

Large and Moderate Deviation Principles for Engel Continued Fractions

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Abstract Large and moderate deviation principles are proved for Engel continued fractions, a new type of continued fraction expansion with non-decreasing partial quotients in number theory.

Keywords Engel continued fractions · Large deviation · Moderate deviation

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1 Introduction

Given a real number, there are various ways to represent it as an expansion of digits, such as continued fractions (see Khintchine [20]) and series expansions (see Galambos [14]) including β -expansions [29], Lüroth expansions [22], Engel expansions [8] and alternating Engel expansions [32]. Perhaps the most well-known representation of real numbers is continued fractions. Over the last thirty years, considerable interests are shown in various continued fraction expansions. Examples of such continued fraction expansions include, for example, backward continued fractions [1], a/b -continued

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fractions [4], Oppenheim continued fractions [9], multidimensional continued fractions [17], α -continued fractions [27] and Rosen continued fractions [5,31]. Most of these continued fraction expansions have invariant and ergodic measures which are absolutely continuous with respect to Lebesgue measure. By means of the ergodic theory and probability theory, some metric and statistical properties of these continued fraction expansions have been well studied when the real number is sampled from the uniform distribution in some interval. Besides, series expansions of real numbers also gave rise to a fruitful study of their ergodic and statistical properties. The history of this topic may begin with Borel, Sierpiński, Kuzmin, Lévy and continue with the seminal work of the Hungarian school (Erdős, Turán, Rényi, Szüsz, Révész, Galambos). Such a theory of using probability theory to answer questions of number theory is called the *probabilistic number theory*; see Elliott [7], Rényi [30] and Tenenbaum [33]. Essentially these classical studies for continued fractions and series expansions mainly focus on the distribution law, the law of large numbers, the central limit theorem and the law of the iterated logarithm for the digit sequence occurring in these expansions. However, it is worth pointing out that these classical limit theorems basically concern that the averages taken over large samples converge to expected values in some sense, but say little or nothing about the rate of convergence. One way to address this is the theory of large deviations in modern probability theory.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and $\{X_n : n \geq 1\}$ be a sequence of real-valued random variables defined on $(\Omega, \mathcal{F}, \mathbf{P})$. A function $I : \mathbb{R} \rightarrow [0, \infty]$ is called a *good rate function* if it is lower semi-continuous and has compact level sets. We say that the sequence $\{X_n : n \geq 1\}$ satisfies a *large deviation principle* (LDP for short) with speed n and good rate function I if for any Borel set Γ ,

$$-\inf_{x \in \Gamma^\circ} I(x) \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{P}(X_n \in \Gamma) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{P}(X_n \in \Gamma) \leq -\inf_{x \in \bar{\Gamma}} I(x),$$

where Γ° and $\bar{\Gamma}$ denote the interior and the closure of Γ , respectively. We use the notation $\mathbf{E}(\xi)$ to denote the expectation of a random variable ξ with respect to the probability measure \mathbf{P} . Gärtner–Ellis theorem (see Theorem 2.3.6 in [6]) tells us that the rate function is often given in term of the Legendre transform of the pressure function. That is, $I(x) = \sup_{\theta \in \mathbb{R}} \{\theta x - \Lambda(\theta)\}$, where the *pressure function* $\Lambda(\cdot)$ is defined as

$$\Lambda(\theta) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{E}(e^{n\theta \cdot X_n}) \tag{1.1}$$

for any $\theta \in \mathbb{R}$ when it exists. For instance, we can study the probability that the empirical mean of a sequence of random variables deviates away from its ergodic mean. These probabilities are exponentially small in general and follow the large deviation principle. Formally, there is no distinction between the large deviation principle and the moderate deviation principle (MDP for short). Usually LDP characterizes the convergence speed of the law of large numbers, while MDP describes the speed of convergence between the law of large numbers and the central limit theorem. For more details about large and moderate deviations, we refer the reader to Dembo and Zeitouni [6], Touchette [34] and Varadhan [35].

The theory of large deviations plays an important role in the framework of the stochastic processes. Nowadays, it has been developed quite rapidly in different directions and many applications arise, for example, in mathematics, statistics, computer science, physics, and in other fields. In this paper, we will investigate the large and moderate deviation principles for Engel continued fractions (see Sect. 2) which is a new type of continued fraction expansion with non-decreasing partial quotients. Recently, such kinds of problems have been considered by Zhu [38], Hu [19], Fang [10, 11], and Fang and Wu [12]. Zhu studied the large deviations for Engel expansions (see [8]), whose digit sequence is also non-decreasing. Erdős et al. [8] proved that the digit sequence of Engel expansions forms a time-homogeneous Markov chain with explicit transition probability functions. And they also gave an explicit computation for the distribution of the digit and its asymptotic analysis. These properties play an important role in Zhu's work. However, the Engel continued fraction system is not Markovian (see Remark 5). That is where the difficulty is. In [10], the author studied the large deviations for modified Engel continued fractions. This new continued fraction system is a small modification of Engel continued fractions, but its partial quotient sequence is strictly increasing. In [11], the author established the large deviations for alternating Engel expansions. This alternating expansion can be treated as the Engel expansion with alternating terms while its digit sequence is strictly increasing. Although these two expansions share the same classical limit theorems (see [36, 37]), the author also remarked that there is a difference between these expansions in the context of large deviations. Here we remark that the rate functions of the large deviations in [10] and [11] are same but different from Zhu's. The authors in [12] considered two interesting discrete Markov processes introduced by Williams [37], which share the same classical limit theorems but have a difference in the context of large deviations. Besides, the theory of large deviations also has been applied to the *analytic number theory*; see Féray et al. [13], Giuliano and Macci [16], Mehrdad and Zhu [25, 26] and Radziwill [28]. It seems that the large and moderate deviations might have the potential to become the useful tools in studying probabilistic and analytic number theory.

This paper is organized as follows. Section 2 is devoted to stating the large and moderate deviation principles for Engel continued fractions. In Sect. 3, we recall the basic properties of the Engel continued fraction expansion and show that the partial quotient sequence of Engel continued fractions is not Markovian (see Remark 5) but very close to a homogeneous Markov chain (see Proposition 3.4), which plays an important role in our proofs. The proofs of our main results are given in Sect. 4.

2 Main Results

In 2002, Hartono et al. [18] introduced a new continued fraction algorithm with non-decreasing partial quotients, called the *Engel continued fraction* (ECF, for short) expansion. The name of this new continued fraction expansion is borrowed from the classical Engel expansion (see Erdős et al. [8]). Now we give the algorithm of this new continued fraction expansion. Let $T_E : (0, 1) \rightarrow (0, 1)$ be the *ECF map* given by

$$T_E x = \frac{1}{\lfloor \frac{1}{x} \rfloor} \cdot \left(\frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \right),$$

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x . Then every real number $x \in (0, 1]$ can be uniquely written as

$$x = \frac{1}{b_1(x) + \frac{1}{b_2(x) + \frac{1}{\ddots + \frac{1}{b_{n-1}(x) + \frac{1}{b_n(x) + \ddots}}}}}, \tag{2.2}$$

where $b_1(x) = \lfloor 1/x \rfloor \in \mathbb{N}$ and $b_{n+1}(x) = b_1(T_E^n x)$ with $b_{n+1}(x) \geq b_n(x)$ for all $n \geq 1$. The representation (2.2) is said to be the *ECF expansion* of x , and $b_n(x)$ are called the *partial quotients* of the ECF expansion of x ($n \in \mathbb{N}$). Sometimes we write the form (2.2) as $[\lfloor b_1(x), b_2(x), \dots, b_n(x), \dots \rfloor]$. Hartono et al. [18] studied the arithmetic and ergodic properties of T_E associated with this new continued fraction expansion. Moreover, they showed that T_E is ergodic with respect to Lebesgue measure and proved that T_E has no finite invariant measure equivalent to the Lebesgue measure, but has infinitely many σ -finite invariant measures. This new class of continued fractions can be applied to designing a pseudorandom bit generator (PRBG) and proposing a new scheme for image cryptosystems; see Masmoudi et al. [23,24].

Now we denote $(\Omega, \mathcal{F}, \mathbf{P})$ by the probability space, where $\Omega = (0, 1]$, \mathcal{F} is the Borel σ -algebra on $(0, 1]$ and \mathbf{P} denotes the Lebesgue measure on $(0, 1]$. Kraaikamp and Wu [21] proved a strong law of large numbers for the partial quotient sequence $\{b_n : n \geq 1\}$, i.e., for \mathbf{P} -almost surely $x \in (0, 1]$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log b_n(x) = 1.$$

This implies that the probability of the event that $\frac{\log b_n(x)}{n}$ deviates away from its ergodic mean 1 tends to zero as n goes to infinity. However, it does not give this decay a speed. This leads to the study of large deviations for the ECF expansion.

Theorem 2.1 *Let $\{b_n : n \geq 1\}$ be the partial quotient sequence of the ECF expansion. Then $\left\{ \frac{\log b_n - n}{n} : n \geq 1 \right\}$ satisfies a LDP with speed n and good rate function*

$$I(x) = \begin{cases} x - \log(x + 1), & \text{if } x > -\frac{\sqrt{5}-1}{2}; \\ -\frac{\sqrt{5}+1}{2}(x + 1) + 2 \log \frac{\sqrt{5}+1}{2}, & \text{if } -1 \leq x \leq -\frac{\sqrt{5}-1}{2}; \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.3}$$

Remark 1 As we have pointed out at the end of introduction, the rate functions of the large deviations in [10] and [11] are same but different from Zhu’s and also different from ours. What is interesting though, is that the rate functions of the large deviations for Engel continued fractions and Engel expansions are similar in structure (comparing

our Theorem 2.1 with Theorem 1.2 of Zhu [38]). We think the main reason is that the digit sequence of modified Engel continued fractions or alternating Engel expansions is strictly increasing, while the digit sequence of Engel expansions or Engel continued fractions is non-decreasing (see Remark 6 for more details).

Remark 2 If we consider the large deviations for the sequence $\left\{ \frac{\log b_{n-n}}{n} : n \geq 1 \right\}$ with $b_1 \geq b$ ($b \in \mathbb{N}$), we will obtain that it also satisfies a LDP with speed n and good rate function

$$I_b(x) = \begin{cases} x - \log(x + 1), & \text{if } x > -1 + \xi_b^{-1}; \\ (1 - \xi_b)(x + 1) + \log \xi_b, & \text{if } -1 \leq x \leq -1 + \xi_b^{-1}; \\ +\infty, & \text{otherwise,} \end{cases}$$

where $\xi_b = (b^2 + 2 + \sqrt{b^2 + 4b}) / (2b)$. This indicates that the initial value of b_1 will affect the rate function of large deviations for the ECF expansion. Moreover, it is easy to check $I_1(x) = I(x)$ by (2.3) and $I_b(x) \rightarrow I_\infty(x)$ as $b \rightarrow \infty$, where I_∞ is defined as

$$I_\infty(x) = \begin{cases} x - \log(x + 1), & \text{if } x > -1; \\ +\infty, & \text{otherwise.} \end{cases}$$

Here we emphasize that I_∞ is the rate function of large deviations for modified ECF expansions [10] or alternating Engel expansions [11] or the empirical mean of independent and identically distributed exponential random variables with parameter 1 (see [6, Exercise 2.2.23] and [34, Example 3.2]).

As an application of Theorem 2.1, we obtain that the Lebesgue measure of the set of points $x \in (0, 1]$ for which $\log b_n(x)/n$ deviates away from its ergodic mean 1 decays to zero exponentially as n tends to infinity.

Corollary 1 *For any $\varepsilon > 0$, there exist two positive constants α and β (both only depending on ε) such that for all $n \geq 1$, we have*

$$\mathbf{P} \left\{ x \in (0, 1] : \left| \frac{\log b_n(x)}{n} - 1 \right| \geq \varepsilon \right\} \leq \alpha e^{-\beta n}.$$

Remark 3 By Borel–Cantelli lemma, we deduce easily that this result implies the strong law of large numbers for the sequence of partial quotients of ECF expansions obtained by Kraaikamp and Wu [21] in 2004.

In 2007, Fan et al. [9] established a central limit theorem for the partial quotient sequence $\{b_n : n \geq 1\}$, which tells us by how much the quantity $\log b_n(x)$ normally exceeds its ergodic mean 1, namely by an order of \sqrt{n} . That is, for every $y \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \mathbf{P} \{ x \in (0, 1] : \log b_n(x) - n \geq \sqrt{n}y \} = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-t^2/2} dt.$$

On the other hand, in Corollary 1 we have seen that for any $\varepsilon > 0$, the probability

$$\mathbf{P}\{x \in (0, 1] : \log b_n(x) - n \geq n\varepsilon\}$$

decays to zero exponentially as n tends to infinity. These two results imply that for any positive sequence a_n with $\sqrt{n} \ll a_n \ll n$, we still have

$$\mathbf{P}\{x \in (0, 1] : \log b_n(x) - n \geq a_n\}$$

tends to zero as n goes to infinity. However, neither the above central limit theorem nor the Theorem 2.1 tells us how fast this convergence is. This question is in the remit of the moderate deviation principle for ECF expansions stated below.

Theorem 2.2 *Let $\{b_n : n \geq 1\}$ be the partial quotient sequence of the ECF expansion and $\{a_n : n \geq 1\}$ be a positive sequence satisfying*

$$a_n \rightarrow \infty, \quad \frac{a_n}{\sqrt{n \log n}} \rightarrow \infty \quad \text{and} \quad \frac{a_n}{n} \rightarrow 0. \tag{2.4}$$

Then $\left\{ \frac{\log b_n - n}{a_n} : n \geq 1 \right\}$ satisfies an MDP with speed $n^{-1}a_n^2$ and good rate function $J(x) = x^2/2$ for any $x \in \mathbb{R}$.

Remark 4 We may obtain this MDP result if the second condition of a_n in (2.4) is replaced by $a_n/\sqrt{n} \rightarrow \infty$ as $n \rightarrow \infty$. However, we need the condition $\frac{a_n}{\sqrt{n \log n}} \rightarrow \infty$ to avoid some technical difficulties. The main reason is that we cannot find some finer estimates when we make use of the Gärtner–Ellis theorem. For this reason, we believe that

$$\liminf_{n \rightarrow \infty} \frac{\log b_n(x) - n}{\sqrt{2n \log \log n}} = -1 \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{\log b_n(x) - n}{\sqrt{2n \log \log n}} = 1$$

hold for \mathbf{P} -almost surely $x \in (0, 1]$ since the law of the iterated logarithms can be obtained by using some moderate deviation inequalities (see [2, 3, 15]).

As an application of Theorem 2.2, we immediately get the following corollary.

Corollary 2 *Let $\{b_n : n \geq 1\}$ be the partial quotient sequence of the ECF expansion and $a_n = n^p$ with $p \in (1/2, 1)$. Then $\left\{ \frac{\log b_n - n}{n^p} : n \geq 1 \right\}$ satisfies an MDP with speed n^{2p-1} and good rate function $J(x) = x^2/2$ for any $x \in \mathbb{R}$.*

3 Preliminary

In this section, we recall some definitions and several arithmetic and statistical properties of the ECF expansion. We first give an elementary arithmetic property of the ECF expansion in representation of real numbers, which is obtained by Hartono et al. [18].

Proposition 3.1 ([18, Theorem 2.1]) *Let $x \in (0, 1]$ be a real number. Then x has a finite ECF expansion (i.e., $T_E^n x = 0$ for some $n \geq 1$) if and only if x is rational.*

Definition 3.1 An n -block (b_1, b_2, \dots, b_n) is said to be admissible for ECF expansions if there exists $x \in (0, 1]$ such that $b_j(x) = b_j$ for all $1 \leq j \leq n$. An infinite sequence $(b_1, b_2, \dots, b_n, \dots)$ is called an admissible sequence if (b_1, b_2, \dots, b_n) is admissible for any $n \geq 1$.

The following proposition, due to Fan et al. [9], gives a characterization of all admissible sequences occurring in the ECF expansion.

Proposition 3.2 ([9, Proposition 2.2]) *A sequence of positive integers $(b_1, b_2, \dots, b_n, \dots)$ is admissible for ECF expansions if and only if for all $n \geq 1$,*

$$b_{n+1} \geq b_n.$$

Definition 3.2 Let (b_1, b_2, \dots, b_n) be an admissible sequence. We call the set

$$B(b_1, b_2, \dots, b_n) = \{x \in (0, 1] : b_1(x) = b_1, b_2(x) = b_2, \dots, b_n(x) = b_n\}$$

the n th order cylinder. In other words, it is the set of points beginning with (b_1, b_2, \dots, b_n) in their ECF expansions.

The following proposition is about the structure and the length of cylinders, which has been obtained in [18] (see also [9]).

Proposition 3.3 *Let (b_1, b_2, \dots, b_n) be an admissible sequence. Then the n th order cylinder $B(b_1, b_2, \dots, b_n)$ is an interval with two endpoints*

$$[[b_1, \dots, b_{n-1}, b_n]] \text{ and } [[b_1, \dots, b_{n-1}, b_n + 1]].$$

Hence that for all $n \geq 1$,

$$\mathbf{P}(B(b_1, b_2, \dots, b_n)) = \frac{\prod_{i=1}^{n-1} b_i}{Q_n(Q_n + Q_{n-1})}, \tag{3.5}$$

where the quantity Q_n satisfies the recursive formula $Q_n = b_n Q_{n-1} + b_{n-1} Q_{n-2}$ under the conventions $Q_{-1} = 0$ and $Q_0 = 1$.

Remark 5 The Markov property states that the distribution of the forthcoming state depends only on the current state and does not depend on the previous ones. So the partial quotient sequence $\{b_n : n \geq 1\}$ does not form a homogeneous Markov chain. In fact, by (3.5), we obtain that $\mathbf{P}(B(1, 1, 2)) = 1/35$, $\mathbf{P}(B(1, 2, 2)) = 1/44$, $\mathbf{P}(B(2, 2, 2)) = 1/88$, $\mathbf{P}(B(1, 1, 2, 2)) = 1/133$, $\mathbf{P}(B(1, 2, 2, 2)) = 1/165$, $\mathbf{P}(B(2, 2, 2, 2)) = 1/330$ and hence that

$$\mathbf{P}(b_4 = 2 \mid b_3 = 2, b_2 = 1, b_1 = 1) = 5/19 \neq 972/3667 = \mathbf{P}(b_4 = 2 \mid b_3 = 2).$$

Although the partial quotient sequence $\{b_n : n \geq 1\}$ does not form a homogeneous Markov chain, we have

Proposition 3.4 *Let $\{b_n : n \geq 1\}$ be the partial quotient sequence of the ECF expansion. Then for any $k \geq j \geq 1$,*

$$\mathbf{P}(b_1 = j) = \frac{1}{j(j + 1)} \tag{3.6}$$

and the conditional probabilities

$$\frac{j}{k(k + 2)} \leq \mathbf{P}(b_{n+1} = k \mid b_n = j) \leq \frac{j + 1}{k(k + 1)} \text{ for all } n \geq 1. \tag{3.7}$$

Proof Equation (3.6) is obvious by taking $n = 1$ in (3.5). Now we prove the inequalities (3.7). For two integers $1 \leq a \leq b$ and real number $0 \leq y < 1$, we define

$$\Phi(a, b, y) = \frac{a(1 + y)}{(b + ay)(b + 1 + ay)}.$$

For any $n \in \mathbb{N}$ and admissible sequence $(b_1, \dots, b_n, d_{n+1})$, by (3.5), we deduce that

$$\frac{\mathbf{P}(B(b_1, \dots, b_n, b_{n+1}))}{\mathbf{P}(B(b_1, \dots, b_n))} = \frac{b_n Q_n(Q_n + Q_{n-1})}{Q_{n+1}(Q_{n+1} + Q_n)} = \Phi(b_n, b_{n+1}, y_n), \tag{3.8}$$

where $y_n = Q_{n-1}/Q_n \geq 0$ and the last equality is from the recursive formula $Q_n = b_n Q_{n-1} + b_{n-1} Q_{n-2}$. This recursive formula of Q_n indicates that $0 \leq b_n y_n \leq 1$. So,

$$\Phi(b_n, b_{n+1}, y_n) \leq \frac{b_n + 1}{b_{n+1}(b_{n+1} + 1)}.$$

Since $b_{n+1} \geq b_n$ and $0 \leq b_n y_n \leq 1$, we obtain that

$$\Phi(b_n, b_{n+1}, y_n) \geq \frac{b_n}{b_{n+1}(b_{n+1} + 2)}.$$

Combing these with (3.8), we have that

$$\frac{b_n}{b_{n+1}(b_{n+1} + 2)} \leq \frac{\mathbf{P}(B(b_1, \dots, b_n, b_{n+1}))}{\mathbf{P}(B(b_1, \dots, b_n))} \leq \frac{b_n + 1}{b_{n+1}(b_{n+1} + 1)}.$$

Taking summations for all admissible sequences (b_1, \dots, b_n) , the inequalities (3.7) are established. □

4 Proofs of Main Results

In this section, we will give the proof of Theorems 2.1 and 2.2. To do this, we first give a simple but useful lemma, which states roughly that the rate of growth for a finite sum of sequences equals the maximal rate of growth of the summands.

Lemma 4.1 *Let $m \geq 2$ be an integer and $\{c_n\}_{n \geq 1}$ be a sequence satisfying $c_n \rightarrow \infty$ as $n \rightarrow \infty$. Then*

$$\limsup_{n \rightarrow \infty} \frac{1}{c_n} \log \left(\sum_{k=1}^m d_n^{(k)} \right) = \max_{1 \leq k \leq m} \limsup_{n \rightarrow \infty} \frac{1}{c_n} \log d_n^{(k)},$$

where $\{d_n^{(1)}\}_{n \geq 1}, \dots, \{d_n^{(m)}\}_{n \geq 1}$ are nonnegative sequences.

Proof Let $m \geq 2$ be an integer. For any $n \geq 1$, since

$$0 \leq \log \left(\sum_{k=1}^m d_n^{(k)} \right) - \max_{1 \leq k \leq m} \log d_n^{(k)} \leq \log m,$$

we deduce that

$$\limsup_{n \rightarrow \infty} \frac{1}{c_n} \log \left(\sum_{k=1}^m d_n^{(k)} \right) = \limsup_{n \rightarrow \infty} \frac{1}{c_n} \max_{1 \leq k \leq m} \log d_n^{(k)} = \max_{1 \leq k \leq m} \limsup_{n \rightarrow \infty} \frac{1}{c_n} \log d_n^{(k)}.$$

□

Lemma 4.2 *Let $\theta < 1$ be a real number. Then for any $j > 1$, we have*

$$\sum_{k=j}^{\infty} \frac{j}{k(k+2)} \left(\frac{k}{j} \right)^{\theta} \geq \frac{j}{j+2} \cdot \frac{1}{1-\theta}$$

and

$$\sum_{k=j}^{\infty} \frac{j+1}{k(k+1)} \left(\frac{k}{j} \right)^{\theta} \leq \left(1 + \frac{1}{j} \right) \cdot \left(1 - \frac{1}{j} \right)^{\theta-1} \cdot \frac{1}{1-\theta}.$$

Proof Let $\theta < 1$ be a real number. For any $j > 1$, we obtain that

$$\sum_{k=j}^{\infty} \frac{1}{k^{2-\theta}} \geq \int_j^{\infty} \frac{1}{x^{2-\theta}} dx = \frac{1}{1-\theta} \cdot j^{\theta-1}. \tag{4.9}$$

Since

$$\frac{k^{\theta}}{k(k+2)} = \frac{1}{k^{2-\theta}} \cdot \frac{k}{k+2} \quad \text{and} \quad \frac{k}{k+2} \geq \frac{j}{j+2} \quad \text{for any } k \geq j,$$

we have that

$$\sum_{k=j}^{\infty} \frac{j}{k(k+2)} \left(\frac{k}{j}\right)^{\theta} = \frac{j}{j^{\theta}} \cdot \sum_{k=j}^{\infty} \frac{k^{\theta}}{k(k+2)} \geq \frac{j}{j^{\theta}} \cdot \frac{j}{j+2} \cdot \sum_{k=j}^{\infty} \frac{1}{k^{2-\theta}}.$$

Combing this with (4.9), we deduce that

$$\sum_{k=j}^{\infty} \frac{j}{k(k+2)} \left(\frac{k}{j}\right)^{\theta} \geq \frac{j}{j^{\theta}} \cdot \frac{j}{j+2} \cdot \frac{1}{1-\theta} \cdot j^{\theta-1} = \frac{j}{j+2} \cdot \frac{1}{1-\theta}.$$

Notice that for any $j > 1$,

$$\sum_{k=j}^{\infty} \frac{k^{\theta}}{k(k+1)} \leq \sum_{k=j}^{\infty} \frac{1}{k^{2-\theta}} \leq \int_{j-1}^{\infty} \frac{1}{x^{2-\theta}} dx = \frac{1}{1-\theta} \cdot (j-1)^{\theta-1},$$

we know that

$$\sum_{k=j}^{\infty} \frac{j+1}{k(k+1)} \left(\frac{k}{j}\right)^{\theta} = \frac{j+1}{j^{\theta}} \cdot \sum_{k=j}^{\infty} \frac{k^{\theta}}{k(k+1)} \leq \left(1 + \frac{1}{j}\right) \cdot \left(1 - \frac{1}{j}\right)^{\theta-1} \cdot \frac{1}{1-\theta}.$$

□

4.1 Proof of Large Deviation Principle

For any $m, n \in \mathbb{N}$, we define

$$\Sigma_{n,m} = \{(b_1, \dots, b_{n-1}, m) : (b_1, \dots, b_{n-1}, m) \text{ is admissible}\}$$

and

$$\Sigma_{n,\leq m} = \{(b_1, \dots, b_{n-1}, j) : (b_1, \dots, b_{n-1}, j) \text{ is admissible for all } 1 \leq j \leq m\}.$$

We believe that the following result is known in the combinatorial theory, while, to our knowledge, we cannot find its proof in any book about combination. For the completeness of this paper, we give its proof here using the mathematical induction.

Lemma 4.3 For any $m, n \in \mathbb{N}$,

$$\sharp \Sigma_{n,m} = C_{n+m-2}^{m-1} \quad \text{and} \quad \sharp \Sigma_{n,\leq m} = C_{n+m-1}^{m-1},$$

where \sharp denotes the number of elements of a finite set and C_n^m means the number of possible combinations of m objects from a set of n objects.

Proof We first prove the first formula by induction and then prove the second formula. In view of Proposition 3.2, we know that the set $\Sigma_{n,1}$ only has one element as $(1, 1, \dots, 1)$. So, $\#\Sigma_{n,1} = 1$. It is easy to check that the set $\Sigma_{n,2}$ has n elements like $(1, \dots, 1, 2), (1, \dots, 2, 2), \dots, (2, \dots, 2, 2)$. Thus, $\#\Sigma_{n,2} = n$. These indicate that the formula $\#\Sigma_{n,m} = C_{n+m-2}^{m-1}$ is true for $m = 1$ and 2 . Now we assume that this formula is true for all $m \leq k$, i.e.,

$$\#\Sigma_{n,1} = C_{n-1}^0, \#\Sigma_{n,2} = C_n^1, \#\Sigma_{n,3} = C_{n+1}^2, \dots, \#\Sigma_{n,k} = C_{n+k-2}^{k-1}.$$

For $m = k + 1$, notice that

$$\#\Sigma_{n,k+1} = \#\Sigma_{n-1,1} + \#\Sigma_{n-1,2} + \#\Sigma_{n-1,3} + \dots + \#\Sigma_{n-1,k} + \#\Sigma_{n-1,k+1},$$

we obtain that

$$\begin{aligned} \#\Sigma_{n,k+1} - \#\Sigma_{n-1,k+1} &= \#\Sigma_{n-1,1} + \#\Sigma_{n-1,2} + \#\Sigma_{n-1,3} + \dots + \#\Sigma_{n-1,k} \\ &= C_{n-1}^0 + C_n^1 + C_{n+1}^2 + \dots + C_{n+k-3}^{k-1} \\ &= C_n^0 + C_n^1 + C_{n+1}^2 + \dots + C_{n+k-3}^{k-1} \\ &= C_{n+1}^1 + C_{n+1}^2 + \dots + C_{n+k-3}^{k-1} \\ &= \dots \\ &= C_{n+k-2}^{k-1}, \end{aligned} \tag{4.10}$$

where third equality follows from $C_{n-1}^0 = C_n^0 = 1$ and the fourth equality is from the basic combination equation $C_n^j + C_n^{j+1} = C_{n+1}^{j+1}$. Being similar to (4.10), we deduce that

$$\#\Sigma_{n-1,k+1} - \#\Sigma_{n-2,k+1} = C_{n+k-3}^{k-1}, \dots, \#\Sigma_{2,k+1} - \#\Sigma_{1,k+1} = C_k^{k-1}.$$

Since $C_n^j = C_n^{n-j}$, we have that

$$\begin{aligned} \#\Sigma_{n,k+1} &= C_{n+k-2}^{k-1} + C_{n+k-3}^{k-1} + \dots + C_{k+1}^{k-1} + C_k^{k-1} + \#\Sigma_{1,k+1} \\ &= C_{n+k-2}^{n-1} + C_{n+k-3}^{n-2} + \dots + C_{k+1}^2 + C_k^1 + 1 \\ &= C_{n+k-2}^{n-1} + C_{n+k-3}^{n-2} + \dots + C_{k+1}^2 + C_{k+1}^1 \\ &= \dots \\ &= C_{n+k-1}^{n-1} = C_{n+k-1}^k, \end{aligned}$$

where the second equality follows from $\Sigma_{1,k+1} = \{(k + 1)\}$ with only one element and the third equality is also from the basic combination equation $C_n^j + C_n^{j+1} = C_{n+1}^{j+1}$. This is to say the formula $\#\Sigma_{n,m} = C_{n+m-2}^{m-1}$ is also true for $m = k + 1$. By induction, we obtain the desired result. Now we are ready to prove $\#\Sigma_{n,\leq m} = C_{n+m-1}^{m-1}$. In fact, it follows from the first result that

$$\begin{aligned} \# \Sigma_{n, \leq m} &= \# \Sigma_{n,1} + \# \Sigma_{n,2} + \# \Sigma_{n,3} + \dots + \# \Sigma_{n,m} \\ &= C_n^0 + C_{n+1}^1 + C_{n+2}^2 + \dots + C_{n+m-2}^{m-1} \\ &= C_{n+1}^1 + C_{n+2}^2 + \dots + C_{n+m-2}^{m-1} \\ &= \dots \\ &= C_{n+m-1}^{m-1}. \end{aligned}$$

□

Lemma 4.4 *Let $\{b_n : n \geq 1\}$ be the partial quotient sequence of the ECF expansion. Then for any $n \in \mathbb{N}$ and $j > 1$,*

$$\mathbf{P}(b_n = 1) \geq A^{-1} \cdot \left(\frac{\sqrt{5} + 1}{2}\right)^{-2n} \quad \text{and} \quad \mathbf{P}(b_n \leq j) \leq A \cdot C_{n+j-1}^{j-1} \cdot \left(\frac{\sqrt{5} + 1}{2}\right)^{-2n},$$

where $A > 1$ is an absolute constant and the combinatorial number C_n^m is as defined in Lemma 4.3.

Proof By the non-decreasing property of b_n , it follows from Proposition 3.3 that

$$\mathbf{P}(b_n = 1) = \mathbf{P}(B(\underbrace{1, 1, \dots, 1}_n)) = \frac{1}{Q_n(Q_n + Q_{n-1})}, \tag{4.11}$$

where Q_n satisfies the recursive formula $Q_n = Q_{n-1} + Q_{n-2}$ under the conventions $Q_{-1} = 0$ and $Q_0 = 1$. This indicates that $\{Q_n\}_{n \geq 0}$ is a sequence of Fibonacci numbers. So,

$$Q_n = \frac{1}{\sqrt{5}} \cdot \left(\left(\frac{1 + \sqrt{5}}{2}\right)^{n+1} - \left(\frac{1 - \sqrt{5}}{2}\right)^{n+1} \right).$$

Since $-1 \leq \left(\frac{1 - \sqrt{5}}{2}\right)^n \leq \frac{\sqrt{5} - 1}{\sqrt{5} + 1}$ for any $n \geq 1$, it is easy to check that

$$\frac{2}{5 + \sqrt{5}} \cdot \left(\frac{1 + \sqrt{5}}{2}\right)^{n+1} \leq Q_n \leq \frac{2}{\sqrt{5}} \cdot \left(\frac{1 + \sqrt{5}}{2}\right)^{n+1}.$$

Combing this with (4.11), notice that $Q_n \geq Q_{n-1}$, we have that

$$A^{-1} \cdot \left(\frac{\sqrt{5} + 1}{2}\right)^{-2n} \leq \frac{1}{2Q_n^2} \leq \mathbf{P}(b_n = 1) \leq \frac{1}{Q_n^2} = A \cdot \left(\frac{\sqrt{5} + 1}{2}\right)^{-2n}, \tag{4.12}$$

where $A > 1$ is an absolute constant. Since

$$\{x \in [0, 1) : b_n(x) \leq j\} = \bigcup_{(b_1, b_2, \dots, b_n) \in \Sigma_{n, \leq j}} B(b_1, b_2, \dots, b_n)$$

and for any $(b_1, b_2, \dots, b_n) \in \Sigma_{n, \leq j}$,

$$\mathbf{P}(B(b_1, b_2, \dots, b_n) \leq \underbrace{\mathbf{P}(B(1, 1, \dots, 1))}_n),$$

combing these with Lemma 4.3 and (4.12), we deduce that

$$\begin{aligned} \mathbf{P}\{x \in [0, 1) : b_n(x) \leq j\} &\leq \#\Sigma_{n, \leq j} \cdot \mathbf{P}(B(1, 1, \dots, 1)) \\ &\leq A \cdot C_{n+j-1}^{j-1} \cdot \left(\frac{\sqrt{5} + 1}{2}\right)^{-2n}. \end{aligned}$$

To prove Theorem 2.1, we also need the following Lemma 4.5. □

Lemma 4.5 *Let $\{b_n : n \geq 1\}$ be the partial quotient sequence of the ECF expansion. Then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{E}(b_n^\theta) = \begin{cases} \max \left\{ -2 \log \frac{\sqrt{5}+1}{2}, \log \frac{1}{1-\theta} \right\}, & \text{if } \theta < 1; \\ +\infty, & \text{if } \theta \geq 1. \end{cases}$$

Proof Let $\theta \geq 1$. Notice that $b_{n+1} \geq b_n$ with $b_1 \geq 1$ for all $n \geq 1$, Eq. (3.6) yields that for any $n \geq 1$,

$$\mathbf{E}(b_n^\theta) \geq \mathbf{E}(b_1^\theta) = \sum_{k=1}^{\infty} \mathbf{P}(b_1 = k) \cdot k^\theta = \sum_{k=1}^{\infty} \frac{k^\theta}{k(k+1)} = +\infty.$$

Therefore, $\lim_{n \rightarrow \infty} \frac{1}{n} \log E(b_n^\theta) = +\infty$.

In the following, we always assume that $\theta < 1$. Since

$$\lim_{j \rightarrow \infty} \frac{j}{j+2} = \lim_{j \rightarrow \infty} \left(1 + \frac{1}{j}\right) \cdot \left(1 - \frac{1}{j}\right)^{\theta-1} = 1,$$

Proposition 3.8 and Lemma 4.2 imply that for any $0 < \varepsilon < 1$, there exists positive integer $N = N(\varepsilon)$ such that for all $j > N$,

$$\frac{1 - \varepsilon}{1 - \theta} \leq \sum_{k=j}^{\infty} \mathbf{P}(b_{n+1} = k \mid b_n = j) \cdot \left(\frac{k}{j}\right)^\theta \leq \frac{1 + \varepsilon}{1 - \theta}. \tag{4.13}$$

By the definition of expectation, we know that

$$\mathbf{E}(b_n^\theta) = \sum_{k=1}^{\infty} \mathbf{P}(b_n = k) \cdot k^\theta = \sum_{k=1}^N \mathbf{P}(b_n = k) \cdot k^\theta + \sum_{k=N+1}^{\infty} \mathbf{P}(b_n = k) \cdot k^\theta.$$

We will prove

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{E}(b_n^\theta) = \max \left\{ -2 \log \frac{\sqrt{5} + 1}{2}, \log \frac{1}{1 - \theta} \right\}.$$

The proof is divided into two parts:

Part 1: Lower bound

- By (4.12), it is clear to see that for any $n \geq 1$,

$$\sum_{k=1}^N \mathbf{P}(b_n = k) \cdot k^\theta \geq \mathbf{P}(b_n = 1) = \mathbf{P}(B(1, 1, \dots, 1)) \geq A^{-1} \cdot \left(\frac{\sqrt{5} + 1}{2} \right)^{-2n}.$$

- For any $n \geq 1$, by the definition of conditional probability, we deduce that

$$\begin{aligned} \sum_{k=N+1}^\infty \mathbf{P}(b_n = k) \cdot k^\theta &= \sum_{k=N+1}^\infty \sum_{j=1}^k \mathbf{P}(b_n = k \mid b_{n-1} = j) \cdot \mathbf{P}(b_{n-1} = j) \cdot k^\theta \\ &= \sum_{j=1}^N \mathbf{P}(b_{n-1} = j) \cdot j^\theta \\ &\quad \cdot \sum_{k=N+1}^\infty \mathbf{P}(b_n = k \mid b_{n-1} = j) \cdot \left(\frac{k}{j} \right)^\theta \\ &\quad + \sum_{j=N+1}^\infty \mathbf{P}(b_{n-1} = j) \cdot j^\theta \\ &\quad \cdot \sum_{k=j}^\infty \mathbf{P}(b_n = k \mid b_{n-1} = j) \cdot \left(\frac{k}{j} \right)^\theta, \end{aligned} \tag{4.14}$$

where the second equality is obtained by inverting the order of summations. Since the first term of the second equality in (4.14) is nonnegative, we have that

$$\begin{aligned} \sum_{k=N+1}^\infty \mathbf{P}(b_n = k) \cdot k^\theta &\geq \sum_{j=N+1}^\infty \mathbf{P}(b_{n-1} = j) \cdot j^\theta \\ &\quad \cdot \sum_{k=j}^\infty \mathbf{P}(b_n = k \mid b_{n-1} = j) \cdot \left(\frac{k}{j} \right)^\theta \\ &\geq \frac{1 - \varepsilon}{1 - \theta} \cdot \sum_{j=N+1}^\infty \mathbf{P}(b_{n-1} = j) \cdot j^\theta, \end{aligned}$$

where the last inequality follows from (4.13). Repeating the above procedure $(n - 1)$ times, we obtain that

$$\begin{aligned} \sum_{k=N+1}^{\infty} \mathbf{P}(b_n = k) \cdot k^\theta &\geq \frac{1 - \varepsilon}{1 - \theta} \cdot \sum_{j=N+1}^{\infty} \mathbf{P}(b_{n-1} = j) \cdot j^\theta \\ &\geq \dots \\ &\geq \left(\frac{1 - \varepsilon}{1 - \theta}\right)^{n-1} \cdot \sum_{j=N+1}^{\infty} \mathbf{P}(b_1 = j) \cdot j^\theta \\ &:= M_1 \cdot \left(\frac{1 - \varepsilon}{1 - \theta}\right)^{n-1}, \end{aligned}$$

where $M_1 := M_1(\varepsilon, \theta) = \sum_{j=N+1}^{\infty} \frac{j^\theta}{j(j+1)}$ is a positive constant since $\theta < 1$ and it also only depends on ε and θ . Therefore,

$$\mathbf{E}(b_n^\theta) \geq A^{-1} \cdot \left(\frac{\sqrt{5} + 1}{2}\right)^{-2n} + M_1 \cdot \left(\frac{1 - \varepsilon}{1 - \theta}\right)^{n-1}.$$

In view of Lemma 4.1, we deduce that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{E}(b_n^\theta) &\geq \liminf_{n \rightarrow \infty} \frac{1}{n} \log \left(A^{-1} \cdot \left(\frac{\sqrt{5} + 1}{2}\right)^{-2n} + M_1 \cdot \left(\frac{1 - \varepsilon}{1 - \theta}\right)^{n-1} \right) \\ &= \max \left\{ -2 \log \frac{\sqrt{5} + 1}{2}, \log \frac{1 - \varepsilon}{1 - \theta} \right\}. \end{aligned}$$

Since $0 < \varepsilon < 1$ is arbitrary, we get

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{E}(b_n^\theta) \geq \max \left\{ -2 \log \frac{\sqrt{5} + 1}{2}, \log \frac{1}{1 - \theta} \right\}.$$

Part 2: Upper bound Let $\gamma = ((\sqrt{5} + 1)/2)^{-2}$.

- In view of Lemma 4.4, we have that

$$\sum_{k=1}^N \mathbf{P}(b_n = k) \cdot k^\theta \leq N \cdot \mathbf{P}(b_n \leq N) \leq N \cdot A \cdot C_{n+N-1}^{N-1} \cdot \gamma^n.$$

- We first estimate the first term of the second equality in (4.14) and then give an upper bound for the second term. Notice that $\sum_{j=1}^N \mathbf{P}(b_{n-1} = j) = \mathbf{P}(b_{n-1} \leq N) \leq A \cdot C_{n+N-2}^{N-1} \cdot \gamma^{n-1}$, we know that

$$\begin{aligned}
 & \sum_{j=1}^N \mathbf{P}(b_{n-1} = j) \cdot j^\theta \cdot \sum_{k=N+1}^\infty \mathbf{P}(b_n = k \mid b_{n-1} = j) \cdot \left(\frac{k}{j}\right)^\theta \\
 & \leq \sum_{j=1}^N \mathbf{P}(b_{n-1} = j) \cdot \sum_{k=N+1}^\infty \frac{j+1}{k(k+1)} \cdot k^\theta \\
 & \leq (N+1) \cdot \sum_{j=1}^N \mathbf{P}(b_{n-1} = j) \cdot \sum_{k=N+1}^\infty \frac{k^\theta}{k(k+1)} \\
 & \leq M_1 \cdot (N+1) \cdot A \cdot C_{n+N-2}^{N-1} \cdot \gamma^{n-1} := M_2 \cdot C_{n+N-2}^{N-1} \cdot \gamma^{n-1}, \tag{4.15}
 \end{aligned}$$

where the first inequality is from Proposition 3.4, the second inequality is from $j \leq N$ and the constant $M_2 = M_1 \cdot (N+1) \cdot A$ only depends on ε and θ . It follows from the right inequality in (4.13) that

$$\begin{aligned}
 & \sum_{j=N+1}^\infty \mathbf{P}(b_{n-1} = j) \cdot j^\theta \cdot \sum_{k=j}^\infty \mathbf{P}(b_n = k \mid b_{n-1} = j) \cdot \left(\frac{k}{j}\right)^\theta \\
 & \leq \frac{1+\varepsilon}{1-\theta} \cdot \sum_{j=N+1}^\infty \mathbf{P}(b_{n-1} = j) \cdot j^\theta.
 \end{aligned}$$

Combing this with (4.14) and (4.15), we obtain that

$$\sum_{k=N+1}^\infty \mathbf{P}(b_n = k) \cdot k^\theta \leq M_2 \cdot C_{n+N-2}^{N-1} \cdot \gamma^{n-1} + \frac{1+\varepsilon}{1-\theta} \cdot \sum_{j=N+1}^\infty \mathbf{P}(b_{n-1} = j) \cdot j^\theta. \tag{4.16}$$

Similarly, we have that

$$\sum_{j=N+1}^\infty \mathbf{P}(b_{n-1} = j) \cdot j^\theta \leq M_2 \cdot C_{n+N-3}^{N-1} \cdot \gamma^{n-2} + \frac{1+\varepsilon}{1-\theta} \cdot \sum_{j=N+1}^\infty \mathbf{P}(b_{n-2} = j) \cdot j^\theta.$$

In view of (4.16), we deduce that

$$\begin{aligned}
 & \sum_{k=N+1}^\infty \mathbf{P}(b_n = k) \cdot k^\theta \\
 & \leq M_2 \cdot C_{n+N-2}^{N-1} \cdot \gamma^{n-1} + \frac{1+\varepsilon}{1-\theta} \cdot M_2 \cdot C_{n+N-3}^{N-1} \cdot \gamma^{n-2} + \left(\frac{1+\varepsilon}{1-\theta}\right)^2 \\
 & \cdot \sum_{j=N+1}^\infty \mathbf{P}(b_{n-2} = j) \cdot j^\theta
 \end{aligned}$$

$$\begin{aligned} &\leq M_2 \cdot C_{n+N-2}^{N-1} \cdot \left(\gamma^{n-1} + \gamma^{n-2} \cdot \frac{1+\varepsilon}{1-\theta} \right) + \left(\frac{1+\varepsilon}{1-\theta} \right)^2 \\ &\cdot \sum_{j=N+1}^{\infty} \mathbf{P}(b_{n-2} = j) \cdot j^\theta, \end{aligned} \tag{4.17}$$

where the last inequality is from $C_{n+N-3}^{N-1} \leq C_{n+N-2}^{N-1}$. Iterating the process in (4.17), we know that

$$\begin{aligned} \sum_{k=N+1}^{\infty} \mathbf{P}(b_n = k) \cdot k^\theta &\leq M_2 \cdot C_{n+N-2}^{N-1} \cdot \sum_{m=0}^{n-2} \gamma^{n-m} \left(\frac{1+\varepsilon}{1-\theta} \right)^m + M_1 \cdot \left(\frac{1+\varepsilon}{1-\theta} \right)^{n-1} \\ &\leq M_2 \cdot C_{n+N-2}^{N-1} \cdot \sum_{m=0}^{n-1} \gamma^{n-m} \left(\frac{1+\varepsilon}{1-\theta} \right)^m \end{aligned} \tag{4.18}$$

where the first inequality is from the definition of M_1 and the last inequality follows from $C_{n+N-2}^{N-1} \geq 1$ and the definition of M_2 . It is not difficult to check that the geometric series

$$\sum_{m=0}^{n-1} \gamma^{n-m} \left(\frac{1+\varepsilon}{1-\theta} \right)^m = M_3 \cdot \left(\left(\frac{1+\varepsilon}{1-\theta} \right)^n - \gamma^n \right),$$

where M_3 is the constant only depending on ε and θ . Combing this with (4.18), we have that

$$\sum_{k=N+1}^{\infty} \mathbf{P}(b_n = k) \cdot k^\theta \leq M \cdot C_{n+N-2}^{N-1} \cdot \left(\left(\frac{1+\varepsilon}{1-\theta} \right)^n - \gamma^n \right),$$

where $M = M_2 \cdot M_3$ is a constant only depending on ε and θ . Therefore,

$$\begin{aligned} \mathbf{E}(b_n^\theta) &= \sum_{k=1}^N \mathbf{P}(b_n = k) \cdot k^\theta + \sum_{k=N+1}^{\infty} \mathbf{P}(b_n = k) \cdot k^\theta \\ &\leq N \cdot A \cdot C_{n+N-1}^{N-1} \cdot \gamma^n + M \cdot C_{n+N-2}^{N-1} \cdot \left(\left(\frac{1+\varepsilon}{1-\theta} \right)^n - \gamma^n \right) \\ &= H(n) \cdot \left(\frac{\sqrt{5}+1}{2} \right)^{-2n} + M \cdot C_{n+N-2}^{N-1} \cdot \left(\frac{1+\varepsilon}{1-\theta} \right)^n, \end{aligned}$$

where C_{n+N-2}^{N-1} and $H(n)$ are both polynomials of n . Consequently, we deduce that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{E}(b_n^\theta) \\ & \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \left(H(n) \cdot \left(\frac{\sqrt{5} + 1}{2} \right)^{-2n} + M \cdot C_{n+N-2}^{N-1} \cdot \left(\frac{1 + \varepsilon}{1 - \theta} \right)^n \right) \\ & = \max \left\{ -2 \log \frac{\sqrt{5} + 1}{2}, \log \frac{1 + \varepsilon}{1 - \theta} \right\}, \end{aligned}$$

where the last equality is obtained by observing that C_{n+N-2}^{N-1} and $H(n)$ are both polynomials of n . Let $\varepsilon \rightarrow 0^+$, we obtain that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{E}(b_n^\theta) \leq \max \left\{ -2 \log \frac{\sqrt{5} + 1}{2}, \log \frac{1}{1 - \theta} \right\}.$$

□

Remark 6 Since the digit sequence $\{q_n : n \geq 1\}$ (the notation follows Zhu [38]) of Engel expansions is also non-decreasing and forms a homogeneous Markov chain (see Erdős et al. [8]), we know that the similar result of Lemma 4.4 in the setting of Engel expansions can be obtained

$$\mathbf{P}(q_n = 2) = 2^{-n} \quad \text{and} \quad \mathbf{P}(q_n \leq j) \leq C_{n+j-1}^{j-1} \cdot 2^{-n}. \tag{4.19}$$

The above method in Lemma 4.5 immediately yields that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{E}(q_n^\theta) = \begin{cases} \max \left\{ -\log 2, \log \frac{1}{1-\theta} \right\}, & \text{if } \theta < 1; \\ +\infty, & \text{if } \theta \geq 1, \end{cases}$$

where the quantity $\log 2$ is from (4.19). This limit plays an important role in the proof of Zhu (see [38, Lemma 1.1]). However, the digit sequence $\{d_n : n \geq 1\}$ of modified Engel continued fractions or alternating Engel expansions is increasing. The similar arguments of Lemmas 4.4 and 4.5 imply that

$$\mathbf{P}(d_n = n) = \frac{1}{n!} \quad \text{and} \quad \mathbf{P}(q_n \leq j) \leq C_{n+j-1}^{j-1} \cdot \frac{1}{n!}.$$

Note that $\lim_{n \rightarrow \infty} (-\log n!)/n = -\infty$, so we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{E}(d_n^\theta) = \begin{cases} \log \left(\frac{1}{1-\theta} \right), & \text{if } \theta < 1; \\ +\infty, & \text{if } \theta \geq 1. \end{cases}$$

Combing this with Gärtner–Ellis theorem, we obtain the large deviations for modified Engel continued fractions and alternating Engel expansions (see [10] and [11]).

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1 For any $\theta \in \mathbb{R}$, we define the function as

$$\Lambda_n(\theta) = \log \mathbf{E} \left(\exp \left(\frac{\log b_n - n}{n} \cdot \theta \right) \right).$$

By Lemma 4.5, we know that the pressure function is given as

$$\Lambda(\theta) = \lim_{n \rightarrow \infty} \frac{1}{n} \Lambda_n(n\theta) = \begin{cases} -\theta - 2 \log \frac{\sqrt{5}+1}{2}, & \text{if } \theta \leq -\frac{\sqrt{5}+1}{2}; \\ -\theta - \log(1 - \theta), & \text{if } -\frac{\sqrt{5}+1}{2} < \theta < 1; \\ +\infty, & \text{if } \theta \geq 1. \end{cases}$$

By Gärtner–Ellis theorem (see [6, Theorem 2.3.6]), we obtain the sequence $\left\{ \frac{\log b_n - n}{n} : n \geq 1 \right\}$ satisfies a LDP with speed n and good rate function

$$I(x) = \sup_{\theta \in \mathbb{R}} \{ \theta x - \Lambda(\theta) \} \text{ for all } x \in \mathbb{R}.$$

Now we show that $I(x)$ is the same as (2.3). Notice that $\theta x - \Lambda(\theta) = -\infty$ for all $x \in \mathbb{R}$ when $\theta \geq 1$, so we only need to compute the followings

$$\Phi(x) = \sup_{-\frac{\sqrt{5}+1}{2} < \theta < 1} \{ \theta x + \theta + \log(1 - \theta) \}$$

and

$$\Psi(x) = \sup_{\theta \leq -\frac{\sqrt{5}+1}{2}} \left\{ \theta x + \theta + 2 \log \frac{\sqrt{5} + 1}{2} \right\}.$$

Thus, $I(x) = \max\{\Phi(x), \Psi(x)\}$ for all $x \in \mathbb{R}$. We first give that

$$\Phi(x) = \begin{cases} x - \log(x + 1), & \text{if } x > -\frac{\sqrt{5}-1}{2}; \\ -\frac{\sqrt{5}+1}{2}(x + 1) + 2 \log \frac{\sqrt{5}+1}{2}, & \text{if } x \leq -\frac{\sqrt{5}-1}{2}. \end{cases}$$

Let $f(\theta) = \theta x + \theta + \log(1 - \theta)$ for any $-(\sqrt{5} + 1)/2 < \theta < 1$. It is clear to check f is strictly concave and that $\theta = x/(x + 1)$ is the unique maximal point of it. From the inequalities $-(\sqrt{5} + 1)/2 < x/(x + 1) < 1$, we know that $x > -(\sqrt{5} - 1)/2$. That is to say, when $x > -(\sqrt{5} - 1)/2$, the function f reaches the maximal value at the point $x/(x + 1) \in (-(\sqrt{5} + 1)/2, 1)$, i.e., $\Phi(x) = x - \log(x + 1)$ in this case. Here we claim that f is decreasing on the interval $(-(\sqrt{5} + 1)/2, 1)$ when $x \leq -(\sqrt{5} - 1)/2$. In fact, it follows from $-1 < x \leq -(\sqrt{5} - 1)/2$ that the maximal

point $x/(x + 1) \leq -(\sqrt{5} + 1)/2$. That is to say, the maximal point $x/(x + 1)$ locates in the left of the interval $(-(\sqrt{5} + 1)/2, 1)$. In other words, the function f is decreasing on the interval $(-(\sqrt{5} + 1)/2, 1)$. If $x \leq -1$, we know that $f(\theta) = \theta(x + 1) + \log(1 - \theta)$ is decreasing. As is mentioned above, the function f reaches the maximal value at the point $-(\sqrt{5} + 1)/2$. Therefore, $\Phi(x)$ is completely determined for any $x \in \mathbb{R}$. Next, we show that

$$\Psi(x) = \begin{cases} -\frac{\sqrt{5}+1}{2}(x + 1) + 2 \log \frac{\sqrt{5}+1}{2}, & \text{if } x \geq -1; \\ +\infty, & \text{if } x < -1. \end{cases}$$

Let $g(\theta) = \theta x + \theta + 2 \log((\sqrt{5} + 1)/2)$ for any $\theta \leq -(\sqrt{5} + 1)/2$. In fact, when $x < -1$, the function g is decreasing and hence that $\Psi(x) = +\infty$ in this case. If $x > -1$, the function g is increasing and hence that g reaches the maximal value at $-(\sqrt{5} + 1)/2$. It is easy to see g is constantly $2 \log((\sqrt{5} + 1)/2)$ when $x = -1$. Now we claim that

$$I(x) = \sup_{\theta \in \mathbb{R}} \{\theta x - \Lambda(\theta)\} = \begin{cases} x - \log(x + 1), & \text{if } x > -\frac{\sqrt{5}-1}{2}; \\ -\frac{\sqrt{5}+1}{2}(x + 1) + 2 \log \frac{\sqrt{5}+1}{2}, & \text{if } -1 \leq x \leq -\frac{\sqrt{5}-1}{2}; \\ +\infty, & \text{if } x < -1. \end{cases}$$

Since $I(x) = \max\{\Phi(x), \Psi(x)\}$ for all $x \in \mathbb{R}$, by the definitions of $\Phi(x)$ and $\Psi(x)$, we easily determine the rate function $I(x)$ when $-1 \leq x \leq -(\sqrt{5} - 1)/2$ and $x < -1$. It remains to compare $\Phi(x)$ with $\Psi(x)$ when $x > -(\sqrt{5} - 1)/2$. To do this, define the function

$$h(x) = x - \log(x + 1) + \frac{\sqrt{5} + 1}{2}(x + 1) - 2 \log \frac{\sqrt{5} + 1}{2}.$$

Then we have that $h(-(\sqrt{5} - 1)/2) = 0$ and the derivative of h satisfies $h'(x) = x/(x + 1) + (\sqrt{5} + 1)/2$ and $h'(x) > 0$ if $x > -(\sqrt{5} - 1)/2$. Thus, $h(x) \geq 0$ when $x > -(\sqrt{5} - 1)/2$. Therefore, the rate function $I(x)$ is completely established. \square

4.2 Proof of Moderate Deviation Principle

As applications of Proposition 3.4 and Lemma 4.2, we obtain the following lemma.

Lemma 4.6 *Let $n \in \mathbb{N}$ and $-1/2 < \theta < 1/2$. Then for any $j \geq 3n - 1$, we have*

$$\sum_{k=j}^{\infty} \mathbf{P}(b_{n+1} = k \mid b_n = j) \cdot \left(\frac{k}{j}\right)^{\theta} \leq \frac{1}{1 - n^{-1}} \cdot \frac{1}{1 - \theta}.$$

Proof For any $j > 1$, it follows from Lemma 4.2 that

$$\sum_{k=j}^{\infty} \frac{j+1}{k(k+1)} \left(\frac{k}{j}\right)^{\theta} \leq \left(1 + \frac{1}{j}\right) \cdot \left(1 - \frac{1}{j}\right)^{\theta-1} \cdot \frac{1}{1-\theta}.$$

Combing this with Proposition 3.4, we deduce that

$$\sum_{k=j}^{\infty} \mathbf{P}(b_{n+1} = k \mid b_n = j) \cdot \left(\frac{k}{j}\right)^{\theta} \leq \left(1 + \frac{1}{j}\right) \cdot \left(1 - \frac{1}{j}\right)^{\theta-1} \cdot \frac{1}{1-\theta}.$$

Now it is sufficient to prove that

$$\left(1 + \frac{1}{j}\right) \cdot \left(1 - \frac{1}{j}\right)^{\theta-1} \leq \frac{1}{1-n^{-1}}$$

for any $j \geq 3n - 1$. Let $n \in \mathbb{N}$. In fact, since $-1/2 < \theta < 1/2$, we deduce that

$$\left(1 + \frac{1}{j}\right) \cdot \left(1 - \frac{1}{j}\right)^{\theta-1} \leq \frac{1+1/j}{(1-1/j)^{3/2}}. \tag{4.20}$$

Note that $(1 - 1/j)^{3/2} \geq 1 - 2/j$ for any $j \geq 1$, we have

$$\frac{(1 - 1/j)^{3/2}}{1 + 1/j} \geq \frac{1 - 2/j}{1 + 1/j} = 1 - \frac{3}{j+1} \geq 1 - \frac{1}{n}$$

for any $j \geq 3n - 1$. Combing this with (4.20), we complete the proof. □

Now we are ready to prove Theorem 2.2.

Proof of Theorem 2.2 Let $\{a_n : n \geq 1\}$ be the sequence of positive numbers satisfying the conditions in (2.4). For any $\lambda \in \mathbb{R}$, we consider the logarithmic moment-generating function (see Dembo and Zeitouni [6, Section 2.3]) of $\frac{\log b_n - n}{a_n}$,

$$\Lambda_n(\lambda) = \log \mathbf{E} \left(\exp \left(\lambda \cdot \frac{\log b_n - n}{a_n} \right) \right).$$

From the Gärtner–Ellis theorem, in order to obtaining the desired result, it suffices to show that for any $\lambda \in \mathbb{R}$,

$$\Lambda(\lambda) = \lim_{n \rightarrow \infty} \frac{n}{a_n^2} \Lambda_n \left(\frac{a_n^2}{n} \lambda \right) = \frac{\lambda^2}{2}.$$

That is,

$$\lim_{n \rightarrow \infty} \frac{n}{a_n^2} \log \mathbf{E} \left(\exp \left\{ \frac{a_n}{n} (\log b_n - n) \lambda \right\} \right) = \frac{\lambda^2}{2}. \tag{4.21}$$

For any $\lambda \in \mathbb{R}$ and $n \geq 1$, let

$$\theta_n := \theta_n(\lambda) = \frac{a_n}{n} \lambda \quad \text{and} \quad \Upsilon_n(\lambda) = \mathbf{E}(\exp\{\theta_n(\log b_n - n)\}).$$

In view of (2.4), it is clear that $\theta_n \rightarrow 0$ as $n \rightarrow \infty$ and $\Upsilon_n(\lambda)$ can be rewritten as

$$\Upsilon_n(\lambda) = e^{-n\theta_n} \mathbf{E}(b_n^{\theta_n}). \tag{4.22}$$

Being similar to the proof of large deviation principle part, to get (4.21), we only need to estimate the expectation $\mathbf{E}(b_n^{\theta_n})$. Since $\theta_n \rightarrow 0$ as $n \rightarrow \infty$, there exists a positive number N (only depending on λ) such that for all $n \geq N$, we have $-1/2 < \theta_n < 1/2$. In the following, we always fix such n . Now we will give the lower and upper bounded estimates of $\mathbf{E}(b_n^{\theta_n})$, respectively.

We first give the lower bound for $\mathbf{E}(b_n^{\theta_n})$. Being similar to the Part 1 in the proof of Lemma 4.5, we know that

$$\begin{aligned} \sum_{k=1}^{\infty} \mathbf{P}(b_n = k) \cdot k^{\theta_n} &= \sum_{j=1}^{\infty} \mathbf{P}(b_{n-1} = j) \cdot j^{\theta_n} \cdot \sum_{k=j}^{\infty} \mathbf{P}(b_n = k \mid b_{n-1} = j) \cdot \left(\frac{k}{j}\right)^{\theta_n} \\ &\geq \sum_{j=1}^{\infty} \mathbf{P}(b_{n-1} = j) \cdot j^{\theta_n} \cdot \sum_{k=j}^{\infty} \frac{j}{k(k+2)} \left(\frac{k}{j}\right)^{\theta_n}, \end{aligned}$$

where the last inequality follows from Proposition 3.4. Combing this with Lemma 4.2, we have that

$$\begin{aligned} \sum_{k=1}^{\infty} \mathbf{P}(b_n = k) \cdot k^{\theta_n} &\geq \sum_{j=1}^{\infty} \mathbf{P}(b_{n-1} = j) \cdot j^{\theta_n} \cdot \frac{j}{j+2} \cdot \frac{1}{1-\theta_n} \\ &\geq \frac{1}{1+2} \cdot \frac{1}{1-\theta_n} \cdot \sum_{j=2}^{\infty} \mathbf{P}(b_{n-1} = j) \cdot j^{\theta_n} \end{aligned}$$

since $j/(j+2) \geq 1/(1+2)$ for any $j \geq 1$. Thus, we deduce that

$$\sum_{k=1}^{\infty} \mathbf{P}(b_n = k) \cdot k^{\theta_n} \geq \frac{1}{1+2} \cdot \frac{1}{1-\theta_n} \cdot \sum_{j=2}^{\infty} \mathbf{P}(b_{n-1} = j) \cdot j^{\theta_n}.$$

Similarly, we obtain that

$$\sum_{k=2}^{\infty} \mathbf{P}(b_{n-1} = k) \cdot k^{\theta_n} \geq \frac{2}{2+2} \cdot \frac{1}{1-\theta_n} \cdot \sum_{j=3}^{\infty} \mathbf{P}(b_{n-2} = j) \cdot j^{\theta_n}.$$

Repeating the above procedure, by (3.6) and $-1/2 < \theta_n < 1/2$, we actually have that

$$\begin{aligned} \sum_{k=1}^{\infty} \mathbf{P}(b_n = k) \cdot k^{\theta_n} &\geq \left(\frac{1}{3} \cdot \frac{2}{4} \cdots \frac{n-1}{n+1}\right) \cdot \left(\frac{1}{1-\theta_n}\right)^{n-1} \cdot \sum_{j=n}^{\infty} \mathbf{P}(b_1 = j) \cdot j^{\theta_n} \\ &\geq M(n) \cdot \frac{2}{n(n+1)} \cdot \left(\frac{1}{1-\theta_n}\right)^{n-1}, \end{aligned}$$

where $M(n) = \sum_{j=n}^{\infty} \frac{1}{(j+1)j^{3/2}}$. Therefore,

$$\mathbf{E}(b_n^{\theta_n}) \geq M(n) \cdot \frac{2}{n(n+1)} \cdot \left(\frac{1}{1-\theta_n}\right)^{n-1}. \tag{4.23}$$

Not that $j(j+1) \leq (j+1)j^{3/2}$ for any $j \geq 1$, we obtain that

$$\frac{1}{(n+1)n^{3/2}} \leq M(n) \leq \frac{1}{n},$$

which implies that

$$\lim_{n \rightarrow \infty} \frac{n}{a_n^2} \log M(n) = 0$$

since $\lim_{n \rightarrow \infty} (n \log n)/a_n^2 = 0$ in view of (2.4). Combing this with (4.22) and (4.23), by Taylor formula, we actually deduce that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{n}{a_n^2} \log \mathbf{E} \left(\exp \left\{ \frac{a_n}{n} (\log b_n - n) \lambda \right\} \right) &\geq \liminf_{n \rightarrow \infty} \left(\frac{n^2}{a_n^2} (-\theta_n) + \frac{n^2}{a_n^2} \log \frac{1}{1-\theta_n} \right) \\ &= \frac{\lambda^2}{2}. \end{aligned}$$

Next we give the upper bound for $\mathbf{E}(b_n^{\theta_n})$. Being similar to the Part 2 in the proof of Lemma 4.5, let $1 + \varepsilon = 1/(1 - n^{-1})$ and $N := N(n) = 3n - 1$. Lemma 4.6 guarantees that the methods of the Part 2 in the proof of Lemma 4.5 are still valid. Therefore, we actually obtain that

$$\mathbf{E}(b_n^{\theta_n}) \leq U(n) \cdot \left(\frac{\sqrt{5} + 1}{2}\right)^{-2n} + V(n) \cdot \left(\frac{1}{1-n^{-1}} \cdot \frac{1}{1-\theta_n}\right)^n,$$

where $U(n)$ and $V(n)$ both are polynomials of n with some degree. Note that

$$\lim_{n \rightarrow \infty} \frac{n}{a_n^2} \log U(n) = \lim_{n \rightarrow \infty} \frac{n}{a_n^2} \log V(n) = \lim_{n \rightarrow \infty} \frac{n^2}{a_n^2} \log \left(1 - \frac{1}{n}\right) = 0$$

and

$$\lim_{n \rightarrow \infty} \frac{n}{a_n^2} \log \left(\frac{\sqrt{5} + 1}{2} \right)^{-2n} = -2 \lim_{n \rightarrow \infty} \frac{n^2}{a_n^2} \log \left(\frac{\sqrt{5} + 1}{2} \right) = -\infty,$$

by Taylor formula, we eventually deduce that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{n}{a_n^2} \log \mathbf{E} \left(\exp \left\{ \frac{a_n}{n} (\log b_n - n) \lambda \right\} \right) \\ & \leq \limsup_{n \rightarrow \infty} \left\{ \frac{n^2}{a_n^2} (-\theta_n) - \frac{n^2}{a_n^2} \log \left(1 - \frac{1}{n} \right) - \frac{n^2}{a_n^2} \log (1 - \theta_n) \right\} = \frac{\lambda^2}{2}. \end{aligned}$$

Thus, the equality (4.21) is established. By the Gärtner–Ellis theorem, we obtain the sequence $\left\{ \frac{\log b_n - n}{a_n} : n \geq 1 \right\}$ satisfies an MDP with speed $n^{-1} a_n^2$ and good rate function

$$J(x) = \sup_{\lambda \in \mathbb{R}} \{ \lambda x - \Lambda(\lambda) \} = x^2/2$$

for any $x \in \mathbb{R}$. □

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