,\end{aligned}$$

where $\Delta_n = \{0 < s_1 < \cdots < s_n \leq T\}$ denotes the simplex in $[0,T]^n$. We choose $\rho = \gamma = \frac{H_2}{H_1+H_2}$ to obtain

$$\mathbb{E}\left[\left|\widehat{\alpha}_\varepsilon^{(k)}(0)\right|^n\right] \leq (n!)^2 C^n \sum_{i,j=1}^n I_{3,i} I_{3,j},$$

where

$$I_{3,j} = \int_{\Delta_n}(t_j - t_{j-1})^{-\frac{H_1 H_2}{H_1+H_2}|k|}\left[t_1(t_2 - t_1)\ldots(t_n - t_{n-1})\right]^{-\frac{H_1 H_2}{H_1+H_2}d}\,\mathrm{d}t,$$

By Lemma 4.5 of [5], we see that if

$$\frac{H_1 H_2}{H_1 + H_2}(|k| + d) \leq 1,$$

then

$$I_{3,j} \leq \frac{C^n T^{\kappa_1 n - \frac{H_1 H_2 |k|}{H_1+H_2}}}{\Gamma\left(n\kappa_1 - \frac{H_1 H_2}{H_1+H_2}|k| + 1\right)},$$

where

$$\kappa_1 = 1 - \frac{dH_1H_2}{H_1+H_2}.$$

Substituting this bound we obtain

$$\begin{aligned}\mathbb{E}\left[\left|\widehat{\alpha}_\varepsilon^{(k)}(0)\right|^n\right] &\le n^2(n!)^2C^n \frac{T^{2\kappa_1 n - \frac{2H_1H_2|k|}{H_1+H_2}}}{\Gamma^2\left(n\kappa_1 - \frac{H_1H_2}{H_1+H_2}|k|+1\right)}\\ &\le (n!)^2C^n \frac{T^{2\kappa_1 n - \frac{2H_1H_2|k|}{H_1+H_2}}}{\left(\Gamma(n\kappa_1 - \frac{H_1H_2}{H_1+H_2}|k|+1)\right)^2}\\ &\le C_T(n!)^{2-2\kappa_1}C^nT^{2\kappa_1 n},\end{aligned}$$

where C is a constant independent of T and n and C_T is a constant independent of n.

For any $\beta > 0$, the above inequality implies

$$\mathbb{E}\left[\left|\widehat{\alpha}^{(k)}(0)\right|^{n\beta}\right] \le C_T(n!)^{\beta(2-2\kappa_1)}C^nT^{2\beta\kappa_1 n}$$

From this bound we conclude that there exists a constant $C_{d,T,k} > 0$ such that

$$\begin{aligned}\mathbb{E}\left[\exp\left\{C_{d,T,k}\left|\widehat{\alpha}^{(k)}(0)\right|^\beta\right\}\right] &= \sum_{n=0}^\infty \frac{C_{d,T,k}^n}{n!}\mathbb{E}\left|\widehat{\alpha}^{(k)}(0)\right|^{n\beta}\\ &\le C_T\sum_{n=0}^\infty C_{d,T,k}^n(n!)^{\beta(2-2\kappa_1)-1}C^nT^{2\beta\kappa_1 n} < \infty,\end{aligned}$$

when $C_{d,T,k}$ is sufficiently small (but strictly positive), where $\beta = \frac{H_1+H_2}{2dH_1H_2}$. □

Proof of part (iii). Without loss of generality, we consider only the case $k = (k_1, 0, \dots, 0)$ and we denote k_i by k. By the definition of k-order derivative local time of independent d-dimensional fractional Brownian motions, we have

$$\begin{aligned}\mathbb{E}\left[\hat{\alpha}_\varepsilon^{(k)}(0)\right] &= \frac{1}{(2\pi)^d}\int_0^T\int_0^T\int_{\mathbb{R}^d}\mathbb{E}\left[e^{i\langle\xi, B_t^{H_1}-\widetilde{B}_s^{H_2}\rangle}\right]e^{-\frac{\varepsilon|\xi|^2}{2}} \mid \xi_1\mid^k \,d\xi dt ds\\ &= \frac{1}{(2\pi)^d}\int_0^T\int_0^T\int_{\mathbb{R}^d} e^{-(\varepsilon+t^{2H_1}+s^{2H_2})\frac{|\xi|^2}{2}} \mid \xi_1\mid^k \,d\xi dt ds.\end{aligned}$$

Thus, we have

$$\mathbb{E}\left[\hat{\alpha}^{(k)}(0)\right] = \frac{1}{(2\pi)^d}\int_0^T\int_0^T\int_{\mathbb{R}^d} e^{-(t^{2H_1}+s^{2H_2})\frac{|\xi|^2}{2}} \mid \xi_1\mid^k \,d\xi dt ds.$$

Integrating with respect to ξ, we find

$$\mathbb{E}\left[\hat{\alpha}^{(k)}(0)\right] = c_{k,d}\int_0^T\int_0^T (t^{2H_1}+s^{2H_2})^{-\frac{(k+d)}{2}}\mathrm{d}t\mathrm{d}s$$

for some constant $c_{k,d}\in(0,\infty)$.

We are going to deal with the above integral. Assume first $0<H_1\le H_2<1$. Making substitution $t=u^{\frac{H_2}{H_1}}$ yields

$$\begin{aligned} I_4 &:= \int_0^T\int_0^T (t^{2H_1}+s^{2H_2})^{-\frac{(k+d)}{2}}\mathrm{d}t\mathrm{d}s \\ &= \int_0^T\int_0^{T^{\frac{H_1}{H_2}}} (u^{2H_2}+s^{2H_2})^{-\frac{k+d}{2}} u^{\frac{H_2}{H_1}-1}\mathrm{d}u\mathrm{d}s. \end{aligned} \tag{2.6}$$

Using polar coordinate $u=r\cos\theta$ and $s=r\sin\theta$, where $0\le\theta\le\frac{\pi}{2}$ and $0\le r\le T$ we have

$$I_4 \ge \int_0^{\frac{\pi}{2}}(\cos\theta)^{\frac{H_2}{H_1}-1}\left(\cos^{2H_2}\theta+\sin^{2H_2}\theta\right)^{-\frac{(k+d)}{2}}\mathrm{d}\theta\int_0^{T^{\frac{H_1}{H_2}}} r^{-(k+d)H_2+\frac{H_2}{H_1}}\mathrm{d}r \tag{2.7}$$

since the planar domain $\left\{(r,\theta), 0\le r\le T\wedge T^{\frac{H_1}{H_2}}, 0\le\theta\le\frac{\pi}{2}\right\}$ is contained in the planar domain $\left\{(s,u), 0\le s\le T, 0\le u\le T^{\frac{H_1}{H_2}}\right\}$. The integral with respect to r appearing in (2.7) is finite only if $-(k+d)H_2+\frac{H_2}{H_1}>-1$, namely only when the condition (1.5) is satisfied. The case $0<H_2\le H_1<1$ can be dealt similarly. This completes the proof of our main theorem. □

3 Appendix

In this section, we recall some known results that are used in this paper. The following lemma is Lemma 8.1 of [1].

Lemma 1 *Let $X_1,\ldots,X_n$ be jointly mean zero Gaussian random variables, and let $Y_1=X_1,\ Y_2=X_2-X_1,\ldots,Y_n=X_n-X_{n-1}$. Then*

$$\mathrm{Var}\left\{\sum_{j=1}^n v_jY_j\right\} \ge \frac{R}{\prod_{j=1}^n\sigma_j^2}\frac{1}{n}\sum_{j=1}^n v_j^2\sigma_j^2,$$

where $\sigma_j^2 = \mathrm{Var}(Y_j)$ *and* R *is the determinant of the covariance matrix of* $\{X_i, i = 1, \dots, n\}$*, which is also given by the following product of conditional variances*

$$R = \mathrm{Var}(X_1)\mathrm{Var}(X_2 \mid X_1) \dots \mathrm{Var}(X_n \mid X_1, \dots, X_{n-1}).$$

The following lemma is from [4], Lemma A.1.

Lemma 2 *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space and let* F *be a square integrable random variable. Suppose that* $\mathcal{G}_1 \subset \mathcal{G}_2$ *are two* σ*-fields contained in* $\mathcal{F}$*. Then*

$$\mathrm{Var}(F \mid \mathcal{G}_1) \geq \mathrm{Var}(F \mid \mathcal{G}_2).$$

The following is Lemma 7.1 of [10] applied to fractional Brownian motion.

Lemma 3 *If* $(B_t, 0 \leq t < \infty)$ *is the fractional Brownian motion of Hurst* H*, then*

$$\mathrm{Var}(X(t)|X(s), |s-t| \geq r) = cr^{2H}.$$

Combining the above three lemmas we have the following

Proposition 1 *Let* $(B_t, 0 \leq t < \infty)$ *be the fractional Brownian motion of Hurst* H *and let* $0 \leq s_1 < \cdots < s_n < \infty$*. Then there is a constant* c *independent of* n *such that*

$$\begin{aligned} &\mathrm{Var}\big(\xi_1 B_{s_1} + \xi_2 \left(B_{s_2} - B_{s_1}\right) + \cdots + \xi_n \left(B_{s_n} - B_{s_{n-1}}\right)\big) \\ &\quad \geq c^n \left[\xi_1^2 \mathrm{Var}(B_{s_1}) + \xi_2^2 \mathrm{Var}\left(B_{s_2} - B_{s_1}\right) + \cdots + \xi_n^2 \mathrm{Var}\left(B_{s_n} - B_{s_{n-1}}\right)\right]. \end{aligned} \tag{3.1}$$

Proof Let $X_i = B_{s_i} - B_{s_{i-1}}$ ($B_{s_{-1}} = 0$ by convention). From Lemma 2 we see

$$\begin{aligned} R_i &:= \mathrm{Var}(X_i \mid X_1, \dots, X_{i-1}) \geq \mathrm{Var}(B_{s_i}|\mathcal{F}_{s_{i-1}}) \\ &\geq c|s_i - s_{i-1}|^{2H} = c\sigma_i^2, \end{aligned}$$

where $\mathcal{F}_t = \sigma(B_s, s \leq t)$. From the definition of R we see $R \geq c^n \prod_{i=1}^n \sigma_i^2$. The proposition is proved by applying Lemma 1. □

The following lemma is Lemma 4.5 of [5].

Lemma 4 *Let* $\alpha \in (-1+\varepsilon, 1)^m$ *with* $\varepsilon > 0$ *and set* $|\alpha| = \sum_{i=1}^m \alpha_i$*. Denote* $T_m(t) = \{(r_1, r_2, \dots, r_m) \in \mathbb{R}^m : 0 < r_1 < \cdots < r_m < t\}$*. Then there is a constant* κ *such that*

$$J_m(t, \alpha) := \int_{T_m(t)} \prod_{i=1}^m (r_i - r_{i-1})^{\alpha_i} \mathrm{d}r \leq \frac{\kappa^m t^{|\alpha|+m}}{\Gamma(|\alpha| + m + 1)},$$

where by convention, $r_0 = 0$.

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