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Asymptotic Behavior of Weighted Power Variations of Fractional Brownian Motion in Brownian Time

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Abstract We study the asymptotic behavior of weighted power variations of fractional Brownian motion in Brownian time $Z_t := X_{Y_t}, t \geqslant 0$, where X is a fractional Brownian motion and Y is an independent Brownian motion.

Keywords Weighted power variations · Limit theorem · Malliavin calculus · Fractional Brownian motion · Fractional Brownian motion in Brownian time

Mathematics Subject Classification 2010 60F05 · 60G15 · 60G22 · 60H05 · 60H07

1 Introduction

Our aim in this paper is to study the asymptotic behavior of weighted power variations of the so-called *fractional Brownian motion in Brownian time* defined as

$$Z_t = X_{Y_t}, \quad t \geqslant 0,$$

where X is a two-sided fractional Brownian motion, with Hurst parameter $H \in (0, 1)$, and Y is a standard (one-sided) Brownian motion independent of X. It is a self-similar process (of order $H/2$) with stationary increments, which is not Gaussian. When $H = 1/2$, one recovers the celebrated *iterated Brownian motion.*

In the present paper we follow and we are inspired by the previous papers [2,4,5,9], and our work may be seen as a natural follow-up of [4,9].

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Let $f : \mathbb{R} \to \mathbb{R}$ be a function belonging to C_b^∞, the class of those functions that are C^∞ and bounded together with their derivatives. Then, for any $t \geqslant 0$ and any integer $p \geqslant 1$, the weighted p-variation of Z is defined as

$$R_n^{(p)}(t) = \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} \frac{1}{2}(f(Z_{k2^{-n}}) + f(Z_{(k+1)2^{-n}}))(Z_{(k+1)2^{-n}} - Z_{k2^{-n}})^p.$$

After proper normalization, we may expect the convergence (in some sense) to a non-degenerate limit (to be determined) of

$$S_n^{(p)}(t) = 2^{n\kappa} \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} \frac{1}{2}(f(Z_{k2^{-n}}) + f(Z_{(k+1)2^{-n}}))[(Z_{(k+1)2^{-n}} - Z_{k2^{-n}})^p - E[(Z_{(k+1)2^{-n}} - Z_{k2^{-n}})^p]], \tag{1.1}$$

for some κ to be discovered. Due to the fact that one cannot separate X from Y inside Z in the definition of $S_n^{(p)}$, working directly with (1.1) seems to be a difficult task (see also [3, Problem 5.1]). This is why, following an idea introduced by Khoshnevisan and Lewis [2] in a study of the case $H = 1/2$, we will rather analyze $S_n^{(p)}$ by means of certain stopping times for Y. The idea is: by stopping Y as it crosses certain levels, and by sampling Z at these times, one can effectively separate X from Y. To be more specific, let us introduce the following collection of stopping times (with respect to the natural filtration of Y), noted

$$\mathscr{T}_n = \{T_{k,n} : k \geqslant 0\}, \quad n \geqslant 0, \tag{1.2}$$

which are in turn expressed in terms of the subsequent hitting times of a dyadic grid cast on the real axis. More precisely, let $\mathscr{D}_n = \{j2^{-n/2} : j \in \mathbb{Z}\}$, $n \geqslant 0$, be the dyadic partition (of $\mathbb{R}$) of order $n/2$. For every $n \geqslant 0$, the stopping times $T_{k,n}$, appearing in (1.2), are given by the following recursive definition: $T_{0,n} = 0$, and

$$T_{k,n} = \inf\{s > T_{k-1,n} : \quad Y(s) \in \mathscr{D}_n \setminus \{Y_{T_{k-1,n}}\}\}, \quad k \geqslant 1.$$

Note that the definition of $T_{k,n}$, and therefore of $\mathscr{T}_n$, only involves the one-sided Brownian motion Y, and that, for every $n \geqslant 0$, the discrete stochastic process

$$\mathscr{Y}_n = \{Y_{T_{k,n}} : k \geqslant 0\}$$

defines a simple and symmetric random walk over $\mathscr{D}_n$. As shown in [2], as n tends to infinity the collection $\{T_{k,n} : 1 \leqslant k \leqslant 2^n t\}$ approximates the common dyadic partition $\{k2^{-n} : 1 \leqslant k \leqslant 2^n t\}$ of order n of the time interval $[0, t]$ (see [2, Lemma 2.2] for a precise statement). Based on this fact, one can introduce the counterpart of (1.1) based on $\mathscr{T}_n$, namely,

$$\tilde{S}_n^{(p)}(t) = 2^{-n\tilde{\kappa}} \sum_{k=0}^{\lfloor 2^n t\rfloor -1} \frac{1}{2}(f(Z_{T_{k,n}}) + f(Z_{T_{k+1,n}}))[(2^{\frac{nH}{2}}(Z_{T_{k+1,n}} - Z_{T_{k,n}}))^p - \mu_p],$$

for some $\tilde{\kappa} > 0$ to be discovered and with $\mu_p := E[N^p]$, where $N \sim \mathcal{N}(0,1)$. At this stage, it is worthwhile noting that we are dealing with symmetric weighted p-variation of Z, and symmetry will play an important role in our analysis as we will see in Lemma 3.1.

In the particular case where $H = \frac{1}{2}$, that is when Z is the *iterated Brownian motion*, the asymptotic behavior of $\tilde{S}_n^{(p)}(\cdot)$ has been studied in [4]. In fact, one can deduce the following two finite-dimensional distributions (f.d.d.) convergences in law from [4, Theorem 1.2].

1) For $f \in C_b^2$ and for any integer $r \geqslant 1$, we have

$$\left(2^{-\frac{3n}{4}} \sum_{k=0}^{\lfloor 2^n t\rfloor -1} \frac{1}{2}(f(Z_{T_{k,n}}) + f(Z_{T_{k+1,n}}))[(2^{\frac{n}{4}}(Z_{T_{k+1,n}} - Z_{T_{k,n}}))^{2r} - \mu_{2r}]\right)_{t\geqslant 0}$$
$$\xrightarrow[n\to\infty]{\text{f.d.d.}} \left(\sqrt{\mu_{4r} - \mu_{2r}^2}\int_{-\infty}^{+\infty} f(X_s)L_t^s(Y)dW_s\right)_{t\geqslant 0}, \tag{1.3}$$

where $L_t^s(Y)$ stands for the local time of Y before time t at level s, W is a two-sided Brownian motion independent of (X, Y) and $\int_{-\infty}^{+\infty} f(X_s)L_t^s(Y)dW_s$ is the Wiener–Itô integral of $f(X_\cdot)L_t^\cdot(Y)$ with respect to W.

2) For $f \in C_b^2$ and for any integer $r \geqslant 2$, we have

$$\left(2^{-\frac{n}{4}} \sum_{k=0}^{\lfloor 2^n t\rfloor -1} \frac{1}{2}(f(Z_{T_{k,n}}) + f(Z_{T_{k+1,n}}))(2^{\frac{n}{4}}(Z_{T_{k+1,n}} - Z_{T_{k,n}}))^{2r-1}\right)_{t\geqslant 0}$$
$$\xrightarrow[n\to\infty]{\text{f.d.d.}} \left(\int_0^{Y_t} f(X_s)(\mu_{2r}d^\circ X_s + \sqrt{\mu_{4r-2} - \mu_{2r}^2}\, dW_s\right)_{t\geqslant 0}, \tag{1.4}$$

where for all $t \in \mathbb{R}$, $\int_0^t f(X_s)d^\circ X_s$ is the Stratonovich integral of $f(X)$ with respect to X defined as the limit in probability of $2^{-\frac{nH}{2}}W_n^{(1)}(f,t)$ as $n \to \infty$, with $W_n^{(1)}(f,t)$ defined in (3.3), W is a two-sided Brownian motion independent of (X, Y) and for $u \in \mathbb{R}$, $\int_0^u f(X_s)dW_s$ is the Wiener–Itô integral of $f(X)$ with respect to W defined in (5.16).

A natural follow-up of (1.3) and (1.4) is to study the asymptotic behavior of $\tilde{S}_n^{(p)}(\cdot)$ when $H \neq \frac{1}{2}$. In fact, the following more general result is our main finding in the present paper.

Theorem 1.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function belonging to C_b^∞ and let W denote a two-sided Brownian motion independent of (X, Y).*

(1) *For $H > \frac{1}{6}$, we have*

$$\sum_{k=0}^{\lfloor 2^n t\rfloor -1} \frac{1}{2}(f(Z_{T_{k,n}}) + f(Z_{T_{k+1,n}}))(Z_{T_{k+1,n}} - Z_{T_{k,n}}) \xrightarrow[n\to\infty]{P} \int_0^{Y_t} f(X_s)d^\circ X_s, \tag{1.5}$$

where for all $t \in \mathbb{R}$, $\int_0^t f(X_s)d^\circ X_s$ is the Stratonovich integral of $f(X)$ with respect to X defined as the limit in probability of $2^{-\frac{nH}{2}} W_n^{(1)}(f,t)$ as $n \to \infty$, with $W_n^{(1)}(f,t)$ defined in (3.3).
For $H = \frac{1}{6}$, we have

$$\sum_{k=0}^{\lfloor 2^n t\rfloor -1} \frac{1}{2}(f(Z_{T_{k,n}}) + f(Z_{T_{k+1,n}}))(Z_{T_{k+1,n}} - Z_{T_{k,n}}) \xrightarrow[n\to\infty]{law} \int_0^{Y_t} f(X_s)d^* X_s, \tag{1.6}$$

where for all $t \in \mathbb{R}$, $\int_0^t f(X_s)d^* X_s$ is the Stratonovich integral of $f(X)$ with respect to X defined as the limit in law of $2^{-\frac{nH}{2}} W_n^{(1)}(f,t)$ as $n \to \infty$.

(2) *For $\frac{1}{6} < H < \frac{1}{2}$ and for any integer $r \geqslant 2$, we have*

$$\begin{aligned}&\left(2^{-\frac{n}{4}} \sum_{k=0}^{\lfloor 2^n t\rfloor -1} \frac{1}{2}(f(Z_{T_{k,n}}) + f(Z_{T_{k+1,n}}))(2^{\frac{nH}{2}}(Z_{T_{k+1,n}} - Z_{T_{k,n}}))^{2r-1}\right)_{t\geqslant 0}\\ &\xrightarrow[n\to\infty]{\text{f.d.d.}} \left(\beta_{2r-1}\int_0^{Y_t} f(X_s)dW_s\right)_{t\geqslant 0},\end{aligned} \tag{1.7}$$

where for $u \in \mathbb{R}$, $\int_0^u f(X_s)dW_s$ is the Wiener–Itô integral of $f(X)$ with respect to W defined in (5.16), $\beta_{2r-1} = \sqrt{\sum_{l=2}^r \kappa_{r,l}^2\, \alpha_{2l-1}^2}$, with α_{2l-1} defined in (2.18) and $\kappa_{r,l}$ defined in (3.4).

(3) *Fix a time $t \geqslant 0$, for $H > \frac{1}{2}$ and for any integer $r \geqslant 1$, we have*

$$\begin{aligned}&2^{-\frac{nH}{2}} \sum_{k=0}^{\lfloor 2^n t\rfloor -1} \frac{1}{2}(f(Z_{T_{k,n}}) + f(Z_{T_{k+1,n}}))(2^{\frac{nH}{2}}(Z_{T_{k+1,n}} - Z_{T_{k,n}}))^{2r-1}\\ &\xrightarrow[n\to\infty]{L^2} \frac{(2r)!}{r!2^r}\int_0^{Y_t} f(X_s)d^\circ X_s,\end{aligned} \tag{1.8}$$

where for all $t \in \mathbb{R}$, $\int_0^t f(X_s)d^\circ X_s$ is defined as in (1.5).

(4) *For* $\frac{1}{4} < H \leqslant \frac{1}{2}$ *and for any integer* $r \geqslant 1$*, we have*

$$\left(2^{-\frac{3n}{4}} \sum_{k=0}^{\lfloor 2^n t\rfloor -1} \frac{1}{2}(f(Z_{T_{k,n}}) + f(Z_{T_{k+1,n}}))[(2^{\frac{nH}{2}}(Z_{T_{k+1,n}} - Z_{T_{k,n}}))^{2r} - \mu_{2r}]\right)_{t\geqslant 0}$$

$$\xrightarrow[n\to\infty]{\text{f.d.d.}} \left(\gamma_{2r} \int_{-\infty}^{+\infty} f(X_s) L_t^s(Y) dW_s\right)_{t\geqslant 0}, \qquad (1.9)$$

where $\int_{-\infty}^{+\infty} f(X_s)L_t^s(Y)dW_s$ *is the Wiener–Itô integral of* $f(X_\cdot)L_t^\cdot(Y)$ *with respect to* W, $\gamma_{2r} := \sqrt{\sum_{a=1}^{r} b_{2r,a}^2 \alpha_{2a}^2}$, *with* α_{2a} *defined in* (2.18) *and* $b_{2r,a}$ *defined in* (7.1)*.*

Theorem 1.1 is also a natural follow-up of [9, Corollary 1.2] where we have studied the asymptotic behavior of the power variations of the fractional Brownian motion in Brownian time. In fact, taking f equal to 1 in (1.8), we deduce the following Corollary.

Corollary 1.2 *Assume that* $H > \frac{1}{2}$*, for any* $t \geqslant 0$ *and any integer* $r \geqslant 1$*, we have*

$$2^{-\frac{nH}{2}} \sum_{k=0}^{\lfloor 2^n t\rfloor -1} (2^{\frac{nH}{2}}(Z_{T_{k+1,n}} - Z_{T_{k,n}}))^{2r-1} \xrightarrow[n\to\infty]{L^2} \frac{(2r)!}{r!2^r} Z_t,$$

thus, we understand the asymptotic behavior of the signed power variations of odd order of the fractional Brownian motion in Brownian time, in the case $H > \frac{1}{2}$*, which was missing in the first point in* [9, Corollary 1.2].

Remark 1.3 1. For $H = \frac{1}{6}$, it has been proved in [8, (3.17)] that

$$\left(\sum_{k=0}^{\lfloor 2^n t\rfloor -1} \frac{1}{2}(f(Z_{T_{k,n}}) + f(Z_{T_{k+1,n}}))(Z_{T_{k+1,n}} - Z_{T_{k,n}})^3\right)_{t\geqslant 0} \xrightarrow[n\to\infty]{f.d.d.} \left(\kappa_3 \int_0^{Y_t} f(X_s)dW_s\right)_{t\geqslant 0},$$

with W a standard two-sided Brownian motion independent of the pair (X, Y) and $\kappa_3 \simeq 2.322$. Thus, (1.7) continues to hold for $H = \frac{1}{6}$ and $r = 2$.

2. In the particular case where $H = 1/2$ (that is, when Z is the *iterated Brownian motion*), we emphasize that the fourth point of Theorem 1.1 allows one to recover (1.3). In fact, since $H = \frac{1}{2}$, then, for any integer $a \geqslant 1$, by (2.18) and its related explanation, $\alpha_{2a}^2 = (2a)!$. So, using the decomposition (7.1) and (2.3), the reader can verify that $\sqrt{\mu_{4r} - \mu_{2r}^2}$ appearing in (1.3) is equal to γ_{2r} appearing in (1.9).
3. The limit process in (1.4) is $\left(\int_0^{Y_t} f(X_s)(\mu_{2r} d^\circ X_s + \sqrt{\mu_{4r-2} - \mu_{2r}^2}\, dW_s\right)_{t\geqslant 0}$. Observe that $\mu_{2r} = E[N^{2r}] = \frac{(2r)!}{r!2^r}$ and since $H = \frac{1}{2}$, then, for any integer $l \geqslant 1$, by (2.18) and its related explanation, $\alpha_{2l-1}^2 = (2l-1)!$. So, using the decomposition (3.4) and (2.3), the reader can verify that $\sqrt{\mu_{4r-2} - \mu_{2r}^2}$ is

equal to β_{2r-1} appearing in (1.7). We deduce that the limit process in (1.4) is $\left(\frac{(2r)!}{r!2^r}\int_0^{Y_t} f(X_s)d^\circ X_s + \beta_{2r-1}\int_0^{Y_t} f(X_s)dW_s\right)_{t\geqslant 0}$. Thus, one can say that, for any integer $r \geqslant 2$, the limit of the weighted $(2r-1)$-variation of Z for $H = \frac{1}{2}$ is intermediate between the limit of the weighted $(2r-1)$-variation of Z for $H > \frac{1}{2}$ and the limit of the weighted $(2r-1)$-variation of Z for $\frac{1}{6} < H < \frac{1}{2}$.

A brief outline of the paper is as follows. In Sect. 2, we give the preliminaries to the proof of Theorem 1.1. In Sect. 3, we start the preparation to our proof. In Sect. 4, we prove (1.5) and (1.6). In Sects. 5, 6 and 7 we prove (1.7), (1.8) and (1.9). Finally, in Sect. 8, we give the proof of a technical lemma.

2 Preliminaries

2.1 Elements of Malliavin Calculus

In this section, we gather some elements of Malliavin calculus we shall need in the sequel. The reader in referred to [6] for details and any unexplained result.

We continue to denote by $X = (X_t)_{t\in\mathbb{R}}$ a two-sided fractional Brownian motion with Hurst parameter $H \in (0, 1)$. That is, X is a zero mean Gaussian process, defined on a complete probability space $(\Omega, \mathscr{A}, P)$, with covariance function,

$$C_H(t,s) = E(X_tX_s) = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t-s|^{2H}), \quad s,t \in \mathbb{R}.$$

We suppose that $\mathscr{A}$ is the σ-field generated by X. For all $n \in \mathbb{N}^*$, we let $\mathscr{E}_n$ be the set of step functions on $[-n, n]$, and $\mathscr{E} := \cup_n \mathscr{E}_n$. Set $\varepsilon_t = \mathbf{1}_{[0,t]}$ (resp. $\mathbf{1}_{[t,0]}$) if $t \geqslant 0$ (resp. $t < 0$). Let $\mathscr{H}$ be the Hilbert space defined as the closure of $\mathscr{E}$ with respect to the inner product

$$\langle \varepsilon_t, \varepsilon_s\rangle_{\mathscr{H}} = C_H(t,s), \quad s,t \in \mathbb{R}. \tag{2.1}$$

The mapping $\varepsilon_t \mapsto X_t$ can be extended to an isometry between $\mathscr{H}$ and the Gaussian space $\mathbb{H}_1$ associated with X. We will denote this isometry by $\varphi \mapsto X(\varphi)$.

Let $\mathscr{F}$ be the set of all smooth cylindrical random variables, i.e., of the form

$$F = \phi(X_{t_1}, \ldots, X_{t_l}),$$

where $l \in \mathbb{N}^*$, $\phi : \mathbb{R}^l \to \mathbb{R}$ is a C^∞-function such that f and its partial derivatives have at most polynomial growth, and $t_1 < \ldots < t_l$ are some real numbers. The derivative of F with respect to X is the element of $L^2(\Omega, \mathscr{H})$ defined by

$$D_sF = \sum_{i=1}^{l} \frac{\partial\phi}{\partial x_i}(X_{t_1}, \ldots, X_{t_l})\varepsilon_{t_i}(s), \quad s \in \mathbb{R}.$$

In particular $D_s X_t = \varepsilon_t(s)$. For any integer $k \geqslant 1$, we denote by $\mathbb{D}^{k,2}$ the closure of $\mathscr{F}$ with respect to the norm

$$\|F\|_{k,2}^2 = E(F^2) + \sum_{j=1}^{k} E[\|D^j F\|_{\mathscr{H}^{\otimes j}}^2].$$

The Malliavin derivative D satisfies the chain rule. If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is C_b^1 and if $F_1, \dots, F_n$ are in $\mathbb{D}^{1,2}$, then $\varphi(F_1, \dots, F_n) \in \mathbb{D}^{1,2}$ and we have

$$D\varphi(F_1, \dots, F_n) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(F_1, \dots, F_n) DF_i.$$

We have the following Leibniz formula, whose proof is straightforward by induction on q. Let $\varphi, \psi \in C_b^q$ $(q \geqslant 1)$, and fix $0 \leqslant u < v$ and $0 \leqslant s < t$. Then $(\varphi(X_s) + \varphi(X_t))\big(\psi(X_u) + \psi(X_v)) \in \mathbb{D}^{q,2}$ and

$$\begin{aligned} & D^q((\varphi(X_s) + \varphi(X_t))(\psi(X_u) + \psi(X_v))) \\ & = \sum_{l=0}^{q} \binom{q}{l} (\varphi^{(l)}(X_s)\varepsilon_s^{\otimes l} + \varphi^{(l)}(X_t)\varepsilon_t^{\otimes l}) \tilde{\otimes} (\psi^{(q-l)}(X_u)\varepsilon_u^{\otimes(q-l)} \\ & \quad + \psi^{(q-l)}(X_v)\varepsilon_v^{\otimes(q-l)}) \end{aligned} \tag{2.2}$$

where $\tilde{\otimes}$ stands for the symmetric tensor product and $\varphi^{(l)}$ (resp. $\psi^{(q-l)}$) means that φ is differentiated l times (resp. ψ is differentiated $q - l$ times). A similar statement holds fo $u < v \leqslant 0$ and $s < t \leqslant 0$.

If a random element $u \in L^2(\Omega, \mathscr{H})$ belongs to the domain of the divergence operator, that is, if it satisfies

$$|E\langle DF, u\rangle_{\mathscr{H}}| \leqslant c_u \sqrt{E\left(F^2\right)} \text{ for any } F \in \mathscr{F},$$

then $I(u)$ is defined by the duality relationship

$$E(FI(u)) = E(\langle DF, u\rangle_{\mathscr{H}}),$$

for every $F \in \mathbb{D}^{1,2}$.

For every $n \geqslant 1$, let $\mathbb{H}_n$ be the nth Wiener chaos of X, that is, the closed linear subspace of $L^2(\Omega, \mathscr{A}, P)$ generated by the random variables $\{H_n(X(h)), h \in \mathscr{H}, \|h\|_{\mathscr{H}} = 1\}$, where H_n is the nth Hermite polynomial. Recall that $H_0 = 0$, $H_p(x) = (-1)^p \exp(\frac{x^2}{2}) \frac{d^p}{dx^p} \exp(-\frac{x^2}{2})$ for $p \geqslant 1$, and that

$$E(H_p(M)H_q(N)) = \begin{cases} p!(E[MN])^p & \text{if } p = q, \\ 0 & \text{otherwise} \end{cases}, \tag{2.3}$$

for jointly Gaussian M, N and integers $p, q \geqslant 1$. The mapping

$$I_n(h^{\otimes n}) = H_n(X(h)) \tag{2.4}$$

provides a linear isometry between the symmetric tensor product $\mathscr{H}^{\odot n}$ and $\mathbb{H}_n$. For $H = \frac{1}{2}$, I_n coincides with the multiple Wiener–Itô integral of order n. The following duality formula holds

$$E(FI_n(h)) = E(\langle D^n F, h\rangle_{\mathscr{H}^{\otimes n}}), \tag{2.5}$$

for any element $h \in \mathscr{H}^{\odot n}$ and any random variable $F \in \mathbb{D}^{n,2}$.

Let $\{e_k, k \geqslant 1\}$ be a complete orthonormal system in $\mathscr{H}$. Given $f \in \mathscr{H}^{\odot n}$ and $g \in \mathscr{H}^{\odot m}$, for every $r = 0, \ldots, n \wedge m$, the contraction of f and g of order r is the element of $\mathscr{H}^{\otimes(n+m-2r)}$ defined by

$$f \otimes_r g = \sum_{k_1,\ldots,k_r=1}^{\infty} \langle f, e_{k_1} \otimes \cdots \otimes e_{k_r}\rangle_{\mathscr{H}^{\otimes r}} \otimes \langle g, e_{k_1} \otimes \cdots \otimes e_{k_r}\rangle_{\mathscr{H}^{\otimes r}}.$$

Finally, we recall the following product formula: If $f \in \mathscr{H}^{\odot n}$ and $g \in \mathscr{H}^{\odot m}$, then

$$I_n(f)I_m(g) = \sum_{r=0}^{n\wedge m} r!\binom{n}{r}\binom{m}{r} I_{n+m-2r}(f \otimes_r g). \tag{2.6}$$

2.2 Some Technical Results

For all $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, we write

$$\delta_{(k+1)2^{-n/2}} = \varepsilon_{(k+1)2^{-n/2}} - \varepsilon_{k2^{-n/2}}.$$

The following lemma will play a pivotal role in the proof of Theorem 1.1. The reader can find an original version of this lemma in [5, Lemma 5, Lemma 6].

Lemma 2.1 1. *If $H \leqslant \frac{1}{2}$, for all integer $q \geqslant 1$, for all $j \in \mathbb{N}$ and $u \in \mathbb{R}$,*

$$\left|\left\langle \varepsilon_u^{\otimes q}, \delta_{(j+1)2^{-n/2}}^{\otimes q}\right\rangle_{\mathscr{H}^{\otimes q}}\right| \leqslant 2^{-nqH}. \tag{2.7}$$

2. *If $H > \frac{1}{2}$, for all integer $q \geqslant 1$, for all $t \in \mathbb{R}_+$ and $j, j' \in \{0, \ldots, \lfloor 2^{n/2}t\rfloor - 1\}$,*

$$\left|\left\langle \varepsilon_{j2^{-n/2}}^{\otimes q}, \delta_{(j'+1)2^{-n/2}}^{\otimes q}\right\rangle_{\mathscr{H}^{\otimes q}}\right| \leqslant 2^q 2^{-\frac{nq}{2}} t^{(2H-1)q}, \tag{2.8}$$

$$\left|\left\langle \varepsilon_{(j+1)2^{-n/2}}^{\otimes q}, \delta_{(j'+1)2^{-n/2}}^{\otimes q}\right\rangle_{\mathscr{H}^{\otimes q}}\right| \leqslant 2^q 2^{-\frac{nq}{2}} t^{(2H-1)q}. \tag{2.9}$$

3. *For all integers $r, n \geqslant 1$ and $t \in \mathbb{R}_+$, and with $C_{H,r}$ a constant depending only on H and r (but independent of t and n),*

(a) *if* $H < 1 - \frac{1}{2r}$,

$$\sum_{k,l=0}^{\lfloor 2^{n/2}t\rfloor -1} \left| \left\langle \delta_{(k+1)2^{-n/2}};\, \delta_{(l+1)2^{-n/2}} \right\rangle_{\mathscr{H}} \right|^r \leqslant C_{H,r}\, t\, 2^{n\left(\frac{1}{2}-rH\right)} \tag{2.10}$$

(b) *if* $H = 1 - \frac{1}{2r}$,

$$\sum_{k,l=0}^{\lfloor 2^{n/2}t\rfloor -1} \left| \left\langle \delta_{(k+1)2^{-n/2}};\, \delta_{(l+1)2^{-n/2}} \right\rangle_{\mathscr{H}} \right|^r \leqslant C_{H,r}\, 2^{n\left(\frac{1}{2}-rH\right)} (t(1+n)+t^2) \tag{2.11}$$

(c) *if* $H > 1 - \frac{1}{2r}$,

$$\sum_{k,l=0}^{\lfloor 2^{n/2}t\rfloor -1} \left| \left\langle \delta_{(k+1)2^{-n/2}};\, \delta_{(l+1)2^{-n/2}} \right\rangle_{\mathscr{H}} \right|^r \leqslant C_{H,r}\Big(t\, 2^{n\left(\frac{1}{2}-rH\right)} + t^{2-(2-2H)r}\, 2^{n(1-r)}\Big). \tag{2.12}$$

4. *For* $H \in (0,1)$. *For all integer* $n \geqslant 1$ *and* $t \in \mathbb{R}_+$,

$$\sum_{k,l=0}^{\lfloor 2^{n/2}t\rfloor -1} \left| \left\langle \varepsilon_{k2^{-n/2}};\, \delta_{(l+1)2^{-n/2}} \right\rangle_{\mathscr{H}} \right| \leqslant 2^{\frac{n}{2}+1} t^{2H+1}, \tag{2.13}$$

$$\sum_{k,l=0}^{\lfloor 2^{n/2}t\rfloor -1} \left| \left\langle \varepsilon_{(k+1)2^{-n/2}};\, \delta_{(l+1)2^{-n/2}} \right\rangle_{\mathscr{H}} \right| \leqslant 2^{\frac{n}{2}+1} t^{2H+1}. \tag{2.14}$$

Proof The proof, which is quite long and technical, is postponed in Sect. 8. □

It has been mentioned in [2] that $\{\|Y_{T_{\lfloor 2^n t\rfloor,n}}\|_4 : n \geqslant 0\}$ is a bounded sequence. More generally, we have the following result.

Lemma 2.2 *For any integer* $k \geqslant 1$, $\{\|Y_{T_{\lfloor 2^n t\rfloor,n}}\|_{2k} : n \geqslant 0\}$ *is a bounded sequence.*

Proof Recall from the introduction that $\{Y_{T_{k,n}} : k \geqslant 0\}$ is a simple and symmetric random walk on $\mathscr{D}_n$, and observe that $Y_{T_{\lfloor 2^n t\rfloor,n}} = \sum_{l=0}^{\lfloor 2^n t\rfloor -1}(Y_{T_{l+1,n}} - Y_{T_{l,n}})$. So, we have

$$
\begin{aligned}
E\big[\big(Y_{T_{\lfloor 2^n t\rfloor,n}}\big)^{2k}\big] &= \sum_{l_1,\dots,l_{2k}=0}^{\lfloor 2^n t\rfloor -1} E\big[\big(Y_{T_{l_1+1,n}} - Y_{T_{l_1,n}}\big)\times\cdots\times\big(Y_{T_{l_{2k}+1,n}} - Y_{T_{l_{2k},n}}\big)\big] \\
&= \sum_{m=1}^{k} \sum_{a_1+\cdots+a_m=2k} C_{a_1,\dots,a_m} \sum_{\substack{l_1,\dots,l_m=0 \\ l_i\neq l_j \text{ for } i\neq j}}^{\lfloor 2^n t\rfloor -1} E\big[\big(Y_{T_{l_1+1,n}} - Y_{T_{l_1,n}}\big)^{a_1}\big] \\
&\qquad \times\cdots\times E\big[\big(Y_{T_{l_m+1,n}} - Y_{T_{l_m,n}}\big)^{a_m}\big], \qquad (2.15)
\end{aligned}
$$

where $\forall i \in \{1,\dots,m\}$ a_i is an even integer, $\forall m \in \{1,\dots,k\}$ $C_{a_1,\dots,a_m} \geqslant 0$, is some combinatorial constant whose explicit value is immaterial here. Now observe that the quantity in (2.15) is equal to

$$
\begin{aligned}
&\sum_{m=1}^{k} \sum_{a_1+\cdots+a_m=2k} C_{a_1,\dots,a_m} \lfloor 2^n t\rfloor\big(\lfloor 2^n t\rfloor - 1\big)\times\cdots\times\big(\lfloor 2^n t\rfloor - m + 1\big)2^{-\frac{n}{2}(a_1+\ldots+a_m)} \\
&= \sum_{m=1}^{k} \sum_{a_1+\cdots+a_m=2k} C_{a_1,\dots,a_m} \lfloor 2^n t\rfloor\big(\lfloor 2^n t\rfloor - 1\big)\times\cdots\times\big(\lfloor 2^n t\rfloor - m + 1\big)2^{-nk},
\end{aligned}
$$

so, since $1 \leqslant m \leqslant k$, we deduce that $\big\{E\big[\big(Y_{T_{\lfloor 2^n t\rfloor,n}}\big)^{2k}\big] : n \geqslant 0\big\}$ is a bounded sequence, which proves the lemma. □

Also, in order to prove the fourth point of Theorem 1.1, we will need estimates on the local time of Y taken from [2], that we collect in the following statement.

Proposition 2.3 1. *For every $x \in \mathbb{R}$, $p \in \mathbb{N}^*$ and $t > 0$, we have*

$$E[(L_t^x(Y))^p] \leqslant 2\, E[(L_1^0(Y))^p]\, t^{p/2} \exp\left(-\frac{x^2}{2t}\right).$$

2. *There exists a positive constant μ such that, for every $a, b \in \mathbb{R}$ with $ab \geqslant 0$ and $t > 0$,*

$$E[|L_t^b(Y) - L_t^a(Y)|^2]^{1/2} \leqslant \mu \sqrt{|b-a|}\, t^{1/4} \exp\left(-\frac{a^2}{4t}\right).$$

3. *There exists a positive random variable $K \in L^8$ such that, for every $j \in \mathbb{Z}$, every $n \geqslant 0$ and every $t > 0$, one has that*

$$\left|\mathcal{L}_{j,n}(t) - L_t^{j2^{-n/2}}(Y)\right| \leqslant 2Kn2^{-n/4}\sqrt{L_t^{j2^{-n/2}}(Y)},$$

where $\mathcal{L}_{j,n}(t) = 2^{-n/2}(U_{j,n}(t) + D_{j,n}(t))$.

2.3 Notation

Throughout all the forthcoming proofs, we shall use the following notation. For all $t \in \mathbb{R}$ and $n \in \mathbb{N}$, we define $X_t^{(n)} := 2^{\frac{nH}{2}} X_{t2^{-\frac{n}{2}}}$. For all $k \in \mathbb{Z}$ and $H \in (0,1)$, we write

$$\rho(k) = \frac{1}{2}(|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H}), \tag{2.16}$$

it is clear that $\rho(-k) = \rho(k)$. Observe that, by (2.1), we have

$$\begin{aligned}\left|\langle \delta_{(k+1)2^{-n/2}}; \delta_{(l+1)2^{-n/2}}\rangle_{\mathscr{H}}\right| &= \left|E\big[\big(X_{(k+1)2^{-n/2}} - X_{k2^{-n/2}}\big)\big(X_{(l+1)2^{-n/2}} - X_{l2^{-n/2}}\big)\big]\right| \\ &= \left|2^{-nH-1}\big(\left|k-l+1\right|^{2H} + \left|k-l-1\right|^{2H} - 2\left|k-l\right|^{2H}\big)\right| \\ &= 2^{-nH}\left|\rho(k-l)\right|. \end{aligned} \tag{2.17}$$

If $H \leqslant \frac{1}{2}$, for all $r \in \mathbb{N}^*$, we define

$$\alpha_r := \sqrt{r! \sum_{a \in \mathbb{Z}} \rho(a)^r}. \tag{2.18}$$

Note that $\sum_{a\in\mathbb{Z}} |\rho(a)|^r < \infty$ if and only if $H < 1 - 1/(2r)$, which is satisfied for all $r \geqslant 1$ if we suppose that $H \leqslant 1/2$ (in the case $H = 1/2$, we have $\rho(0) = 1$ and $\rho(a) = 0$ for all $a \neq 0$. So, for any $r \in \mathbb{N}^*$, we have $\sum_{a\in\mathbb{Z}} |\rho(a)|^r = 1$).

For simplicity, throughout the paper we remove the subscript $\mathscr{H}$ in the inner product defined in (2.1), that is, we write $\langle\ ;\ \rangle$ instead of $\langle\ ;\ \rangle_{\mathscr{H}}$.

For any sufficiently smooth function $f : \mathbb{R} \to \mathbb{R}$, the notation $\partial^l f$ means that f is differentiated l times. We denote for any $j \in \mathbb{Z}$, $\Delta_{j,n} f(X) := \frac{1}{2}(f(X_{j2^{-n/2}}) + f(X_{(j+1)2^{-n/2}}))$.

In the proofs contained in this paper, C shall denote a positive, finite constant that may change value from line to line.

3 Preparation to the proof of Theorem 1.1

3.1 A Key Algebraic Lemma

For each integer $n \geqslant 1$, $k \in \mathbb{Z}$ and real number $t \geqslant 0$, let $U_{j,n}(t)$ (resp. $D_{j,n}(t)$) denote the number of *upcrossings* (resp. *downcrossings*) of the interval $[j2^{-n/2}, (j+1)2^{-n/2}]$ within the first $\lfloor 2^n t \rfloor$ steps of the random walk $\{Y_{T_{k,n}}\}_{k\geqslant 0}$, that is,

$$\begin{aligned} U_{j,n}(t) &= \sharp\big\{k = 0, \ldots, \lfloor 2^n t\rfloor - 1 : \\ &\quad Y_{T_{k,n}} = j2^{-n/2} \text{ and } Y_{T_{k+1,n}} = (j+1)2^{-n/2}\big\}; \\ D_{j,n}(t) &= \sharp\big\{k = 0, \ldots, \lfloor 2^n t\rfloor - 1 : \\ &\quad Y_{T_{k,n}} = (j+1)2^{-n/2} \text{ and } Y_{T_{k+1,n}} = j2^{-n/2}\big\}. \end{aligned}$$

The following lemma taken from [2, Lemma 2.4] is going to be the key when studying the asymptotic behavior of the weighted power variation $V_n^{(r)}(f,t)$ of order $r \geqslant 1$, defined as:

$$V_n^{(r)}(f,t) = \sum_{k=0}^{\lfloor 2^n t \rfloor -1} \frac{1}{2}(f(Z_{T_{k,n}})+f(Z_{T_{k+1,n}}))[(2^{\frac{nH}{2}}(Z_{T_{k+1,n}}-Z_{T_{k,n}}))^r-\mu_r], \quad t \geqslant 0, \tag{3.1}$$

where $\mu_r := E[N^r]$, with $N \sim \mathcal{N}(0,1)$. Its main feature is to separate X from Y, thus providing a representation of $V_n^{(r)}(f,t)$ which is amenable to analysis.

Lemma 3.1 *Fix $f \in C_b^\infty$, $t \geqslant 0$ and $r \in \mathbb{N}^*$. Then*

$$\begin{aligned} V_n^{(r)}(f,t) = \sum_{j\in\mathbb{Z}} \frac{1}{2}\big(f\big(X_{j2^{-\frac{n}{2}}}\big)+f\big(X_{(j+1)2^{-\frac{n}{2}}}\big)\big)\big[\big(2^{\frac{nH}{2}}\big(X_{(j+1)2^{-\frac{n}{2}}}-X_{j2^{-\frac{n}{2}}}\big)\big)^r - \mu_r\big] \\ \times\big(U_{j,n}(t)+(-1)^r D_{j,n}(t)\big). \end{aligned} \tag{3.2}$$

3.2 Transforming the Weighted Power Variations of Odd Order

By [2, Lemma 2.5], one has

$$U_{j,n}(t) - D_{j,n}(t) = \begin{cases} 1_{\{0\leqslant j<j^*(n,t)\}} & \text{if } j^*(n,t) > 0 \\ 0 & \text{if } j^*(n,t) = 0 \\ -1_{\{j^*(n,t)\leqslant j<0\}} & \text{if } j^*(n,t) < 0 \end{cases},$$

where $j^*(n,t) = 2^{n/2} Y_{T_{\lfloor 2^n t\rfloor,n}}$. As a consequence, $V_n^{(2r-1)}(f,t)$ is equal to

$$\begin{cases} \displaystyle\sum_{j=0}^{j^*(n,t)-1} \tfrac{1}{2}\big(f\big(X^+_{j2^{-n/2}}\big)+f\big(X^+_{(j+1)2^{-n/2}}\big)\big)\big(X^{n,+}_{j+1}-X^{n,+}_j\big)^{2r-1} & \text{if } j^*(n,t) > 0 \\ 0 & \text{if } j^*(n,t) = 0 \\ \displaystyle\sum_{j=0}^{|j^*(n,t)|-1} \tfrac{1}{2}\big(f(X^-_{j2^{-n/2}})+f(X^-_{(j+1)2^{-n/2}})\big)\big(X^{n,-}_{j+1}-X^{n,-}_j\big)^{2r-1} & \text{if } j^*(n,t) < 0 \end{cases},$$

where $X_t^+ := X_t$ for $t \geqslant 0$, $X_{-t}^- := X_t$ for $t < 0$, $X_t^{n,+} := 2^{\frac{nH}{2}} X^+_{2^{-\frac{n}{2}}t}$ for $t \geqslant 0$ and $X_{-t}^{n,-} := 2^{\frac{nH}{2}} X^-_{2^{-\frac{n}{2}}(-t)}$ for $t < 0$.

Let us now introduce the following sequence of processes $W^{(2r-1)}_{\pm,n}$, in which H_p stands for the pth Hermite polynomial ($H_1(x) = x$, $H_2(x) = x^2 - 1$, etc.):

$$
\begin{aligned}
W_{\pm,n}^{(2r-1)}(f,t) &= \sum_{j=0}^{\lfloor 2^{n/2}t\rfloor-1} \frac{1}{2}\left(f\left(X^{\pm}_{j2^{-\frac{n}{2}}}\right)\right.\\
&\quad \left.+f\left(X^{\pm}_{(j+1)2^{-\frac{n}{2}}}\right)\right)H_{2r-1}\big(X^{n,\pm}_{j+1}-X^{n,\pm}_{j}\big), \quad t\geqslant 0 \qquad (3.3)\\
W_n^{(2r-1)}(f,t) &:= \begin{cases} W_{+,n}^{(2r-1)}(f,t) & \text{if } t\geqslant 0\\ W_{-,n}^{(2r-1)}(f,-t) & \text{if } t<0 \end{cases}.
\end{aligned}
$$

We then have, using the decomposition

$$x^{2r-1} = \sum_{i=1}^{r} \kappa_{r,i} H_{2i-1}(x), \tag{3.4}$$

(with $\kappa_{r,r}=1$, and $\kappa_{r,1}=\frac{(2r)!}{r!2^r}=E[N^{2r}]$, with $N\sim\mathcal{N}(0,1)$. If interested, the reader can find the explicit value of $\kappa_{r,i}$, for $1<i<r$, e.g., in [9, Corollary 1.2]),

$$V_n^{(2r-1)}(f,t) = \sum_{i=1}^{r} \kappa_{r,i} W_n^{(2i-1)}\left(f, Y_{T_{\lfloor 2^n t\rfloor,n}}\right). \tag{3.5}$$

4 Proofs of (1.5) and (1.6)

4.1 Proof of (1.5)

In [8, Theorem 2.1], we have proved that for $H>\frac{1}{6}$ and $f\in C_b^\infty$, the following change-of-variable formula holds true

$$F(Z_t)-F(0) = \int_0^t f(Z_s)d^\circ Z_s, \quad t\geqslant 0 \tag{4.1}$$

where F is a primitive of f and $\int_0^t f(Z_s)d^\circ Z_s$ is the limit in probability of $2^{-\frac{nH}{2}}V_n^{(1)}(f,t)$ as $n\to\infty$, with $V_n^{(1)}(f,t)$ defined in (3.1). On the other hand, it has been proved in [5, Theorem 4] (see also [10, Theorem 1.3] for an extension of this formula to the bi-dimensional case) that for all $t\in\mathbb{R}$, the following change-of-variable formula holds true for $H>\frac{1}{6}$

$$F(X_t)-F(0) = \int_0^t f(X_s)d^\circ X_s, \tag{4.2}$$

where $\int_0^t f(X_s)d^\circ X_s$ is the Stratonovich integral of $f(X)$ with respect to X defined as the limit in probability of $2^{-\frac{nH}{2}}W_n^{(1)}(f,t)$ as $n\to\infty$, with $W_n^{(1)}(f,t)$ defined in (3.3). Thanks to (4.2), we deduce that

$$F(Z_t) - F(0) = \int_0^{Y_t} f(X_s)d^\circ X_s, \quad t \geqslant 0$$

by combining this last equality with (4.1), we get $\int_0^t f(Z_s)d^\circ Z_s = \int_0^{Y_t} f(X_s)d^\circ X_s$. So, we deduce that, for $H > \frac{1}{6}$,

$$\sum_{k=0}^{\lfloor 2^n t\rfloor -1} \frac{1}{2}(f(Z_{T_{k,n}}) + f(Z_{T_{k+1,n}}))(Z_{T_{k+1,n}} - Z_{T_{k,n}}) \xrightarrow[n\to\infty]{P} \int_0^{Y_t} f(X_s)d^\circ X_s,$$

thus (1.5) holds true.

4.2 Proof of (1.6)

In [8, Theorem 2.1], we have proved that for $H = \frac{1}{6}$ and $f \in C_b^\infty$, the following change-of-variable formula holds true

$$F(Z_t) - F(0) + \frac{\kappa_3}{12}\int_0^{Y_t} f''(X_s)dW_s \overset{\text{(law)}}{=} \int_0^t f(Z_s)d^\circ Z_s, \quad t \geqslant 0 \qquad (4.3)$$

where F is a primitive of f, W is a standard two-sided Brownian motion independent of the pair (X, Y), $\kappa_3 \simeq 2.322$ and $\int_0^t f(Z_s)d^\circ Z_s$ is the limit in law of $2^{-\frac{nH}{2}} V_n^{(1)}(f,t)$ as $n \to \infty$, with $V_n^{(1)}(f,t)$ defined in (3.1). On the other hand, it has been proved in (2.19) in [7] that for all $t \in \mathbb{R}$, the following change-of-variable formula holds true for $H = \frac{1}{6}$

$$F(X_t) - F(0) + \frac{\kappa_3}{12}\int_0^{t} f''(X_s)dW_s = \int_0^t f(X_s)d^* X_s, \qquad (4.4)$$

where κ_3 and W are the same as in (4.3), $\int_0^t f(X_s)d^*X_s$ is the Stratonovich integral of $f(X)$ with respect to X defined as the limit in law of $2^{-\frac{nH}{2}} W_n^{(1)}(f,t)$ as $n \to \infty$, with $W_n^{(1)}(f,t)$ defined in (3.3). Thanks to (4.4), we deduce that

$$F(Z_t) - F(0) + \frac{\kappa_3}{12}\int_0^{Y_t} f''(X_s)dW_s = \int_0^{Y_t} f(X_s)d^* X_s, \quad t \geqslant 0.$$

By combining this last equality with (4.3), we get $\int_0^t f(Z_s)d^\circ Z_s \overset{law}{=} \int_0^{Y_t} f(X_s)d^* X_s$. So, we deduce that, for $H = \frac{1}{6}$,

$$\sum_{k=0}^{\lfloor 2^n t\rfloor -1} \frac{1}{2}(f(Z_{T_{k,n}}) + f(Z_{T_{k+1,n}}))(Z_{T_{k+1,n}} - Z_{T_{k,n}}) \xrightarrow[n\to\infty]{law} \int_0^{Y_t} f(X_s)d^* X_s,$$

thus (1.6) holds true.

5 Proof of (1.7)

Thanks to (3.1) and (3.5), for any integer $r \geqslant 2$, we have

$$\begin{aligned} 2^{-n/4} V_n^{(2r-1)}(f,t) &= 2^{-n/4} \sum_{k=0}^{\lfloor 2^n t \rfloor -1} \frac{1}{2}\big(f(Z_{T_{k,n}}) \\ &\quad + f(Z_{T_{k+1,n}})\big)\big(2^{\frac{nH}{2}}(Z_{T_{k+1,n}} - Z_{T_{k,n}})\big)^{2r-1} \\ &= 2^{-n/4} \sum_{l=1}^{r} \kappa_{r,l} W_n^{(2l-1)}(f, Y_{T_{\lfloor 2^n t\rfloor,n}}) \end{aligned} \tag{5.1}$$

The proof of (1.7) will be done in several steps.

5.1 Step 1: Limit of $2^{-n/4} \sum_{l=2}^{r} \kappa_{r,l} W_n^{(2l-1)}(f,t)$

Observe that, by (3.4), we have

$$\sum_{j=0}^{\lfloor 2^{n/2} t\rfloor -1} \frac{1}{2}\big(f(X^{\pm}_{j2^{-\frac{n}{2}}}) + f(X^{\pm}_{(j+1)2^{-\frac{n}{2}}})\big)(X^{n,\pm}_{j+1} - X^{n,\pm}_{j})^{2r-1} = \sum_{l=1}^{r} \kappa_{r,l} W^{(2l-1)}_{\pm,n}(f,t).$$

We have the following proposition:

Proposition 5.1 *If $H \in (\frac{1}{6}, \frac{1}{2})$, if $r \geqslant 2$ then, for any $f \in C_b^\infty$,*

$$\left(X_x, 2^{-\frac{n}{4}} \sum_{l=2}^{r} \kappa_{r,l} W^{(2l-1)}_{\pm,n}(f,t)\right)_{x\in\mathbb{R},\, t\geqslant 0} \xrightarrow[n\to\infty]{f.d.d.} \left(X_x, \beta_{2r-1} \int_0^t f(X_s^{\pm}) dW_s^{\pm}\right)_{x\in\mathbb{R},\, t\geqslant 0}, \tag{5.2}$$

where $\beta_{2r-1} = \sqrt{\sum_{l=2}^{r} \kappa_{r,l}^2 \alpha_{2l-1}^2}$, α_{2l-1} is given by (2.18), $W_t^+ = W_t$ if $t > 0$ and $W_t^- = W_{-t}$ if $t < 0$, with W a two-sided Brownian motion independent of (X, Y), and where $\int_0^t f(X_s^{\pm}) dW_s^{\pm}$ must be understood in the Wiener–Itô sense.

Proof For all $t \geqslant 0$, we define $F^{(2r-1)}_{\pm,n}(f,t) := 2^{-\frac{n}{4}} \sum_{l=2}^{r} \kappa_{r,l} W^{(2l-1)}_{\pm,n}(f,t)$. In what follows we may study separately the finite-dimensional distributions convergence in law of $\big(X, F^{(2r-1)}_{+,n}(f,\cdot), F^{(2r-1)}_{-,n}(f,\cdot)\big)$ when n is even and when n is odd. For the sake of simplicity, we will only consider the even case, the analysis when n is odd being *mutatis mutandis* the same. So, assume that n is even and let m be another even integer such that $n \geqslant m \geqslant 0$. We shall apply a coarse gaining argument. We have

$$F_{\pm,n}^{(2r-1)}(f,t) = 2^{-n/4} \sum_{i=1}^{\lfloor 2^{m/2}t \rfloor} \sum_{j=(i-1)2^{\frac{n-m}{2}}}^{i2^{\frac{n-m}{2}}-1} \frac{1}{2}(f(X^{\pm}_{j2^{-n/2}}) + f(X^{\pm}_{(j+1)2^{-n/2}}))$$
$$\times \left(\sum_{l=2}^{r} \kappa_{r,l} H_{2l-1}(X^{n,\pm}_{j+1} - X^{n,\pm}_{j}) \right) \tag{5.3}$$
$$+2^{-n/4} \sum_{j=\lfloor 2^{m/2}t \rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t \rfloor -1} \frac{1}{2}(f(X^{\pm}_{j2^{-n/2}}) + f(X^{\pm}_{(j+1)2^{-n/2}}))$$
$$\times \left(\sum_{l=2}^{r} \kappa_{r,l} H_{2l-1}(X^{n,\pm}_{j+1} - X^{n,\pm}_{j}) \right). \tag{5.4}$$

Observe that $2^{\frac{n-m}{2}}$ is an integer precisely because we have assumed that n and m are even numbers. We have

$$(5.3) = A^{\pm}_{n,m}(t) + B^{\pm}_{n,m}(t) + C^{\pm}_{n,m}(t),$$

where

$$A^{\pm}_{n,m}(t) = 2^{-n/4} \sum_{i=1}^{\lfloor 2^{m/2}t \rfloor} \frac{1}{2}(f(X^{\pm}_{i2^{-m/2}}) + f(X^{\pm}_{(i+1)2^{-m/2}})) \sum_{j=(i-1)2^{\frac{n-m}{2}}}^{i2^{\frac{n-m}{2}}-1}$$
$$\times \left(\sum_{l=2}^{r} \kappa_{r,l} H_{2l-1}(X^{n,\pm}_{j+1} - X^{n,\pm}_{j}) \right)$$
$$B^{\pm}_{n,m}(t) = 2^{-n/4} \sum_{i=1}^{\lfloor 2^{m/2}t \rfloor} \sum_{j=(i-1)2^{\frac{n-m}{2}}}^{i2^{\frac{n-m}{2}}-1} \frac{1}{2}(f(X^{\pm}_{(j+1)2^{-n/2}}) - f(X^{\pm}_{(i+1)2^{-m/2}}))$$
$$\times \left(\sum_{l=2}^{r} \kappa_{r,l} H_{2l-1}(X^{n,\pm}_{j+1} - X^{n,\pm}_{j}) \right)$$
$$C^{\pm}_{n,m}(t) = 2^{-n/4} \sum_{i=1}^{\lfloor 2^{m/2}t \rfloor} \sum_{j=(i-1)2^{\frac{n-m}{2}}}^{i2^{\frac{n-m}{2}}-1} \frac{1}{2}(f(X^{\pm}_{j2^{-n/2}}) - f(X^{\pm}_{i2^{-m/2}}))$$
$$\times \left(\sum_{l=2}^{r} \kappa_{r,l} H_{2l-1}(X^{n,\pm}_{j+1} - X^{n,\pm}_{j}) \right)$$

Here is a sketch of what remains to be done in order to complete the proof of (5.2). Firstly, we will prove **(a)** the f.d.d. convergence in law of $(X, A^{+}_{n,m}, A^{-}_{n,m})$ to $(X, \beta_{2r-1} \int_0^{\cdot} f(X^{+}_s) dW^{+}_s, \beta_{2r-1} \int_0^{\cdot} f(X^{-}_s) dW^{-}_s)$ as $n \to \infty$ and then $m \to \infty$. Secondly, we will show that **(b)** $B^{\pm}_{n,m}(t)$ converges to 0 in $L^2(\Omega)$ as $n \to \infty$ and then $m \to \infty$. By applying the same techniques, we would also obtain that the same holds with

$C^{\pm}_{n,m}(t)$. Thirdly, we will prove that **(c)** (5.4) converges to 0 in $L^2(\Omega)$ as $n\to\infty$ and then $m\to\infty$. Once this has been done, one can easily deduce the f.d.d. convergence in law of $(X, F^{(2r-1)}_{+,n}(f,\cdot), F^{(2r-1)}_{-,n}(f,\cdot))$ to $(X, \beta_{2r-1}\int_0^\cdot f(X_s^+)dW_s^+, \beta_{2r-1}\int_0^\cdot f(X_s^-)dW_s^-)$ as $n\to\infty$, which is equivalent to (5.2).

(a) Finite-dimensional distributions convergence in law of $(X, A^+_{n,m}, A^-_{n,m})$

Fix m. Showing the f.d.d. convergence in law of $(X, A^+_{n,m}, A^-_{n,m})$ as $n\to\infty$ can be easily reduced to checking the f.d.d. convergence in law of the following random-vector valued process:

$$\left(X_x : x\in\mathbb{R},\ 2^{-n/4}\sum_{j=(i-1)2^{\frac{n-m}{2}}}^{i2^{\frac{n-m}{2}}-1} H_{2l-1}(X^{n,+}_{j+1}-X^{n,+}_j)\ ,\ 2^{-n/4}\times\sum_{j=(i-1)2^{\frac{n-m}{2}}}^{i2^{\frac{n-m}{2}}-1} H_{2l-1}(X^{n,-}_{j+1}-X^{n,-}_j) : 2\leqslant l\leqslant r,\ 1\leqslant i\leqslant \lfloor 2^{m/2}t\rfloor\right).$$

Thanks to (3.27) in [9] (see also (3.4) in [9] and page 1073 in [5]), we have

$$\left(2^{-n/4}\sum_{j=(i-1)2^{\frac{n-m}{2}}}^{i2^{\frac{n-m}{2}}-1} H_{2l-1}(X^{n,+}_{j+1}-X^{n,+}_j), 2^{-n/4}\sum_{j=(i-1)2^{\frac{n-m}{2}}}^{i2^{\frac{n-m}{2}}-1} H_{2l-1}(X^{n,-}_{j+1}-X^{n,-}_j) : 2\leqslant l\leqslant r,\ 1\leqslant i\leqslant \lfloor 2^{m/2}t\rfloor\right) \xrightarrow[n\to\infty]{\text{law}}$$

$$\left(\alpha_{2l-1}\big(B^{l,+}_{(i+1)2^{-m/2}}-B^{l,+}_{i2^{-m/2}}\big)\ ,\ \alpha_{2l-1}\big(B^{l,-}_{(i+1)2^{-m/2}}-B^{l,-}_{i2^{-m/2}}\big) : 2\leqslant l\leqslant r, 1\leqslant i\leqslant \lfloor 2^{m/2}t\rfloor\right)$$

where $(B^{(2)},\ldots,B^{(r)})$ is a $(r-1)$-dimensional two-sided Brownian motion and α_{2l-1} is defined in (2.18), for all $t\geqslant 0$, $B^{r,+}_t := B^{(r)}_t$, $B^{r,-}_t := B^{(r)}_{-t}$.

Since $E[X_x H_{2r-1}(X^{n,\pm}_{j+1}-X^{n,\pm}_j)]=0$ when $r\geqslant 2$ (Hermite polynomials of different orders are orthogonal), Peccati–Tudor Theorem (see, e.g., [6, Theorem 6.2.3]) applies and yields

$$\left(X_x, 2^{-n/4}\sum_{j=(i-1)2^{\frac{n-m}{2}}}^{i2^{\frac{n-m}{2}}-1} H_{2l-1}(X^{n,+}_{j+1}-X^{n,+}_j), 2^{-n/4}\sum_{j=(i-1)2^{\frac{n-m}{2}}}^{i2^{\frac{n-m}{2}}-1} H_{2l-1}(X^{n,+}_{j+1}-X^{n,+}_j) : 2\leqslant l\leqslant r,\ 1\leqslant i\leqslant \lfloor 2^{m/2}t\rfloor\right)_{x\geqslant 0} \xrightarrow[n\to\infty]{\text{f.d.d.}}$$

$$\Big(X_x, \alpha_{2l-1}(B^{l,+}_{(i+1)2^{-m/2}} - B^{l,+}_{i2^{-m/2}}), \alpha_{2l-1}(B^{l,-}_{(i+1)2^{-m/2}} - B^{l,-}_{i2^{-m/2}}) : \\ 2 \leqslant l \leqslant r, 1 \leqslant i \leqslant \lfloor 2^{m/2} t \rfloor\Big)_{x \geqslant 0},$$

with $(B^{(2)}, \dots, B^{(r-1)})$ is independent of X (and independent of Y as well). We then have, as $n \to \infty$ and m is fixed,

$$(X, A^+_{n,m}, A^-_{n,m}) \xrightarrow{\text{f.d.d.}} \\ \left(X, \beta_{2r-1} \sum_{i=1}^{\lfloor 2^{m/2} . \rfloor} \frac{1}{2}(f(X^+_{i2^{-m/2}}) + f(X^+_{(i+1)2^{-m/2}}))(W^+_{(i+1)2^{-m/2}} - W^+_{i2^{-m/2}}), \right. \\ \left. \beta_{2r-1} \sum_{i=1}^{\lfloor 2^{m/2} . \rfloor} \frac{1}{2}(f(X^-_{i2^{-m/2}}) + f(X^-_{(i+1)2^{-m/2}}))(W^-_{(i+1)2^{-m/2}} - W^-_{i2^{-m/2}}) \right),$$

with $\beta_{2r-1} := \sqrt{\sum_{l=2}^r \kappa_{r,l}^2 \alpha_{2l-1}^2}$ and W is a two-sided Brownian motion independent of X (and independent of Y as well). One can write

$$\sum_{i=1}^{\lfloor 2^{m/2} t \rfloor} \frac{1}{2}(f(X^\pm_{i2^{-m/2}}) + f(X^\pm_{(i+1)2^{-m/2}}))(W^\pm_{(i+1)2^{-m/2}} - W^\pm_{i2^{-m/2}}) = K^\pm_m(t) + L^\pm_m(t),$$

with

$$K^\pm_m(t) = \sum_{i=1}^{\lfloor 2^{m/2} t \rfloor} f(X^\pm_{i2^{-m/2}}) \left(W^\pm_{(i+1)2^{-m/2}} - W^\pm_{i2^{-m/2}}\right),$$
$$L^\pm_m(t) = \sum_{i=1}^{\lfloor 2^{m/2} t \rfloor} \frac{1}{2}(f(X^\pm_{(i+1)2^{-m/2}}) - f(X^\pm_{i2^{-m/2}}))(W^\pm_{(i+1)2^{-m/2}} - W^\pm_{i2^{-m/2}}).$$

It is clear that $K^\pm_m(t) \underset{m \to \infty}{\overset{L^2}{\longrightarrow}} \int_0^t f(X^\pm_s) dW^\pm_s$. On the other hand, $L^\pm_m(t)$ converges to 0 in L^2 as $m \to \infty$. Indeed, by independence,

$$E[L^\pm_m(t)^2] \\ = \frac{1}{4} \sum_{i,j=1}^{\lfloor 2^{m/2} t \rfloor} E[(f(X^\pm_{(i+1)2^{-m/2}}) - f(X^\pm_{i2^{-m/2}}))(f(X^\pm_{(j+1)2^{-m/2}}) - f(X^\pm_{j2^{-m/2}}))] \\ \times E[(W^\pm_{(i+1)2^{-m/2}} - W^\pm_{i2^{-m/2}})(W^\pm_{(j+1)2^{-m/2}} - W^\pm_{j2^{-m/2}})]$$

$$= \frac{1}{4} \sum_{i=1}^{\lfloor 2^{m/2} t \rfloor} E[(f(X^{\pm}_{(i+1)2^{-m/2}}) - f(X^{\pm}_{i2^{-m/2}}))^2] \times E[(W^{\pm}_{(i+1)2^{-m/2}} - W^{\pm}_{i2^{-m/2}})^2]$$

$$= \frac{2^{-m/2}}{4} \sum_{i=1}^{\lfloor 2^{m/2} t \rfloor} E[f'(X_{\theta_i})^2 (X^{\pm}_{(i+1)2^{-m/2}}) - X^{\pm}_{i2^{-m/2}})^2], \tag{5.5}$$

where θ_i denotes a random real number satisfying $i2^{-m/2} < \theta_i < (i+1)2^{-m/2}$. Since $f \in C_b^\infty$ and by Cauchy–Schwarz inequality, we deduce that

$$\begin{aligned}(5.5) &\leqslant C_f 2^{-m/2} \sum_{i=1}^{\lfloor 2^{m/2} t \rfloor} E[(X^{\pm}_{(i+1)2^{-m/2}}) - (X^{\pm}_{i2^{-m/2}})^4]^{1/2} \\ &= C_f 2^{-m/2} \lfloor 2^{m/2} t \rfloor 2^{-mH} \sqrt{3} \\ &\leqslant C_f 2^{-mH} t,\end{aligned}$$

from which the claim follows. Summarizing, we just showed that

$$(X, A^{+}_{n,m}, A^{-}_{n,m}) \xrightarrow{\text{f.d.d.}} (X, \beta_{2r-1} \int_0^{\cdot} f(X_s^+) dW_s^+, \beta_{2r-1} \int_0^{\cdot} f(X_s^-) dW_s^-)$$

as $n \to \infty$ then $m \to \infty$.

(b) $B^{\pm}_{n,m}(t)$ converges to 0 in $L^2(\Omega)$ as $n \to \infty$ and then $m \to \infty$.

It suffices to prove that for all $k \in \{2, \dots, r\}$,

$$B^{\pm,k}_{n,m}(t) \xrightarrow{L^2} 0, \tag{5.6}$$

as $n \to \infty$ and then $m \to \infty$, where $B^{\pm,k}_{n,m}(t)$ is defined as follows

$$\begin{aligned}B^{\pm,k}_{n,m}(t) := 2^{-n/4} \sum_{i=1}^{\lfloor 2^{m/2} t \rfloor} \sum_{j=(i-1)2^{\frac{n-m}{2}}}^{i2^{\frac{n-m}{2}}-1} \frac{1}{2}(f(X^{\pm}_{(j+1)2^{-n/2}}) \\ - f(X^{\pm}_{(i+1)2^{-m/2}})) H_{2k-1}(X^{n,\pm}_{j+1} - X^{n,\pm}_j).\end{aligned}$$

With obvious notation, we have that

$$B^{\pm,k}_{n,m}(t) = 2^{-n/4} \sum_{i=1}^{\lfloor 2^{m/2} t \rfloor} \sum_{j=(i-1)2^{\frac{n-m}{2}}}^{i2^{\frac{n-m}{2}}-1} \Delta^{n,m}_{i,j} f(X^{\pm}) H_{2k-1}(X^{n,\pm}_{j+1} - X^{n,\pm}_j).$$

It suffices to prove the convergence to 0 of $B^{+,k}_{n,m}(t)$, the proof for $B^{-,k}_{n,m}(t)$ being exactly the same. In fact, the reader can find this proof in the proof of [5, Theorem 1, (1.15)] at page 1073.

(c) (5.4) **converges to** 0 **in** $L^2(\Omega)$ **as** $n\to\infty$ **and then** $m\to\infty$.

It suffices to prove that for all $k\in\{2,\dots,r\}$, $J^{\pm,k}_{n,m}(t)\xrightarrow{L^2}0$ as $n\to\infty$ and then $m\to\infty$, where $J^{\pm,k}_{n,m}(t)$ is defined as follows,

$$J^{\pm,k}_{n,m}(t)=2^{-n/4}\sum_{j=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor-1}\frac{1}{2}\big(f(X^{\pm}_{j2^{-n/2}})+f(X^{\pm}_{(j+1)2^{-n/2}})\big)H_{2k-1}(X^{n,\pm}_{j+1}-X^{n,\pm}_{j})$$
$$=2^{-n/4}\sum_{j=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor-1}\Theta^n_j f(X^{\pm})H_{2k-1}(X^{n,\pm}_{j+1}-X^{n,\pm}_{j}),$$

with obvious notation. We will only prove the convergence to 0 of $J^{+,k}_{n,m}(t)$, the proof for $J^{-,k}_{n,m}(t)$ being exactly the same. Using the relationship between Hermite polynomials and multiple stochastic integrals, namely $H_r(2^{nH/2}(X^+_{(j+1)2^{-n/2}}-X^+_{j2^{-n/2}}))=2^{nrH/2}I_r(\delta^{\otimes r}_{(j+1)2^{-n/2}})$, we obtain, using (2.6) as well,

$$E[(J^{+,k}_{n,m}(t))^2]=\Bigg|2^{-n/2}2^{nH(2k-1)}\sum_{j,j'=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor-1}\sum_{l=0}^{2k-1}l!\binom{2k-1}{l}^2$$
$$\times E\Big[\Theta^n_j f(X^+)\Theta^n_{j'}f(X^+)I_{2(2k-1)-2l}\Big(\delta^{\otimes(2k-1-l)}_{(j+1)2^{-n/2}}\otimes\delta^{\otimes(2k-1-l)}_{(j'+1)2^{-n/2}}\Big)\Big]$$
$$\times\langle\delta_{(j+1)2^{-n/2}};\delta_{(j'+1)2^{-n/2}}\rangle^l\Bigg|$$
$$\leqslant 2^{-n/2}2^{nH(2k-1)}\sum_{j,j'=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor-1}\sum_{l=0}^{2k-1}l!\binom{2k-1}{l}^2$$
$$\times\Big|E\Big[\Theta^n_j f(X^+)\Theta^n_{j'}f(X^+)I_{2(2k-1)-2l}\Big(\delta^{\otimes(2k-1-l)}_{(j+1)2^{-n/2}}\otimes\delta^{\otimes(2k-1-l)}_{(j'+1)2^{-n/2}}\Big)\Big]\Big|$$
$$\times\big|\langle\delta_{(j+1)2^{-n/2}};\delta_{(j'+1)2^{-n/2}}\rangle\big|^l$$
$$=\sum_{l=0}^{2k-1}l!\binom{2k-1}{l}^2 Q^{+,l}_{n,m}(t),\tag{5.7}$$

with obvious notation. Thanks both to the duality formula (2.5) and to (2.2), we have

$$d^{(+,l)}_n(j,j'):=E\Big[\Theta^n_j f(X^+)\Theta^n_{j'}f(X^+)I_{2(2k-1)-2l}\Big(\delta^{\otimes(2k-1-l)}_{(j+1)2^{-n/2}}\otimes\delta^{\otimes(2k-1-l)}_{(j'+1)2^{-n/2}}\Big)\Big]$$
$$=E\Big[\Big\langle D^{2(2k-1-l)}(\Theta^n_j f(X^+)\Theta^n_{j'}f(X^+));\ \delta^{\otimes(2k-1-l)}_{(j+1)2^{-n/2}}\otimes\delta^{\otimes(2k-1-l)}_{(j'+1)2^{-n/2}}\Big\rangle\Big]$$
$$=\frac{1}{4}\sum_{a=0}^{2(2k-1-l)}\binom{2(2k-1-l)}{a}E\Big[\Big\langle\Big(f^{(a)}\Big(X^+_{j2^{-n/2}}\Big)\varepsilon^{\otimes a}_{j2^{-n/2}}+f^{(a)}\Big(X^+_{(j+1)2^{-n/2}}\Big)$$

$$\times\varepsilon^{\otimes a}_{(j+1)2^{-n/2}}\Big)\tilde{\otimes}\Big(f^{\left(2(2k-1-l)-a\right)}\left(X^{+}_{j'2^{-n/2}}\right)\varepsilon^{\otimes(2(2k-1-l)-a)}_{j'2^{-n/2}}$$
$$+f^{\left(2(2k-1-l)-a\right)}\left(X^{+}_{(j'+1)2^{-n/2}}\right)\varepsilon^{\otimes(2(2k-1-l)-a)}_{(j'+1)2^{-n/2}}\Big)\,;\,\delta^{\otimes(2k-1-l)}_{(j+1)2^{-n/2}}\otimes\delta^{\otimes(2k-1-l)}_{(j'+1)2^{-n/2}}\Big\rangle\Big].$$

At this stage, the proof of the claim (**c**) is going to be different according to the value of l:

- If $l = 2k-1$ in (5.7) then

$$\begin{aligned}
Q^{+,2k-1}_{n,m}(t) &= 2^{-n/2}2^{nH(2k-1)}\sum_{j,j'=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor-1}\left|E\big[\Theta^n_j f(X^+)\Theta^n_{j'}f(X^+)\big]\right| \\
&\times\left|\langle\delta_{(j+1)2^{-n/2}};\delta_{(j'+1)2^{-n/2}}\rangle\right|^{2k-1} \\
&\leqslant C_f 2^{-n/2}2^{nH(2k-1)}\sum_{j,j'=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor-1}\left|\left\langle\delta_{(j+1)2^{-n/2}};\delta_{(j'+1)2^{-n/2}}\right\rangle\right|^{2k-1} \\
&= C_f 2^{-n/2}\sum_{j,j'=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor-1}\left|\frac{1}{2}\left(\left|j-j'+1\right|^{2H}+\left|j-j'-1\right|^{2H}-2\left|j-j'\right|^{2H}\right)\right|^{2k-1} \\
&= C_f 2^{-n/2}\sum_{j=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor-1}\;\sum_{p=j-\lfloor 2^{n/2}t\rfloor+1}^{j-\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}\left|\frac{1}{2}\left(\left|p+1\right|^{2H}+\left|p-1\right|^{2H}-2\left|p\right|^{2H}\right)\right|^{2k-1}
\end{aligned}\tag{5.8}$$

where we have the first inequality since f belongs to C^∞_b and the last one follows by the change of variable $p = j - j'$. Using the notation (2.16), and by a Fubini argument, we get that the quantity given in (5.8) is equal to

$$C_f 2^{-n/2}\sum_{p=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}-\lfloor 2^{n/2}t\rfloor+1}^{\lfloor 2^{n/2}t\rfloor-\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}-1}\left|\rho(p)\right|^{2k-1}\Big(\left(p+\lfloor 2^{n/2}t\rfloor\right)\wedge\lfloor 2^{n/2}t\rfloor$$
$$-\left(p+\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}\right)\vee\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}\Big).\tag{5.9}$$

By separating the cases when $0\leqslant p\leqslant\lfloor 2^{n/2}t\rfloor-\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}-1$ or when $\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}-\lfloor 2^{n/2}t\rfloor+1\leqslant p<0$ we deduce that

$$\begin{aligned}
0 &\leqslant\left(\frac{(p+\lfloor 2^{n/2}t\rfloor)}{2^{n/2}}\wedge\frac{(\lfloor 2^{n/2}t\rfloor)}{2^{n/2}}-\frac{(p+\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}})}{2^{n/2}}\vee\lfloor 2^{m/2}t\rfloor 2^{-m/2}\right) \\
&\leqslant\lfloor 2^{n/2}t\rfloor 2^{-n/2}-\lfloor 2^{m/2}t\rfloor 2^{-m/2}=\left|\lfloor 2^{n/2}t\rfloor 2^{-n/2}-\lfloor 2^{m/2}t\rfloor 2^{-m/2}\right| \\
&\leqslant\left|\lfloor 2^{n/2}t\rfloor 2^{-n/2}-t\right|+\left|t-\lfloor 2^{m/2}t\rfloor 2^{-m/2}\right|\leqslant 2^{-n/2}+2^{-m/2}.
\end{aligned}$$

As a result, the quantity given in (5.9) is bounded by

$$C_f \sum_{p\in\mathbb{Z}} \big|\rho(p)\big|^{2k-1}\big(2^{-n/2}+2^{-m/2}\big),$$

with $\sum_{p\in\mathbb{Z}} |\rho(p)|^{2k-1} < \infty$ (because $H < 1/2 \leqslant 1 - \frac{1}{4k-2}$). Finally, we have

$$Q_{n,m}^{+,2k-1}(t) \leqslant C\big(2^{-n/2}+2^{-m/2}\big). \tag{5.10}$$

- Preparation to the cases $0 \leqslant l \leqslant 2k-2$: In order to handle the terms $Q_{n,m}^{+,l}(t)$ whenever $0 \leqslant l \leqslant 2k-2$, we will make use of the following decomposition:

$$\big|d_n^{(+,l)}(j,j')\big| \leqslant \frac{1}{4}\big(\Omega_n^{(1,l)}(j,j')+\Omega_n^{(2,l)}(j,j')+\Omega_n^{(3,l)}(j,j')+\Omega_n^{(4,l)}(j,j')\big), \tag{5.11}$$

where

$$\begin{aligned}
\Omega_n^{(1,l)}(j,j') &= \sum_{a=0}^{2(2k-1-l)} \binom{2(2k-1-l)}{a}\Big|E\Big[f^{(a)}\big(X^+_{j2^{-n/2}}\big)f^{(2(2k-1-l)-a)}\big(X^+_{j'2^{-n/2}}\big)\Big]\Big|\\
&\quad\times\Big|\Big\langle \varepsilon^{\otimes a}_{j2^{-n/2}}\tilde{\otimes}\varepsilon^{\otimes(2(2k-1-l)-a)}_{j'2^{-n/2}};\,\delta^{\otimes(2k-1-l)}_{(j+1)2^{-n/2}}\otimes\delta^{\otimes(2k-1-l)}_{(j'+1)2^{-n/2}}\Big\rangle\Big|\\
\Omega_n^{(2,l)}(j,j') &= \sum_{a=0}^{2(2k-1-l)} \binom{2(2k-1-l)}{a}\Big|E\Big[f^{(a)}\big(X^+_{j2^{-n/2}}\big)f^{\left(2(2k-1-l)-a\right)}\big(X^+_{(j'+1)2^{-n/2}}\big)\Big]\Big|\\
&\quad\times\Big|\Big\langle \varepsilon^{\otimes a}_{j2^{-n/2}}\tilde{\otimes}\varepsilon^{\otimes(2(2k-1-l)-a)}_{(j'+1)2^{-n/2}};\,\delta^{\otimes(2k-1-l)}_{(j+1)2^{-n/2}}\otimes\delta^{\otimes(2k-1-l)}_{(j'+1)2^{-n/2}}\Big\rangle\Big|\\
\Omega_n^{(3,l)}(j,j') &= \sum_{a=0}^{2(2k-1-l)} \binom{2(2k-1-l)}{a}\Big|E\Big[f^{(a)}\big(X^+_{(j+1)2^{-n/2}}\big)f^{\left(2(2k-1-l)-a\right)}\big(X^+_{j'2^{-n/2}}\big)\Big]\Big|\\
&\quad\times\Big|\Big\langle \varepsilon^{\otimes a}_{(j+1)2^{-n/2}}\tilde{\otimes}\varepsilon^{\otimes(2(2k-1-l)-a)}_{j'2^{-n/2}};\,\delta^{\otimes(2k-1-l)}_{(j+1)2^{-n/2}}\otimes\delta^{\otimes(2k-1-l)}_{(j'+1)2^{-n/2}}\Big\rangle\Big|\\
\Omega_n^{(4,l)}(j,j') &= \sum_{a=0}^{2(2k-1-l)} \binom{2(2r-1-l)}{a}\Big|E\Big[f^{(a)}\big(X^+_{(j+1)2^{-n/2}}\big)f^{\left(2(2k-1-l)-a\right)}\big(X^+_{(j'+1)2^{-n/2}}\big)\Big]\Big|\\
&\quad\times\Big|\Big\langle \varepsilon^{\otimes a}_{(j+1)2^{-n/2}}\tilde{\otimes}\varepsilon^{\otimes(2(2k-1-l)-a)}_{(j'+1)2^{-n/2}};\,\delta^{\otimes(2k-1-l)}_{(j+1)2^{-n/2}}\otimes\delta^{\otimes(2k-1-l)}_{(j'+1)2^{-n/2}}\Big\rangle\Big|.
\end{aligned}$$

- For $1 \leqslant l \leqslant 2k-2$: Since f belongs to C_b^∞ and thanks to (2.7), we deduce that

$$d_n^{(+,l)}(j,j') \leqslant C\big(2^{-nH}\big)^{2(2k-1-l)}.$$

As a consequence of this previous inequality, we have

$$
\begin{aligned}
&Q_{n,m}^{+,l}(t)\\
&\leqslant C\big(2^{-nH}\big)^{2(2k-2)}\,2^{-n/2}\,2^{nH(2k-1)}\sum_{j,j'=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor-1}\big|\langle\delta_{(j+1)2^{-n/2}};\delta_{(j'+1)2^{-n/2}}\rangle\big|^l\\
&\leqslant C\big(2^{-nH}\big)^{2(2k-2)}\,2^{nH(2k-1)}2^{-nHl}\Big(\sum_{p\in\mathbb{Z}}|\rho(p)|^l\Big)\big(2^{-n/2}+2^{-m/2}\big)\\
&\leqslant C\,2^{-nH(2k-2)}\big(2^{-n/2}+2^{-m/2}\big),
\end{aligned}
\tag{5.12}
$$

where we have the second inequality by the same arguments that have been used previously in the case $l=2k-1$.

- For $l=0$: Thanks to the decomposition (5.11) we get

$$
Q_{n,m}^{+,0}(t)\leqslant\frac{1}{4}2^{-n/2}2^{nH(2k-1)}\sum_{k'=1}^{4}\sum_{j,j'=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor-1}\Omega_n^{(k',0)}(j,j')
\tag{5.13}
$$

We will study only the term corresponding to $\Omega_n^{(2,0)}(j,j')$ in (5.13), which is representative to the difficulty. It is given by

$$
\begin{aligned}
&\frac{1}{4}2^{-n/2}2^{nH(2k-1)}\sum_{j,j'=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor-1}\sum_{a=0}^{2(2k-1)}\binom{2(2k-1)}{a}\Big|E\big[f^{(a)}\big(X^+_{j2^{-n/2}}\big)\\
&\times f^{(2(2k-1)-a)}\big(X^+_{(j'+1)2^{-n/2}}\big)\big]\Big|\Big|\Big\langle\varepsilon^{\otimes a}_{j2^{-n/2}}\tilde{\otimes}\varepsilon^{\otimes(2(2k-1)-a)}_{(j'+1)2^{-n/2}};\delta^{\otimes(2k-1)}_{(j+1)2^{-n/2}}\otimes\delta^{\otimes(2k-1)}_{(j'+1)2^{-n/2}}\Big\rangle\Big|\\
&\leqslant C2^{-n/2}2^{nH(2k-1)}\sum_{j,j'=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor-1}\sum_{a=0}^{2(2k-1)}\Big|\Big\langle\varepsilon^{\otimes a}_{j2^{-n/2}}\tilde{\otimes}\varepsilon^{\otimes(2(2k-1)-a)}_{(j'+1)2^{-n/2}};\\
&\delta^{\otimes(2k-1)}_{(j+1)2^{-n/2}}\otimes\delta^{\otimes(2k-1)}_{(j'+1)2^{-n/2}}\Big\rangle\Big|.
\end{aligned}
$$

We define $E_n^{(a,k)}(j,j') := \big|\big\langle\varepsilon^{\otimes a}_{j2^{-n/2}}\tilde{\otimes}\varepsilon^{\otimes(2(2k-1)-a)}_{(j'+1)2^{-n/2}};\delta^{\otimes(2k-1)}_{(j+1)2^{-n/2}}\otimes\delta^{\otimes(2k-1)}_{(j'+1)2^{-n/2}}\big\rangle\big|$. By (2.7), we thus get, with $\tilde{c}_a$ some combinatorial constants,

$$
\begin{aligned}
E_n^{(a,k)}(j,j')\leqslant\tilde{c}_a\,2^{-nH(4k-3)}\Big(&\big|\big\langle\varepsilon_{j2^{-n/2}};\delta_{(j+1)2^{-n/2}}\big\rangle\big|+\big|\langle\varepsilon_{j2^{-n/2}};\delta_{(j'+1)2^{-n/2}}\rangle\big|\\
&+\big|\big\langle\varepsilon_{(j'+1)2^{-n/2}};\delta_{(j+1)2^{-n/2}}\big\rangle\big|+\big|\big\langle\varepsilon_{(j'+1)2^{-n/2}};\delta_{(j'+1)2^{-n/2}}\big\rangle\big|\Big).
\end{aligned}
$$

For instance, we can write

$$\begin{aligned}
&\sum_{j,j'=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor-1} \left|\left\langle \varepsilon_{(j'+1)2^{-n/2}};\, \delta_{(j+1)2^{-n/2}}\right\rangle\right| \\
&= 2^{-nH-1} \sum_{j,j'=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor-1} \left|(j+1)^{2H} - j^{2H} + \left|j'-j+1\right|^{2H} - \left|j'-j\right|^{2H}\right| \\
&\leqslant 2^{-nH-1} \sum_{j,j'=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor-1} \left((j+1)^{2H} - j^{2H}\right) \\
&\quad + 2^{-nH-1} \sum_{\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}} \leqslant j \leqslant j' \leqslant \lfloor 2^{n/2}t\rfloor-1} \left((j'-j+1)^{2H} - (j'-j)^{2H}\right) \\
&\quad + 2^{-nH-1} \sum_{\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}} \leqslant j' < j \leqslant \lfloor 2^{n/2}t\rfloor-1} \left((j-j')^{2H} - (j-j'-1)^{2H}\right) \\
&\leqslant \frac{3}{2}\, 2^{-nH}\left(\lfloor 2^{n/2}t\rfloor - \lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}\right)\lfloor 2^{n/2}t\rfloor^{2H} \leqslant \frac{3t^{2H}}{2}\left(2^{n/2}t - \lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{j,j'=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor-1} \left|\left\langle \varepsilon_{j2^{-n/2}};\, \delta_{(j+1)2^{-n/2}}\right\rangle\right| \leqslant \frac{3t^{2H}}{2}\left(2^{n/2}t - \lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}\right); \\
&\sum_{j,j'=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor-1} \left|\left\langle \varepsilon_{j2^{-n/2}};\, \delta_{(j'+1)2^{-n/2}}\right\rangle\right| \leqslant \frac{3t^{2H}}{2}\left(2^{n/2}t - \lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}\right); \\
&\sum_{j,j'=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor-1} \left|\left\langle \varepsilon_{(j'+1)2^{-n/2}};\, \delta_{(j'+1)2^{-n/2}}\right\rangle\right| \leqslant \frac{3t^{2H}}{2}\left(2^{n/2}t - \lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}\right).
\end{aligned}$$

As a consequence, we deduce

$$Q_{n,m}^{(+,0)}(t) \leqslant C\, 2^{-nH(2k-2)}\left(t - \lfloor 2^{m/2}t\rfloor 2^{\frac{-m}{2}}\right) \leqslant C\, 2^{-nH(2k-2)} 2^{-m/2}. \quad (5.14)$$

Combining (5.10), (5.12) and (5.14) finally shows

$$\begin{aligned}
E\left[\left(J_{n,m}^{+,k}(t)\right)^2\right] \leqslant C\Big(&2^{-n/2} + 2^{-m/2} + 2^{-nH(2k-2)}\left(2^{-n/2} + 2^{-m/2}\right) \\
&+ 2^{-nH(2k-2)} 2^{-m/2}\Big).
\end{aligned}$$

So, we deduce that $J_{n,m}^{+,k}(t)$ converges to 0 in $L^2(\Omega)$ as $n \to \infty$ and then $m \to \infty$. Finally, thanks to (a), (b) and (c), (5.2) holds true. □

5.2 Step 2: Limit of $2^{-n/4}W_n^{(1)}(f, Y_{T_{\lfloor 2^n t\rfloor,n}})$

Thanks to (1.5), for $H > \frac{1}{6}$, $2^{-\frac{nH}{2}} W_n^{(1)}(f, Y_{T_{\lfloor 2^n t\rfloor,n}}) \xrightarrow[n\to\infty]{P} \int_0^{Y_t} f(X_s)d^\circ X_s$. Thus, since $H < \frac{1}{2}$, we deduce that

$$2^{-n/4}W_n^{(1)}\left(f, Y_{T_{\lfloor 2^n t\rfloor,n}}\right) \xrightarrow[n\to\infty]{P} 0. \tag{5.15}$$

5.3 Step 3: Moment bounds for $W_n^{(2r-1)}(f, \cdot)$

We recall the following result from [8]. Fix an integer $r \geqslant 1$ as well as a function $f \in C_b^\infty$. There exists a constant $c > 0$ such that, for all real numbers $s < t$ and all $n \in \mathbb{N}$,

$$E\big[\big(W_n^{(2r-1)}(f,t) - W_n^{(2r-1)}(f,s)\big)^2\big] \leqslant c \max\left(|s|^{2H}, |t|^{2H}\right)\left(|t-s|2^{n/2}+1\right).$$

5.4 Step 4: Last step in the proof of (1.7)

Following [2], we introduce the following natural definition for two-sided stochastic integrals: for $u \in \mathbb{R}$, let

$$\int_0^u f(X_s)dW_s = \begin{cases} \int_0^u f(X_s^+)dW_s^+ & \text{if } u \geqslant 0 \\ \int_0^{-u} f(X_s^-)dW_s^- & \text{if } u < 0 \end{cases}, \tag{5.16}$$

where W^+ and W^- are defined in Proposition 5.1, X^+ and X^- are defined in Sect. 4, and $\int_0^\cdot f(X_s^\pm)dW_s^\pm$ must be understood in the Wiener–Itô sense.

Using (3.5), (5.15), the conclusion of Step 3 (to pass from $Y_{T_{\lfloor 2^n t\rfloor,n}}$ to Y_t) and since by [2, Lemma 2.3], we have $Y_{T_{\lfloor 2^n t\rfloor,n}} \xrightarrow{L^2} Y_t$ as $n \to \infty$, we deduce that the limit of $2^{-n/4}V_n^{(2r-1)}(f,t)$ is the same as that of

$$2^{-n/4}\sum_{l=2}^{r}\kappa_{r,l}W_n^{(2l-1)}(f, Y_t).$$

Thus, the proof of (1.7) follows directly from (5.2), the definition of the integral in (5.16), as well as the fact that X, W and Y are independent.

6 Proof of (1.8)

We suppose that $H > \frac{1}{2}$. The proof of (1.8) will be done in several steps:

6.1 Step 1: Limits and moment bounds for $W_n^{(2i-1)}(f, \cdot)$

We recall the following Itô-type formula from [5, Theorem 4] (see also [10, Theorem 1.3] for an extension of this formula to the bi-dimensional case). For all $t \in \mathbb{R}$, the following change-of-variable formula holds true for $H > \frac{1}{2}$

$$F(X_t) - F(0) = \int_0^t f(X_s) d^\circ X_s, \tag{6.1}$$

where F is a primitive of f and $\int_0^t f(X_s) d^\circ X_s$ is the Stratonovich integral of $f(X)$ with respect to X defined as the limit in probability of $2^{-\frac{nH}{2}} W_n^{(1)}(f, t)$ as $n \to \infty$.

For the rest of the proof, we suppose that $f \in C_b^\infty$. The following proposition will play a pivotal role in the proof of (1.8).

Proposition 6.1 *There exists a positive constant C, independent of n and t, such that for all $i \geqslant 1$ and $t \in \mathbb{R}$, we have*

$$E\big[\big(2^{-\frac{nH}{2}} W_n^{(2i-1)}(f, t)\big)^2\big] \leqslant C\,\psi(t, H, i, n), \tag{6.2}$$

where, we have

$$\begin{aligned}\psi(t, H, i, n) &:= \big|t\big|^{(2H-1)(4i-3)}\,\big|t\big|^{2H+1}\,2^{-n(2i-2)(1-H)}\\ &+ C\sum_{a=1}^{2i-2}\Big(\big[\big|t\big|(1+n)+t^2\big]\,\big|t\big|^{2(2H-1)(2i-1-a)}\,2^{-\frac{n}{2}(2H-1)}\,2^{-n(1-H)\big[2i-1-a\big]}\\ &+ \big|t\big|^{2(1-(1-H)a)}\,\big|t\big|^{2(2H-1)(2i-1-a)}\,2^{-n(1-H)[2i-2]}\mathbf{1}_{\big\{H>1-\frac{1}{2a}\big\}}\Big)\\ &+ C\big[\big|t\big|(1+n)+t^2\big]2^{-n\big(H-\frac{1}{2}\big)}\\ &+ C\,\big|t\big|^{2(1-(1-H)(2i-1))}\,2^{-n(1-H)(2i-2)}\mathbf{1}_{\big\{H>1-\frac{1}{(4i-2)}\big\}}.\end{aligned}$$

Proof Set $\phi_n(j, j') := \Delta_{j,n} f(X) \Delta_{j',n} f(X)$, where we recall that $\Delta_{j,n} f(X) := \frac{1}{2}(f(X_{j2^{-n/2}}) + f(X_{(j+1)2^{-n/2}}))$. Fix $t \geqslant 0$ (the proof in the case $t < 0$ is similar), for all $i \geqslant 1$, we have

$$\begin{aligned}E\big[\big(2^{-\frac{nH}{2}} W_n^{(2i-1)}(f, t)\big)^2\big] &= E\big[\big(2^{-\frac{nH}{2}} W_{+,n}^{(2i-1)}(f, t)\big)^2\big]\\ &= 2^{-nH}\sum_{j,j'=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\Big(\phi_n(j, j') H_{2i-1}\big(X_{j+1}^{n,+} - X_j^{n,+}\big) H_{2i-1}\big(X_{j'+1}^{n,+} - X_{j'}^{n,+}\big)\Big)\end{aligned}$$

$$= 2^{-nH(1-(2i-1))} \sum_{j,j'=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\Big(\phi_n(j,j') I_{2i-1}\big(\delta_{(j+1)2^{-n/2}}^{\otimes 2i-1}\big) I_{2i-1}\big(\delta_{(j'+1)2^{-n/2}}^{\otimes 2i-1}\big)\Big)$$

$$= 2^{-nH(2-2i)} \sum_{a=0}^{2i-1} a! \binom{2i-1}{a}^2 \sum_{j,j'=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\Big(\phi_n(j,j')$$

$$\times I_{4i-2-2a}\big(\delta_{(j+1)2^{-n/2}}^{\otimes 2i-1-a} \otimes \delta_{(j'+1)2^{-n/2}}^{\otimes 2i-1-a}\big)\Big) \langle \delta_{(j+1)2^{-n/2}}, \delta_{(j'+1)2^{-n/2}}\rangle^a$$

$$= 2^{-nH(2-2i)} \sum_{a=0}^{2i-1} a! \binom{2i-1}{a}^2 \sum_{j,j'=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\Big(\Big\langle D^{4i-2-2a}\big(\phi_n(j,j')\big),$$

$$\delta_{(j+1)2^{-n/2}}^{\otimes 2i-1-a} \otimes \delta_{(j'+1)2^{-n/2}}^{\otimes 2i-1-a}\Big\rangle\Big) \langle \delta_{(j+1)2^{-n/2}}, \delta_{(j'+1)2^{-n/2}}\rangle^a$$

$$= \sum_{a=0}^{2i-1} a! \binom{2i-1}{a}^2 Q_n^{(i,a)}(t), \tag{6.3}$$

with obvious notation at the last equality and with the third equality following from (2.4), the fourth one from (2.6) and the fifth one from (2.5). We have the following estimates.

- Case $a = 2i - 1$

$$\begin{aligned} \left|Q_n^{(i,2i-1)}(t)\right| &\leqslant 2^{-nH(2-2i)} \sum_{j,j'=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\big(|\phi_n(j,j')|\big) \\ &\quad \times \left|\langle \delta_{(j+1)2^{-n/2}}, \delta_{(j'+1)2^{-n/2}}\rangle\right|^{2i-1} \\ &\leqslant C 2^{-nH(2-2i)} \sum_{j,j'=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \left|\langle \delta_{(j+1)2^{-n/2}}, \delta_{(j'+1)2^{-n/2}}\rangle\right|^{2i-1}. \end{aligned}$$

Now, we distinguish three cases:
(a) If $H < 1 - \frac{1}{(4i-2)}$: by (2.10) we have

$$\left|Q_n^{(i,2i-1)}(t)\right| \leqslant C\, t\, 2^{-nH(2-2i)}\, 2^{n\left(\frac{1}{2}-(2i-1)H\right)} = C\, t\, 2^{-n\left(H-\frac{1}{2}\right)}.$$

(b) If $H = 1 - \frac{1}{(4i-2)}$: by (2.11) we have

$$\begin{aligned} \left|Q_n^{(i,2i-1)}(t)\right| &\leqslant C\left[t(1+n)+t^2\right] 2^{-nH(2-2i)}\, 2^{n\left(\frac{1}{2}-(2i-1)H\right)} \\ &= C\left[t(1+n)+t^2\right] 2^{-n\left(H-\frac{1}{2}\right)}. \end{aligned}$$

(c) If $H > 1 - \frac{1}{(4i-2)}$: by (2.12) we have

$$\left|Q_n^{(i,2i-1)}(t)\right| \leqslant C\, t\, 2^{-nH(2-2i)}\, 2^{n\left(\frac{1}{2}-(2i-1)H\right)} + C\, t^{2-(2-2H)(2i-1)}\, 2^{-nH(2-2i)}\, 2^{n(1-(2i-1))} = C\, t\, 2^{-n(H-\frac{1}{2})} + C\, t^{2(1-(1-H)(2i-1))}\, 2^{-n(1-H)(2i-2)}.$$

So, we deduce that

$$\left|Q_n^{(i,2i-1)}(t)\right| \leqslant C\left[\left|t\right|(1+n)+t^2\right]2^{-n\left(H-\frac{1}{2}\right)} + C\left|t\right|^{2\left(1-(1-H)(2i-1)\right)}\, 2^{-n(1-H)(2i-2)}\mathbf{1}_{\left\{H>1-\frac{1}{(4i-2)}\right\}} \tag{6.4}$$

- Preparation to the cases where $0 \leqslant a \leqslant 2i-2$
  Thanks to (2.2) we have

$$\begin{aligned}
D^{4i-2-2a}(\phi_n(j,j')) &= D^{4i-2-2a}(\Delta_{j,n}f(X)\Delta_{j',n}f(X)) \leqslant C\sum_{l=0}^{4i-2-2a} \\
&\Big(f^{(l)}(X_{j2^{-n/2}})\varepsilon_{j2^{-n/2}}^{\otimes l} + f^{(l)}(X_{(j+1)2^{-n/2}})\varepsilon_{(j+1)2^{-n/2}}^{\otimes l})\tilde{\otimes} \\
&(f^{(4i-2-2a-l)}(X_{j'2^{-n/2}})\varepsilon_{j'2^{-n/2}}^{\otimes 4i-2-2a-l} + f^{(4i-2-2a-l)}(X_{(j'+1)2^{-n/2}})\varepsilon_{(j'+1)2^{-n/2}}^{\otimes 4i-2-2a-l}) \\
&= C\sum_{l=0}^{4i-2-2a}(f^{(l)}(X_{j2^{-n/2}})f^{(4i-2-2a-l)}(X_{j'2^{-n/2}})\varepsilon_{j2^{-n/2}}^{\otimes l}\tilde{\otimes}\varepsilon_{j'2^{-n/2}}^{\otimes 4i-2-2a-l} + f^{(l)}(X_{j2^{-n/2}}) \\
&\times f^{(4i-2-2a-l)}(X_{(j'+1)2^{-n/2}})\varepsilon_{j2^{-n/2}}^{\otimes l}\tilde{\otimes}\varepsilon_{(j'+1)2^{-n/2}}^{\otimes 4i-2-2a-l} \\
&+f^{(l)}\big(X_{(j+1)2^{-n/2}}\big)f^{(4i-2-2a-l)}(X_{j'2^{-n/2}}) \\
&\times\varepsilon_{(j+1)2^{-n/2}}^{\otimes l}\tilde{\otimes}\varepsilon_{j'2^{-n/2}}^{\otimes 4i-2-2a-l} + f^{(l)}\big(X_{(j+1)2^{-n/2}}\big)f^{(4i-2-2a-l)}(X_{(j'+1)2^{-n/2}})\varepsilon_{(j+1)2^{-n/2}}^{\otimes l}\tilde{\otimes} \\
&\varepsilon_{(j'+1)2^{-n/2}}^{\otimes 4i-2-2a-l}\Big)
\end{aligned} \tag{6.5}$$

So, we have

- Case $1 \leqslant a \leqslant 2i-2$

$$\begin{aligned}
&\left|Q_n^{(i,a)}(t)\right| \\
&\leqslant C2^{-nH(2-2i)}\sum_{l=0}^{4i-2-2a}\sum_{j,j'=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \\
&\left|\left\langle\left(\varepsilon_{j2^{-n/2}}^{\otimes l}+\varepsilon_{(j+1)2^{-n/2}}^{\otimes l}\right)\tilde{\otimes}\left(\varepsilon_{j'2^{-n/2}}^{\otimes 4i-2-2a-l}+\varepsilon_{(j'+1)2^{-n/2}}^{\otimes 4i-2-2a-l}\right),\right.\right.
\end{aligned}$$

$$\left.\delta_{(j+1)2^{-n/2}}^{\otimes 2i-1-a} \otimes \delta_{(j'+1)2^{-n/2}}^{\otimes 2i-1-a}\right\rangle\Big|\Big|\left\langle \delta_{(j+1)2^{-n/2}}, \delta_{(j'+1)2^{-n/2}}\right\rangle\Big|^{a}$$
$$\leqslant C\, t^{(2H-1)(4i-2-2a)}\, 2^{-nH(2-2i)} \left(2^{-\frac{n}{2}}\right)^{4i-2-2a}$$
$$\times \sum_{j,j'=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \Big|\left\langle \delta_{(j+1)2^{-n/2}}, \delta_{(j'+1)2^{-n/2}}\right\rangle\Big|^{a},$$

where we have the first inequality because $f \in C_b^\infty$ and thanks to (6.5), and the second one thanks to (2.8) and (2.9). Now, we distinguish three cases:
(a) If $H < 1 - \frac{1}{2a}$: by (2.10) we have

$$\begin{aligned}\left|Q_n^{(i,a)}(t)\right| &\leqslant C\, t\, t^{(2H-1)(4i-2-2a)}\, 2^{-nH(2-2i)} 2^{-n(2i-1-a)}\, 2^{n\left(\frac{1}{2}-aH\right)}\\ &= C\, t^{2(2H-1)(2i-1-a)+1}\, 2^{-\frac{n}{2}(2H-1)}\, 2^{-n(1-H)[2i-1-a]}.\end{aligned}$$

(b) If $H = 1 - \frac{1}{2a}$: by (2.11) we have

$$\begin{aligned}&\left|Q_n^{(i,a)}(t)\right|\\ &\leqslant C\left[t(1+n)+t^2\right] t^{(2H-1)(4i-2-2a)}\, 2^{-nH(2-2i)} 2^{-n(2i-1-a)}\, 2^{n\left(\frac{1}{2}-aH\right)}\\ &= C\left[t(1+n)+t^2\right] t^{2(2H-1)(2i-1-a)}\, 2^{-\frac{n}{2}(2H-1)}\, 2^{-n(1-H)[2i-1-a]}.\end{aligned}$$

(c) If $H > 1 - \frac{1}{2a}$: by (2.12) we have

$$\begin{aligned}\left|Q_n^{(i,a)}(t)\right| &\leqslant C\, t\, t^{(2H-1)(4i-2-2a)}\, 2^{-nH(2-2i)} 2^{-n(2i-1-a)}\, 2^{n\left(\frac{1}{2}-aH\right)}\\ &\quad + C\, t^{2-(2-2H)a}\, t^{(2H-1)(4i-2-2a)}\, 2^{-nH(2-2i)} 2^{-n(2i-1-a)}\, 2^{n(1-a)}\\ &= C\, t^{2(2H-1)(2i-1-a)+1}\, 2^{-\frac{n}{2}(2H-1)}\, 2^{-n(1-H)[2i-1-a]}\\ &\quad + C\, t^{2(1-(1-H)a)}\, t^{2(2H-1)(2i-1-a)}\, 2^{-n(1-H)[2i-2]}.\end{aligned}$$

So, we deduce that

$$\begin{aligned}\left|Q_n^{(i,a)}(t)\right| &\leqslant C\left[|t|(1+n)+t^2\right] |t|^{2(2H-1)(2i-1-a)}\, 2^{-\frac{n}{2}(2H-1)}\, 2^{-n(1-H)\left[2i-1-a\right]}\\ &\quad + C\, |t|^{2(1-(1-H)a)}\, |t|^{2(2H-1)(2i-1-a)}\, 2^{-n(1-H)\left[2i-2\right]} \mathbf{1}_{\left\{H>1-\frac{1}{2a}\right\}}\end{aligned} \tag{6.6}$$

- Case $a = 0$

$$Q_n^{(i,0)}(t) = 2^{-nH(2-2i)} \sum_{j,j'=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\left(\left\langle D^{4i-2}\left(\phi_n(j,j')\right), \delta_{(j+1)2^{-n/2}}^{\otimes 2i-1} \otimes \delta_{(j'+1)2^{-n/2}}^{\otimes 2i-1}\right\rangle\right).$$

By (6.5) we deduce that

$$\left|Q_n^{(i,0)}(t)\right| \leqslant C2^{-nH(2-2i)} \sum_{l=0}^{4i-2} \sum_{j,j'=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \left|\left\langle \left(\varepsilon_{j2^{-n/2}}^{\otimes l} + \varepsilon_{(j+1)2^{-n/2}}^{\otimes l}\right) \right.\right.$$

$$\left.\left. \tilde{\otimes} \left(\varepsilon_{j'2^{-n/2}}^{\otimes 4i-2-l} + \varepsilon_{(j'+1)2^{-n/2}}^{\otimes 4i-2-l}\right), \delta_{(j+1)2^{-n/2}}^{\otimes 2i-1} \otimes \delta_{(j'+1)2^{-n/2}}^{\otimes 2i-1} \right\rangle\right|. \tag{6.7}$$

We define

$$E_n^{(i,l)}(j,j') := \left|\left\langle \left(\varepsilon_{j2^{-n/2}}^{\otimes l} + \varepsilon_{(j+1)2^{-n/2}}^{\otimes l}\right) \tilde{\otimes} \left(\varepsilon_{j'2^{-n/2}}^{\otimes 4i-2-l} + \varepsilon_{(j'+1)2^{-n/2}}^{\otimes 4i-2-l}\right), \right.\right.$$

$$\left.\left. \delta_{(j+1)2^{-n/2}}^{\otimes 2i-1} \otimes \delta_{(j'+1)2^{-n/2}}^{\otimes 2i-1} \right\rangle\right|.$$

Observe that by (2.8) and (2.9), we have

$$\begin{aligned} E_n^{(i,l)}(j,j') \leqslant & Ct^{(2H-1)(4i-3)} \left(2^{-\frac{n}{2}}\right)^{4i-3} \left(\left|\left\langle\left(\varepsilon_{j2^{-n/2}} + \varepsilon_{(j+1)2^{-n/2}}\right), \delta_{(j'+1)2^{-n/2}}\right\rangle\right| \right. \\ & + \left|\left\langle\left(\varepsilon_{j'2^{-n/2}} + \varepsilon_{(j'+1)2^{-n/2}}\right), \delta_{(j+1)2^{-n/2}}\right\rangle\right| + \left|\left\langle\left(\varepsilon_{j2^{-n/2}} \right.\right.\right. \\ & \left.\left.\left. + \varepsilon_{(j+1)2^{-n/2}}\right), \delta_{(j+1)2^{-n/2}}\right\rangle\right| \\ & \left. + \left|\left\langle\left(\varepsilon_{j'2^{-n/2}} + \varepsilon_{(j'+1)2^{-n/2}}\right), \delta_{(j'+1)2^{-n/2}}\right\rangle\right|\right). \end{aligned}$$

By combining these previous estimates with (6.7), (2.13) and (2.14), we deduce that

$$\begin{aligned} \left|Q_n^{(i,0)}(t)\right| &\leqslant C\ |t|^{(2H-1)(4i-3)}\ |t|^{2H+1}\ 2^{-nH(2-2i)} \left(2^{-\frac{n}{2}}\right)^{4i-3} 2^{\frac{n}{2}} \\ &= C\ |t|^{(2H-1)(4i-3)}\ |t|^{2H+1}\ 2^{-n(2i-2)(1-H)}. \end{aligned} \tag{6.8}$$

By combining (6.3) with (6.4), (6.6) and (6.8), we deduce that (6.2) holds true.

□

6.2 Step 2: Limit of $2^{-\frac{nH}{2}} W_n^{(2i-1)}(f, Y_{T_{\lfloor 2^n t\rfloor,n}})$

Let us prove that for $i \geqslant 2$,

$$2^{-\frac{nH}{2}} W_n^{(2i-1)}\left(f, Y_{T_{\lfloor 2^n t\rfloor,n}}\right) \xrightarrow[n\to\infty]{L^2} 0. \tag{6.9}$$

Due to the independence between X and Y and thanks to (6.2), we have

$$\begin{aligned} E\left[\left(2^{-\frac{nH}{2}} W_n^{(2i-1)}\left(f, Y_{T_{\lfloor 2^n t\rfloor,n}}\right)\right)^2\right] &= E\left[E\left[\left(2^{-\frac{nH}{2}} W_n^{(2i-1)}\left(f, Y_{T_{\lfloor 2^n t\rfloor,n}}\right)\right)^2 \middle| Y\right]\right] \\ &\leqslant CE\left[\psi\left(Y_{T_{\lfloor 2^n t\rfloor,n}}, H, i, n\right)\right]. \end{aligned}$$

It suffices to prove that

$$E\big[\psi\big(Y_{T_{\lfloor 2^n t\rfloor,n}}, H, i, n\big)\big] \underset{n\to\infty}{\longrightarrow} 0. \tag{6.10}$$

For simplicity, we write $Y_n(t)$ instead of $Y_{T_{\lfloor 2^n t\rfloor,n}}$. We have

$$\begin{aligned} E[\psi(Y_n(t), H, i, n)] &= E[|Y_n(t)|^{(2H-1)(4i-3)}\,|Y_n(t)|^{2H+1}]\,2^{-n(2i-2)(1-H)} \\ &+ C\sum_{a=1}^{2i-2}\Bigg(E[[|Y_n(t)|(1+n)+(Y_n(t))^2] \\ &|Y_n(t)|^{2(2H-1)(2i-1-a)}]\,2^{-\frac{n}{2}(2H-1)}\,2^{-n(1-H)[2i-1-a]} \\ &+ E[|Y_n(t)|^{2(1-(1-H)a)}\,|Y_n(t)|^{2(2H-1)(2i-1-a)}]\,2^{-n(1-H)[2i-2]}\mathbf{1}_{\{H>1-\frac{1}{2a}\}}\Bigg) \\ &+ CE[|Y_n(t)|(1+n)+(Y_n(t))^2]2^{-n(H-\frac{1}{2})} \\ &\quad + CE[|Y_n(t)|^{2(1-(1-H)(2i-1))}]\,2^{-n(1-H)(2i-2)} \\ &\times \mathbf{1}_{\{H>1-\frac{1}{(4i-2)}\}}. \end{aligned} \tag{6.11}$$

Let us prove that, for all $1 \leqslant a \leqslant 2i-2$

$$E[|Y_n(t)|^{2(1-(1-H)a)}\,|Y_n(t)|^{2(2H-1)(2i-1-a)}]\,2^{-n(1-H)[2i-2]}\mathbf{1}_{\{H>1-\frac{1}{2a}\}} \underset{n\to\infty}{\longrightarrow} 0,$$

(the proof of the convergence to 0 of the other terms in (6.11) is similar). In fact, by Hölder inequality, we have

$$\begin{aligned} &E[|Y_n(t)|^{2(1-(1-H)a)}\,|Y_n(t)|^{2(2H-1)(2i-1-a)}]\mathbf{1}_{\{H>1-\frac{1}{2a}\}} \\ &\quad \leqslant E[|Y_n(t)|^{4(1-(1-H)a)}g]^{\frac{1}{2}}\,E[|Y_n(t)|^{4(2H-1)(2i-1-a)}]^{\frac{1}{2}}\mathbf{1}_{\{H>1-\frac{1}{2a}\}}. \end{aligned}$$

Observe that for $H > 1-\frac{1}{2a}$ we have $2 < 4(1-(1-H)a) < 4$. So, by Hölder inequality, we deduce that $E[|Y_n(t)|^{4(1-(1-H)a)}]^{\frac{1}{2}} \leqslant E[(Y_n(t))^4]^{\frac{1}{2}(1-(1-H)a)} \leqslant C$ for all $n \in \mathbb{N}$, where we have the last inequality by Lemma 2.2. On the other hand since $H > \frac{1}{2}$ we have $4(2H-1)(2i-1-a) > 0$, and it is clear that there exists an integer $k_0 > 1$ such that $\frac{2k_0}{4(2H-1)(2i-1-a)} > 1$. Thus, by Hölder inequality, we have $E[|Y_n(t)|^{4(2H-1)(2i-1-a)}]^{\frac{1}{2}} \leqslant E[(Y_n(t))^{2k_0}]^{\frac{(2H-1)(2i-1-a)}{k_0}} \leqslant C$ for all $n \in \mathbb{N}$, where we have the last inequality by Lemma 2.2. Finally, we deduce that

$$
\begin{aligned}
&E[|Y_n(t)|^{2(1-(1-H)a)}\;|Y_n(t)|^{2(2H-1)(2i-1-a)}]\;2^{-n(1-H)[2i-2]}\mathbf{1}_{\{H>1-\frac{1}{2a}\}}\\
&\leqslant C2^{-n(1-H)[2i-2]}\underset{n\to\infty}{\to}0.
\end{aligned}
$$

Thus, (6.10) holds true.

6.3 Step 3: Limit of $V_n^{(1)}(f,\cdot)$

Recall that for all $t\geqslant 0$ and $r\geqslant 1$,

$$
V_n^{(r)}(f,t):=\sum_{k=0}^{\lfloor 2^n t\rfloor-1}\frac{1}{2}(f(Z_{T_{k,n}})+f(Z_{T_{k+1,n}}))[(2^{\frac{nH}{2}}(Z_{T_{k+1,n}}-Z_{T_{k,n}}))^r-\mu_r].
$$

We claim that

$$
2^{-\frac{nH}{2}}V_n^{(1)}(f,t)\underset{n\to\infty}{\overset{L^2}{\longrightarrow}}\int_0^{Y_t}f(X_s)d^{\circ}X_s. \tag{6.12}
$$

We will make use of the following Taylor's type formula (if interested the reader can find a proof of this formula, e.g., in [1] page 1788). Fix $f\in C_b^{\infty}$, let F be a primitive of f. For any $a,\ b\in\mathbb{R}$,

$$
\begin{aligned}
F(b)-F(a)=&\frac{1}{2}(f(a)+f(b))(b-a)-\frac{1}{24}(f''(a)+f''(b))(b-a)^3\\
&+O(|b-a|^5),
\end{aligned}
$$

where $|O(|b-a|^5)|\leqslant C_F|b-a|^5$, C_F being a constant depending only on F. One can thus write

$$
\begin{aligned}
F(Z_{T_{\lfloor 2^n t\rfloor,n}})-F(0)&=\sum_{k=0}^{\lfloor 2^n t\rfloor-1}(F(Z_{T_{k+1,n}})-F(Z_{T_{k,n}}))\\
&=2^{-\frac{nH}{2}}V_n^{(1)}(f,t)-\frac{2^{-\frac{3nH}{2}}}{12}V_n^{(3)}(f'',t)+\sum_{k=0}^{\lfloor 2^n t\rfloor-1}O(|Z_{T_{k+1,n}}-Z_{T_{k,n}}|^5).
\end{aligned}
\tag{6.13}
$$

Thanks to the Minkowski inequality, we have

$$
\left\|\sum_{k=0}^{\lfloor 2^n t\rfloor-1}O(|Z_{T_{k+1,n}}-Z_{T_{k,n}}|^5)\right\|_2\leqslant C_F\sum_{k=0}^{\lfloor 2^n t\rfloor-1}\left\||Z_{T_{k+1,n}}-Z_{T_{k,n}}|^5\right\|_2.
$$

Due to the independence between X and Y, the self-similarity and the stationarity of increments of X, we have

$$\left\| \left| Z_{T_{k+1,n}} - Z_{T_{k,n}} \right|^5 \right\|_2 = (E[(Z_{T_{k+1,n}} - Z_{T_{k,n}})^{10}])^{\frac{1}{2}} = (E[E[(Z_{T_{k+1,n}} - Z_{T_{k,n}})^{10} \mid Y]])^{\frac{1}{2}}$$
$$= (2^{-5nH} E[X_1^{10}])^{\frac{1}{2}} = 2^{-\frac{5nH}{2}} \|X_1^5\|_2.$$

Finally, thanks to the previous calculation and since $H > \frac{1}{2}$, we deduce that

$$\left\| \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} O\left(\left| Z_{T_{k+1,n}} - Z_{T_{k,n}} \right|^5 \right) \right\|_2 \leqslant C_F \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} 2^{-\frac{5nH}{2}} \left\| X_1^5 \right\|_2$$
$$\leqslant C_F \left\| X_1^5 \right\|_2 t\, 2^{n\left(1 - \frac{5H}{2}\right)} \underset{n\to\infty}{\longrightarrow} 0. \tag{6.14}$$

By (3.5), we have $2^{-\frac{3nH}{2}} V_n^{(3)}(f,t) = 2^{-\frac{3nH}{2}} W_n^{(3)}(f, Y_{T_{\lfloor 2^n t \rfloor, n}}) + 32^{-\frac{3nH}{2}} W_n^{(1)}(f, Y_{T_{\lfloor 2^n t \rfloor, n}})$. By (6.9), we have that $2^{-\frac{3nH}{2}} W_n^{(3)}(f, Y_{T_{\lfloor 2^n t \rfloor, n}})$ converges to 0 in L^2 as $n \to \infty$. By (6.2) and thanks to the independence of X and Y, we deduce that

$$E\left[\left(2^{-\frac{3nH}{2}} W_n^{(1)} \left(f, Y_{T_{\lfloor 2^n t \rfloor, n}} \right) \right)^2 \right]$$
$$\leqslant C 2^{-2nH} \left(2^{-n\left(H - \frac{1}{2}\right)} \left[(1+n) E\left[\left| Y_{T_{\lfloor 2^n t \rfloor, n}} \right| \right] + E\left[\left(Y_{T_{\lfloor 2^n t \rfloor, n}} \right)^2 \right] \right] \right.$$
$$\left. + E\left[\left| Y_{T_{\lfloor 2^n t \rfloor, n}} \right|^{2H} \right] + E\left[\left| Y_{T_{\lfloor 2^n t \rfloor, n}} \right|^{4H} \right] \right),$$

by Hölder inequality and thanks to Lemma 2.2, we can prove easily that the last quantity converges to 0 as $n \to \infty$. Finally, we get

$$2^{-\frac{3nH}{2}} V_n^{(3)}(f,t) \underset{n\to\infty}{\overset{L^2}{\longrightarrow}} 0. \tag{6.15}$$

Now, let us prove that

$$F(Z_{T_{\lfloor 2^n t \rfloor, n}}) - F(0) \underset{n\to\infty}{\overset{L^2}{\longrightarrow}} F(Z_t) - F(0). \tag{6.16}$$

In fact, as it has been mentioned in the introduction, $T_{\lfloor 2^n t \rfloor, n} \overset{a.s.}{\longrightarrow} t$ as $n \to \infty$ (see [2, Lemma 2.2] for a precise statement), and thanks to the continuity of F as well as the continuity of the paths of Z, we have

$$F(Z_{T_{\lfloor 2^n t \rfloor, n}}) - F(0) \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} F(Z_t) - F(0). \tag{6.17}$$

In addition, by the mean value theorem, and since f is bounded, we have that $\left| F(Z_{T_{\lfloor 2^n t \rfloor, n}}) - F(0) \right| \leqslant \sup_{x\in\mathbb{R}} |f(x)| |Z_{T_{\lfloor 2^n t \rfloor, n}}|$, so, we deduce that

$$\|F(Z_{T_{\lfloor 2^n t \rfloor, n}}) - F(0)\|_4 \leqslant \sup_{x\in\mathbb{R}} |f(x)| \|Z_{T_{\lfloor 2^n t \rfloor, n}}\|_4.$$

Due to independence between X and Y, and to the self-similarity of X, we have $\|Z_{T_{\lfloor 2^n t\rfloor,n}}\|_4 = \|X_{Y_{T_{\lfloor 2^n t\rfloor,n}}}\|_4 = \||Y_{T_{\lfloor 2^n t\rfloor,n}}|^H X_1\|_4 = \||Y_{T_{\lfloor 2^n t\rfloor,n}}|^H\|_4\|X_1\|_4$. By Hölder inequality, we have $\||Y_{T_{\lfloor 2^n t\rfloor,n}}|^H\|_4 \leqslant (\|Y_{T_{\lfloor 2^n t\rfloor,n}}\|_4)^H$. Finally, we have

$$\|F(Z_{T_{\lfloor 2^n t\rfloor,n}}) - F(0)\|_4 \leqslant \sup_{x\in\mathbb{R}} |f(x)|\|X_1\|_4(\|Y_{T_{\lfloor 2^n t\rfloor,n}}\|_4)^H.$$

Thanks to Lemma 2.2 and to the previous inequality, we deduce that the sequence $\left(F(Z_{T_{\lfloor 2^n t\rfloor,n}}) - F(0)\right)_{n\in\mathbb{N}}$ is bounded in L^4. Combining this fact with (6.17) we deduce that (6.16) holds true.

Finally, combining (6.13) with (6.14), (6.15) and (6.16), we deduce that

$$2^{-\frac{nH}{2}} V_n^{(1)}(f,t) \xrightarrow[n\to\infty]{L^2} F(Z_t) - F(0).$$

By (6.1), we have $F(X_t) - F(0) = \int_0^t f(X_s)d^\circ X_s$ which implies that $F(Z_t) - F(0) = \int_0^{Y_t} f(X_s)d^\circ X_s$. So, we deduce finally that (6.12) holds true.

6.4 Step 4: Last step in the proof of (1.8)

Thanks to (3.5), we have

$$V_n^{(2r-1)}(f,t) = \sum_{i=1}^{r} \kappa_{r,i} W_n^{(2i-1)}(f, Y_{T_{\lfloor 2^n t\rfloor,n}}).$$

For $r = 1$, (1.8) holds true by (6.12). For $r \geqslant 2$, we have $2^{-\frac{nH}{2}} V_n^{(2r-1)}(f,t) = \kappa_{r,1} 2^{-\frac{nH}{2}} V_n^{(1)}(f,t) + \sum_{i=2}^{r} \kappa_{r,i} 2^{-\frac{nH}{2}} W_n^{(2i-1)}(f, Y_{T_{\lfloor 2^n t\rfloor,n}})$. Combining this equality with (6.9) and (6.12), we deduce that (1.8) holds true.

7 Proof of (1.9)

Recall that for all $t \geqslant 0$ and $r \geqslant 1$,

$$V_n^{(2r)}(f,t) := \sum_{k=0}^{\lfloor 2^n t\rfloor - 1} \frac{1}{2}(f(Z_{T_{k,n}}) + f(Z_{T_{k+1,n}}))[(2^{\frac{nH}{2}}(Z_{T_{k+1,n}} - Z_{T_{k,n}}))^{2r} - \mu_{2r}],$$

and for all $i \in \mathbb{Z}$, $\Delta_{i,n} f(X) := \frac{1}{2}(f(X_{i2^{-n/2}}) + f(X_{(i+1)2^{-n/2}})$. Thanks to Lemma 3.1, we have

$$\begin{aligned}2^{-\frac{n}{2}} V_n^{(2r)}(f,t) &= 2^{-\frac{n}{2}} \sum_{i\in\mathbb{Z}} \Delta_{i,n} f(X)[(X_{i+1}^{(n)} - X_i^{(n)})^{2r} - \mu_{2r}](U_{i,n}(t) + D_{i,n}(t))\\ &= \sum_{i\in\mathbb{Z}} \Delta_{i,n} f(X)[(X_{i+1}^{(n)} - X_i^{(n)})^{2r} - \mu_{2r}]\mathcal{L}_{i,n}(t),\end{aligned}$$

with obvious notation at the last line. Fix $t \geqslant 0$. In order to study the asymptotic behavior of $2^{-\frac{n}{2}} V_n^{(2r)}(t)$ as n tends to infinity (after using the adequate normalization according to the value of the Hurst parameter H) , we shall consider (separately) the cases when n is even and when n is odd.

When n is even, for any even integers $n \geqslant m \geqslant 0$ and any integer $p \geqslant 0$, one can decompose $2^{-\frac{n}{2}} V_n^{(2r)}(t)$ as

$$2^{-\frac{n}{2}} V_n^{(2r)}(t) = A_{m,n,p}^{(2r)}(t) + B_{m,n,p}^{(2r)}(t) + C_{m,n,p}^{(2r)}(t) + D_{m,n,p}^{(2r)}(t) + E_{n,p}^{(2r)}(t),$$

where

$$\begin{aligned}
A_{m,n,p}^{(2r)}(t) &= \sum_{-p2^{m/2}+1 \leqslant j \leqslant p2^{m/2}} \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} \Delta_{i,n} f(X)[(X_{i+1}^{(n)} - X_i^{(n)})^{2r} - \mu_{2r}] \\
&\quad \times (\mathcal{L}_{i,n}(t) - L_t^{i2^{-n/2}}(Y)) \\
B_{m,n,p}^{(2r)}(t) &= \sum_{-p2^{m/2}+1 \leqslant j \leqslant p2^{m/2}} \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} \Delta_{i,n} f(X)[(X_{i+1}^{(n)} - X_i^{(n)})^{2r} - \mu_{2r}] \\
&\quad \times (L_t^{i2^{-n/2}}(Y) - L_t^{j2^{-m/2}}(Y)) \\
C_{m,n,p}^{(2r)}(t) &= \sum_{-p2^{m/2}+1 \leqslant j \leqslant p2^{m/2}} L_t^{j2^{-m/2}}(Y) \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} (\Delta_{i,n} f(X) - \Delta_{j,m} f(X)) \\
&\quad \times [(X_{i+1}^{(n)} - X_i^{(n)})^{2r} - \mu_{2r}] \\
D_{m,n,p}^{(2r)}(t) &= \sum_{-p2^{m/2}+1 \leqslant j \leqslant p2^{m/2}} \Delta_{j,m} f(X) L_t^{j2^{-m/2}}(Y) \\
&\quad \times \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} [(X_{i+1}^{(n)} - X_i^{(n)})^{2r} - \mu_{2r}] \\
E_{n,p}^{(2r)}(t) &= \sum_{i \geqslant p2^{n/2}} \Delta_{i,n} f(X)[(X_{i+1}^{(n)} - X_i^{(n)})^{2r} - \mu_{2r}]\mathcal{L}_{i,n}(t) \\
&\quad + \sum_{i < -p2^{n/2}} \Delta_{i,n} f(X)[(X_{i+1}^{(n)} - X_i^{(n)})^{2r} - \mu_{2r}]\mathcal{L}_{i,n}(t).
\end{aligned}$$

We can see that since we have taken even integers $n \geqslant m \geqslant 0$ then $2^{m/2}, 2^{\frac{n-m}{2}}$ and $2^{n/2}$ are integers as well. This justifies the validity of the previous decomposition.

When n is odd, for any odd integers $n \geqslant m \geqslant 0$ we can work with the same decomposition for $V_n^{(2r)}(t)$. The only difference is that we have to replace the sum $\sum_{-p2^{m/2}+1 \leqslant j \leqslant p2^{m/2}}$ in $A_{m,n,p}^{(2r)}(t)$, $B_{m,n,p}^{(2r)}(t)$, $C_{m,n,p}^{(2r)}(t)$ and $D_{m,n,p}^{(2r)}(t)$ by

$\sum_{-p2^{\frac{m+1}{2}}+1\leqslant j\leqslant p2^{\frac{m+1}{2}}}$. And instead of $\sum_{i\geqslant p2^{n/2}}$ and $\sum_{i<-p2^{n/2}}$ in $E^{(2r)}_{n,p}(t)$, we must consider $\sum_{i\geqslant p2^{\frac{n+1}{2}}}$ and $\sum_{i<-p2^{\frac{n+1}{2}}}$ respectively. The analysis can then be done *mutatis mutandis*.

Suppose that $\frac{1}{4}<H\leqslant\frac{1}{2}$. Firstly, we will prove that $2^{-\frac{n}{4}}A^{(2r)}_{m,n,p}(t)$, $2^{-\frac{n}{4}}B^{(2r)}_{m,n,p}(t)$, $2^{-\frac{n}{4}}C^{(2r)}_{m,n,p}(t)$ and $2^{-\frac{n}{4}}E^{(2r)}_{n,p}(t)$ converge to 0 in L^2 by letting n, then m, then p tends to infinity. Secondly, we will study the f.d.d. convergence in law of $(2^{-\frac{n}{4}}D^{(2r)}_{m,n,p}(t))_{t\geqslant 0}$, which will then be equivalent to the f.d.d. convergence in law of $(2^{-\frac{3n}{4}}V^{(2r)}_n(t))_{t\geqslant 0}$.

(1) $2^{-\frac{n}{4}}A^{(2r)}_{m,n,p}(t)\xrightarrow[n\to\infty]{L^2}0$:

We have, for all $r\in\mathbb{N}^*$,

$$x^{2r}=\sum_{a=1}^{r}b_{2r,a}H_{2a}(x)+\mu_{2r},\tag{7.1}$$

where H_n is the nth Hermite polynomial, $\mu_{2r}=E[N^{2r}]$ with $N\sim\mathcal{N}(0,1)$, and $b_{2r,a}$ are some explicit constants (if interested, the reader can find these explicit constants, e.g., in [9, Corollary 1.2]). We deduce that

$$\begin{aligned}A^{(2r)}_{m,n,p}(t)&=\sum_{a=1}^{r}b_{2r,a}\sum_{-p2^{m/2}+1\leqslant j\leqslant p2^{m/2}}\ \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1}\Delta_{i,n}f(X)H_{2a}(X^{(n)}_{i+1}-X^{(n)}_i)\\&\quad\times(\mathcal{L}_{i,n}(t)-L^{i2^{-n/2}}_t(Y))\\&=\sum_{a=1}^{r}b_{2r,a}A^{(2r)}_{m,n,p,a}(t),\end{aligned}\tag{7.2}$$

with obvious notation at the last line. It suffices to prove that for any fixed m and p and for all $a\in\{1,\dots,r\}$

$$2^{-\frac{n}{4}}A^{(2r)}_{m,n,p,a}(t)\xrightarrow[n\to\infty]{L^2}0.\tag{7.3}$$

Set $\phi_n(i,i'):=\Delta_{i,n}f(X)\Delta_{i',n}f(X)$. Thanks to (2.4), (2.5), (2.6) and to the independence of X and Y, we have

$$\begin{aligned}E\left[(2^{-\frac{n}{4}}A^{(2r)}_{m,n,p,a}(t))^2\right]&=\Bigg|2^{2nHa-\frac{n}{2}}\sum_{-p2^{m/2}+1\leqslant j,j'\leqslant p2^{m/2}}\ \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1}\ \sum_{i'=(j'-1)2^{\frac{n-m}{2}}}^{j'2^{\frac{n-m}{2}}-1}E\Big[\phi_n(i,i')\\&\quad\times I_{2a}\left(\delta_{(i+1)2^{-\frac{n}{2}}}\right)I_{2a}\left(\delta_{(i'+1)2^{-\frac{n}{2}}}\right)\Big]E\left[\left(\mathcal{L}_{i,n}(t)-L^{i2^{-\frac{n}{2}}}_t(Y)\right)\left(\mathcal{L}_{i',n}(t)-L^{i'2^{-\frac{n}{2}}}_t(Y)\right)\right]\Bigg|\\&\leqslant 2^{2nHa-\frac{n}{2}}\sum_{-p2^{m/2}+1\leqslant j,j'\leqslant p2^{m/2}}\ \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1}\ \sum_{i'=(j'-1)2^{\frac{n-m}{2}}}^{j'2^{\frac{n-m}{2}}-1}\left|E\left[\phi_n(i,i')I_{2a}\left(\delta^{\otimes 2a}_{(i+1)2^{-\frac{n}{2}}}\right)\right.\right.\end{aligned}$$

$$\times I_{2a}\left(\delta^{\otimes 2a}_{(i'+1)2^{-\frac{n}{2}}}\right)\Big]\Big|\left\|\mathcal{L}_{i,n}(t)-L_t^{i2^{-\frac{n}{2}}}(Y)\right\|_2\times\left\|\mathcal{L}_{i',n}(t)-L_t^{i'2^{-\frac{n}{2}}}(Y)\right\|_2$$

$$\leqslant 2^{2nHa-\frac{n}{2}}\sum_{l=0}^{2a} l!\binom{2a}{l}^2\sum_{-p2^{m/2}+1\leqslant j,j'\leqslant p2^{m/2}}\sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1}\sum_{i'=(j'-1)2^{\frac{n-m}{2}}}^{j'2^{\frac{n-m}{2}}-1}\Big|E\big[\phi_n(i,i')$$

$$\times I_{4a-2l}\left(\delta^{\otimes 2a-l}_{(i+1)2^{-n/2}}\otimes\delta^{\otimes 2a-l}_{(i'+1)2^{-n/2}}\right)\Big]\Big|\,\left|\langle\delta_{(i+1)2^{-n/2}},\delta_{(i'+1)2^{-n/2}}\rangle\right|^l\left\|\mathcal{L}_{i,n}(t)-L_t^{i2^{-\frac{n}{2}}}(Y)\right\|_2$$

$$\times\left\|\mathcal{L}_{i',n}(t)-L_t^{i'2^{-\frac{n}{2}}}(Y)\right\|_2$$

$$=2^{2nHa-\frac{n}{2}}\sum_{l=0}^{2a} l!\binom{2a}{l}^2\sum_{-p2^{m/2}+1\leqslant j,j'\leqslant p2^{m/2}}\sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1}\sum_{i'=(j'-1)2^{\frac{n-m}{2}}}^{j'2^{\frac{n-m}{2}}-1}\Big|E\Big[\Big\langle D^{4a-2l}(\phi_n(i,i')),$$

$$\delta^{\otimes 2a-l}_{(i+1)2^{-n/2}}\otimes\delta^{\otimes 2a-l}_{(i'+1)2^{-n/2}}\Big\rangle\Big]\Big|\,\left|\langle\delta_{(i+1)2^{-n/2}},\delta_{(i'+1)2^{-n/2}}\rangle\right|^l\left\|\mathcal{L}_{i,n}(t)-L_t^{i2^{-\frac{n}{2}}}(Y)\right\|_2$$

$$\times\left\|\mathcal{L}_{i',n}(t)-L_t^{i'2^{-\frac{n}{2}}}(Y)\right\|_2$$

$$=\sum_{l=0}^{2a} l!\binom{2a}{l}^2\Upsilon_n^{(l,a)}(t), \tag{7.4}$$

by obvious notation at the last line. By the points 2 and 3 of Proposition 2.3, see also (3.14) in [9] for the detailed proof, we have

$$\left\|\mathcal{L}_{i,n}(t)-L_t^{i2^{-\frac{n}{2}}}(Y)\right\|_2\leqslant 2\sqrt{\mu}\,\|K\|_4\;t^{1/8}n2^{-n/4}\,2^{-n/8}\,|i|^{1/4}$$
$$+2\,\|K\|_4\,\|L_t^0(Y)\|_2^{1/2}n2^{-n/4}.$$

Since $-p2^{m/2}+1\leqslant j\leqslant p2^{m/2}$ and $(j-1)2^{\frac{n-m}{2}}\leqslant i\leqslant j2^{\frac{n-m}{2}}-1$, we deduce that $-p2^{n/2}\leqslant i\leqslant p2^{n/2}-1$. So, $|i|\leqslant p2^{n/2}$. Consequently we have that $|i|^{1/4}\leqslant p^{1/4}2^{n/8}$, which shows that $\|\mathcal{L}_{i,n}(t)-L_t^{i2^{-\frac{n}{2}}}(Y)\|_2\leqslant C(p^{1/4}+1)n2^{-\frac{n}{4}}$. Finally, we deduce that

$$\left\|\mathcal{L}_{i,n}(t)-L_t^{i2^{-\frac{n}{2}}}(Y)\right\|_2\times\left\|\mathcal{L}_{i',n}(t)-L_t^{i'2^{-\frac{n}{2}}}(Y)\right\|_2\leqslant C(p^{1/4}+1)^2n^22^{-\frac{n}{2}}. \tag{7.5}$$

Now, observe that, by the same arguments that has been used to show (6.5) and since $f\in C_b^\infty$, we have

$$\Theta^{(a,l)}_{i,i',n}:=\Big|E\Big[\Big\langle D^{4a-2l}(\phi_n(i,i')),\delta^{\otimes 2a-l}_{(i+1)2^{-n/2}}\otimes\delta^{\otimes 2a-l}_{(i'+1)2^{-n/2}}\Big\rangle\Big]\Big|$$
$$\leqslant C\sum_{k=0}^{4a-2l}\binom{4a-2l}{k}\Big|\Big\langle\left(\varepsilon^{\otimes k}_{i2^{-n/2}}+\varepsilon^{\otimes k}_{(i+1)2^{-n/2}}\right)\tilde{\otimes}\left(\varepsilon^{\otimes 4a-2l-k}_{i'2^{-n/2}}+\varepsilon^{\otimes 4a-2l-k}_{(i'+1)2^{-n/2}}\right),$$
$$\delta^{\otimes 2a-l}_{(i+1)2^{-n/2}}\otimes\delta^{\otimes 2a-l}_{(i'+1)2^{-n/2}}\Big\rangle\Big|.$$

Since $H\leqslant\frac{1}{2}$, thanks to (2.7), we have $\Theta^{(a,l)}_{i,i',n}\leqslant C2^{-nH(4a-2l)}$. So, by combining (7.4) with (7.5), for $l=0$, we have

$$\Upsilon_n^{(0,a)}(t)\leqslant C2^{2nHa-\frac{n}{2}}\sum_{i,i'=-p2^{\frac{n}{2}}}^{p2^{\frac{n}{2}}-1}(2^{-4nHa}(p^{\frac{1}{4}}+1)^2n^22^{-\frac{n}{2}})\leqslant Cp(p^{\frac{1}{4}}+1))^2n^22^{-2nHa}, \tag{7.6}$$

for $l\neq 0$, we have

$$\Upsilon_n^{(l,a)}(t)\leqslant C\left(p^{\frac{1}{4}}+1\right)^2n^22^{2nHa-n}\left(2^{-nH(4a-2l)}\sum_{i,i'=-p2^{\frac{n}{2}}}^{p2^{\frac{n}{2}}-1}\left|\left\langle\delta_{(i+1)2^{-n/2}},\delta_{(i'+1)2^{-n/2}}\right\rangle\right|^l\right).$$

By the same arguments that has been used in the proof of (2.10), one can prove that for $H<1-\frac{1}{2l}$, we have

$$\sum_{i,i'=-p2^{\frac{n}{2}}}^{p2^{\frac{n}{2}}-1}\left|\left\langle\delta_{(i+1)2^{-n/2}},\delta_{(i'+1)2^{-n/2}}\right\rangle\right|^l\leqslant C_{H,l}\,p\,2^{n\left(\frac{1}{2}-lH\right)}. \tag{7.7}$$

For $H=\frac{1}{2}$, thanks to (2.17) and to the discussion of the case $H=\frac{1}{2}$ after (2.18), we have

$$\sum_{i,i'=-p2^{\frac{n}{2}}}^{p2^{\frac{n}{2}}-1}|\langle\delta_{(i+1)2^{-n/2}},\delta_{(i'+1)2^{-n/2}}\rangle|\leqslant 2^{-\frac{n}{2}}(p2^{\frac{n}{2}}-(-p2^{\frac{n}{2}}))=2p,$$

thus, (7.7) holds true for $l=1$ and $H=\frac{1}{2}$. So, since $H\leqslant\frac{1}{2}$, we deduce that

$$\begin{aligned}\sum_{l=1}^{2a}\Upsilon_n^{(l,a)}(t)&\leqslant Cp\left(p^{\frac{1}{4}}+1\right)^2n^2\sum_{l=1}^{2a}2^{2nHa-n}\left(2^{-nH(4a-2l)}2^{n\left(\frac{1}{2}-lH\right)}\right)\\&=Cp\left(p^{\frac{1}{4}}+1\right)^2n^22^{-\frac{n}{2}}\sum_{l=1}^{2a}2^{-nH(2a-l)}.\end{aligned} \tag{7.8}$$

By combining (7.4) with (7.6) and (7.8), we deduce that (7.3) holds true for $H\leqslant\frac{1}{2}$.

(2) $2^{-\frac{n}{4}}B_{m,n,p}^{(2r)}(t)\xrightarrow{L^2}0$ as $m\to\infty$, uniformly on n :

Using (7.1), we get

$$\begin{aligned}B_{m,n,p}^{(2r)}(t)&=\sum_{a=1}^{r}b_{2r,a}\sum_{-p2^{m/2}+1\leqslant j\leqslant p2^{m/2}}\sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1}\Delta_{i,n}f(X)H_{2a}\left(X_{i+1}^{(n)}-X_i^{(n)}\right)\\&\quad\times\left(L_t^{i2^{-n/2}}(Y)-L_t^{j2^{-m/2}}(Y)\right)\\&=\sum_{a=1}^{r}b_{2r,a}B_{m,n,p,a}^{(2r)}(t),\end{aligned} \tag{7.9}$$

with obvious notation at the last line. It suffices to prove that for any fixed p and for all $a \in \{1, \ldots, r\}$

$$2^{-\frac{n}{4}} B_{m,n,p,a}^{(2r)}(t) \underset{m\to\infty}{\overset{L^2}{\longrightarrow}} 0, \tag{7.10}$$

uniformly on n. By the same arguments that has been used to prove (7.4), we get

$$\begin{aligned} &E\left[\left(2^{-\frac{n}{4}} B_{m,n,p,a}^{(2r)}(t)\right)^2\right] \\ &\leqslant 2^{2nHa-\frac{n}{2}} \sum_{l=0}^{2a} l! \binom{2a}{l}^2 \sum_{-p2^{m/2}+1\leqslant j,j'\leqslant p2^{m/2}} \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} \sum_{i'=(j'-1)2^{\frac{n-m}{2}}}^{j'2^{\frac{n-m}{2}}-1} \left|E\left[\left\langle D^{4a-2l}(\phi_n(i,i')) \right.\right.\right. , \\ &\left.\left.\left.\delta_{(i+1)2^{-n/2}}^{\otimes 2a-l} \otimes \delta_{(i'+1)2^{-n/2}}^{\otimes 2a-l}\right\rangle\right]\right| \left|\langle \delta_{(i+1)2^{-n/2}}, \delta_{(i'+1)2^{-n/2}}\rangle\right|^l \left|E\left[\left(L_t^{i2^{-n/2}}(Y) - L_t^{j2^{-m/2}}(Y)\right)\right.\right. \\ &\left.\left.\times \left(L_t^{i'2^{-n/2}}(Y) - L_t^{j'2^{-m/2}}(Y)\right)\right]\right|, \end{aligned}$$

by Proposition 2.3 (point 2) and Cauchy–Schwarz, we have

$$\begin{aligned} &\left|E\left[\left(L_t^{i2^{-n/2}}(Y) - L_t^{j2^{-m/2}}(Y)\right)\left(L_t^{i'2^{-n/2}}(Y) - L_t^{j'2^{-m/2}}(Y)\right)\right]\right| \\ &\quad\leqslant \mu^2\sqrt{t}\sqrt{|i2^{-n/2} - j2^{-m/2}||i'2^{-n/2} - j'2^{-m/2}|} \leqslant \mu^2\sqrt{t}2^{-m/2}. \end{aligned}$$

So, we deduce that

$$\begin{aligned} &E\left[\left(2^{-\frac{n}{4}} B_{m,n,p,a}^{(2r)}(t)\right)^2\right] \leqslant C2^{-\frac{m}{2}} 2^{2nHa-\frac{n}{2}} \sum_{l=0}^{2a} l! \binom{2a}{l}^2 \sum_{-p2^{m/2}+1\leqslant j,j'\leqslant p2^{m/2}} \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} \\ &\sum_{i'=(j'-1)2^{\frac{n-m}{2}}}^{j'2^{\frac{n-m}{2}}-1} \left|E\left[\left\langle D^{4a-2l}(\phi_n(i,i')), \delta_{(i+1)2^{-n/2}}^{\otimes 2a-l} \otimes \delta_{(i'+1)2^{-n/2}}^{\otimes 2a-l}\right\rangle\right]\right| \left|\langle \delta_{(i+1)2^{-n/2}}, \delta_{(i'+1)2^{-n/2}}\rangle\right|^l \\ &= \sum_{l=0}^{2a} l! \binom{2a}{l}^2 \Lambda_{n,m}^{(l,a)}(t), \end{aligned} \tag{7.11}$$

by obvious notation at the last line.

By the same arguments that has been used in the proof of (7.3), we have, for $\frac{1}{4} < H \leqslant \frac{1}{2}$, and $l = 0$

$$\Lambda_{n,m}^{(0,a)}(t) \leqslant C2^{-\frac{m}{2}} 2^{2nHa-\frac{n}{2}} \sum_{i,i'=-p2^{\frac{n}{2}}}^{p2^{\frac{n}{2}}-1} \left(2^{-4nHa}\right) \leqslant Cp^2 2^{-\frac{m}{2}} 2^{-n\left(2Ha-\frac{1}{2}\right)} \leqslant Cp^2 2^{-\frac{m}{2}}, \tag{7.12}$$

for $l \neq 0$, we have

$$\Lambda_{n,m}^{(l,a)}(t) \leqslant C 2^{-\frac{m}{2}} 2^{2nHa-\frac{n}{2}} \left(2^{-nH(4a-2l)} \sum_{i,i'=-p2^{\frac{n}{2}}}^{p2^{\frac{n}{2}}-1} |\langle \delta_{(i+1)2^{-n/2}}, \delta_{(i'+1)2^{-n/2}} \rangle|^l \right).$$

So, thanks to (7.7), we deduce that

$$\sum_{l=1}^{2a} \Lambda_{n,m}^{(l,a)}(t) \leqslant C\, p\, 2^{-\frac{m}{2}} 2^{2nHa-\frac{n}{2}} \left(\sum_{l=1}^{2a} 2^{-nH(4a-2l)} 2^{n\left(\frac{1}{2}-lH\right)} \right)$$
$$= C\, p\, 2^{-\frac{m}{2}} \left(\sum_{l=1}^{2a} 2^{-nH(2a-l)} \right) \leqslant C\, p\, 2^{-\frac{m}{2}}. \tag{7.13}$$

By combining (7.11) with (7.12) and (7.13), we deduce that (7.10) holds true for $\frac{1}{4} < H \leqslant \frac{1}{2}$.

(3) $2^{-\frac{n}{4}} C_{m,n,p}^{(2r)}(t) \xrightarrow{L^2} 0$ as $n \to \infty$, then $m \to \infty$:

Using (7.1), we get

$$C_{m,n,p}^{(2r)}(t) = \sum_{a=1}^{r} b_{2r,a} \sum_{-p2^{m/2}+1 \leqslant j \leqslant p2^{m/2}} L_t^{j2^{-m/2}}(Y) \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} (\Delta_{i,n} f(X) - \Delta_{j,m} f(X))$$
$$\times H_{2a}\left(X_{i+1}^{(n)} - X_i^{(n)}\right)$$
$$= \sum_{a=1}^{r} b_{2r,a} \sum_{-p2^{m/2}+1 \leqslant j \leqslant p2^{m/2}} L_t^{j2^{-m/2}}(Y) \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} \frac{1}{2}\left(f\left(X_{i2^{-\frac{n}{2}}}\right) - f\left(X_{j2^{-\frac{m}{2}}}\right)\right)$$
$$\times H_{2a}\left(X_{i+1}^{(n)} - X_i^{(n)}\right)$$
$$+ \sum_{a=1}^{r} b_{2r,a} \sum_{-p2^{m/2}+1 \leqslant j \leqslant p2^{m/2}} L_t^{j2^{-m/2}}(Y) \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} \frac{1}{2}\left(f\left(X_{(i+1)2^{-\frac{n}{2}}}\right)\right.$$
$$\left. - f\left(X_{(j+1)2^{-\frac{m}{2}}}\right)\right) H_{2a}\left(X_{i+1}^{(n)} - X_i^{(n)}\right)$$
$$= \sum_{a=1}^{r} b_{2r,a} \left(C_{m,n,p,a}^{(1)}(t) + C_{m,n,p,a}^{(2)}(t) \right),$$

with obvious notation. It suffices to prove that for any fixed p and for all $a \in \{1, \ldots, r\}$

$$2^{-\frac{n}{4}} C_{m,n,p,a}^{(2)}(t) \xrightarrow{L^2} 0, \tag{7.14}$$

as $n \to \infty$, then $m \to \infty$. By obvious notation, we have

$$C^{(2)}_{m,n,p,a}(t) = \sum_{-p2^{m/2}+1 \leqslant j \leqslant p2^{m/2}} L_t^{j2^{-m/2}}(Y) \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} \Delta^{n,m}_{i,j} f(X) H_{2a}\left(X^{(n)}_{i+1} - X^{(n)}_i\right).$$

Thanks to the independence of X and Y, and to the first point of Proposition 2.3, we have

$$\begin{aligned}
E\left[\left(2^{-\frac{n}{4}} C^{(2)}_{m,n,p,a}(t)\right)^2\right] &= 2^{-\frac{n}{2}} \Bigg| \sum_{-p2^{m/2}+1 \leqslant j,j' \leqslant p2^{m/2}} E\left(L_t^{j2^{-m/2}}(Y) L_t^{j'2^{-m/2}}(Y)\right) \\
&\quad \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} \sum_{i'=(j'-1)2^{\frac{n-m}{2}}}^{j'2^{\frac{n-m}{2}}-1} E\left(\Delta^{n,m}_{i,j} f(X) \Delta^{n,m}_{i',j'} f(X) H_{2a}\left(X^{(n)}_{i+1} - X^{(n)}_i\right) H_{2a}\left(X^{(n)}_{i'+1} - X^{(n)}_{i'}\right)\right) \Bigg| \\
&\leqslant C 2^{-\frac{n}{2}} \sum_{-p2^{m/2}+1 \leqslant j,j' \leqslant p2^{m/2}} \\
&\quad \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} \sum_{i'=(j'-1)2^{\frac{n-m}{2}}}^{j'2^{\frac{n-m}{2}}-1} \left| E\left(\Delta^{n,m}_{i,j} f(X) \Delta^{n,m}_{i',j'} f(X) H_{2a}\left(X^{(n)}_{i+1} - X^{(n)}_i\right) H_{2a}\left(X^{(n)}_{i'+1} - X^{(n)}_{i'}\right)\right)\right|,
\end{aligned}$$

by the same arguments that has been used previously for several times, we deduce that

$$\begin{aligned}
E\left[\left(2^{-\frac{n}{4}} C^{(2)}_{m,n,p,a}(t)\right)^2\right] &\leqslant 2^{-n/2} 2^{2nHa} \sum_{-p2^{m/2}+1 \leqslant j,j' \leqslant p2^{m/2}} \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} \sum_{i'=(j'-1)2^{\frac{n-m}{2}}}^{j'2^{\frac{n-m}{2}}-1} \\
&\quad \sum_{l=0}^{2a} l! \binom{2a}{l}^2 \left| E\left[\Delta^{n,m}_{i,j} f(X) \Delta^{n,m}_{i',j'} f(X) I_{4a-2l}\left(\delta^{\otimes(2a-l)}_{(j+1)2^{-n/2}} \otimes \delta^{\otimes(2a-l)}_{(j'+1)2^{-n/2}}\right)\right]\right| \\
&\quad \times |\langle \delta_{(j+1)2^{-n/2}}; \delta_{(j'+1)2^{-n/2}} \rangle|^l \\
&= 2^{-n/2} 2^{2nHa} \sum_{l=0}^{2a} l! \binom{2a}{l}^2 O^l_{n,m}(t),
\end{aligned} \tag{7.15}$$

with obvious notation. Following the proof of (5.6), we get that

- If $l = 2a$ then the term $O^{2a}_{n,m}(t)$ in (7.15) can be bounded by

$$\frac{1}{4} \sup_{|x-y| \leqslant 2^{-m/2}} E\left(\left|f(X_x) - f(X_y)\right|^2\right) \sum_{i,i'=-p2^{\frac{n}{2}}}^{p2^{\frac{n}{2}}-1} \left|\left\langle \delta_{(i+1)2^{-n/2}}; \delta_{(i'+1)2^{-n/2}} \right\rangle\right|^{2a}.$$

Since $H \leqslant \frac{1}{2}$ and thanks to (7.7), observe that

$$O^{2a}_{n,m}(t) \leqslant C\, p\, 2^{n\left(\frac{1}{2}-2Ha\right)} \sup_{|x-y|\leqslant 2^{-m/2}} E\left(\left|f(X_x)-f(X_y)\right|^2\right). \tag{7.16}$$

- If $1 \leqslant l \leqslant 2a-1$ then, by (7.7) among other things used in the proof of (5.6), we have

$$\begin{aligned} O^{l}_{n,m}(t) &\leqslant C\left(2^{-nH}\right)^{(4a-2l)} \sum_{i,i'=-p2^{\frac{n}{2}}}^{p2^{\frac{n}{2}}-1} |\langle \delta_{(i+1)2^{-n/2}}; \delta_{(i'+1)2^{-n/2}}\rangle|^l \\ &\leqslant C\, p\, 2^{-nH(4a-2l)}\, 2^{n\left(\frac{1}{2}-lH\right)}. \end{aligned} \tag{7.17}$$

- If $l = 0$ then

$$O^{0}_{n,m}(t) \leqslant C\left(2^{-nH}\right)^{4a}\left(2p2^{\frac{n}{2}}\right)^2 \leqslant C\, p^2\, 2^{-4nHa} 2^n. \tag{7.18}$$

By combining (7.15) with (7.16), (7.17) and (7.18), we get

$$\begin{aligned} E[(2^{-\frac{n}{4}} C^{(2)}_{m,n,p,a}(t))^2] \leqslant C\Bigg(&\sup_{|x-y|\leqslant 2^{-m/2}} E(|f(X_x)-f(X_y)|^2) + p\left(\sum_{l=1}^{2a-1} 2^{-nH(2a-l)}\right) \\ &+ p^2 2^{-n\left(2Ha-\frac{1}{2}\right)}\Bigg), \end{aligned}$$

it is then clear that, since $\frac{1}{4} < H \leqslant \frac{1}{2}$, the last quantity converges to 0 as $n \to \infty$ and then $m \to \infty$. Finally, we have proved that (7.14) holds true.

(4) $2^{-\frac{n}{4}} E^{(2r)}_{n,p}(t) \xrightarrow{L^2} 0$ as $p \to \infty$, uniformly on n :
Using (7.1), we get

$$\begin{aligned} E^{(2r)}_{n,p}(t) &= \sum_{a=1}^{r} b_{2r,a}\Bigg(\sum_{i\geqslant p2^{n/2}} \Delta_{i,n} f(X) H_{2a}\left(X^{(n)}_{i+1}-X^{(n)}_i\right)\mathcal{L}_{i,n}(t) \\ &\quad + \sum_{i<-p2^{n/2}} \Delta_{i,n} f(X) H_{2a}\left(X^{(n)}_{i+1}-X^{(n)}_i\right)\mathcal{L}_{i,n}(t)\Bigg) \\ &= \sum_{a=1}^{r} b_{2r,a} E^{(2r)}_{n,p,a}(t), \end{aligned} \tag{7.19}$$

with obvious notation at the last line. It suffices to prove that for all $a \in \{1,\ldots,r\}$

$$2^{-\frac{n}{4}} E^{(2r)}_{n,p,a}(t) \xrightarrow[p\to\infty]{L^2} 0, \tag{7.20}$$

uniformly on n. By the same arguments that has been used previously, we have

$$\begin{aligned} &E\left[\left(2^{-\frac{n}{4}}E_{n,p,a}^{(2r)}(t)\right)^2\right] \\ &\leqslant 22^{2nHa-\frac{n}{2}}\sum_{l=0}^{2a} l!\binom{2a}{l}^2 \sum_{i,i'\geqslant p2^{n/2}} \left|E\left[\left\langle D^{4a-2l}(\phi_n(i,i')), \delta_{(i+1)2^{-n/2}}^{\otimes 2a-l}\otimes\delta_{(i'+1)2^{-n/2}}^{\otimes 2a-l}\right\rangle\right]\right| \\ &\times\left|\left\langle\delta_{(i+1)2^{-n/2}},\delta_{(i'+1)2^{-n/2}}\right\rangle\right|^l \left|E\left[\mathcal{L}_{i,n}(t)\mathcal{L}_{i',n}(t)\right]\right| \\ &+22^{2nHa-\frac{n}{2}}\sum_{l=0}^{2a} l!\binom{2a}{l}^2 \sum_{i,i'<-p2^{n/2}} \left|E\left[\left\langle D^{4a-2l}(\phi_n(i,i')), \delta_{(i+1)2^{-n/2}}^{\otimes 2a-l}\otimes\delta_{(i'+1)2^{-n/2}}^{\otimes 2a-l}\right\rangle\right]\right| \\ &\times\left|\left\langle\delta_{(i+1)2^{-n/2}},\delta_{(i'+1)2^{-n/2}}\right\rangle\right|^l \left|E\left[\mathcal{L}_{i,n}(t)\mathcal{L}_{i',n}(t)\right]\right|. \end{aligned} \tag{7.21}$$

It suffices to prove the convergence to 0 of the quantity given in (7.21). We have,

$$\begin{aligned} &2^{2nHa-\frac{n}{2}}\sum_{l=0}^{2a} l!\binom{2a}{l}^2 \sum_{i,i'\geqslant p2^{n/2}} \left|E\left[\left\langle D^{4a-2l}(\phi_n(i,i')), \delta_{(i+1)2^{-n/2}}^{\otimes 2a-l}\otimes\delta_{(i'+1)2^{-n/2}}^{\otimes 2a-l}\right\rangle\right]\right| \\ &\times|\langle\delta_{(i+1)2^{-n/2}},\delta_{(i'+1)2^{-n/2}}\rangle|^l \left|E[\mathcal{L}_{i,n}(t)\mathcal{L}_{i',n}(t)]\right| \\ &=\sum_{l=0}^{2a} l!\binom{2a}{l}^2\Omega_{n,p}^{(l,a)}(t), \end{aligned}$$

with obvious notation at the last line. It is enough to prove that, for all $l\in\{0,\ldots,2a\}$:

$$\Omega_{n,p}^{(l,a)}(t) \underset{p\to\infty}{\longrightarrow} 0, \tag{7.22}$$

uniformly on n. By the same arguments that has been used in the proof of (7.3), for $\frac{1}{4}<H\leqslant\frac{1}{2}$, we have

For $l=0$:

$$\Omega_{n,p}^{(0,a)}(t)\leqslant C2^{2nHa-\frac{n}{2}}2^{-4nHa}\sum_{i,i'\geqslant p2^{n/2}}\left|E[\mathcal{L}_{i,n}(t)\mathcal{L}_{i',n}(t)]\right|.$$

By the third point of Proposition 2.3, we have

$$\left|\mathcal{L}_{i,n}(t)\right|\leqslant L_t^{i2^{-n/2}}(Y)+2Kn2^{-n/4}\sqrt{L_t^{i2^{-n/2}}(Y)}$$

so that

$$E\left[\mathcal{L}_{i,n}(t)^2\right]\leqslant 2E\left[L_t^{i2^{-n/2}}(Y)^2\right]+8n^2 2^{-n/2}\|K^2\|_2\|L_t^{i2^{-n/2}}(Y)\|_2,$$

which implies

$$\left\|\mathcal{L}_{i,n}(t)\right\|_2 \leqslant C\left\|L_t^{i2^{-n/2}}(Y)\right\|_2 + Cn2^{-n/4}\left\|L_t^{i2^{-n/2}}(Y)\right\|_2^{\frac{1}{2}}. \tag{7.23}$$

On the other hand, thanks to the point 1 of Proposition 2.3, we have

$$E\left[L_t^{i2^{-n/2}}(Y)^2\right] \leqslant Ct\exp\left(-\frac{\left(i2^{-n/2}\right)^2}{2t}\right). \tag{7.24}$$

Consequently, we get

$$\left\|L_t^{i2^{-n/2}}(Y)\right\|_2 \leqslant Ct^{1/2}\exp\left(-\frac{\left(i2^{-n/2}\right)^2}{4t}\right). \tag{7.25}$$

By combining (7.23) with (7.24) and (7.25), we deduce that

$$\begin{aligned}\left\|\mathcal{L}_{i,n}(t)\right\|_2 &\leqslant C\exp\left(-\frac{\left(i2^{-n/2}\right)^2}{4t}\right) + Cn2^{-n/4}\exp\left(-\frac{\left(i2^{-n/2}\right)^2}{8t}\right)\\ &\leqslant C\exp\left(-\frac{\left(i2^{-n/2}\right)^2}{4t}\right) + C\exp\left(-\frac{\left(i2^{-n/2}\right)^2}{8t}\right).\end{aligned} \tag{7.26}$$

Observe that, by Cauchy–Schwarz inequality, we have

$$\Omega_{n,p}^{(0,a)}(t) \leqslant C\left(2^{-\frac{n}{2}}\sum_{i\geqslant p2^{n/2}}\left\|\mathcal{L}_{i,n}(t)\right\|_2\right)\left(2^{-2nHa}\sum_{i'\geqslant p2^{n/2}}\left\|\mathcal{L}_{i',n}(t)\right\|_2\right).$$

Thanks to (7.26), we get

$$\begin{aligned}2^{-\frac{n}{2}}\sum_{i\geqslant p2^{n/2}}\left\|\mathcal{L}_{i,n}(t)\right\|_2 \leqslant C2^{-n/2}&\sum_{i\geqslant p2^{n/2}}\exp\left(-\frac{\left(i2^{-n/2}\right)^2}{4t}\right)\\ &+ C2^{-n/2}\sum_{i\geqslant p2^{n/2}}\exp\left(-\frac{\left(i2^{-n/2}\right)^2}{8t}\right).\end{aligned}$$

But, for $k \in \{4, 8\}$,

$$2^{-n/2}\sum_{i\geqslant p2^{n/2}}\exp\left(-\frac{\left(i2^{-n/2}\right)^2}{kt}\right) \leqslant \int_{p-1}^{\infty}\exp\left(\frac{-x^2}{kt}\right)dx.$$

On the other hand, since $H > \frac{1}{4}$, we have

$$2^{-2nHa} \sum_{i' \geqslant p2^{n/2}} \left\| \mathcal{L}_{i',n}(t) \right\|_2 \leqslant 2^{-n\left(2Ha-\frac{1}{2}\right)} 2^{-\frac{n}{2}} \sum_{i' \geqslant p2^{n/2}} \left\| \mathcal{L}_{i',n}(t) \right\|_2$$
$$\leqslant C 2^{-\frac{n}{2}} \sum_{i' \geqslant p2^{n/2}} \left\| \mathcal{L}_{i',n}(t) \right\|_2$$

Finally, we deduce that

$$\Omega_{n,p}^{(0,a)}(t) \leqslant C \left(\int_{p-1}^{\infty} \exp\left(\frac{-x^2}{4t}\right) dx + \int_{p-1}^{\infty} \exp\left(\frac{-x^2}{8t}\right) dx \right)^2 \underset{p\to\infty}{\longrightarrow} 0, \tag{7.27}$$

uniformly on n.

For $l \neq 0$: By the same arguments that has been used in the proof of (7.3) and thanks to (2.17), the Cauchy–Schwarz inequality and (7.26), we have

$$\begin{aligned}
\Omega_{n,p}^{(l,a)}(t) &\leqslant C 2^{2nHa-\frac{n}{2}} \left(2^{-nH(4a-2l)} \sum_{i,i' \geqslant p2^{n/2}} \left| \langle \delta_{(i+1)2^{-n/2}}, \delta_{(i'+1)2^{-n/2}} \rangle \right|^l \left| E\left[\mathcal{L}_{i,n}(t) \mathcal{L}_{i',n}(t) \right] \right| \right) \\
&\leqslant C 2^{2nHa-\frac{n}{2}} 2^{-nH(4a-2l)} 2^{-nHl} \left(\sum_{i,i' \geqslant p2^{n/2}} \left| \rho(i-i') \right|^l \left\| \mathcal{L}_{i,n}(t) \right\|_2 \left\| \mathcal{L}_{i',n}(t) \right\|_2 \right) \\
&\leqslant C 2^{-nH(2a-l)} \left(2^{-\frac{n}{2}} \sum_{i \geqslant p2^{n/2}} \left\| \mathcal{L}_{i,n}(t) \right\|_2 \right) \left(\sum_{a\in\mathbb{Z}} |\rho(a)|^l \right) \\
&\leqslant C 2^{-\frac{n}{2}} \sum_{i \geqslant p2^{n/2}} \left\| \mathcal{L}_{i,n}(t) \right\|_2 \\
&\leqslant C \left(\int_{p-1}^{\infty} \exp\left(\frac{-x^2}{4t}\right) dx + \int_{p-1}^{\infty} \exp\left(\frac{-x^2}{8t}\right) dx \right) \underset{p\to\infty}{\longrightarrow} 0,
\end{aligned} \tag{7.28}$$

uniformly on n, and we have the fourth inequality because , since $H \leqslant \frac{1}{2} \leqslant 1 - \frac{1}{2l}$, $\sum_{a\in\mathbb{Z}} |\rho(a)|^l < \infty$. By combining (7.27) and (7.28), we deduce that (7.22) holds true for $\frac{1}{4} < H \leqslant \frac{1}{2}$.

(5) The convergence in law of $D_{m,n,p}^{(2r)}(t)$ as $n \to \infty$, then $m \to \infty$, then $p \to \infty$:

Let us prove that

$$\left(2^{-\frac{n}{4}} D_{m,n,p}^{(2r)}(t)\right)_{t\geqslant 0} \overset{f.d.d.}{\longrightarrow} \left(\gamma_{2r} \int_{-\infty}^{+\infty} f(X_s) L_t^s(Y) dW_s\right)_{t\geqslant 0}, \tag{7.29}$$

as $n \to \infty$, then $m \to \infty$, then $p \to \infty$, where γ_{2r} and $\int_{-\infty}^{+\infty} f(X_s) L_t^s(Y) dW_s$ are defined in the point (3) of Theorem 1.1. In fact, using the decomposition (7.1), we have

$$
\begin{aligned}
2^{-\frac{n}{4}} D_{m,n,p}^{(2r)}(t) &= 2^{-\frac{n}{4}} \sum_{-p2^{m/2}+1 \leqslant j \leqslant p2^{m/2}} \Delta_{j,m} f(X) L_t^{j2^{-m/2}}(Y) \\
&\quad \times \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} \left[\left(X_{i+1}^{(n)} - X_i^{(n)} \right)^{2r} - \mu_{2r} \right] \\
&= 2^{-\frac{n}{4}} \sum_{-p2^{m/2}+1 \leqslant j \leqslant p2^{m/2}} \Delta_{j,m} f(X) L_t^{j2^{-m/2}}(Y) \\
&\quad \times \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} \sum_{a=1}^{r} b_{2r,a} H_{2a} \left(X_{i+1}^{(n)} - X_i^{(n)} \right).
\end{aligned}
$$

It was been proved in (3.27) in [9] that

$$
\left(2^{-n/4} \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} H_{2a}\left(X_{i+1}^{(n)} - X_i^{(n)}\right),\ 1 \leqslant a \leqslant r : -p2^{m/2}+1 \leqslant j \leqslant p2^{m/2} \right) \xrightarrow{\text{law}}
$$
$$
\left(\alpha_{2a} \left(B_{(j+1)2^{-m/2}}^{(a)} - B_{j2^{-m/2}}^{(a)} \right),\ 1 \leqslant a \leqslant r : -p2^{m/2}+1 \leqslant j \leqslant p2^{m/2} \right)
$$

where $(B^{(1)}, \ldots, B^{(r)})$ is a r-dimensional two-sided Brownian motion and α_{2a} is defined in (2.18). Since for any $x \in \mathbb{R}$, $E[X_x H_{2a}(X_{j+1}^{n,\pm} - X_j^{n,\pm})] = 0$ (Hermite polynomials of different orders are orthogonal), and thanks to the independence between X and Y, Peccati–Tudor Theorem (see, e.g., [6, Theorem 6.2.3]) applies and yields

$$
\left(X_x, Y_y, 2^{-n/4} \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} H_{2a}\left(X_{i+1}^{(n)} - X_i^{(n)}\right),\ 1 \leqslant a \leqslant r : -p2^{m/2}+1 \leqslant j \leqslant p2^{m/2} \right)_{x,y \in \mathbb{R}} \xrightarrow{\text{f.d.d.}}
$$
$$
\left(X_x, Y_y, \alpha_{2a} \left(B_{(j+1)2^{-m/2}}^{(a)} - B_{j2^{-m/2}}^{(a)} \right),\ 1 \leqslant a \leqslant r : -p2^{m/2}+1 \leqslant j \leqslant p2^{m/2} \right)_{x,y \in \mathbb{R}}
$$

where $(B^{(1)}, \ldots, B^{(r)})$ is a r-dimensional two-sided Brownian motion independent of X and Y. Hence, for any fixed m and p, we have

$$
\begin{aligned}
&\left(2^{-\frac{n}{4}} D_{m,n,p}^{(2r)}(t) \right)_{t \geqslant 0} \underset{n \to \infty}{\xrightarrow{f.d.d.}} \gamma_{2r} \\
&\quad \times \left(\sum_{-p2^{m/2}+1 \leqslant j \leqslant p2^{m/2}} \Delta_{j,m} f(X) L_t^{j2^{-m/2}}(Y) \left(W_{(j+1)2^{-m/2}} - W_{j2^{-m/2}} \right) \right)_{t \geqslant 0},
\end{aligned}
\tag{7.30}
$$

where $\gamma_{2r} := \sqrt{\sum_{a=1}^{r} b_{2r,a}^2 \alpha_{2a}^2}$ and W is a two-sided Brownian motion. Fix $t \geqslant 0$, observe that

$$\sum_{-p2^{m/2}+1\leqslant j\leqslant p2^{m/2}} \Delta_{j,m} f(X) L_t^{j2^{-m/2}}(Y)\left(W_{(j+1)2^{-m/2}} - W_{j2^{-m/2}}\right)$$
$$= \sum_{-p2^{m/2}+1\leqslant j\leqslant p2^{m/2}} f\left(X_{j2^{-\frac{m}{2}}}\right) L_t^{j2^{-m/2}}(Y)\left(W_{(j+1)2^{-m/2}} - W_{j2^{-m/2}}\right)$$
$$+ \sum_{-p2^{m/2}+1\leqslant j\leqslant p2^{m/2}} \frac{1}{2}\left(f\left(X_{(j+1)2^{-\frac{m}{2}}}\right)\right.$$
$$\left. - f\left(X_{j2^{-\frac{m}{2}}}\right)\right) L_t^{j2^{-m/2}}(Y)\left(W_{(j+1)2^{-m/2}} - W_{j2^{-m/2}}\right)$$
$$= \sum_{-p2^{m/2}+1\leqslant j\leqslant p2^{m/2}} f\left(X_{j2^{-\frac{m}{2}}}\right) L_t^{j2^{-m/2}}(Y)\left(W_{(j+1)2^{-m/2}} - W_{j2^{-m/2}}\right)$$
$$+ N_{m,p}(t), \tag{7.31}$$

with obvious notation at the last line. Since $E[\int_{-\infty}^{+\infty} \left(f(X_s)L_t^s(Y)\right)^2 ds] \leqslant C \int_{-\infty}^{+\infty} E[(L_t^s(Y))^2]ds \leqslant C \int_{-\infty}^{+\infty} \exp\left(\frac{-s^2}{2t}\right)ds < \infty$, where we have the second inequality by the point 1 of Proposition 2.3, and thanks to the independence between (X, Y) and W and the a.s. continuity of $s \to f(X_s)$ and $s \to L_t^s(Y)$, we deduce that

$$\sum_{-p2^{m/2}+1\leqslant j\leqslant p2^{m/2}} f\left(X_{j2^{-\frac{m}{2}}}\right) L_t^{j2^{-m/2}}(Y)\left(W_{(j+1)2^{-m/2}} - W_{j2^{-m/2}}\right)$$
$$\xrightarrow[m\to\infty]{L^2} \int_{-p}^{+p} f(X_s)L_t^s(Y)dW_s \xrightarrow[p\to\infty]{L^2} \int_{-\infty}^{+\infty} f(X_s)L_t^s(Y)dW_s. \tag{7.32}$$

Now, let us prove that, for any fixed p,

$$N_{m,p}(t) \xrightarrow[m\to\infty]{L^2} 0. \tag{7.33}$$

In fact, since $f(X_{(j+1)2^{-\frac{m}{2}}}) - f(X_{j2^{-\frac{m}{2}}}) = f'(X_{\theta_j})(X_{(j+1)2^{-\frac{m}{2}}} - X_{j2^{-\frac{m}{2}}})$ where θ_j is a random real number satisfying $j2^{-\frac{m}{2}} < \theta_j < (j+1)2^{-\frac{m}{2}}$, and thanks to the independence of X , Y and W, the independence of the increments of W, and the point 1 of Proposition 2.3, we have

$$E\left[(N_{m,p}(t))^2\right] = \frac{1}{4} \sum_{-p2^{m/2}+1\leqslant j,j'\leqslant p2^{m/2}} E\left[\left(f\left(X_{(j+1)2^{-\frac{m}{2}}}\right) - f\left(X_{j2^{-\frac{m}{2}}}\right)\right)\right.$$
$$\left.\times\left(f\left(X_{(j'+1)2^{-\frac{m}{2}}}\right) - f\left(X_{j'2^{-\frac{m}{2}}}\right)\right) L_t^{j2^{-m/2}}(Y) L_t^{j'2^{-m/2}}(Y)\right]$$
$$\times E\left[\left(W_{(j+1)2^{-m/2}} - W_{j2^{-m/2}}\right)\left(W_{(j'+1)2^{-m/2}} - W_{j'2^{-m/2}}\right)\right]$$
$$= \frac{2^{-\frac{m}{2}}}{4} \sum_{-p2^{m/2}+1\leqslant j\leqslant p2^{m/2}} E\left[\left(f'(X_{\theta_j})\left(X_{(j+1)2^{-\frac{m}{2}}} - X_{j2^{-\frac{m}{2}}}\right)\right)^2\right.$$

$$
\begin{aligned}
&\times E\left[\left(L_t^{j2^{-m/2}}(Y)\right)^2\right] \\
&\leqslant C2^{-\frac{m}{2}} \sum_{-p2^{m/2}+1\leqslant j\leqslant p2^{m/2}} E\left[\left(X_{(j+1)2^{-\frac{m}{2}}} - X_{j2^{-\frac{m}{2}}}\right)^2\right] \\
&= C2^{-mH}2^{-\frac{m}{2}}2p2^{\frac{m}{2}} = Cp2^{-mH} \underset{m\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

Thus (7.33) holds true. Thanks to (7.30), (7.31), (7.32) and (7.33), we deduce that (7.29) holds true.

Finally, by combining (7.3) with (7.10), (7.14), (7.20) and (7.29), we deduce that (1.9) holds true.

8 Proof of Lemma 2.1

1. We have, $\langle \varepsilon_u^{\otimes q}, \delta_{(j+1)2^{-n/2}}^{\otimes q}\rangle_{\mathscr{H}^{\otimes q}} = \langle \varepsilon_u, \delta_{(j+1)2^{-n/2}}\rangle_{\mathscr{H}}^q$. Thanks to (2.1), we have

$$\langle \varepsilon_u, \delta_{(j+1)2^{-n/2}}\rangle_{\mathscr{H}} = E\left(X_u\left(X_{(j+1)2^{-n/2}} - X_{j2^{-n/2}}\right)\right).$$

Observe that, for all $0 \leqslant s \leqslant t$ and $u \in \mathbb{R}$,

$$E\left(X_u(X_t - X_s)\right) = \frac{1}{2}\left(t^{2H} - s^{2H}\right) + \frac{1}{2}\left(|s-u|^{2H} - |t-u|^{2H}\right).$$

Since for $H \leqslant 1/2$ one has $|b^{2H} - a^{2H}| \leqslant |b-a|^{2H}$ for any $a, b \in \mathbb{R}_+$, we immediately deduce (2.7).

2. By (2.1), for all $j, j' \in \{0, \ldots, \lfloor 2^{n/2}t\rfloor - 1\}$,

$$
\begin{aligned}
&\left|\langle \varepsilon_{j2^{-n/2}}, \delta_{(j'+1)2^{-n/2}}\rangle_{\mathscr{H}}\right| = \left|E\left[X_{j2^{-n/2}}\left(X_{(j'+1)2^{-n/2}} - X_{j'2^{-n/2}}\right)\right]\right| \\
&\quad = \left|2^{-nH-1}\left(|j'+1|^{2H} - |j'|^{2H}\right) + 2^{-nH-1}\left(|j-j'|^{2H} - |j-j'-1|^{2H}\right)\right| \\
&\quad \leqslant 2^{-nH-1}\left||j'+1|^{2H} - |j'|^{2H}\right| + 2^{-nH-1}\left||j-j'|^{2H} - |j-j'-1|^{2H}\right|.
\end{aligned} \tag{8.1}
$$

We consider the function $f : [a,b] \to \mathbb{R}$ defined by

$$f(x) = |x|^{2H}.$$

Applying the mean value theorem to f, we have that

$$\left||b|^{2H} - |a|^{2H}\right| \leqslant 2H\left(|a| \vee |b|\right)^{2H-1}|b-a| \leqslant 2\left(|a| \vee |b|\right)^{2H-1}|b-a|. \tag{8.2}$$

We deduce from (8.2) that

$$2^{-nH-1}\left|\left|j'+1\right|^{2H}-\left|j'\right|^{2H}\right|\leqslant 2^{-nH}\left|j'+1\right|^{2H-1}$$
$$\leqslant 2^{-nH}\left|\lfloor 2^{n/2}t\rfloor\right|^{2H-1}\leqslant 2^{-n/2}t^{2H-1},$$

similarly we have,

$$2^{-nH-1}\left|\left|j-j'\right|^{2H}-\left|j-j'-1\right|^{2H}\right|\leqslant 2^{-nH}\left|\lfloor 2^{n/2}t\rfloor\right|^{2H-1}\leqslant 2^{-n/2}t^{2H-1}.$$

Combining the last two inequalities with (8.1), and since $\langle \varepsilon_{j2^{-n/2}}^{\otimes q}, \delta_{(j'+1)2^{-n/2}}^{\otimes q}\rangle_{\mathscr{H}^{\otimes q}} = \langle \varepsilon_{j2^{-n/2}}, \delta_{(j'+1)2^{-n/2}}\rangle_{\mathscr{H}}^{q}$, we deduce that (2.8) holds true. The proof of (2.9) may be done similarly.

3. By (2.1) we have

$$\left|\langle \delta_{(k+1)2^{-n/2}};\delta_{(l+1)2^{-n/2}}\rangle_{\mathscr{H}}\right|^{r}=\left|E\left[\left(X_{(k+1)2^{-n/2}}-X_{k2^{-n/2}}\right)\left(X_{(l+1)2^{-n/2}}-X_{l2^{-n/2}}\right)\right]\right|^{r}$$
$$=\left|2^{-nH-1}\left(\left|k-l+1\right|^{2H}+\left|k-l-1\right|^{2H}-2\left|k-l\right|^{2H}\right)\right|^{r}=2^{-nrH}\left|\rho(k-l)\right|^{r},$$

where we have the last equality by the notation (2.16). So, we deduce that

$$\sum_{k,l=0}^{\lfloor 2^{n/2}t\rfloor-1}\left|\left\langle \delta_{(k+1)2^{-n/2}};\delta_{(l+1)2^{-n/2}}\right\rangle_{\mathscr{H}}\right|^{r}=2^{-nrH}\sum_{k,l=0}^{\lfloor 2^{n/2}t\rfloor-1}\left|\rho(k-l)\right|^{r}$$
$$=2^{-nrH}\sum_{k=0}^{\lfloor 2^{n/2}t\rfloor-1}\sum_{p=k-\lfloor 2^{n/2}t\rfloor+1}^{k}\left|\rho(p)\right|^{r}$$
$$=2^{-nrH}\sum_{p=1-\lfloor 2^{n/2}t\rfloor}^{\lfloor 2^{n/2}t\rfloor-1}\left|\rho(p)\right|^{r}\left(\left(p+\lfloor 2^{n/2}t\rfloor\right)\wedge\lfloor 2^{n/2}t\rfloor-p\vee 0\right)$$
$$\leqslant 2^{-nrH}\lfloor 2^{n/2}t\rfloor\sum_{p=1-\lfloor 2^{n/2}t\rfloor}^{\lfloor 2^{n/2}t\rfloor-1}\left|\rho(p)\right|^{r}\leqslant 2^{n\left(\frac{1}{2}-rH\right)}t\sum_{p=1-\lfloor 2^{n/2}t\rfloor}^{\lfloor 2^{n/2}t\rfloor-1}\left|\rho(p)\right|^{r},\tag{8.3}$$

where we have the second equality by the change of variable $p=k-l$ and the third equality by a Fubini argument. Observe that $|\rho(p)|^r\sim(H(2H-1))^r p^{(2H-2)r}$ as $p\to+\infty$. So, we deduce that

(a) if $H<1-\frac{1}{2r}$: $\sum_{p\in\mathbb{Z}}|\rho(p)|^r<\infty$, by combining this fact with (8.3) we deduce that (2.10) holds true.

(b) If $H=1-\frac{1}{2r}$: $|\rho(p)|^r\sim\frac{(H(2H-1))^r}{p}$ as $p\to+\infty$. So, we deduce that there exists a constant $C_{H,r}>0$ independent of n and t such that for all integer

$n \geqslant 1$ and all $t \in \mathbb{R}_+$

$$\sum_{p=1-\lfloor 2^{n/2}t\rfloor}^{\lfloor 2^{n/2}t\rfloor-1} \left|\rho(p)\right|^r \leqslant C_{H,r}\left(1+\sum_{p=2}^{\lfloor 2^{n/2}t\rfloor}\frac{1}{p}\right) \leqslant C_{H,r}\left(1+\int_1^{2^{n/2}t}\frac{1}{x}dx\right)$$
$$= C_{H,r}\left(1+\frac{n\log(2)}{2}+\log(t)\right) \leqslant C_{H,r}\big(1+n+t\big).$$

By combining this last inequality with (8.3) we deduce that (2.11) holds true.

(c) If $H > 1-\frac{1}{2r}$: $|\rho(p)|^r \sim \frac{(H(2H-1))^r}{p^{(2-2H)r}}$ as $p \to +\infty$ where $0 < (2-2H)r < 1$. So, we deduce that there exists a constant $K_{H,r} > 0$ independent of n and t such that for all integer $n \geqslant 1$ and all $t \in \mathbb{R}_+$

$$\sum_{p=1-\lfloor 2^{n/2}t\rfloor}^{\lfloor 2^{n/2}t\rfloor-1} \left|\rho(p)\right|^r \leqslant K_{H,r}\left(1+\sum_{p=1}^{\lfloor 2^{n/2}t\rfloor}\frac{1}{p^{(2-2H)r}}\right)$$
$$\leqslant K_{H,r}\left(1+\int_0^{2^{n/2}t}\frac{1}{x^{(2-2H)r}}dx\right)$$
$$= K_{H,r}\left(1+\frac{2^{\frac{n}{2}\left(1-(2-2H)r\right)}t^{1-(2-2H)r}}{1-(2-2H)r}\right)$$
$$\leqslant C_{H,r}\big(1+2^{\frac{n}{2}\left(1-(2-2H)r\right)}t^{1-(2-2H)r}\big),$$

where $C_{H,r} = K_{H,r} \vee \frac{K_{H,r}}{1-(2-2H)r}$. By combining the last inequality with (8.3) we deduce that (2.12) holds true.

4. As it has been proved in (8.1), we have

$$\left|\langle \varepsilon_{k2^{-n/2}}, \delta_{(l+1)2^{-n/2}}\rangle_{\mathscr{H}}\right| = \left|E\big[X_{k2^{-n/2}}\big(X_{(l+1)2^{-n/2}} - X_{l2^{-n/2}}\big)\big]\right|$$
$$\leqslant 2^{-nH-1}\left|\left|l+1\right|^{2H} - \left|l\right|^{2H}\right| + 2^{-nH-1}\left|\left|k-l\right|^{2H} - \left|k-l-1\right|^{2H}\right|,$$

so, by a telescoping argument we get

$$\sum_{k,l=0}^{\lfloor 2^{n/2}t\rfloor-1} \left|\langle \varepsilon_{k2^{-n/2}}; \delta_{(l+1)2^{-n/2}}\rangle_{\mathscr{H}}\right|$$
$$\leqslant 2^{\frac{n}{2}-1}t^{2H+1} + 2^{-nH-1}\sum_{k,l=0}^{\lfloor 2^{n/2}t\rfloor-1}\left|\left|k-l\right|^{2H} - \left|k-l-1\right|^{2H}\right|, \quad (8.4)$$

by using the change of variable $p = k-l$ and a Fubini argument, among other things that has been used in the previous proof, we deduce that

$$2^{-nH-1}\sum_{k,l=0}^{\lfloor 2^{n/2}t\rfloor-1}\left||k-l|^{2H}-|k-l-1|^{2H}\right|\leqslant 2^{\frac{n}{2}}t^{2H+1}.$$

By combining this last inequality with (8.4) we deduce that (2.13) holds true. The proof of (2.14) may be done similarly.

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