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A General Darling–Erdős Theorem in Euclidean Space

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Abstract We provide an improved version of the Darling–Erdős theorem for sums of i.i.d. random variables with mean zero and finite variance. We extend this result to multidimensional random vectors. Our proof is based on a new strong invariance principle in this setting which has other applications as well such as an integral test refinement of the multidimensional Hartman–Wintner LIL. We also identify a borderline situation where one has weak convergence to a shifted version of the standard limiting distribution in the classical Darling–Erdős theorem.

Keywords Darling–Erdős theorem · Extreme value distribution · Hartman–Wintner LIL · Integral test · Strong invariance principle · Multidimensional version · Double truncation

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1 Introduction

Let $X, X_1, X_2, \ldots$ be i.i.d. random variables and set $S_n = \sum_{j=1}^{n} X_j, n \geq 1$. Further set $Lt = \log_e(t \vee e), LLt = L(Lt)$ and $LLLt = L(LLt), t \geq 0$. In 1956, Darling and Erdős proved that under the assumption $\mathbb{E}|X|^3 < \infty, \mathbb{E}X^2 = 1$ and $\mathbb{E}X = 0$, the following convergence in distribution result holds,

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$$a_n \max_{1 \leq k \leq n} |S_k|/\sqrt{k} - b_n \overset{d}{\to} \tilde{Y}, \tag{1.1}$$

where $a_n = \sqrt{2LLn}, b_n = 2LLn + LLLn/2 - \log(\pi)/2$ and $\tilde{Y}$ is a random variable which has an extreme value distribution with distribution function $y \mapsto \exp(-\exp(-y))$.

The above third moment assumption was later relaxed in [14,16] to $\mathbb{E}|X|^{2+\delta} < \infty$ for some $\delta > 0$, but the question remained open whether a finite second moment would be already sufficient.

This was finally answered in [6], where it is shown that (1.1) holds if and only if

$$\mathbb{E}X^2 I\{|X| \geq t\} = o\left((LLt)^{-1}\right) \text{ as } t \to \infty.$$

Moreover, it is shown in [6] that the above result holds more generally under the assumption of a finite second moment if one replaces the normalizers $\sqrt{k}$ by $\sqrt{B_k}$, where $B_n = \sum_{j=1}^{n} \sigma_j^2$ and

$$\sigma_n^2 := \mathbb{E}X^2 I\left\{|X| \leq \sqrt{n}/(LLn)^p\right\}$$

for some $p \geq 2$. So we have under the classical assumption $\mathbb{E}X^2 < \infty$ and $\mathbb{E}X = 0$,

$$a_n \max_{1 \leq k \leq n} |S_k|/\sqrt{B_k} - b_n \overset{d}{\to} \tilde{Y}.$$

For some further related work on the classical Darling–Erdős theorem, the reader is referred to [2,4,10] and the references in these articles.

The Darling–Erdős theorem is also related to finding an integral test refining the Hartman–Wintner LIL, a problem which was already addressed by Feller in 1946. Here, one can relatively easily prove that the classical Kolmogorov–Erdős–Petrowski integral test for Brownian motion holds for sums of i.i.d. mean zero random variables if one has $\mathbb{E}|X|^{2+\delta} < \infty$ for some $\delta > 0$. In this case, one has for any non-decreasing function $\phi:]0, \infty[\to]0, \infty[$,

$$\mathbb{P}\left\{|S_n| \leq \sqrt{n}\phi(n) \text{ eventually}\right\} = 1 \text{ or } 0,$$

according as $\sum_{n=1}^{\infty} n^{-1}\phi(n) \exp(-\phi^2(n)/2)$ is finite or infinite.

Feller proved that this result remains valid under the second moment assumption if one replaces $\sqrt{n}$ by $\sqrt{B_n}$ defined as above with $p \geq 4$. Similarly, as for the Darling–Erdős theorem, this implies that the Kolmogorov–Erdős–Petrowski integral test holds in its original form if

$$\mathbb{E}X^2 I\{|X| \geq t\} = O\left((LLt)^{-1}\right) \text{ as } t \to \infty.$$

The proof in [11] was based on a skillful double truncation argument which only worked for symmetric distributions. Finally, in [6] an extension of this argument to

the general non-symmetric case was found so that we now know that most results in [11] are correct (see also [1] for more historical background).

There is still one question in the paper [11] which has not yet been addressed, namely whether it is possible to make the theorem "slightly more elegant" by replacing the sequence $\sqrt{B_n}$ by $\sqrt{n}\sigma_n$. Feller writes that he "was unable to devise a proof simple enough to be justified by the slight improvement of the theorem" (see p. 632 in [11]). We believe that we have found a simple enough proof of Feller's claim (see Step 3 in the proof of Theorem 2.3).

This leads to the following improved version of the Darling–Erdős theorem under the finite second moment assumption:

$$a_n \max_{1 \leq k \leq n} |S_k|/\sqrt{k}\sigma_k - b_n \xrightarrow{d} \tilde{Y}. \tag{1.2}$$

At the same time, we can show that there is a much wider choice for the truncation level in the definition of σ_n^2. For instance, it is possible to define σ_n^2 as $\mathbb{E}X^2 I\{|X| \leq \sqrt{n}\}$.

This improved version of the Darling–Erdős theorem will actually follow from a general result for d-dimensional random vectors which will be given in the following section.

2 Statement of Main Results

We now consider i.i.d. d-dimensional random vectors $X, X_1, X_2, \ldots$ such that $\mathbb{E}|X|^2 < \infty$ and $\mathbb{E}X = 0$, where we denote the Euclidean norm by $|\cdot|$. The corresponding matrix norm will be denoted by $\|\cdot\|$, that is, we set

$$\|A\| := \sup_{|x| \leq 1} |Ax|$$

for any (d, d)-matrix A. It is well known that $\|A\|$ = the largest eigenvalue of A if A is symmetric and nonnegative definite.

Let again $S_n := \sum_{j=1}^{n} X_j, n \geq 1$. Horváth [12] obtained in 1994 the following multidimensional version of the Darling–Erdős theorem assuming that $\mathbb{E}|X|^{2+\delta} < \infty$ for some $\delta > 0$ and that $\mathrm{Cov}(X)$ (= the covariance matrix of X) is equal to the d-dimensional identity matrix I,

$$a_n \max_{1 \leq k \leq n} |S_k|/\sqrt{k} - b_{d,n} \xrightarrow{d} \tilde{Y}, \tag{2.1}$$

where $\tilde{Y}$ has the same distribution as in dimension 1,

$$b_{d,n} := 2LLn + dLLLn/2 - \log(\Gamma(d/2)),$$

and $\Gamma(t), t > 0$ is the Gamma function. Recall that $\Gamma(1/2) = \sqrt{\pi}$ so that this extends the one-dimensional Darling–Erdős theorem.

We are ready to formulate our general result. We consider non-decreasing sequences c_n of positive real numbers satisfying for large n,

$$\exp\left(-(\log n)^{\epsilon_n}\right) \leq c_n/\sqrt{n} \leq \exp\left((\log n)^{\epsilon_n}\right), \tag{2.2}$$

where $\epsilon_n \to 0$.

Further, let for each n, Γ_n be the symmetric nonnegative definite matrix such that

$$\Gamma_n^2 = \left[\mathbb{E}X^{(i)}X^{(j)}I\{|X| \leq c_n\}\right]_{1 \leq i,j \leq d}, \quad n \geq 1. \tag{2.3}$$

If the covariance matrix of $X = (X^{(1)}, \ldots, X^{(d)})$ is positive definite, the matrices Γ_n will be invertible for large enough n. Replacing c_n by $c_{n \vee n_0}$ for a suitable $n_0 \geq 1$ if necessary, we can assume w.l.o.g. that *all* matrices Γ_n are invertible.

Theorem 2.1 *Let $X, X_n, n \geq 1$ be i.i.d. mean zero random vectors in $\mathbb{R}^d$ such that $E|X|^2 < \infty$ and $\mathrm{Cov}(X) = I$. Then, we have for any sequence $\{c_n\}$ satisfying condition* (2.2),

$$a_n \max_{1 \leq k \leq n} |\Gamma_k^{-1} S_k|/\sqrt{k} - b_{d,n} \xrightarrow{d} \tilde{Y}, \tag{2.4}$$

where $\tilde{Y}: \Omega \to \mathbb{R}$ is a random variable such that

$$\mathbb{P}\{\tilde{Y} \leq t\} = \exp(-\exp(-t)), t \in \mathbb{R}.$$

Under the additional assumption

$$\mathbb{E}|X|^2 I\{|X| \geq t\} = o\left((LLt)^{-1}\right) \text{ as } t \to \infty, \tag{2.5}$$

we also have

$$a_n \max_{1 \leq k \leq n} |S_k|/\sqrt{k} - b_{d,n} \xrightarrow{d} \tilde{Y}. \tag{2.6}$$

It is easy to see that condition (2.5) is satisfied if $\mathbb{E}|X|^2 LL|X| < \infty$. This latter condition, however, is more restrictive than (2.5).

It is natural to ask whether this condition is also necessary as in the one-dimensional case (see Theorem 2 in [6]). This question becomes much more involved in the multidimensional case, and we get a slightly weaker result, namely that the following condition

$$\mathbb{E}|X|^2 I\{|X| \geq t\} = O\left((LLt)^{-1}\right) \text{ as } t \to \infty \tag{2.7}$$

is necessary for (2.6).

To prove this result, we show that if condition (2.7) is not satisfied, then $a_n \max_{1 \leq k \leq n} |S_k|/\sqrt{k} - b_{d,n}$ cannot converge in distribution to any variable of the form $\tilde{Y} + c$, where $c \in \mathbb{R}$.

If one allows this larger class of limiting distributions, condition (2.7) is optimal. There are examples where $\mathbb{E}|X|^2 I\{|X| \geq t\} = O((LLt)^{-1})$ and $a_n \max_{1 \leq k \leq n} |S_k|/\sqrt{k} - b_{d,n}$ converges in distribution to $\tilde{Y} + c$ for some $c \neq 0$ (see Theorem 6.1 below).

Theorem 2.2 *Let $X, X_n, n \geq 1$ be i.i.d. mean zero random vectors in $\mathbb{R}^d$ such that $E|X|^2 < \infty$ and $\mathrm{Cov}(X) = I$ and suppose that there exists a $c \in \mathbb{R}$ such that*

$$a_n \max_{1 \leq k \leq n} |S_k|/\sqrt{k} - b_{d,n} \xrightarrow{d} \tilde{Y} + c, \tag{2.8}$$

where $\tilde{Y}$ is as in Theorem 2.1. Then, condition (2.7) holds.

Our basic tool for proving the above results is a new strong invariance principle for sums of i.i.d. random vectors which is valid under a finite second moment assumption. If one has an approximation with an almost sure error term of order $o(\sqrt{n/LLn})$, one can obtain the Darling–Erdős theorem directly from the normally distributed case. The problem is that it is impossible to get such an approximation under the sole assumption of a finite second moment. In [6], it was shown that one needs the "good" approximation of order $o(\sqrt{n/LLn})$ only if the sums $|S_n|$ are large and it was shown that in dimension 1 one can obtain approximations which are particularly efficient for the random subsequence where the sums are large. Using recent results on d-dimensional strong approximations (see [9,15]), we are now able to obtain an analogue of the approximation in [6] in the d-dimensional setting [see Lemma 3.1 and relation (3.6) below]. As an additional new feature, we also show that an approximation by $\Gamma_n \sum_{j=1}^n Z_j$ is possible where $Z_j, j \geq 1$ are i.i.d. $\mathcal{N}(0, I)$-distributed random vectors with $\mathcal{N}(0, \Sigma)$ denoting the d-dimensional normal distribution with mean zero and covariance matrix Σ. This type of approximation leads to the improved versions of the Darling–Erdős theorem and Feller's integral test as indicated in Sect. 1.

Theorem 2.3 *Let $X, X_n, n \geq 1$ be i.i.d. mean zero random vectors in $\mathbb{R}^d$ such that $E|X|^2 < \infty$ and $\mathrm{Cov}(X) = \Gamma^2$, where Γ is a symmetric nonnegative definite (d,d)-matrix. Let c_n be a non-decreasing sequence of positive real numbers satisfying condition (2.2) for large n and let Γ_n be defined as in (2.3). If the underlying p-space $(\Omega, \mathcal{F}, \mathbb{P})$ is rich enough one can construct independent $\mathcal{N}(0, I)$-distributed random vectors $Z_n, n \geq 1$ such that we have for the partial sums $T_n := \sum_{j=1}^n Z_j, n \geq 1$,*

(a) $S_n - \Gamma T_n = o(\sqrt{nLLn})$ *as $n \to \infty$ with prob. 1,*
(b) $\mathbb{P}\{|S_n - \Gamma_n T_n| \geq 2\sqrt{n}/LLn, |S_n| \geq \frac{4}{3}\|\Gamma\|\sqrt{nLLn}$ *infinitely often$\} = 0$, and*
(c) $\mathbb{P}\{|S_n - \Gamma_n T_n| \geq 2\sqrt{n}/LLn, |\Gamma T_n| \geq \frac{4}{3}\|\Gamma\|\sqrt{nLLn}$ *infinitely often$\} = 0$.*

Combining our strong invariance principle with the Kolmogorov–Erdős–Petrowski integral test for d-dimensional Brownian motion, one obtains by the same arguments as in Section 5 of [6] the following result,

Theorem 2.4 *Let $X, X_n, n \geq 1$ be i.i.d. mean zero random vectors in $\mathbb{R}^d$ such that $E|X|^2 < \infty$ and $\mathrm{Cov}(X) = \Gamma^2$, where Γ is a symmetric positive definite (d,d)-matrix. Let c_n be a non-decreasing sequence of positive real numbers satisfying condition (2.2) for large n and let Γ_n be defined as in (2.3). Then, we have for any non-decreasing function $\phi:]0, \infty[\to]0, \infty[$,*

$$\mathbb{P}\left\{|\Gamma_n^{-1} S_n| \leq \sqrt{n}\phi(n) \text{ eventually}\right\} = 1 \text{ or } = 0$$

according as

$$I_d(\phi) := \sum_{n=1}^{\infty} n^{-1} \phi^d(n) \exp(-\phi^2(n)/2) < \infty \ or = \infty.$$

Note that we can assume w.l.o.g. that all matrices Γ_n are invertible since they converge to Γ which is invertible.

Let $\lambda_n(\Lambda_n)$ be the smallest (largest) eigenvalue of Γ_n, $n \geq 1$. Assuming that $\mathrm{Cov}(X) = I$, we can infer from Theorem 2.4,

$$I_d(\phi) < \infty \Rightarrow \mathbb{P}\left\{|S_n| \leq \Lambda_n \sqrt{n}\phi(n) \text{ eventually}\right\} = 1 \qquad (2.9)$$

and

$$I_d(\phi) = \infty \Rightarrow \mathbb{P}\left\{|S_n| > \lambda_n \sqrt{n}\phi(n) \text{ infinitely often}\right\} = 1, \qquad (2.10)$$

which is the d-dimensional version of the result conjectured by Feller [11].

The proof of our strong invariance principle (= Theorem 2.3) will be given in Sect. 3. In the two subsequent Sects. 4 and 5, we will show how Theorems 2.1 and 2.2 follow from the strong invariance principle. In Sect. 6, we return to the real-valued case and show that if $\mathbb{E}X^2 I\{|X| \geq t\} \sim c(LLt)^{-1}$ that then (1.1) still remains valid if we replace $\tilde{Y}$ by $\tilde{Y} - c$. Finally, we answer a question which was posed in [13].

3 Proof of the Strong Invariance Principle

Our proof is divided into three steps.

Step 1 We recall a double truncation argument which goes back to Feller [11] for symmetric random variables. This was later extended to non-symmetric random variables in [6] and finally to random elements in Hilbert space in [7]. To formulate the relevant result, we need some extra notation. We set

$$
\begin{aligned}
X'_n &:= X_n I\left\{|X_n| \leq \sqrt{n}/(LLn)^5\right\}, & \overline{X}'_n &:= X'_n - \mathbb{E}X'_n; \\
X''_n &:= X_n I\left\{\sqrt{n}/(LLn)^5 < |X_n| \leq \sqrt{nLLn}\right\}, & \overline{X}''_n &:= X''_n - \mathbb{E}X''_n; \\
X'''_n &:= X_n I\left\{\sqrt{nLLn} < |X_n|\right\}, & \overline{X}'''_n &:= X'''_n - \mathbb{E}X'''_n;
\end{aligned}
$$

and we denote the corresponding sums by $S'_n, \overline{S}'_n, S''_n, \overline{S}''_n, S'''_n, \overline{S}'''_n$.

Then, we have (see [7], Lemmas 11 and 12)

$$S_n - \overline{S}'_n = o(\sqrt{nLLn}) \text{ a.s.} \qquad (3.1)$$

and

$$\mathbb{P}\left\{|S_n - \overline{S}'_n| \geq \sqrt{n}/(LLn), |S_n| \geq \|\Gamma\|\sqrt{nLLn} \text{ i.o.}\right\} = 0. \qquad (3.2)$$

Step 2 Let Σ_n be the sequence of symmetric nonnegative definite matrices such that Σ_n^2 is the covariance matrix of X'_n for $n \geq 1$. Furthermore, let $A(t)$ be the symmetric nonnegative definite matrices satisfying

$$A(t)^2 = \left[\mathbb{E} X^{(j)} X^{(k)} I\{|X| \leq t\} \right]_{1 \leq j,k \leq d}, t \geq 0.$$

It is easy to see that $A(t)^2, t \geq 0$ is monotone, that is, $A(t)^2 - A(s)^2$ is nonnegative definite if $0 \leq s \leq t$.

This implies that $A(t), t \geq 0$ is monotone as well (see Theorem V.1.9 in [3]). Consequently, $A(c_n), n \geq 1$ is a monotone sequence of symmetric nonnegative definite matrices whenever c_n is non-decreasing. Moreover, $A(c_n)$ converges to Γ if $c_n \to \infty$.

We have the following strong approximation result, where we set

$$\tilde{\Gamma}_n := A\left(\sqrt{n}/(LLn)^5 \right), n \geq 1.$$

Lemma 3.1 *If the underlying p-space is rich enough, one can construct independent random vectors $Z_i \sim \mathcal{N}(0, I)$ such that*

$$\overline{S}'_n - \sum_{i=1}^{n} \tilde{\Gamma}_i Z_i = o\left(\sqrt{n}/LLn \right) \text{ a.s.} \tag{3.3}$$

and

$$S_n - \sum_{i=1}^{n} \Gamma Z_i = o\left(\sqrt{nLLn} \right) \text{ a.s.} \tag{3.4}$$

Proof (i) We first show that one can construct independent $\mathcal{N}(0, I)$-distributed random vectors such that

$$\overline{S}'_n - \sum_{i=1}^{n} \Sigma_i Z_i = o\left(\sqrt{n}/LLn \right) \text{ a.s.}$$

By Corollary 3.2 from [9] and the fact that $\mathbb{E}|\overline{X}'_n|^3 \leq 8\mathbb{E}|X'_n|^3$, it is enough to show

$$\sum_{n=1}^{\infty} \mathbb{E}\left|X'_n\right|^3 \Big/ \left(\frac{\sqrt{n}}{LLn} \right)^3 < \infty.$$

Using the simple inequality,

$$\mathbb{E}|X'_n|^3 \leq \mathbb{E}|X|^{2+\delta} I\{|X| \leq \sqrt{n}\} \sqrt{n}^{1-\delta} (LLn)^{-5(1-\delta)}, \quad 0 < \delta < 1,$$

we find (setting $\delta = 2/5$) that the above series is

$$\leq \sum_{n=1}^{\infty} \mathbb{E}|X|^{12/5} I\{|X| \leq \sqrt{n}\}/\sqrt{n}^{12/5}$$

Using a standard argument (see for instance, the proof of part (a) of Lemma 3.3 in [9]), one can show that this last series is finite whenever $\mathbb{E}|X|^2 < \infty$.

(ii) To complete the proof of (3.3), it is now sufficient to show that

$$\sum_{i=1}^{n} (\Sigma_i - \tilde{\Gamma}_i) Z_i = o(\sqrt{n}/LLn) \text{ a.s.}$$

By a standard argument, this follows if

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\left|\left(\Sigma_n - \tilde{\Gamma}_n\right) Z_n\right|^2}{n/(LLn)^2} < \infty. \tag{3.5}$$

Since $Z_n \sim \mathcal{N}(0, I)$, we have

$$\mathbb{E}\left|\left(\Sigma_n - \tilde{\Gamma}_n\right) Z_n\right|^2 \leq d \left\|\Sigma_n - \tilde{\Gamma}_n\right\|^2 \leq d\|\Sigma_n^2 - \tilde{\Gamma}_n^2\|,$$

where we have used Theorem X.1.1 in [3] for the last inequality. From the definition of Σ_n and $\tilde{\Gamma}_n$, it is obvious that

$$\langle x, (\tilde{\Gamma}_n^2 - \Sigma_n^2)x \rangle = \left(\mathbb{E}\langle X, x \rangle I \left\{|X| \leq \sqrt{n}/(LLn)^5\right\}\right)^2, \quad x \in \mathbb{R}^d.$$

The last expression equals $\left(\mathbb{E}\langle X, x \rangle I\{|X| > \sqrt{n}/(LLn)^5\}\right)^2$ since $\mathbb{E}\langle X, x \rangle = 0$. Hence $\left\|\Sigma_n^2 - \tilde{\Gamma}_n^2\right\| \leq \left(\mathbb{E}|X| I\{|X| > \sqrt{n}/(LLn)^5\}\right)^2 \leq (\mathbb{E}|X|^2)^2 n^{-1} (LLn)^{10}$.

It is easy now to see that the series in (3.5) is finite.

(iii) Finally note that

$$S_n - \sum_{i=1}^{n} \Gamma Z_i = (S_n - \overline{S}_n') + \left(\overline{S}_n' - \sum_{i=1}^{n} \tilde{\Gamma}_i Z_i\right) + \sum_{i=1}^{n} (\tilde{\Gamma}_i - \Gamma) Z_i,$$

where the first two terms are of almost sure order $o(\sqrt{nLLn})$ by (3.1) and (3.3), respectively. Since $\tilde{\Gamma}_n \to \Gamma$ as $n \to \infty$, we also have that

$$\sum_{i=1}^{n} (\tilde{\Gamma}_i - \Gamma) Z_i = o(\sqrt{nLLn}) \text{ a.s.},$$

and we can conclude that indeed $S_n - \sum_{i=1}^{n} \Gamma Z_i = o(\sqrt{nLLn})$ a.s. Lemma 3.1 has been proven. □

Step 3 Combining Lemma 3.1 with relations (3.1) and (3.2), we find that

$$\mathbb{P}\left\{\left|S_n - \sum_{j=1}^{n} \tilde{\Gamma}_j Z_j\right| \geq 3\sqrt{n}/(2LLn), |\Gamma T_n| \geq \frac{5}{4}\|\Gamma\|\sqrt{nLLn} \text{ i.o.}\right\} = 0. \quad (3.6)$$

We next show that

$$\mathbb{P}\left\{\left|\sum_{j=1}^{n} (\Gamma_n - \tilde{\Gamma}_j) Z_j\right| \geq \sqrt{n}/(2LLn), |T_n| \geq \frac{5}{4}\sqrt{nLLn} \text{ i.o.}\right\} = 0, \quad (3.7)$$

where

$$\Gamma_n := A(c_n), n \geq 1$$

and c_n is an arbitrary non-decreasing sequence of positive real numbers satisfying condition (2.2) for large n.

Using that $\{|\Gamma T_n| \geq \|\Gamma\|x\} \subset \{|T_n| \geq x\}, x > 0$, we get from (3.6) and (3.7):

$$\mathbb{P}\left\{|S_n - \Gamma_n T_n| \geq 2\sqrt{n}/LLn, |\Gamma T_n| \geq \frac{5}{4}\|\Gamma\|\sqrt{nLLn} \text{ i.o.}\right\} = 0. \quad (3.8)$$

Further recall that $S_n - \Gamma T_n = o(\sqrt{nLLn})$ a.s. (see Lemma 3.1). Consequently, we can infer from (3.8) that

$$\mathbb{P}\left\{|S_n - \Gamma_n T_n| \geq 2\sqrt{n}/LLn, |S_n| \geq \frac{4}{3}\|\Gamma\|\sqrt{nLLn} \text{ i.o.}\right\} = 0. \quad (3.9)$$

We see that the proof of Theorem 2.3 is complete once we have established (3.7). Toward this end, we need the following inequality which is valid for normally distributed random vectors $Y: \Omega \to \mathbb{R}^d$ with mean zero and covariance matrix Σ:

$$\mathbb{P}\{|Y| \geq x\} \leq \exp\left(-x^2/(8\sigma^2)\right), x \geq 2\mathbb{E}|Y|^2, \quad (3.10)$$

where σ^2 is the largest eigenvalue of Σ (see Lemma 4 in [7]).

From (3.10), we trivially get that

$$\mathbb{P}\{|Y| \geq x\} \leq 2\exp\left(-x^2/\left(8\mathbb{E}|Y|^2\right)\right), x \geq 0. \quad (3.11)$$

Though this last inequality is clearly suboptimal, it will nevertheless be more than sufficient for the proof of (3.7).

Proof of (3.7) To simplify notation, we set $d_n := \sqrt{n}/(LLn)^5$, $n \geq 1$, $n_k := 2^k$ and $\ell_k := [2^{k-1}/(Lk)^5]$, $k \geq 0$. By the Borel–Cantelli lemma, it is enough to show that

$$\sum_{k=1}^{\infty} \mathbb{P}\left(\bigcup_{n=n_{k-1}}^{n_k}\left\{\left|\sum_{j=1}^{n}(\Gamma_n - \tilde{\Gamma}_j)Z_j\right| \geq \sqrt{n}/(2LLn), |T_n| \geq \frac{5}{4}\sqrt{nLLn}\right\}\right) < \infty.$$

Set $\tilde{\mathbb{N}} := \tilde{\mathbb{N}}_1 \cap \tilde{\mathbb{N}}_2$, where

$$\tilde{\mathbb{N}}_1 := \left\{k \colon \|\Gamma_{n_k} - \tilde{\Gamma}_{\ell_k}\| \leq \|\Gamma\|(Lk)^{-5/2}\right\}$$

and

$$\tilde{\mathbb{N}}_2 := \left\{k \colon \|\Gamma_{n_k} - \Gamma_{n_{k-1}}\| \leq (Lk)^{-2}\right\}.$$

Then, it is easy to see that the above series is finite if

$$\sum_{k \in \tilde{\mathbb{N}}} \mathbb{P}\left\{\max_{n_{k-1} \leq n \leq n_k}\left|\sum_{j=1}^{n}(\Gamma_n - \tilde{\Gamma}_j)Z_j\right| \geq 2^{(k-1)/2}/(2Lk)\right\} < \infty \qquad (3.12)$$

and

$$\sum_{k \notin \tilde{\mathbb{N}}} \mathbb{P}\left\{\max_{n_{k-1} \leq n \leq n_k}|T_n| \geq 2^{(k-1)/2}(Lk)^{1/2}\right\} < \infty. \qquad (3.13)$$

To bound the series in (3.12), we first note that employing the Lévy inequality for sums of independent symmetric random vectors, one obtains

$$\mathbb{P}\left\{\max_{n_{k-1} \leq n \leq n_k}\left|\sum_{j=1}^{n}(\Gamma_n - \tilde{\Gamma}_j)Z_j\right| \geq 2^{(k-1)/2}/(2Lk)\right\}$$

$$\leq \mathbb{P}\left\{\max_{n_{k-1} \leq n \leq n_k}\left|\sum_{j=1}^{n}(\Gamma_{n_k} - \tilde{\Gamma}_j)Z_j\right| \geq 2^{(k-1)/2}/(4Lk)\right\}$$

$$+ \mathbb{P}\left\{\max_{n_{k-1} \leq n \leq n_k}\left|(\Gamma_{n_k} - \Gamma_n)\sum_{j=1}^{n}Z_j\right| \geq 2^{(k-1)/2}/(4Lk)\right\}$$

$$\leq 2\mathbb{P}\left\{\left|\sum_{j=1}^{n_k}(\Gamma_{n_k} - \tilde{\Gamma}_j)Z_j\right| \geq 2^{(k-1)/2}/(4Lk)\right\}$$

$$+ 2\mathbb{P}\left\{\|\Gamma_{n_k} - \Gamma_{n_{k-1}}\|\left|\sum_{j=1}^{n_k}Z_j\right| \geq 2^{(k-1)/2}/(4Lk)\right\} =: 2p_{k,1} + 2p_{k,2},$$

where we have also used the fact that

$$\|\Gamma_{n_k} - \Gamma_n\| \le \|\Gamma_{n_k} - \Gamma_{n_{k-1}}\| \text{ if } n_{k-1} \le n \le n_k.$$

This follows easily from the monotonicity of the sequence Γ_n.

To bound $p_{k,1}$, we first note that by Theorem X.1.1 in [3] for $1 \le j \le \ell_k$,

$$\|\Gamma_{n_k} - \tilde{\Gamma}_j\|^2 \le \|\Gamma_{n_k}^2 - \tilde{\Gamma}_j^2\| = \sup_{|x| \le 1} \mathbb{E}\langle x, X \rangle^2 I \left\{ d_j \wedge c_{n_k} < |X| \le d_j \vee c_{n_k} \right\}$$

$$\le \sup_{|x| \le 1} \mathbb{E}\langle x, X \rangle^2 = \|\Gamma\|^2. \tag{3.14}$$

Apply (3.11) with $Y = \sum_{j=1}^{n_k}(\Gamma_{n_k} - \tilde{\Gamma}_j)Z_j$. Then, clearly $\mathbb{E}Y = 0$ and, moreover, by independence of the random vectors Z_j, we have for $k \in \tilde{\mathbb{N}}$,

$$\mathbb{E}|Y|^2 = \sum_{j=1}^{n_k} \mathbb{E}|(\Gamma_{n_k} - \tilde{\Gamma}_j)Z_j|^2 \le d \sum_{j=1}^{n_k} \|\Gamma_{n_k} - \tilde{\Gamma}_j\|^2$$

$$\le d\|\Gamma\|^2 \left(\ell_k + (n_k - \ell_k)(Lk)^{-5} \right) \le 2d\|\Gamma\|^2 n_k (Lk)^{-5}.$$

We conclude that

$$p_{k,1} \le 2 \exp\left(-(Lk)^3 / \left(512d\|\Gamma\|^2 \right) \right), k \in \tilde{\mathbb{N}}_1.$$

Similarly, we obtain

$$p_{k,2} \le \mathbb{P}\left\{ \left| \sum_{j=1}^{n_k} Z_j \right| \ge 2^{(k-1)/2} Lk/4 \right\} \le 2\exp\left(-(Lk)^2/(256d) \right), \quad k \in \tilde{\mathbb{N}}_2.$$

It is now clear that the series in (3.12) is finite.

To show that the series in (3.13) is finite, we note that by (3.11) and the Lévy inequality,

$$\mathbb{P}\left\{ \max_{n_{k-1} \le n \le n_k} |T_n| \ge 2^{(k-1)/2}(Lk)^{1/2} \right\} \le 2\mathbb{P}\left\{ |T_{n_k}| \ge 2^{(k-1)/2}(Lk)^{1/2} \right\} \le 4k^{-\eta},$$

where $\eta = (16d)^{-1}$ and it is enough to check that

$$\sum_{k \notin \tilde{\mathbb{N}}} k^{-\eta} < \infty.$$

To verify that this series is finite, observe that by the argument used in (3.14) we have,

$$\| \Gamma_{n_k} - \Gamma_{n_{k-1}} \|^2 \leq \sup_{|x| \leq 1} \mathbb{E} \langle x, X \rangle^2 I \left\{ c_{n_{k-1}} < |X| \leq c_{n_k} \right\}$$

$$\leq \mathbb{E}|X|^2 I \left\{ c_{n_{k-1}} < |X| \leq c_{n_k} \right\},$$

which implies

$$\sum_{k=1}^{\infty} \| \Gamma_{n_k} - \Gamma_{n_{k-1}} \|^2 \leq \mathbb{E}|X|^2 < \infty.$$

We conclude that

$$\sum_{k \notin \tilde{\mathbb{N}}_2} (Lk)^{-2} < \infty.$$

So the proof of (3.7) is complete if we show that

$$\sum_{k \notin \tilde{\mathbb{N}}_1} k^{-\eta} < \infty. \tag{3.15}$$

We need another lemma.

Lemma 3.2 *Consider two sequences $c_{k,i}, k \geq 1$ of positive real numbers satisfying for large enough k,*

$$2^{k/2} \exp(-k^\delta) \leq c_{k,i} \leq 2^{k/2} \exp(k^\delta), i = 1, 2, \tag{3.16}$$

where $0 < \delta < 1$. Set $\Gamma_{k,i} := A(c_{k,i}), i = 1, 2, k \geq 1$. Then, we have,

$$\sum_{k=1}^{\infty} k^{-\delta} \| \Gamma_{k,1} - \Gamma_{k,2} \|^2 < \infty.$$

Proof Using the same argument as in (3.14), we have for large k,

$$\| \Gamma_{k,1} - \Gamma_{k,2} \|^2 \leq \mathbb{E}|X|^2 I \left\{ 2^{k/2} \exp(-k^\delta) < |X| \leq 2^{k/2} \exp(k^\delta) \right\} \leq \sum_{j=[k-3k^\delta]}^{[k+3k^\delta]} \beta_j,$$

where $\beta_j := \mathbb{E}|X|^2 I \{ 2^{j-1} < |X|^2 \leq 2^j \}, j \geq 1$.

We can conclude that for some $k_0 \geq 1$ and a suitable $j_0 \geq 0$,

$$\sum_{k=k_0}^{\infty} k^{-\delta} \| \Gamma_{k,1} - \Gamma_{k,2} \|^2 \leq \sum_{j=j_0}^{\infty} \beta_j \sum_{k=m_1(j)}^{m_2(j)} k^{-\delta}, \tag{3.17}$$

where

$$m_1(j) = \min\left\{k \geq k_0 : [k + 3k^\delta] \geq j\right\}$$

and

$$m_2(j) = \max\left\{k \geq k_0 : [k - 3k^\delta] \leq j\right\}.$$

It is easy to see that $m_1(j) \geq j - 3j^\delta \geq j/2$ and $m_2(j) \leq j + 4j^\delta$ for large j. Consequently, we have for large j,

$$\sum_{k=m_1(j)}^{m_2(j)} k^{-\delta} \leq 2^\delta (m_2(j) - m_1(j) + 1)j^{-\delta} \leq 2^{3+\delta} < \infty. \qquad (3.18)$$

We obviously have $\sum_{j=1}^{\infty} \beta_j < \infty$ (as $\mathbb{E}|X|^2 < \infty$). Combining relations (3.17) and (3.18), we obtain the assertion of the lemma. □

We apply the above lemma with $c_{k,1} = c_{n_k}, c_{k,2} = d_{\ell_k}, k \geq 1$. From condition (2.2), we readily obtain that for large k,

$$2^{k/2} \exp(-k^{\epsilon'_k}) \leq c_{k,1} \leq 2^{k/2} \exp(k^{\epsilon'_k}),$$

where $\epsilon'_k := \epsilon_{n_k} \to 0$ so that condition (3.16) is satisfied for any $\delta > 0$. This is also the case for the sequence $c_{k,2}$. So we can choose $\delta = \eta/2$, and it follows that

$$\infty > \sum_{k \notin \tilde{\mathbb{N}}_1} k^{-\eta/2} \|\Gamma_{n_k} - \tilde{\Gamma}_{\ell_k}\|^2 \geq \|\Gamma\|^2 \sum_{k \notin \tilde{\mathbb{N}}_1} k^{-\eta/2}(Lk)^{-5},$$

which shows that (3.15) holds.

4 Proof of Theorem 2.1

We first prove (2.4). Set $k_n = [\exp((Ln)^\alpha)]$, where $0 < \alpha < 1$. Then, it follows from the d-dimensional version of the Hartman–Wintner LIL that for any given $\epsilon > 0$, with prob. 1,

$$|\Gamma_k^{-1} S_k|/\sqrt{k} \leq \lambda_k^{-1} \sqrt{2LLk}(1 + \epsilon), k \geq k_0(\omega, \epsilon),$$

where λ_k is the smallest eigenvalue of Γ_k. As $\lambda_k \nearrow 1$, we can conclude that for large enough n,

$$\max_{1 \leq k < k_n} |\Gamma_k^{-1} S_k|/\sqrt{k} \leq \sqrt{2\alpha LLn}(1 + 2\epsilon),$$

which is $\leq \sqrt{2LLn}$ if we choose ϵ small enough. It follows that

$$a_n \max_{1 \leq k < k_n} |\Gamma_k^{-1} S_k| / \sqrt{k} - b_{d,n} \to -\infty \text{ a.s.} \qquad (4.1)$$

So (2.4) holds if and only if

$$a_n \max_{k \in K_n} |\Gamma_k^{-1} S_k| / \sqrt{k} - b_{d,n} \xrightarrow{d} \tilde{Y},$$

where $K_n := \{k_n + 1, \ldots, n\}$.

We split K_n into two random subsets:

$$K_{n,1}(\cdot) := \left\{ k \in K_n : |S_k - \Gamma_k T_k| \leq 2\sqrt{k}/(LLk) \right\}, \quad K_{n,2}(\cdot) := K_n \backslash K_{n,1}(\cdot).$$

In view of Theorem 2.3(b) (where we set $\Gamma = I$), there are with prob. 1 only finitely many k's such that

$$|S_k - \Gamma_k T_k| > 2\sqrt{k}/LLk \text{ and } |\Gamma_k^{-1} S_k| \geq 4\sqrt{kLLk}/(3\lambda_k),$$

where λ_k is again the smallest eigenvalue of Γ_k.

As $\lambda_k \nearrow 1$, we can conclude that with prob. 1 there are only finitely many k's such that

$$|S_k - \Gamma_k T_k| > 2\sqrt{k}/LLk \text{ and } |\Gamma_k^{-1} S_k| \geq \sqrt{2kLLk},$$

and it follows that

$$a_n \max_{k \in K_{n,2}(\cdot)} |\Gamma_k^{-1} S_k| / \sqrt{k} - b_{d,n} \to -\infty \text{ a.s.}$$

We see that (2.4) is equivalent to

$$a_n \max_{k \in K_{n,1}(\cdot)} |\Gamma_k^{-1} S_k| / \sqrt{k} - b_{d,n} \xrightarrow{d} \tilde{Y}.$$

From the definition of the sets $K_{n,1}(\cdot)$, we easily get that

$$a_n \max_{k \in K_{n,1}(\cdot)} |\Gamma_k^{-1} S_k| / \sqrt{k} - a_n \max_{k \in K_{n,1}(\cdot)} |T_k| / \sqrt{k} \to 0 \text{ a.s.}$$

By Slutsky's lemma (2.4) holds if and only if

$$a_n \max_{k \in K_{n,1}(\cdot)} |T_k| / \sqrt{k} - b_{d,n} \xrightarrow{d} \tilde{Y}.$$

Looking at Theorem 2.3(c), we can also conclude that

$$a_n \max_{k \in K_{n,2}(\cdot)} |T_k|/\sqrt{k} - b_{d,n} \to -\infty \text{ a.s.}$$

and the proof of (2.4) further reduces to showing

$$a_n \max_{k \in K_n} |T_k|/\sqrt{k} - b_{d,n} \xrightarrow{d} \tilde{Y}.$$

Using the same argument as in (4.1), we also see that

$$a_n \max_{1 \le k < k_n} |T_k|/\sqrt{k} - b_{d,n} \to -\infty \text{ a.s.}$$

and we have shown that (2.4) holds if

$$a_n \max_{1 \le k \le n} |T_k|/\sqrt{k} - b_{d,n} \xrightarrow{d} \tilde{Y}.$$

This is the Darling–Erdős theorem for normally distributed random vectors which follows from (2.1). Thus, (2.4) has been proven.

We now turn to the proof of (2.6). By Slutsky's lemma and (4.1), it is enough to show that

$$\Delta_n := a_n \left| \max_{k_n \le k \le n} |S_k|/\sqrt{k} - \max_{k_n \le k \le n} |\Gamma_k^{-1} S_k|/\sqrt{k} \right| \xrightarrow{\mathbb{P}} 0,$$

Using the triangular inequality, it is easy to see that

$$\Delta_n \le a_n \max_{k_n \le k \le n} \left| (I - \Gamma_k) \Gamma_k^{-1} S_k/\sqrt{k} \right| \le a_n \|I - \Gamma_{k_n}\| \max_{1 \le k \le n} |\Gamma_k^{-1} S_k|/\sqrt{k}.$$

From (2.4), it follows that $(\max_{1 \le k \le n} |\Gamma_k^{-1} S_k|/\sqrt{k})/\sqrt{LLn}$ is stochastically bounded. By assumption (2.5), we also have that $\|I - \Gamma_{k_n}\| = o((LLn)^{-1})$. Recalling that $a_n = \sqrt{2LLn}$, we see that $\Delta_n \xrightarrow{\mathbb{P}} 0$ and our proof of Theorem 2.1 is complete. □

5 Proof of Theorem 2.2

Using the same arguments as in the proof of Theorem 2.1, we can infer from (2.8) via relations (3.4) and (3.6) that

$$M_n := a_n \max_{k_n \le k \le n} |T_k'|/\sqrt{k} - b_{d,n} \xrightarrow{d} \tilde{Y} + c, \tag{5.1}$$

where $T_k' = \sum_{j=1}^{k} \tilde{\Gamma}_j Z_j$ and the random vectors Z_j are i.i.d. with $\mathcal{N}(0, I)$-distribution and $k_n \le \exp((Ln)^\alpha)$ for some $0 < \alpha < 1$.

Our first lemma gives an upper bound of $\mathbb{P}\{M_n > t\}$ via the corresponding probability for the maximum of a subcollection of the random variables $\{|T_k'|/\sqrt{k}\colon k_n \leq k \leq n\}$ (see Lemma 4.3 in [5] for a related result).

Let $0 < \xi < 1$ be fixed. Set

$$m_j = [\exp(j\xi/LLn)], \ j \geq 1 \text{ and } N = N_n = [LnLLn/\xi].$$

Then, $m_N \leq n \leq m_{N+1}$. Also note that the sequence m_j depends on n and ξ. Next, set

$$j_n := \min\{j\colon m_j \geq Ln\} \text{ and } k_n = m_{j_n}$$

so that $j_n \sim \xi^{-1}(LLn)^2$ and $k_n \sim Ln$ as $n \to \infty$.
Finally to simplify notation, we set $f_n(y) = (b_{d,n} + y)/a_n$, $y \in \mathbb{R}$ so that

$$\mathbb{P}\{M_n > y\} = \mathbb{P}\left\{\max_{k_n \leq k \leq n} |T_k'|/\sqrt{k} > f_n(y)\right\}.$$

Lemma 5.1 *Given $0 < \delta < 1$, we have for $y \in \mathbb{R}$ and $n \geq n_0 = n_0(\xi, \delta, y)$,*

$$(1 - \delta)\,\mathbb{P}\{M_n > y + \delta\} \leq \mathbb{P}\left\{\max_{j_n \leq j \leq N} |T_{m_j}'|/\sqrt{m_j} > f_n(y)\right\} + \mathbb{P}\{|Z_1| \geq f_n(y)\},$$

provided that $0 < \xi \leq \delta^3/(36d)$.

Proof Noting that

$$\mathbb{P}\{M_n > y + \delta\} = \mathbb{P}\left\{\max_{k_n \leq k \leq n} |T_k'|/\sqrt{k} > f_n(y + \delta)\right\}$$

$$\leq \mathbb{P}\left\{\max_{k_n \leq k \leq n} |T_k'|/\sqrt{k} > f_n(y + \delta), \max_{j_n \leq j \leq N} |T_{m_j}'|/\sqrt{m_j} \leq f_n(y)\right\}$$

$$+ \mathbb{P}\left\{\max_{j_n \leq j \leq N} |T_{m_j}'|/\sqrt{m_j} > f_n(y)\right\},$$

it is enough to show that

$$\mathbb{P}\left\{\max_{k_n \leq k \leq n} |T_k'|/\sqrt{k} > f_n(y + \delta), \max_{j_n \leq j \leq N} |T_{m_j}'|/\sqrt{m_j} \leq f_n(y)\right\} \qquad (5.2)$$

$$\leq \delta\,\mathbb{P}\left\{\max_{k_n \leq k \leq n} |T_k'|/\sqrt{k} > f_n(y + \delta)\right\} + \mathbb{P}\{|Z_1| \geq f_n(y)\},$$

if ξ is sufficiently small.
Consider the following stopping time,

$$\tau := \inf\left\{k \geq k_n\colon |T_k'|/\sqrt{k} > f_n(y + \delta)\right\}.$$

Then, it is obvious that the probability in (5.2) is bounded above by

$$
\sum_{j=j_n+1}^{N-1} \sum_{k=m_{j-1}+1}^{m_j-1} \mathbb{P}\left\{ \tau = k, \max_{j_n \le j \le N} |T'_{m_j}|/\sqrt{m_j} \le f_n(y) \right\} + \mathbb{P}\{m_{N-1} < \tau \le n\}
$$

(5.3)

Furthermore, we have for $j_n + 1 \le j \le N - 1$,

$$
\sum_{k=m_{j-1}+1}^{m_j-1} \mathbb{P}\left\{ \tau = k, \max_{j_n \le j \le N} |T'_{m_j}|/\sqrt{m_j} \le f_n(y) \right\}
$$

$$
\le \sum_{k=m_{j-1}+1}^{m_j-1} \mathbb{P}\{\tau = k\} \mathbb{P}\left\{ \frac{|T'_k - T'_{m_{j+1}}|}{\sqrt{m_{j+1} - k}} > \frac{\sqrt{k} f_n(y + \delta) - \sqrt{m_{j+1}} f_n(y)}{\sqrt{m_{j+1} - k}} \right\}.
$$

Next observe that

$$
\max_{m_{j-1} < k \le m_j} \mathbb{P}\left\{ \frac{|T'_k - T'_{m_{j+1}}|}{\sqrt{m_{j+1} - k}} > \frac{\sqrt{k} f_n(y + \delta) - \sqrt{m_{j+1}} f_n(y)}{\sqrt{m_{j+1} - k}} \right\}
$$

$$
\le \mathbb{P}\left\{ |Z_1| > \frac{\sqrt{m_{j-1}} f_n(y + \delta) - \sqrt{m_{j+1}} f_n(y)}{\sqrt{m_{j+1} - m_{j-1}}} \right\}.
$$

After some calculation, we find that for large enough n,

$$
\frac{\sqrt{m_{j-1}} f_n(y + \delta) - \sqrt{m_{j+1}} f_n(y)}{\sqrt{m_{j+1} - m_{j-1}}} \ge \frac{\delta}{3\sqrt{\xi}} - 4\sqrt{\xi} \ge \frac{\delta}{6\sqrt{\xi}}
$$

where the last inequality holds since $\xi \le \delta/24$. We trivially have by Markov's inequality,

$$
\mathbb{P}\{|Z_1| \ge \delta/(6\sqrt{\xi})\} \le 36\xi \mathbb{E}[|Z_1|^2]/\delta^2 = 36\xi d/\delta^2,
$$

which is $\le \delta$ by our condition on ξ.

It follows that

$$
\sum_{j=j_n+1}^{N-1} \sum_{k=m_{j-1}+1}^{m_j-1} \mathbb{P}\left\{ \tau = k, \max_{j_n \le j \le N} |T'_{m_j}|/\sqrt{m_j} \le f_n(y) \right\} \le \delta \, \mathbb{P}\{k_n \le \tau \le m_{N-1}\}.
$$

(5.4)

Concerning the second term in (5.3), simply note that

$$\mathbb{P}\left\{m_{N-1} < \tau \le n\right\}$$

$$\le \mathbb{P}\left\{m_{N-1} < \tau \le n, |T'_{m_{N+1}}|/\sqrt{m_{N+1}} \le f_n(y)\right\} + \mathbb{P}\left\{|T'_{m_{N+1}}|/\sqrt{m_{N+1}} > f_n(y)\right\}$$

$$= \sum_{k=m_{N-1}+1}^{n} \mathbb{P}\left\{\tau = k, |T'_{m_{N+1}}|/\sqrt{m_{N+1}} \le f_n(y)\right\} + \mathbb{P}\{|Z_1| > f_n(y)\}.$$

Arguing as above, we readily obtain,

$$\mathbb{P}\left\{m_{N-1} < \tau \le n\right\} \le \delta\mathbb{P}\left\{m_{N-1} < \tau \le n\right\} + \mathbb{P}\{|Z_1| > f_n(y)\}. \qquad (5.5)$$

Combining relations (5.4) and (5.5) and recalling (5.3), we see that

$$\mathbb{P}\left\{\max_{k_n \le k \le n} |T'_k|/\sqrt{k} > f_n(y+\delta), \max_{j_n \le j \le N} |T'_{m_j}|/\sqrt{m_j} \le f_n(y)\right\}$$

$$\le \delta\,\mathbb{P}\{k_n \le \tau \le n\} + \mathbb{P}\{|Z_1| > f_n(y)\}.$$

This implies (5.2) since

$$\mathbb{P}\{k_n \le \tau \le n\} = \mathbb{P}\left\{\max_{k_n \le k \le n} |T'_k|/\sqrt{k} > f_n(y+\delta)\right\},$$

and the proof of Lemma 5.1 is complete.   □

We finally need the following lemma,

Lemma 5.2 *Let Y be a d-dimensional random vector with distribution $\mathcal{N}(0, \Sigma)$, where $d \ge 2$. Assume that the largest eigenvalue of Σ is equal to 1 and has multiplicity $d-1$. Denote the remaining (smallest) eigenvalue of Σ by σ^2. Then, we have:*

$$\mathbb{P}\{|Y| \ge t\} \le \frac{2}{\sqrt{1-\sigma^2}}\mathbb{P}\{|Z| \ge t\}, t > 0,$$

where $Z: \Omega \to \mathbb{R}^{d-1}$ has a normal$(0, I_{d-1})$-distribution.

Proof If $d \ge 3$, Lemma 5.2 follows by integrating the inequality given in Lemma 1(a) of [7].

To prove Lemma 5.2 if $d = 2$, we proceed similarly as in [7]. Choose an orthonormal basis e_1, e_2 of $\mathbb{R}^2$ consisting of two eigenvectors corresponding to the eigenvalues 1 and $\sigma^2 \in]0, 1[$ of Σ. Then,

$$Y = \sum_{i=1}^{2} \langle Y, e_i \rangle e_i =: \eta_1 e_1 + \sigma\eta_2 e_2$$

where $\eta_i, 1 \le i \le 2$ are independent standard normal random variables.

It is then obvious that

$$Y^2 = \eta_1^2 + \sigma^2 \eta_2^2 =: R_1 + R_2,$$

where R_1 and R_2/σ^2 have Chi-square distributions with 1 degree of freedom. Denote the densities of $R_1 + R_2$, R_1 and R_2 by h, h_1, h_2.

Then, $h_2(y) = h_1(y/\sigma^2)/\sigma^2$ and

$$h(z) = \sigma^{-2} \int_0^z h_1(z - y)h_1(y/\sigma^2)\mathrm{d}y, z \geq 0.$$

Using that $h_1(y) = (2\pi)^{-1/2}y^{-1/2}e^{-y/2}$, $y > 0$, we can infer that

$$h(z)/h_1(z) = \frac{1}{2\pi\sigma} \int_0^z (1 - y/z)^{-1/2}y^{-1/2}e^{-(\sigma^{-2}-1)y/2}\mathrm{d}y$$

$$\leq \frac{1}{\sqrt{2\pi}\sigma} \int_0^{z/2} y^{-1/2}e^{-(\sigma^{-2}-1)y/2}\mathrm{d}y + \frac{e^{-(\sigma^{-2}-1)z/4}}{\sqrt{2\pi}\sigma\sqrt{z}} \int_{z/2}^z (1 - y/z)^{-1/2}\mathrm{d}y$$

$$\leq (2\pi\sigma^2(\sigma^{-2} - 1))^{-1/2} + \sqrt{2z}e^{-(\sigma^{-2}-1)z/4}(\pi\sigma)^{-1}.$$

Employing the trivial inequality $e^{-x/2} \leq x^{-1/2}$, $x > 0$, it follows that

$$h(z)/h_1(z) \leq \left[(2\pi)^{-1/2} + \sqrt{8/\pi}\right](1 - \sigma^2)^{-1/2} \leq 2(1 - \sigma^2)^{-1/2}, z \geq 0.$$

We can conclude that for $t \geq 0$,

$$\mathbb{P}\{|Y| \geq t\} = \int_{t^2}^{\infty} h(z)\mathrm{d}z \leq \frac{2}{\sqrt{1 - \sigma^2}} \int_{t^2}^{\infty} h_1(z)\mathrm{d}z = \frac{2}{\sqrt{1 - \sigma^2}}\mathbb{P}\{|Z| \geq t\}$$

and Lemma 5.2 has been proven.   □

Recall that $d_n = \sqrt{n}/(LLn)^5$ and $\tilde{\Gamma}_n = A(d_n), n \geq 1$. Let $\{v_1, \ldots, v_d\}$ be an orthonormal basis of $\mathbb{R}^d$. Then, it is easy to see that

$$\mathbb{E}|X|^2 I\{|X| > d_n\} = \sum_{i=1}^{d} \mathbb{E}\langle X, v_i\rangle^2 I\{|X| > d_n\}$$

$$= \sum_{i=1}^{d} \left\langle v_i, (I - \tilde{\Gamma}_n^2)v_i \right\rangle \leq d\|I - \tilde{\Gamma}_n^2\|.$$

It is now obvious that condition (2.7) is equivalent to

$$\|I - \tilde{\Gamma}_n^2\| = O((LLn)^{-1}) \text{ as } n \to \infty,$$

Furthermore, $\|I - \tilde{\Gamma}_n^2\|$ is equal to $1 - \tilde{\lambda}_n^2$, where $\tilde{\lambda}_n$ is the smallest eigenvalue of $\tilde{\Gamma}_n$ since $\tilde{\Gamma}_n^2$ is symmetric and $I - \tilde{\Gamma}_n^2$ is nonnegative definite. So it remains to be shown that (5.1) implies

$$1 - \tilde{\lambda}_n^2 = O\left((LLn)^{-1}\right) \text{ as } n \to \infty. \tag{5.6}$$

or, equivalently, to show that if (5.6) does not hold, we cannot have (5.1).

To that end, we apply Lemma 5.1 with $\delta = 1/2$ and we get for $y \in \mathbb{R}$,

$$
\begin{aligned}
&\mathbb{P}\{M_n \geq y\} \\
&\leq 2\mathbb{P}\left\{\max_{j_n \leq j \leq N} |T'_{m_j}|/\sqrt{m_j} > f_n(y - 1/2)\right\} + 2\mathbb{P}\{|Z_1| \geq f_n(y - 1/2)\} \\
&\leq 2N\mathbb{P}\{|\tilde{\Gamma}_n Z_1| \geq f_n(y - 1/2\} + 2\mathbb{P}\{|Z_1| \geq f_n(y - 1/2)\}. \tag{5.7}
\end{aligned}
$$

Here we have used the monotonicity of the sequence $\tilde{\Gamma}_k, k \geq 1$ which implies that $\tilde{\Gamma}_n - \text{Cov}(T'_{m_j}/\sqrt{m_j})$ is nonnegative definite for $j_n \leq j \leq N$. This allows us to conclude that for $j_n \leq j \leq N$,

$$\mathbb{P}\{|T'_{m_j}|/\sqrt{m_j} > x\} \leq \mathbb{P}\{|\tilde{\Gamma}_n Z_1| > x\}, x \in \mathbb{R}.$$

Let D_n be the d-dimensional diagonal matrix with $D_n(i, i) = 1, 1 \leq i \leq d - 1$ and $D_n(d, d) = \tilde{\lambda}_n$. Then, clearly

$$\mathbb{P}\{|\tilde{\Gamma}_n Z_1| \geq f_n(y - 1/2)\} \leq \mathbb{P}\{|D_n Z_1| \geq f_n(y - 1/2)\}$$

and we can infer from Lemma 5.2 that

$$\mathbb{P}\{|\tilde{\Gamma}_n Z_1| \geq f_n(y - 1/2)\} \leq 2(1 - \tilde{\lambda}_n^2)^{-1/2}\mathbb{P}\left\{|Z'| \geq f_n(y - 1/2)\right\}, \tag{5.8}$$

where Z' is a $(d - 1)$-dimensional normal mean zero random vector with covariance matrix equal to the identity matrix.

Using the fact that the square of the Euclidean norm of a d-dimensional $\mathcal{N}(0, I)$-distributed random vector X has a gamma distribution with parameters $d/2$ and 2, one can show that there exist positive constants $C_1(d), C_2(d)$ so that

$$C_1(d)t^{d-2}\exp(-t^2/2) \leq \mathbb{P}\{|X| \geq t\} \leq C_2(d)t^{d-2}\exp(-t^2/2), t \geq 2d. \tag{5.9}$$

(see Lemma 1 and Lemma 3 in [8], where more precise bounds are given if $d \geq 3$. If $d = 1$, this follows directly from well-known bounds for the tail probabilities of the one-dimensional normal distribution. If $d = 2$, the random variable $|X|^2$ has an exponential distribution and (5.9) is trivial).

We can conclude that for large enough n,

$$\mathbb{P}\{|Z'| \geq f_n(y - 1/2)\} \leq C_2(d - 1) f_n(y - 1/2)^{d-3} \exp(-(f_n(y - 1/2)^2/2)$$
$$\leq C_3(d) f_n(y - 1/2)^{-1} \mathbb{P}\{|Z_1| \geq f_n(y - 1/2)\},$$

where we set $C_3(d) = C_2(d - 1)/C_1(d)$. Returning to inequality (5.8) and noting that $f_n(y - 1/2) \geq \sqrt{\log \log n}$ if n is large, we get in this case,

$$\mathbb{P}\{|\tilde{\Gamma}_n Z_1| \geq f_n(y - 1/2)\} \leq 2C_3(d)\{(1 - \tilde{\lambda}_n^2) \log \log n\}^{-1/2} \mathbb{P}\{|Z_1| \geq f_n(y - 1/2)\}.$$

Applying (5.9) once more, we find that

$$\mathbb{P}\{|Z_1| \geq f_n(y - 1/2)\} = O(N^{-1}) = O\left((\log n \log \log n)^{-1}\right) \text{ as } n \to \infty.$$

Recalling (5.7) we can conclude that if

$$\limsup_{n \to \infty}(1 - \tilde{\lambda}_n^2) \log \log n = \infty,$$

we have for any $y \in \mathbb{R}$,

$$\liminf_{n \to \infty} \mathbb{P}\{M_n > y\} = 0.$$

Consequently, M_n cannot converge in distribution to any variable of the form $\tilde{Y} + c$. □

Remark 1. Denote the distribution of $a_n \max_{1 \leq k \leq n} |S_k|/\sqrt{k} - b_{d,n}$ by Q_n. From (3.4) and (3.6), it follows that this sequence is tight if and only if the distributions of M_n form a tight sequence. The above argument actually shows that this last sequence cannot be tight if condition (2.7) is not satisfied. Moreover, it is not difficult to prove via Theorem 2.1 that (2.7) implies that the sequence $\{Q_n : n \geq 1\}$ is tight. Thus, we have

$$\{Q_n : n \geq 1\} \text{ is tight } \iff (2.7).$$

2. Also note that

$$\mathbb{P}\left\{a_n \max_{1 \leq k \leq n} |\Gamma_k^{-1} S_k|/\sqrt{k} - b_n \leq 0\right\} \leq \mathbb{P}\left\{a_n \Lambda_n^{-1} \max_{1 \leq k \leq n} |S_k|/\sqrt{k} - b_n \leq 0\right\},$$

where Λ_n is the *largest* eigenvalue of Γ_n which in turn is defined as in (2.3) [here we can choose any sequence c_n satisfying condition (2.2)].
Using this inequality, one can show by the same argument as on p. 255 in [6] that (2.6) implies

$$1 - \Lambda_n^2 = o\left((LLn)^{-1}\right) \text{ as } n \to \infty.$$

This is of course weaker than (2.5) if $d \geq 2$.

6 Some Further Results

We first prove the following Darling–Erdős type theorem with a shifted limiting distribution.

Theorem 6.1 *Let X, X_n, $n \geq 1$ be i.i.d. real-valued random variables with $\mathbb{E}X^2 = 1$ and $\mathbb{E}X = 0$. Assume that for some $c > 0$,*

$$\mathbb{E}X^2 I\{|X| \geq t\} \sim c(LLt)^{-1} \text{ as } t \to \infty.$$

Then, we have,

$$a_n \max_{1 \leq k \leq n} |S_k|/\sqrt{k} - b_n \xrightarrow{d} \tilde{Y} - c,$$

where $\tilde{Y}$ and b_n are defined as in (1.1).

Proof (i) Set $\sigma_n^2 = \mathbb{E}X^2 I\{|X| \leq \sqrt{n}\}$ and let $1 \leq k_n \leq \exp((Ln)^\alpha)$ for some $0 < \alpha < 1$. Then, we have by Theorem 2.1 and the argument in (4.1),

$$a_n \max_{k_n \leq k \leq n} |S_k|/\sqrt{k}\sigma_k - b_n \xrightarrow{d} \tilde{Y},$$

which trivially implies for any sequence ρ_n of positive real numbers converging to 1,

$$\rho_n a_n \max_{k_n \leq k \leq n} |S_k|/\sqrt{k}\sigma_k - \rho_n b_n \xrightarrow{d} \tilde{Y} \tag{6.1}$$

Set $k_n = [\exp((Ln)^\alpha]$, where $0 < \alpha < 1$. Then, it is easy to see that

$$\mathbb{P}\left\{a_n \max_{1 \leq k \leq n} \frac{|S_k|}{\sqrt{k}} - b_n \leq y\right\}$$

$$\leq \mathbb{P}\left\{\sigma_{k_n} a_n \max_{k_n \leq k \leq n} \frac{|S_k|}{\sqrt{k}\sigma_k} - b_n \leq y\right\}$$

$$= \mathbb{P}\left\{\sigma_{k_n} a_n \max_{k_n \leq k \leq n} \frac{|S_k|}{\sqrt{k}\sigma_k} - \sigma_{k_n} b_n \leq y + (1 - \sigma_{k_n})b_n\right\}$$

Noticing that $(1 - \sigma_{k_n})b_n \sim (1 - \sigma_{k_n}^2)(2LLn)/(1 + \sigma_{k_n}) \sim c(LLk_n)^{-1}LLn$ (since $\sigma_{k_n}^2 \to \mathbb{E}X^2 = 1$), it is clear that $(1 - \sigma_{k_n})b_n \to c/\alpha$ as $n \to \infty$.
By (6.1) (with $\rho_n = \sigma_{k_n}$), this last sequence of probabilities converges to $\mathbb{P}\{\tilde{Y} \leq y + c/\alpha\}$.
Since this holds for any $0 < \alpha < 1$, it follows that

$$\limsup_{n \to \infty} \mathbb{P}\left\{a_n \max_{1 \leq k \leq n} \frac{|S_k|}{\sqrt{k}} - b_n \leq y\right\} \leq \mathbb{P}\{\tilde{Y} \leq y + c\}. \tag{6.2}$$

(ii) Similarly, we have,

$$\mathbb{P}\left\{a_n \max_{1 \le k \le n} \frac{|S_k|}{\sqrt{k}} - b_n \le y\right\}$$
$$\ge \mathbb{P}\left\{\sigma_n a_n \max_{1 \le k \le n} \frac{|S_k|}{\sqrt{k}\sigma_k} - \sigma_n b_n \le y + (1 - \sigma_n)b_n\right\},$$

where $(1 - \sigma_n)b_n \to c$ as $n \to \infty$.

Applying (6.1) (with $k_n = 1$ and $\rho_n = \sigma_n$), we obtain that

$$\liminf_{n \to \infty} \mathbb{P}\left\{a_n \max_{1 \le k \le n} \frac{|S_k|}{\sqrt{k}} - b_n \le y\right\} \ge \mathbb{P}\{\tilde{Y} \le y + c\}$$

and Theorem 6.1 has been proven. □

We finally mention the following result for real-valued random variables given in [13] where it is shown that if $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$ and $\mathbb{E}X^2 LL|X| < \infty$, then one has

$$2LLn\left(\sup_{k \ge n} \frac{|S_k|}{\sqrt{2kLLk}} - 1\right) - \frac{3}{2}LLLn + LLLLn + \log(3/\sqrt{8}) \xrightarrow{d} \tilde{Y}. \quad (6.3)$$

The authors asked whether this result can hold under the finite second moment assumption.

Using Theorem 2.3 in combination with Theorem 1.1 in [13], we obtain the following general result:

$$2LLn\left(\sup_{k \ge n} \frac{|S_k|}{\sqrt{2kLLk}\sigma_k} - 1\right) - \frac{3}{2}LLLn + LLLLn + \log(3/\sqrt{8}) \xrightarrow{d} \tilde{Y},$$

where $\sigma_n^2 = \mathbb{E}X^2 I\{|X| \le c_n\}$ and c_n is a non-decreasing sequence of positive real numbers satisfying condition (2.2). As in [6], this implies that (6.3) holds if and only if condition (2.5) is satisfied.

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