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Complex Outliers of Hermitian Random Matrices

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Abstract In this paper, we study the asymptotic behavior of the outliers of the sum a Hermitian random matrix and a finite rank matrix which is not necessarily Hermitian. We observe several possible convergence rates and outliers locating around their limits at the vertices of regular polygons as in Benaych-Georges and Rochet (Probab Theory Relat Fields, 2015), as well as possible correlations between outliers at macroscopic distance as in Knowles and Yin (Ann Probab 42(5):1980–2031, 2014) and Benaych-Georges and Rochet (2015). We also observe that a single spike can generate several outliers in the spectrum of the deformed model, as already noticed in Benaych-Georges and Nadakuditi (Adv Math 227(1):494–521, 2011) and Belinschi et al. (Outliers in the spectrum of large deformed unitarily invariant models 2012, arXiv:1207.5443v1). In the particular case where the perturbation matrix is Hermitian, our results complete the work of Benaych-Georges et al. (Electron J Probab 16(60):1621–1662, 2011), as we consider fluctuations of outliers lying in "holes" of the limit support, which happen to exhibit surprising correlations.

Keywords Random matrices · Spiked models · Extreme eigenvalue statistics · Gaussian fluctuations

Mathematics Subject Classification 15B52 · 60F05

1 Introduction

It is known that adding a finite rank perturbation to a large matrix barely changes the global behavior of its spectrum. Nevertheless, some of the eigenvalues, called

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outliers, can deviate away from the bulk of the spectrum, depending on the strength of the perturbation. This phenomenon, well known as the *BBP transition*, was first brought to light for empirical covariance matrices by Johnstone in [23], by Baik et al. in [3], and then shown under several hypothesis in the Hermitian case in [6–9,13–15,17,24,25,29,30]. Non-Hermitian models have been also studied: i.i.d. matrices in [12,27,32], elliptic matrices in [28], and matrices from the Single Ring Theorem in [10]. In [10], and lately in [27], the authors have also studied the fluctuations of the outliers and, due to non-Hermitian structure, obtained unusual results: The distribution of the fluctuations highly depends on the shape of the Jordan canonical form of the perturbation; in particular, the convergence rate depends on the size of the Jordan blocks. Also, the outliers tend to locate around their limit at the vertices of a regular polygon. At last, they observe correlations between the fluctuations of outliers at a macroscopic distance with each other.

In this paper, we show that the same kind of phenomenon occurs when we perturb an Hermitian matrix $\mathbf{H}$ with a non-Hermitian one $\mathbf{A}$. More precisely, we study finite rank perturbations for Hermitian random matrices $\mathbf{H}$ whose spectral measure tends to a compactly supported measure μ and the perturbation $\mathbf{A}$ is just a complex matrix with a finite rank. With further assumptions, we prove that outliers of $\mathbf{H}+\mathbf{A}$ may appear at a macroscopic distance from the bulk and, following the ideas of [10], we show that they fluctuate with convergence rates which depend on the matrix $\mathbf{A}$ through its Jordan canonical form. Remind that any complex matrix is similar to a block diagonal matrix with diagonal blocks of the type

$$\mathbf{R}_p(\theta) := \begin{pmatrix} \theta & 1 & & (0) \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ (0) & & & \theta \end{pmatrix},$$

so that $\mathbf{A} \sim \operatorname{diag}\left(\mathbf{R}_{p_1}(\theta_1), \ldots, \mathbf{R}_{p_q}(\theta_q)\right)$, this last matrix being called the *Jordan Canonical Form* of $\mathbf{A}$ [22, Chapter 3]. We show, up to some hypothesis, that for any eigenvalue θ of $\mathbf{A}$, if we denote by

$$\underbrace{p_1, \ldots, p_1}_{\beta_1 \text{ times}} > \underbrace{p_2, \ldots, p_2}_{\beta_2 \text{ times}} > \cdots > \underbrace{p_\alpha, \ldots, p_\alpha}_{\beta_\alpha \text{ times}}$$

the sizes of the blocks associated with θ in the Jordan Canonical Form of $\mathbf{A}$ and introduce the (possibly empty) set

$$\mathscr{S}_\theta := \left\{ \xi \in \mathbb{C},\ G_\mu(\xi) = \frac{1}{\theta} \right\}$$

where $G_\mu(z) := \int \frac{1}{z-x}\mu(\mathrm{d}x)$ is the Cauchy transform of the measure μ, then there are exactly $\beta_1 p_1 + \cdots + \beta_\alpha p_\alpha$ outliers of $\mathbf{H}+\mathbf{A}$ tending to each element of $\mathscr{S}_\theta$. We also prove that for each element ξ in $\mathscr{S}_\theta$, there are exactly $\beta_1 p_1$ outliers tending to ξ at rate

$N^{-1/(2p_1)}$, $\beta_2 p_2$ outliers tending to ξ at rate $N^{-1/(2p_2)}$, etc... (see Fig. 2). Furthermore, the limit joint distribution of the fluctuations is explicit, not necessarily Gaussian, and might show correlations even between outliers at a macroscopic distance with each other. This phenomenon of correlations between the fluctuations of two outliers with distinct limits has already been proved for non-Gaussian Wigner matrices when **A** is Hermitian (see [25]), while in our case, Gaussian Wigner matrices can have such correlated outliers: Indeed, the correlations that we bring to light here are due to the fact that the eigenspaces of **A** are not necessarily orthogonal or that one single spike generates several outliers. Indeed, we observe that the outliers may outnumber the rank of **A**. This had already been noticed in [8, Remark 2.11] when the support of the limit spectral measure of **H** has some "holes" or in the different model of [5], where the authors study the case where **A** is Hermitian but with full rank and is invariant in distribution by unitary conjugation. Here, the phenomenon can be proved to occur even when the support of the limit spectral measure of **H** is connected. At last, if we apply our results in the particular case where **A** is Hermitian, we also see that two outliers at a macroscopic distance with each other are correlated if they both are generated by the same spike (which can occur only if the limit support is disconnected) and are independent otherwise (see Fig. 3). From this point of view, this completes the work of [6], where fluctuations of outliers lying in "holes" of the limit support had not been studied.

The fact to consider a non-Hermitian deformation on a Hermitian random matrix has already been studied in theoretical physics (see [18–21]) in the particular case where **H** is a GOE/GUE matrix and **A** is a nonnegative Hermitian matrix times i (the square root of -1). They proved a weaker version of Theorem 2.3 in this specific case but did not study the fluctuations.

The proofs of this paper rely essentially on the ideas of the paper [10] about outliers in the Single Ring Theorem and on the results proved in [6,30,31]. More precisely, the study of the fluctuations reproduces the outlines of the proofs of [10] as long as the model fulfills some conditions. Thanks to [30,31], we show that these conditions are satisfied for Wigner matrices. At last, using [6] and the Weingarten calculus, we show the same for Hermitian matrices invariant in distribution by unitary conjugation. In the appendix, as a tool for the outliers study, we prove a result on the fluctuations of the entries of such matrices.

2 General Results

At first, we formulate the results in general settings, and we shall give, in the next section, examples of random matrices on which these results apply.

2.1 Convergence of the Outliers

2.1.1 Setup and Assumptions

For all $N \geq 1$, let $\mathbf{H}_N$ be an Hermitian random $N \times N$ matrix whose *empirical spectral measure*, as N goes to infinity, converges weakly in probability to a compactly supported measure μ

$$\mu_N := \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i(\mathbf{H})} \longrightarrow \mu. \tag{1}$$

We shall suppose that μ is non-trivial in the sense that μ is not a single Dirac measure. Also, we suppose that $\mathbf{H}_N$ does not possess any *natural outliers*, i.e.,

Assumption 2.1 As N goes to infinity, with probability tending to one,

$$\sup_{\lambda\in\operatorname{Spec}(\mathbf{H}_N)} \operatorname{dist}(\lambda, \operatorname{supp}(\mu)) \longrightarrow 0.$$

For all $N \geq 1$, let $\mathbf{A}_N$ be an $N \times N$ random matrix independent from $\mathbf{H}_N$ (which does not satisfies necessarily $\mathbf{A}_N^* = \mathbf{A}_N$) whose rank is bounded by an integer r (independent from N). We know that we can write

$$\mathbf{A}_N := \mathbf{U}\begin{pmatrix}\mathbf{A}_0 & 0\\ 0 & 0\end{pmatrix}\mathbf{U}^* \tag{2}$$

where $\mathbf{U}$ is an $N \times N$ unitary matrix and $\mathbf{A}_0$ is $2r \times 2r$ matrix. We notice that $\mathbf{A}_N$ only depends on the $2r$-first columns of $\mathbf{U}$ so that, we shall write

$$\mathbf{A}_N \ := \ \mathbf{U}_{2r}\mathbf{A}_0\mathbf{U}_{2r}^*,$$

where the $N \times 2r$ matrix $\mathbf{U}_{2r}$ designates the $2r$-first columns of $\mathbf{U}$. We shall assume that $\mathbf{A}_0$ is deterministic and independent from N. We shall denote by $\theta_1, \ldots, \theta_j$ the distinct nonzero eigenvalues of $\mathbf{A}_0$ and $k_1, \ldots, k_j$ their respective multiplicity[1] (note that $\sum_{i=1}^{j} k_i \leq r$).

We consider the additive perturbation

$$\widetilde{\mathbf{H}}_N := \mathbf{H}_N + \mathbf{A}_N, \tag{3}$$

We set

$$G_\mu(z) := \int \frac{1}{z-x}\mu(\mathrm{d}x). \tag{4}$$

the Cauchy transform of the measure μ. We introduce, for all $i \in \{1, \ldots, j\}$, the finite, possibly empty, set

$$\mathscr{S}_{\theta_i} := \left\{\xi \in \mathbb{C}\setminus\operatorname{supp}(\mu),\ G_\mu(\xi) = \frac{1}{\theta_i}\right\}, \ \text{ and } \ m_i \ := \ \operatorname{Card}\mathscr{S}_{\theta_i} \tag{5}$$

We make the following assumption

[1] The *multiplicity* of an eigenvalue is defined as its order as a root of the characteristic polynomial, which is greater than or equal to the dimension of the associated eigenspace.

Assumption 2.2 For any $\delta > 0$, as N goes to infinity, we have

$$\sup_{\operatorname{dist}(z,\operatorname{supp}(\mu))>\delta} \left\| \mathbf{U}_{2r}^* \left(z\mathbf{I} - \mathbf{H}_N \right)^{-1} \mathbf{U}_{2r} - G_\mu(z)\mathbf{I} \right\|_{\mathrm{op}} \xrightarrow{(\mathbb{P})} 0.$$

2.1.2 Result

Theorem 2.3 (Convergence of the outliers) *For $\theta_1, \ldots, \theta_j, k_1, \ldots, k_j, \mathscr{S}_{\theta_1}, \ldots, \mathscr{S}_{\theta_j}$, and $m_1, \ldots, m_j$ as defined above, with probability tending to one, $\widetilde{\mathbf{H}}_N := \mathbf{H}_N + \mathbf{A}_N$ possesses exactly $\sum_{i=1}^{j} k_i m_i$ eigenvalues at a macroscopic distance of* supp μ *(outliers). More precisely, for all small enough $\delta > 0$, for all large enough N, for all $i \in \{1, \ldots, j\}$, if we set*

$$\mathscr{S}_{\theta_i} = \{\xi_{i,1}, \ldots, \xi_{i,m_i}\},$$

there are m_i eigenvalues $\widetilde{\lambda}_{i,1}, \ldots, \widetilde{\lambda}_{i,m_i}$ of $\widetilde{\mathbf{H}}_N$ in $\{z,\ \operatorname{dist}(z, \operatorname{supp}(\mu)) > \delta\}$ satisfying

$$\widetilde{\lambda}_{i,n} = \xi_{i,n} + o(1), \quad \textit{for all } n \in \{1, \ldots, m_i\},$$

after a proper labeling.

Remark 2.4 If all the $\mathscr{S}_{\theta_i}$'s are empty, there is possibly no outlier at all. This condition is the analogous of the *phase transition* condition in [8, Theorem 2.1] in the case where the θ_i's are real, which is if

$$\frac{1}{\theta_i} \notin \left] \lim_{x \to a^-} G_\mu(x),\ \lim_{x \to b^+} G_\mu(x) \right[$$

where a (respectively, b) designates the infimum (respectively, the supremum) of the support of μ, then θ_i does not generate any outlier. In our case, if $|\theta_i|$ is large enough, $\mathscr{S}_{\theta_i}$ is necessarily non-empty[2], which means that a strong enough perturbation always creates outliers.

Remark 2.5 We notice that the outliers can outnumber the rank of $\mathbf{A}$. This phenomenon was already observed in [8] in the case where the support of the limit spectral distribution has a disconnected support (see also [5]). In our case, the phenomenon occurs even for connected support (see Fig. 1).

2.2 Fluctuations of the Outliers

To study the fluctuations, one needs to understand the limit distribution of

$$\sqrt{N} \left\| \mathbf{U}_{2r}^* \left(z\mathbf{I} - \mathbf{H}_N \right)^{-1} \mathbf{U}_{2r} - G_\mu(z)\mathbf{I} \right\|_{\mathrm{op}}. \tag{6}$$

[2] due to the fact that the Cauchy transform of a compactly supported measure can always be inverted in a neighborhood of infinity.

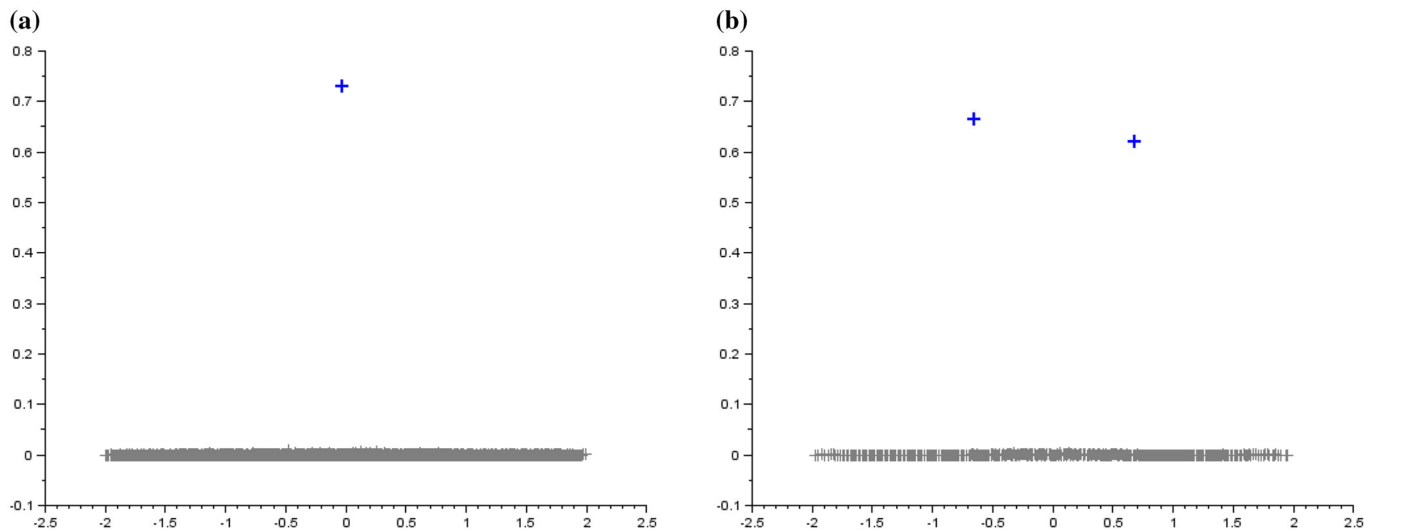

Fig. 1 Spectrums of two Hermitian matrices with the same limit bulk but different limit spectral densities on this bulk, perturbed by the same matrix: both do not have the same number of outliers (the blue crosses "+"). **a** Spectrum of an Hermitian matrix of size $N = 2000$, whose spectral measure tends to the *semicircle law* $\mu_{\mathrm{sc}}(\mathrm{d}x) := \frac{1}{2\pi}\sqrt{4-x^2}\mathbb{1}[-2,2](\mathrm{d}x)$ (such as Wigner matrix), with perturbation matrix $\mathbf{A} = \mathrm{diag}(i\sqrt{2}, 0\ldots, 0)$. **b** Spectrum of an Hermitian matrix of size $N = 2000$, whose spectral measure tends to $\frac{2}{5}\delta_{-1}(\mathrm{d}x) + \frac{2}{5}\delta_1(\mathrm{d}x) + \frac{1}{5}\mu_{\mathrm{sc}}(\mathrm{d}x)$ and with perturbation matrix $\mathbf{A} = \mathrm{diag}(i\sqrt{2}, 0\ldots, 0)$

In the particular case where $\mathbf{H}_N$ is a Wigner matrix, we know from [30] that this quantity is tight but does not necessarily converge. Hence, we shall need additional assumptions.

2.2.1 Setup and Assumptions

As $\mathbf{A}_N$ is not Hermitian, we need to introduce the Jordan Canonical Form (JCF) to describe the fluctuations. More precisely, we shall consider the JCF of $\mathbf{A}_0$ which does not depend on N. We know that, in a proper basis, $\mathbf{A}_0$ is a direct sum of *Jordan blocks*, i.e., blocks of the form

$$\mathbf{R}_p(\theta) = \begin{pmatrix} \theta & 1 & & (0) \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ (0) & & & \theta \end{pmatrix}, \; p \times p \text{ matrix}, \qquad \theta \in \mathbb{C}, p \geq 1 \tag{7}$$

Let us denote by $\theta_1, \ldots, \theta_q$ the distinct eigenvalues of $\mathbf{A}_0$ such that $\mathscr{S}_\theta \neq \emptyset$ (see (5) for the definition of $\mathscr{S}_\theta$), and for each $i = 1, \ldots, q$, we introduce a positive integer α_i, some positive integers $p_{i,1} > \cdots > p_{i,\alpha_i}$ corresponding to the distinct sizes of the blocks relative to the eigenvalue θ_i and $\beta_{i,1}, \ldots, \beta_{i,\alpha_i}$ such that for all j, $\mathbf{R}_{p_{i,j}}(\theta_i)$ appears $\beta_{i,j}$ times, so that, for a certain $2r \times 2r$ non-singular matrix $\mathbf{Q}$, we have:

$$\mathbf{J} = \mathbf{Q}^{-1}\mathbf{A}_0\mathbf{Q} = \hat{\mathbf{A}} \bigoplus \bigoplus_{i=1}^{q} \bigoplus_{j=1}^{\alpha_i} \underbrace{\begin{pmatrix} \mathbf{R}_{p_{i,j}}(\theta_i) & & \\ & \ddots & \\ & & \mathbf{R}_{p_{i,j}}(\theta_i) \end{pmatrix}}_{\beta_{i,j} \text{ blocks}} \tag{8}$$

where $\oplus$ is defined, for square block matrices, by $\mathbf{M} \oplus \mathbf{N} := \begin{pmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{N} \end{pmatrix}$ and $\hat{\mathbf{A}}$ is a matrix such that its eigenvalues θ are such that $\mathscr{S}_\theta = \emptyset$ or null.

The asymptotic orders of the fluctuations of the eigenvalues of $\mathbf{H}_N + \mathbf{A}_N$ depend on the sizes $p_{i,j}$ of the blocks. Actually, for each θ_i and each $\xi_{i,n} \in \mathscr{S}_{\theta_i} = \{\xi_{i,1}, \ldots, \xi_{i,m_i}\}$, we know, by Theorem 2.3, there are $\sum_{j=1}^{\alpha_i} p_{ij} \times \beta_{i,j}$ eigenvalues $\widetilde{\lambda}$ of $\mathbf{H}_N + \mathbf{A}_N$ which tend to $\xi_{i,n}$: We shall write them with a $\xi_{i,n}$ on the top left corner, as follows:

$$^{\xi_{i,n}}\widetilde{\lambda}.$$

Theorem 2.10 below will state that for each block with size $p_{i,j}$ corresponding to θ_i in the JCF of $\mathbf{A}_0$, there are $p_{i,j}$ eigenvalues (we shall write them with $p_{i,j}$ on the bottom left corner : ${}^{\xi_{i,n}}_{p_{i,j}}\widetilde{\lambda}$) whose convergence rate will be $N^{-1/(2p_{i,j})}$. As there are

$\beta_{i,j}$ blocks of size $p_{i,j}$, there are actually $p_{i,j} \times \beta_{i,j}$ eigenvalues tending to $\xi_{i,n}$ with convergence rate $N^{-1/(2p_{i,j})}$ (we shall write them ${}_{p_{i,j}}^{\xi_{i,n}}\widetilde{\lambda}_{s,t}$ with $s \in \{1, \dots, p_{i,j}\}$ and $t \in \{1, \dots, \beta_{i,j}\}$). It would be convenient to denote by $\Lambda_{i,j,n}$ the vector with size $p_{i,j} \times \beta_{i,j}$ defined by

$$\Lambda_{i,j,n} := \left(N^{1/(2p_{i,j})} \cdot \left({}_{p_{i,j}}^{\xi_{i,n}}\widetilde{\lambda}_{s,t} - \xi_{i,n}\right)\right)_{\substack{1\le s\le p_{i,j} \\ 1\le t\le \beta_{i,j}}}. \tag{9}$$

In addition, we make an assumption on the convergence of (6).

Assumption 2.6 (1) The vector $\left(\sqrt{N}\mathbf{U}_{2r}^*\left(\left(\xi_{i,n} - \mathbf{H}_N\right)^{-1} - \frac{1}{\theta_i}\right)\mathbf{U}_{2r}\right)_{\substack{1\le i\le q \\ 1\le n\le m_i}}$ converges in distribution, and none of its entries tends to zero.

(2) For all $k \ge 1$, all $i \in \{1, \dots, q\}$, and all $n \in \{1, \dots, m_i\}$,

$$\sqrt{N}\mathbf{U}_{2r}^*\left(\left(\xi_{i,n} - \mathbf{H}_N\right)^{-(k+1)} - \int \frac{\mu(\mathrm{d}x)}{(\xi_{i,n} - x)^{k+1}}\right)\mathbf{U}_{2r}$$

is tight.

or

(0') For all $i \in \{1, \dots, q\}$ and all $n \in \{1, \dots, m_i\}$, as N goes to infinity,

$$\sqrt{N}\left(\frac{1}{N}\operatorname{Tr}\left(\xi_{i,n} - \mathbf{H}_N\right)^{-1} - \frac{1}{\theta_i})\right) \longrightarrow 0.$$

(1') The vector $\left(\sqrt{N}\mathbf{U}_{2r}^*\left(\left(\xi_{i,n} - \mathbf{H}_N\right)^{-1} - \frac{1}{N}\operatorname{Tr}\left(\xi_{i,n} - \mathbf{H}_N\right)^{-1}\right)\mathbf{U}_{2r}\right)_{\substack{1\le i\le q \\ 1\le n\le m_i}}$ converges in distribution, and none of its entries tends to zero.

(2') For all $k \ge 1$ and for all $i \in \{1, \dots, q\}$,

$$\sqrt{N}\mathbf{U}_{2r}^*\left(\left(\xi_{i,n} - \mathbf{H}_N\right)^{-(k+1)} - \frac{1}{N}\operatorname{Tr}\left(\xi_{i,n} - \mathbf{H}_N\right)^{-(k+1)}\right)\mathbf{U}_{2r}$$

is tight.

As in [10], we define now the family of random matrices that we shall use to characterize the limit distribution of the $\Lambda_{i,j,n}$'s. For each $i = 1, \dots, q$, let $I(\theta_i)$ (respectively, $J(\theta_i)$) denote the set, with cardinality $\sum_{j=1}^{\alpha_i}\beta_{i,j}$, of indices in $\{1, \dots, r\}$ corresponding to the first (respectively, last) columns of the blocks $\mathbf{R}_{p_{i,j}}(\theta_i)$ ($1 \le j \le \alpha_i$) in (8).

Remark 2.7 Note that the columns of $\mathbf{Q}$ (respectively, of $(\mathbf{Q}^{-1})^*$) whose index belongs to $I(\theta_i)$ (respectively, $J(\theta_i)$) are eigenvectors of $\mathbf{A}_0$ (respectively, of $\mathbf{A}_0^*$) associated with θ_i (respectively, $\overline{\theta_i}$). See [10, Remark 2.7].

Now, let

$$\left(m_{k,\ell}^{\theta_i,n}\right)_{\substack{1\le i\le q\\ 1\le n\le m_i\\ (k,\ell)\in J(\theta_i)\times I(\theta_i)}} \tag{10}$$

be the multivariate random variable defined as the limit joint distribution of

$$\left(\sqrt{N}\mathbf{e}_k^*\mathbf{Q}^{-1}\mathbf{U}_{2r}^*\left(\left(\xi_{i,n}-\mathbf{H}_N\right)^{-1}-\frac{1}{\theta_i}\right)\mathbf{U}_{2r}\mathbf{Q}\mathbf{e}_\ell\right)_{\substack{1\le i\le q\\ 1\le n\le m_i\\ (k,\ell)\in J(\theta_i)\times I(\theta_i)}}$$
$$\xrightarrow[\text{jointly}]{(\mathrm{d})}\left(m_{k,\ell}^{\theta_i,n}\right)_{\substack{1\le i\le q\\ 1\le n\le m_i\\ (k,\ell)\in J(\theta_i)\times I(\theta_i)}} \tag{11}$$

(which does exist by Assumption 2.6) and where $\mathbf{e}_1,\ldots,\mathbf{e}_r$ are the column vectors of the canonical basis of $\mathbb{C}^r$).

For each i, j, let $K(i,j)$ (respectively, $K(i,j)^-$) be the set, with cardinality $\beta_{i,j}$ (respectively, $\sum_{j'=1}^{j-1}\beta_{i,j'}$), of indices in $J(\theta_i)$ corresponding to a block of the type $\mathbf{R}_{p_{i,j}}(\theta_i)$ (respectively, to a block of the type $\mathbf{R}_{p_{i,j'}}(\theta_i)$ for $j'<j$). In the same way, let $L(i,j)$ (respectively, $L(i,j)^-$) be the set, with the same cardinality as $K(i,j)$ (respectively, as $K(i,j)^-$), of indices in $I(\theta_i)$ corresponding to a block of the type $\mathbf{R}_{p_{i,j}}(\theta_i)$ (respectively, to a block of the type $\mathbf{R}_{p_{i,j'}}(\theta_i)$ for $j'<j$). Note that $K(i,j)^-$ and $L(i,j)^-$ are empty if $j=1$. Let us define the random matrices for each $n\in\{1,\ldots,m_i\}$

$$\begin{aligned}
\mathrm{M}_{j,n}^{\theta_i,\mathrm{I}} &:= [m_{k,\ell}^{\theta_i,n}]_{\substack{k\in K(i,j)^-\\ \ell\in L(i,j)^-}} & \mathrm{M}_{j,n}^{\theta_i,\mathrm{II}} &:= [m_{k,\ell}^{\theta_i,n}]_{\substack{k\in K(i,j)^-\\ \ell\in L(i,j)}}\\
\mathrm{M}_{j,n}^{\theta_i,\mathrm{III}} &:= [m_{k,\ell}^{\theta_i,n}]_{\substack{k\in K(i,j)\\ \ell\in L(i,j)^-}} & \mathrm{M}_{j,n}^{\theta_i,\mathrm{IV}} &:= [m_{k,\ell}^{\theta_i,n}]_{\substack{k\in K(i,j)\\ \ell\in L(i,j)}}
\end{aligned} \tag{12}$$

and then let us define the $\beta_{i,j}\times\beta_{i,j}$ matrix $\mathbf{M}_{j,n}^{\theta_i}$ as

$$\mathbf{M}_{j,n}^{\theta_i} := \theta_i\left(\mathrm{M}_{j,n}^{\theta_i,\mathrm{IV}}-\mathrm{M}_{j,n}^{\theta_i,\mathrm{III}}\left(\mathrm{M}_{j,n}^{\theta_i,\mathrm{I}}\right)^{-1}\mathrm{M}_{j,n}^{\theta_i,\mathrm{II}}\right) \tag{13}$$

Remark 2.8 It follows from the fact that the matrix $\mathbf{Q}$ is invertible, that $\mathrm{M}_{j,n}^{\theta_i,\mathrm{I}}$ is a.s. invertible and so is $\mathbf{M}_{j,n}^{\theta_i}$.

Remark 2.9 In the particular case where $\mathbf{A}_0$ is Hermitian (which means that $\mathbf{Q}^{-1}=\mathbf{Q}^*$ and the θ_i's are real), then the matrices $\mathbf{M}_{j,n}^{\theta_i}$ are also Hermitian.

Now, we can formulate the result on the fluctuations.

2.2.2 Result

Theorem 2.10 *(1) As N goes to infinity, the random vector*

$$\left(\Lambda_{i,j,n}\right)_{\substack{1\le i\le q\\ 1\le j\le \alpha_i\\ 1\le n\le m_i}}$$

defined at (9) *converges to the distribution of a random vector*

$$\left(\Lambda^{\infty}_{i,j,n}\right)_{\substack{1\le i\le q\\ 1\le j\le \alpha_i\\ 1\le n\le m_i}}$$

with joint distribution defined by the fact that for each $1 \le i \le q$, $1 \le j \le \alpha_i$ *and* $1 \le n \le m_i$, $\Lambda^{\infty}_{i,j,n}$ *is the collection of the* ${p_{i,j}}^{th}$ *roots of the eigenvalues of some random matrix* $\mathbf{M}^{\theta_i}_{j,n}$.

(2) The distributions of the random matrices $\mathbf{M}^{\theta_i}_{j,n}$ *are absolutely continuous with respect to the Lebesgue measure, and the random vector* $\left(\Lambda^{\infty}_{i,j,n}\right)_{\substack{1\le i\le q\\ 1\le j\le \alpha_i}}$ *has no deterministic coordinate.*

Theorem 2.10 is illustrated in Fig. 2 with an example. We clearly see appearing regular polygons.

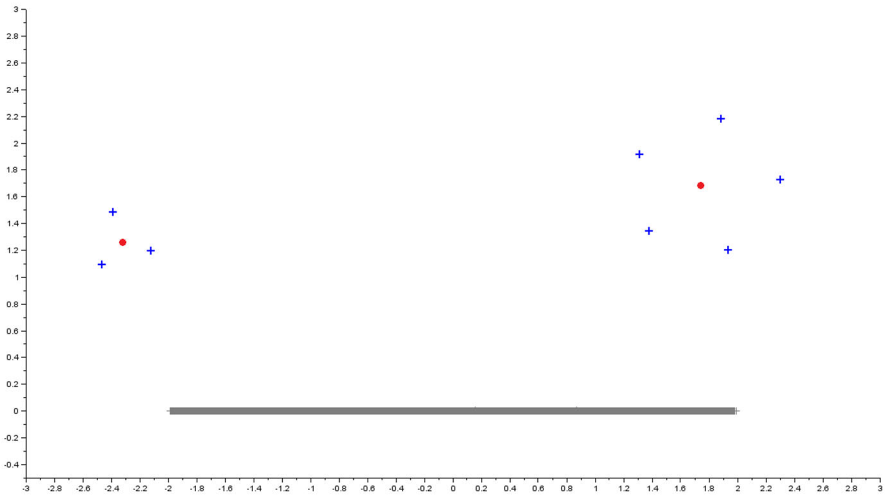

Fig. 2 Spectrum of a Wigner matrix of size $N = 5000$ with perturbation matrix $\mathbf{A} = \mathrm{diag}\,(\mathbf{R}_5(1.5+2\,\mathrm{i}), \mathbf{R}_3(-2+1.5\,\mathrm{i}), 0, \ldots, 0)$. We see the *blue crosses* "+" (outliers) forming, respectively, a regular pentagon and an equilateral triangle around the *red dots* "•" (their limit). We also see a significant difference between the two rates of convergence, $N^{-1/10}$ and $N^{-1/6}$ (Color figure online)

3 Applications

In this section, we give examples of random matrices which satisfy the assumptions of Theorem 2.3 and Theorem 2.10.

3.1 Wigner Matrices

Let $\mathbf{H}_N = \frac{1}{\sqrt{N}}\mathbf{W}_N$ be a symmetric/Hermitian Wigner matrix with independent entries up to the symmetry. More precisely, we assume that

Assumption 3.1 Real symmetric case :

- $\left(\mathbf{W}_N\right)_{i,j}$, $1 \leq i \leq j \leq N$, are independent,
- The $(\mathbf{W}_N)_{i,j}$'s for $i \neq j$ (respectively, $i = j$) are identically distributed,
- $\mathbb{E}(\mathbf{W}_N)_{1,1} = \mathbb{E}(\mathbf{W}_N)_{1,2} = 0$, $\mathbb{E}(\mathbf{W}_N)_{1,1}^2 = 2\sigma^2$, $\mathbb{E}(\mathbf{W}_N)_{1,2}^2 = \sigma^2$,
- $c_3 := \mathbb{E}\left|(\mathbf{W}_N)_{1,1}\right|^3 < \infty$, $m_5 := \mathbb{E}\left|(\mathbf{W}_N)_{1,2}\right|^5 < \infty$.

Hermitian case :

- $\left(\operatorname{Re}\mathbf{W}_N\right)_{i,j}$, $\left(\operatorname{Im}\mathbf{W}_N\right)_{i,j}$, $1 \leq i < j \leq N$, $\left(\mathbf{W}_N\right)_{i,i}$, $1 \leq i \leq N$, are independent.
- The $(\operatorname{Re}\mathbf{W}_N)_{i,j}$'s, $(\operatorname{Im}\mathbf{W}_N)_{i,j}$'s for $i \neq j$ (respectively, $(\mathbf{W}_N)_{i,i}$'s), are identically distributed,
- $\mathbb{E}(\mathbf{W}_N)_{1,1} = \mathbb{E}(\mathbf{W}_N)_{1,2} = 0$, $\mathbb{E}(\mathbf{W}_N)_{1,1}^2 = \sigma^2$, $\mathbb{E}(\operatorname{Re}\mathbf{W}_N)_{1,2}^2 = \dfrac{\sigma^2}{2}$,
- $c_3 := \mathbb{E}\left|(\mathbf{W}_N)_{1,1}\right|^3 < \infty$, $m_5 := \mathbb{E}\left|(\mathbf{W}_N)_{1,2}\right|^5 < \infty$.

In this case, we have the following version of Theorem 2.3

Theorem 3.2 *(Convergence of the outliers for Wigner matrices) Let $\theta_1, \ldots, \theta_j$ be the eigenvalues of $\mathbf{A}_N$ such that $|\theta_i| > \sigma$. Then, with probability tending to one, for all large enough N, there are exactly j eigenvalues $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_j$ of $\widetilde{\mathbf{H}}_N := \frac{1}{\sqrt{N}}\mathbf{W}_N + \mathbf{A}_N$ at a macroscopic distance of $[-2\sigma, 2\sigma]$ (outliers). More precisely, for all small enough $\delta > 0$, for all large enough N, for all $i \in \{1, \ldots, j\}$,*

$$\widetilde{\lambda}_i = \theta_i + \frac{\sigma^2}{\theta_i} + o(1),$$

after a proper labeling.

Proof We just need to check that Assumptions 2.1 and 2.2 are satisfied.

- As long as the entries of $\mathbf{W}_N$ have a finite fourth moment, we know (see [2, Theorem5.2]) that Assumption 2.1 is satisfied.
- Now, we need to show that for any $\delta > 0$, as N goes to infinity,

$$\sup_{\text{dist}(z,\text{supp}(\mu))>\delta} \left\| \mathbf{U}_{2r}^* \left(z\mathbf{I} - \mathbf{H}_N \right)^{-1} \mathbf{U}_{2r} - G_{\mu_{\text{sc}}}(z)\mathbf{I} \right\|_{\text{op}} \xrightarrow{(\mathbb{P})} 0.$$

Since we are dealing with $2r \times 2r$-sized matrices, it suffices to prove that for any unite vectors $\mathbf{u}$,$\mathbf{v}$ of $\mathbb{C}^N$, for any $\delta > 0$ and any $\eta > 0$, as N goes to infinity,

$$\mathbb{P}\Big(\sup_{\text{dist}(z,\text{supp}(\mu))>\delta} \left| \mathbf{u}^* \big((z\mathbf{I} - \mathbf{H}_N)^{-1} - G_{\mu_{\text{sc}}}(z)\mathbf{I} \big) \mathbf{v} \right| > \eta \Big) \longrightarrow 0.$$

Moreover, as both $G_{\mu_{\text{sc}}}(z)$ and $\left\| (z\mathbf{I} - \mathbf{H}_N)^{-1} \right\|_{\text{op}}$ go to 0 when $|z|$ goes to infinity, we know there is a large enough constant M such that we just need to prove that

$$\mathbb{P}\Big(\sup_{\substack{\text{dist}(z,\text{supp}(\mu))>\delta \\ |z| \leq M}} \left| \mathbf{u}^* \big((z\mathbf{I} - \mathbf{H}_N)^{-1} - G_{\mu_{\text{sc}}}(z)\mathbf{I} \big) \mathbf{v} \right| > \eta \Big) \longrightarrow 0.$$

Then, for any $\eta' > 0$, the compact set $K = \{z,\ \text{dist}(z, \text{supp}(\mu)) > \delta$ and $|z| \leq M\}$ admits a η'-net, which is a finite set $\{z_1, \ldots, z_p\}$ of K such that

$$\forall z \in K,\ \exists i \in \{1, \ldots, p\}, \quad |z - z_i| < \eta',$$

so that, using the uniform boundedness of the derivative of $G_{\mu_{\text{sc}}}(z)$ and $\mathbf{u}^* (z - \mathbf{H}_N)^{-1} \mathbf{v}$ on K, for a small enough η', we just need to prove that

$$\mathbb{P}\Big(\max_{i=1}^{p} \left| \mathbf{u}^* \big((z_i\mathbf{I} - \mathbf{H}_N)^{-1} - G_{\mu_{\text{sc}}}(z_i)\mathbf{I} \big) \mathbf{v} \right| > \eta/2 \Big) \longrightarrow 0.$$

Then, we properly decompose each function $x \mapsto \frac{1}{z_i - x}$ as a sum of a smooth compactly supported function and one that vanishes on a neighborhood of $[-2\sigma, 2\sigma]$ and conclude using [30, (ii) Theorem 1.6]

Moreover, in the Wigner case, we have

$$G_{\mu_{\text{sc}}}(z) = \frac{z - \sqrt{z^2 - 4\sigma^2}}{2\sigma^2},$$

where $\sqrt{z^2 - 4\sigma^2}$ is the branch of the square root with branch cut $[-2\sigma, 2\sigma]$ so that for any z outside $[-2\sigma, 2\sigma]$, the equation $G_{\mu_{\text{sc}}}(z) = \frac{1}{\theta}$ possesses one solution if and only if $|\theta| > \sigma$ and the unique solution is

$$\theta + \frac{\sigma^2}{\theta},$$

which means that in the Wigner case, the outliers cannot outnumber the rank of the perturbation, and the *phase transition* condition is simply : $|\theta| > \sigma$. Actually, in [5] (see Remark 3.2), the authors explain that if μ is $\boxplus$-infinitely divisible, then the sets $\mathscr{S}_{\theta_i}$'s have at most one element, which means that for Wigner matrices, it is not possible to observe the phenomenon of "outliers outnumber the rank of **A**." □

Remark 3.3 One can find an other proof of Theorem 3.2 in [28] as a particular case of the Theorem 2.4 (see [28, Remark 2.5]) due to the fact that a Wigner matrix can be seen as a particular Elliptic matrix. Nevertheless, the authors of [28] do not deal with the matter of the fluctuations.

To study the fluctuations of the outliers in the Wigner case, we must make an additional assumption on the perturbation $\mathbf{A}_N$.

Assumption 3.4 The matrix $\mathbf{A}_N$ has only a finite number (independent of N) of entries which are nonzero.

Remark 3.5 Assumption 3.4 is equivalent to suppose that $\mathbf{U}_{2r}$ (the $2r$-first columns of $\mathbf{U}$) possesses only a finite number K (independent of N) of nonzero rows. Actually, this assumption is the analogous "the eigenvectors of **A** do not spread out" hypothesis corresponding to the "case a)" in [14].

Remark 3.6 If **U** is Haar-distributed and independent from **W**, we can avoid making Assumption 3.4 (see Sect. 3.2). One can also slightly weaken Assumption 3.4 by assuming that the $2r$-first rows of **U** correspond to the N first coordinates of a collection of non-random vectors $\mathbf{u}_1, \ldots, \mathbf{u}_{2r}$ in $\ell^2(\mathbb{N})$ (see [30, Theorem 1.7]).

Theorem 3.7 (Fluctuations for Wigner matrices) *With assumptions 3.1 and 3.4, Theorem 2.10 holds. Moreover, the distribution of the random vector*

$$\left(m_{k,\ell}^{\theta_i}\right)_{\substack{1\leq i\leq q\\(k,\ell)\in J(\theta_i)\times I(\theta_i)}},$$

defined by (10), is

$$\left(\mathbf{e}_k^*\mathbf{Q}^{-1}\mathbf{U}_{K,2r}^*\Upsilon(\xi_i)\mathbf{U}_{K,2r}\mathbf{Q}\mathbf{e}_\ell\right)_{\substack{1\leq i\leq q\\(k,\ell)\in J(\theta_i)\times I(\theta_i)}},$$

where $\xi_i := \theta_i + \dfrac{\sigma^2}{\theta_i}$ *and where* $\Upsilon(z)$ *is a* $K\times K$ *random field defined by*

$$\Upsilon(z) := (G_{\mu_{\mathrm{sc}}}(z))^2\left(\mathbf{W}^{(K)} + \mathbf{Y}(z)\right) \tag{14}$$

where $\mathbf{W}^{(K)}$ *is the* $K\times K$ *upper-left corner submatrix of a matrix* $\widetilde{\mathbf{W}}_N$ *such that* $\widetilde{\mathbf{W}}_N \overset{(\mathrm{d})}{=} \mathbf{W}_N$ *and* $\mathbf{Y}(z)$ *is a* $K\times K$ *Gaussian random field defined by [31, (2.7),(2.8),(2.9),(2.10),(2.11),(2.12)] in the real case and [31, (2.42),(2.43),(2.44), (2.45),(2.46),(2.47)] in the complex case.*

Remark 3.8 This provides an example of non-universal fluctuations, in the sense that the $\left(m_{k,\ell}^{\theta_i}\right)$'s are not necessarily Gaussian. However, when $\mathbf{H}_N$ is a GOE or GUE matrix, the $\left(m_{k,\ell}^{\theta_i}\right)$'s are centered Gaussian variables such that

$$\mathbb{E}\left(m_{k,\ell}^{\theta_i}\, m_{k',\ell'}^{\theta_{i'}}\right) = \psi_{\mathrm{sc}}(\xi_i, \xi_{i'})\, \left(\mathbf{e}_k^* \mathbf{Q}^{-1}(\mathbf{Q}^{-1})^* \mathbf{e}_{k'}\; \mathbf{e}_\ell^* \mathbf{Q}^* \mathbf{Q} \mathbf{e}_{\ell'} \;+\; \delta_{k,\ell'}\delta_{k',\ell}\right),$$
$$\mathbb{E}\left(m_{k,\ell}^{\theta_i}\, \overline{m_{k',\ell'}^{\theta_{i'}}}\right) = \psi_{\mathrm{sc}}(\xi_i, \overline{\xi_{i'}})\, \left(\mathbf{e}_k^* \mathbf{Q}^{-1}(\mathbf{Q}^{-1})^* \mathbf{e}_{k'}\; \mathbf{e}_\ell^* \mathbf{Q}^* \mathbf{Q} \mathbf{e}_{k'} \;+\; \delta_{k,\ell'}\delta_{k',\ell}\right), \quad (15)$$

for the GOE, and

$$\mathbb{E}\left(m_{k,\ell}^{\theta_i}\, m_{k',\ell'}^{\theta_{i'}}\right) = \psi_{\mathrm{sc}}(\xi_i, \xi_{i'})\, \delta_{k,\ell'}\delta_{k',\ell},$$
$$\mathbb{E}\left(m_{k,\ell}^{\theta_i}\, \overline{m_{k',\ell'}^{\theta_{i'}}}\right) = \psi_{\mathrm{sc}}(\xi_i, \overline{\xi_{i'}})\, \mathbf{e}_k^* \mathbf{Q}^{-1}(\mathbf{Q}^{-1})^* \mathbf{e}_{k'}\; \mathbf{e}_{\ell'}^* \mathbf{Q}^* \mathbf{Q}\, \mathbf{e}_\ell, \quad (16)$$

for the GUE, where

$$\psi_{\mathrm{sc}}(z, w) := G_{\mu_{\mathrm{sc}}}^2(z) G_{\mu_{\mathrm{sc}}}^2(w) \left(\sigma^2 + \sigma^4 \varphi_{\mathrm{sc}}(z, w)\right),$$
$$\varphi_{\mathrm{sc}}(z, w) := \int \frac{1}{z - x} \frac{1}{w - x} \mu_{\mathrm{sc}}(\mathrm{d}x).$$

We notice that if $\mathbf{Q}^{-1} \neq \mathbf{Q}^*$, then we might observe correlations between the fluctuations of outliers at a macroscopic distance with each other. This phenomenon has already been observed in [25] for non-Gaussian Wigner matrices, whereas, here, the phenomenon may still occur for GUE matrices. Actually, (15) and (16) can be simplified due to the fact

$$\sigma^2 G_{\mu_{\mathrm{sc}}}^2(z) - z G_{\mu_{\mathrm{sc}}}(z) + 1 \;=\; 0,$$

so that $\varphi_{\mathrm{sc}}(z, w) = -\dfrac{G_{\mu_{\mathrm{sc}}}(z) - G_{\mu_{\mathrm{sc}}}(w)}{z - w}$ satisfies

$$\sigma^2 G_{\mu_{\mathrm{sc}}}(z) G_{\mu_{\mathrm{sc}}}(w) \varphi_{\mathrm{sc}}(z, w) = \varphi_{\mathrm{sc}}(z, w) - G_{\mu_{\mathrm{sc}}}(z) G_{\mu_{\mathrm{sc}}}(w). \quad (17)$$

Hence,

$$\begin{aligned}
&G_{\mu_{\mathrm{sc}}}^2(\xi_i) G_{\mu_{\mathrm{sc}}}^2(\xi_{i'}) \left(\sigma^2 + \sigma^4 \varphi_{\mathrm{sc}}(\xi_i, \xi_{i'})\right) \\
&\quad = \sigma^2 G_{\mu_{\mathrm{sc}}}(\xi_i) G_{\mu_{\mathrm{sc}}}(\xi_{i'}) \Big[G_{\mu_{\mathrm{sc}}}(\xi_i) G_{\mu_{\mathrm{sc}}}(\xi_{i'}) + \sigma^2 G_{\mu_{\mathrm{sc}}}(\xi_i) G_{\mu_{\mathrm{sc}}}(\xi_{i'}) \varphi_{\mathrm{sc}}(\xi_i, \xi_{i'}) \Big] \\
&\quad = \sigma^2 G_{\mu_{\mathrm{sc}}}(\xi_i) G_{\mu_{\mathrm{sc}}}(\xi_{i'}) \varphi_{\mathrm{sc}}(\xi_i, \xi_{i'}) \\
&\quad = \varphi_{\mathrm{sc}}(\xi_i, \xi_{i'}) - G_{\mu_{\mathrm{sc}}}(\xi_i) G_{\mu_{\mathrm{sc}}}(\xi_{i'}) \;=\; \Phi_{\mathrm{sc}}(\xi_i, \xi_{i'}).
\end{aligned}$$

and we fall back on the expression of the variance for the UCI model (see Sect. 3.2), which is expected since the GUE belongs to the UCI model.

Proof We show that the assumptions 3.1 and 3.4 imply Assumption 2.6, more precisely (1) and (2). For (1), we simply use [31, Theorem 2.1/2.5] to show that

$$\sqrt{N}\mathbf{U}_{2r}^* \left((z - \mathbf{H}_N)^{-1} - G_{\mu_{\mathrm{sc}}}(z)\mathbf{I} \right) \mathbf{U}_{2r},$$

converges weakly (as it is done in [30]). The limit distribution is also given by [31, Theorem 2.1/2.5].

Then, for (2), we know by [31, (i) of Theorem 2.3/2.7] (respectively, [31, (iii) of Proposition 2.1]) that, for all $k \geq 1$, the diagonal entries (respectively, the off-diagonal entries) of the matrix

$$\sqrt{N}\big((z - \mathbf{H}_N)^{-k-1} - \int (z-x)^{-k-1} \mu_{\mathrm{sc}}(\mathrm{d}x)\mathbf{I} \big)$$

converge in distribution so that

$$\sqrt{N}\mathbf{U}_{2r}^* \left((z - \mathbf{H}_N)^{-k-1} - \int (z-x)^{-k-1} \mu_{\mathrm{sc}}(\mathrm{d}x)\mathbf{I} \right) \mathbf{U}_{2r}$$

is tight.

3.2 Hermitian Matrices Whose Distribution is Invariant by Unitary Conjugation

Let $\mathbf{H}_N$ be an Hermitian matrix such that for any unitary $N \times N$ matrix $\mathbf{U}_N$, we have

$$\mathbf{U}_N \mathbf{H}_N \mathbf{U}_N^* \overset{(\mathrm{d})}{=} \mathbf{H}_N. \tag{18}$$

$\mathbf{H}_N$ can be written $\mathbf{H}_N = \mathbf{U}_N \mathbf{D}_N \mathbf{U}_N^*$ where $\mathbf{D}_N$ is diagonal, $\mathbf{U}_N$ is Haar-distributed, and $\mathbf{U}_N$ and $\mathbf{D}_N$ are independent. We also assume that $\mathbf{H}_N$ satisfies (1) and Assumption 2.1. We shall call such matrices *UCI matrices* (for unitary conjugation invariance). In this case, as we can, we can write

$$\widetilde{\mathbf{H}}_N = \mathbf{H}_N + \mathbf{A}_N = \mathbf{U}_N \left(\mathbf{D}_N + \mathbf{U}_N^* \mathbf{A}_N \mathbf{U}_N \right) \mathbf{U}_N^*,$$

so that, without any loss of generality, we can simply assume that $\mathbf{H}_N$ is a diagonal matrix, and $\mathbf{A}_N$ is a matrix of the form

$$\mathbf{A}_N = \mathbf{U}_{2r} \mathbf{A}_0 \mathbf{U}_{2r}^*$$

where $\mathbf{U}_{2r}$ is the $2r$-first columns of an Haar-distributed matrix independent from $\mathbf{H}_N$.

Theorem 3.9 (Convergence of the outliers for UCI matrices) *If* $\mathbf{H}_N$ *is an UCI matrix, then Theorem 2.3 holds.*

Remark 3.10 Unlike the Wigner case, Theorem 2.3 does not need to be reformulated. In this case, we do observe the phenomenon of the outliers outnumbering the rank of $\mathbf{A}_N$.

Proof We just need to check that Assumption 2.2 is satisfied. To do so, one can apply a slightly modified version of [6, Lemma 2.2], where we replace all the "$\mathrm{dist}(z, [a, b]) > \delta$" by "$\mathrm{dist}(z, \mathrm{supp}(\mu)) > \delta$," which does not change the ideas of the proof. □

For the fluctuations, we need to assume that for all $i \in \{1, \ldots, q\}$ and all $n \in \{1, \ldots, m_i\}$, as N goes to infinity,

$$\sqrt{N}\left(\frac{1}{N}\operatorname{Tr}\left(\xi_{i,n} - \mathbf{H}_N\right)^{-1} - \frac{1}{\theta_i})\right) \longrightarrow 0. \tag{19}$$

Remark 3.11 Actually, in [6], the authors make the same assumption ([6, Hypothesis 3.1]).

Theorem 3.12 *(Fluctuations for UCI matrices) If $\mathbf{H}_N$ is an UCI matrix, then it satisfies Theorem 2.10. More precisely, the*

$$\left(m_{k,\ell}^{\theta_i,n}\right)_{\substack{1\le i\le q\\ 1\le n\le m_i\\ (k,\ell)\in J(\theta_i)\times I(\theta_i)}},$$

defined by (10) *are centered Gaussian variables such that*

$$\mathbb{E}\left(m_{k,\ell}^{\theta_i,n}\, m_{k',\ell'}^{\theta_{i'},n'}\right) = \Phi(\xi_{i,n}, \xi_{i',n'})\, \delta_{k,\ell'}\delta_{k',\ell},$$
$$\mathbb{E}\left(m_{k,\ell}^{\theta_i,n'}\, \overline{m_{k',\ell'}^{\theta_{i'},n'}}\right) = \Phi(\xi_{i,n}, \overline{\xi_{i',n'}})\; \mathbf{e}_k^*\mathbf{Q}^{-1}(\mathbf{Q}^{-1})^*\mathbf{e}_{k'}\; \mathbf{e}_{\ell'}^*\mathbf{Q}^*\mathbf{Q}\,\mathbf{e}_\ell,$$

where

$$\begin{aligned}\Phi(z, w) &:= \int \frac{1}{z-x}\frac{1}{w-x}\mu(\mathrm{d}x) - \int \frac{1}{z-x}\mu(\mathrm{d}x)\int \frac{1}{w-x}\mu(\mathrm{d}x)\\ &= \begin{cases} -\frac{G_\mu(z)-G_\mu(w)}{z-w} - G_\mu(z)G_\mu(w) & : \text{ if } z \neq w,\\ -G'_\mu(z) - (G_\mu(z))^2 & : \text{ otherwise.}\end{cases}\end{aligned}$$

Remark 3.13 Remind that we supposed that μ is not a single Dirac measure, so that Φ is not equal to zero.

Remark 3.14 If $\mathbf{A}_N$ is Hermitian, the size of all the Jordan blocks is equal to 1 and the fluctuations are real random variables (see Remark 2.9). We find back that, in the Hermitian case, fluctuations between outliers at a macroscopic distance are independent (see [6]) except if the two outliers come from the same eigenvalue of $\mathbf{A}$ (i.e., they both belong to the same set $\mathscr{S}_\theta$). In this case, the fluctuations of outliers belonging to the same set $\mathscr{S}_\theta$ are all correlated. This phenomenon is illustrated by Fig. 3a, b.

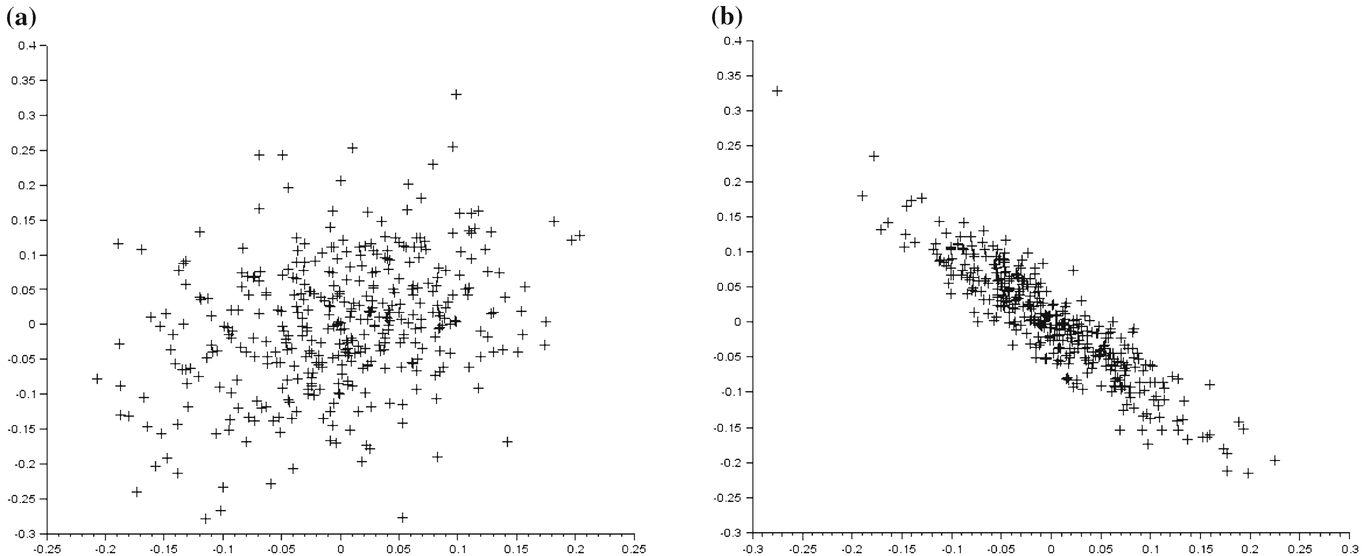

Fig. 3 Correlation between the fluctuations of two outliers $\xi_1 \neq \xi_2$ for a sample of 400 matrices of size 500, in the case where both **H** and **A** are Hermitian and where the support of μ is disconnected. Here, $\mu(\mathrm{d}x)$ is taken equal to $\frac{1}{2}(\delta_{-1}(\mathrm{d}x) + \mathbb{1}_{[1,2]}(\mathrm{d}x))$. (a) Uncorrelated case : $G_\mu(\xi_1) \neq G_\mu(\xi_2)$, which means that ξ_1 and ξ_2 do not belong to the same set $\mathscr{S}_\theta$. (b) Correlated case : $G_\mu(\xi_1) = G_\mu(\xi_2)$, which means that ξ_1 and ξ_2 belong to the same set $\mathscr{S}_\theta$

Proof We just need to check that $\mathbf{H}_N$ satisfies (1′), (2′) of Assumption 2.6 (since (0′) is assumed below). Actually, for any $k \geq 1$ and any $i \in \{1, \ldots, q\}$, the diagonal matrix

$$\left(\xi_{i,n} - \mathbf{H}_N\right)^{-(k+1)} - \frac{1}{N} \operatorname{Tr} \left(\xi_{i,n} - \mathbf{H}_N\right)^{-(k+1)},$$

fulfill the assumptions of Theorem 5.3, so that (2′) is true. Then, (1′) is true thanks to Theorem 5.5. This theorem also gives us the covariance.

4 Proofs

4.1 Convergence of the Outliers: Proof of Theorem 2.3

In [5], the authors give an interpretation of why the limit is necessarily a solution of $G_\mu(z) = \frac{1}{\theta}$ with the subordinate functions of the free additive convolution of measures in the particular case where one of the measures is δ_0 (see [5, Example 4.1]). Actually, our definition of the sets $\mathscr{S}_{\theta_i}$'s corresponds to the one of the set O_θ in [5, Definition 4.1]. A quick (but inaccurate) way to see why the limit is $G_\mu^{-1}(\frac{1}{\theta})$ and to understand the approach of the proof is to write

$$\det\left(z - (\mathbf{H}_N + \mathbf{A}_N)\right) \;=\; \det\left(z - \mathbf{H}_N\right) \det\left(\mathbf{I} - (z - \mathbf{H}_N)^{-1}\, \mathbf{A}_N\right),$$

then if $(z - \mathbf{H}_N)^{-1} \sim G_\mu(z)\mathbf{I}$, we can write

$$\det\left(z - (\mathbf{H}_N + \mathbf{A}_N)\right) \sim \det\left(z - \mathbf{H}_N\right) G_\mu(z) \det\left(\frac{1}{G_\mu(z)}\mathbf{I} - \mathbf{A}_N\right)$$

so that if z is an outlier of $\mathbf{H}_N + \mathbf{A}_N$, $\frac{1}{G_\mu(z)}$ must be an eigenvalue of $\mathbf{A}_N$.

To do it properly, we introduce the following function[3],

$$f(z) = \det\left(\mathbf{I} - \mathbf{U}_{2r}^* \left(z\mathbf{I} - \mathbf{H}\right)^{-1} \mathbf{U}_{2r}\mathbf{A}_0\right) \;=\; \frac{\det\left(z - (\mathbf{H}_N + \mathbf{A}_N)\right)}{\det\left(z - \mathbf{H}_N\right)}, \qquad (20)$$

we know that the zeros of f are eigenvalues of $\widetilde{\mathbf{H}}_N$ which are not eigenvalues of $\mathbf{H}_N$. Then, we introduce the function

$$f_0(z) := \det\left(\mathbf{I} - G_\mu(z)\mathbf{A}_0\right), \qquad (21)$$

and the proof of Theorem 2.3 relies on the two following lemmas.

[3] we used a classical trick of finite rank perturbation models which $\det(\mathbf{I}_m + \mathbf{AB}) = \det(\mathbf{I}_n + \mathbf{BA})$ for any $m \times n$ matrix $\mathbf{A}$ and $n \times m$ matrix $\mathbf{B}$

Lemma 4.1 *As N goes to infinity, we have*

$$\sup_{\mathrm{dist}(z,\mathrm{supp}(\mu))>\delta} |f(z) - f_0(z)| \xrightarrow{(\mathbb{P})} 0.$$

Lemma 4.2 *Let K be a compact set, and let $\varepsilon > 0$ such that*

- $\mathrm{dist}(K, \bigcup_{i=1}^{j} \mathscr{S}_{\theta_i}) \geq \varepsilon$,
- $\mathrm{dist}(K, \mathrm{supp}(\mu)) \geq \varepsilon$.

Then, with a probability tending to one,

$$\inf_{z\in K} \left|\det\left(\mathbf{I} - (z - \mathbf{H}_N)^{-1} \mathbf{A}_N\right)\right| > 0.$$

If these lemmas are true, the end of the proof goes as follows. We know that, with a probability tending to one, there is $\varepsilon > 0$, such that

- there is a constant $M > 0$ such that $\mathbf{H}_N + \mathbf{A}_N$ has no eigenvalues in the area $\{z,\ |z| > M\}$,
- $\mathrm{Spec}(\mathbf{H}_N) \subset \{z,\ \mathrm{dist}(z, \mathrm{supp}(\mu)) < \varepsilon\}$,

We set

$$\mathscr{S} := \bigcup_{i=1}^{j} \mathscr{S}_{\theta_i}$$

, and we define

$$\mathscr{S}^{\varepsilon} := \bigcup_{i=1}^{j} \bigcup_{\xi\in\mathscr{S}_{\theta_i}} \{z,\ |z - \xi| < \varepsilon\} \tag{22}$$

with the convention that $\mathscr{S}^{\varepsilon} = \emptyset$ if $\mathscr{S} = \emptyset$. Up to a smaller choice of ε, we can suppose that none of the disk centered in the element of the $\mathscr{S}_{\theta_i}$'s and of radius ε intersects each other nor intersect $\{z,\ \mathrm{dist}(z, \mathrm{supp}(\mu)) < \varepsilon\}$. Then, using Lemma 4.2, with

$$K := \{z,\ |z| \leq M\} \setminus \left(\mathscr{S}^{\varepsilon} \cup \{z,\ \mathrm{dist}(z, \mathrm{supp}(\mu)) < \varepsilon\}\right),$$

we deduce all the eigenvalues of $\widetilde{\mathbf{H}}_N$ are contained in $\mathscr{S}^{\varepsilon} \cup \{z,\ \mathrm{dist}(z, \mathrm{supp}(\mu)) < \varepsilon\}$. Indeed, if z is an eigenvalue of $\widetilde{\mathbf{H}}_N$ such that $\mathrm{dist}(z, \mathrm{supp}(\mu)) > \varepsilon$, z must be a zero of f.

Moreover, for each $i \in \{1, \ldots, j\}$ and each $\xi \in \mathscr{S}_{\theta_i}$, we know that from Lemma 4.1

$$\sup_{z,|z-\xi|=\varepsilon} |f(z) - f_0(z)| \longrightarrow 0, \quad \text{and} \quad \inf_{z,|z-\xi|=\varepsilon} |f_0(z)| > 0.$$

We deduce by Rouché Theorem (see [4, p.131]) that f and f_0, for all large enough N, have the same number of zeros inside the domain $\{z,\ |z-\xi| < \varepsilon\}$, for each ξ in the $\mathscr{S}_{\theta_i}$'s.

Now, we just need to prove the two previous lemmas.

Proof of Lemma 4.1 We know that, for some positive constant C,

$$\sup_{\mathrm{dist}(z,\mathrm{supp}(\mu))>\delta} |f(z)-f_0(z)| \le C \sup_{\mathrm{dist}(z,\mathrm{supp}(\mu))>\delta} \left\| \mathbf{U}_{2r}^* \left((z-\mathbf{H}_N)^{-1} - G_\mu(z)\mathbf{I} \right) \mathbf{U}_{2r} \right\|_{\mathrm{op}},$$

and we conclude with Assumption 2.2.

Proof of Lemma 4.2 We write, thanks to Assumption 2.2,

$$\begin{aligned} \det\left(\mathbf{I} - (z-\mathbf{H}_N)^{-1}\mathbf{A}_N\right) &= \det\left(\mathbf{I} - \mathbf{U}_{2r}^* (z-\mathbf{H}_N)^{-1}\mathbf{U}_{2r}\mathbf{A}_0\right) \\ &= \det\left(\mathbf{I} - G_\mu(z)\mathbf{A}_0 + o(1)\right) \\ &= \prod_{i=1}^{k}\left(1 - G_\mu(z)\theta_i\right) + o(1). \end{aligned}$$

Then, since $z \in K$, it easy to show that for each i, $|1 - G_\mu(z)\theta_i| > 0$.

4.2 Fluctuations

The proof of Theorem 2.10 is the same than [10, Theorem 2.10], and all we need to do here is to prove this analogous version of [10, Lemma 5.1].

Lemma 4.3 *For all $j \in \{1, \dots, \alpha_i\}$ and all $n \in \{1, \dots, m_i\}$, let $F_{j,n}^{\theta_i}(z)$ be the rational function defined by*

$$F_{j,n}^{\theta_i}(z) := f\left(\xi_{i,n} + \frac{z}{N^{1/(2p_{i,j})}}\right). \tag{23}$$

Then, there exists a collection of positive constants $(\gamma_{i,j})_{\substack{1\le i\le q\\ 1\le j\le\alpha_i}}$ and a collection of nonvanishing random variables $(C_{i,j,n})_{\substack{1\le i\le q\\ 1\le j\le\alpha_i\\ 1\le n\le m_i}}$ independent of z, such that we have the convergence in distribution (for the topology of the uniform convergence over any compact set)

$$\left(N^{\gamma_{i,j}} F_{j,n}^{\theta_i}(\cdot)\right)_{\substack{1\le i\le q\\ 1\le j\le\alpha_i\\ 1\le n\le m_i}} \underset{N\to\infty}{\longrightarrow} \left(z\in\mathbb{C} \mapsto z^{\pi_{i,j}} \cdot C_{i,j,n} \cdot \det\left(z^{p_{i,j}} - \mathbf{M}_{j,n}^{\theta_i}\right)\right)_{\substack{1\le i\le q\\ 1\le j\le\alpha_i\\ 1\le n\le m_i}}$$

where $\mathbf{M}_{j,n}^{\theta_i}$ is the random matrix introduced at (11) and $\pi_{i,j} := \sum_{l>j}\beta_{i,l}p_{i,l}$.

Once this lemma proven, the Theorem 2.10 follows (see section 5.1 of [10] for more details). To prove Lemma 4.3, we shall proceed as it is done in [10] to prove Lemma 5.1. First, we write, for a fixed $\theta_i (= \theta)$, a fixed $n \in \{1, \ldots, m_i\}$ and a fixed $j \in \{1, \ldots, \alpha_i\}$ (which shall be implicit) and fixed $p_{i,j} (= p)$, recall that $\mathbf{A}_0 = \mathbf{QJQ}^{-1}$,

$$\begin{aligned} F_{j,n}^{\theta}(z) &= \det\left(\mathbf{I} - \left(\xi_n + \frac{z}{N^{1/(2p)}} - \mathbf{H}_N\right)^{-1} \mathbf{U}_{2r}\mathbf{QJQ}^{-1}\mathbf{U}_{2r}^*\right) \\ &= \det\left(\mathbf{I} - G_\mu\left(\xi_n + \frac{z}{N^{1/(2p)}}\right)\mathbf{J} - \frac{1}{\sqrt{N}}\mathbf{Z}_N\left(\xi_n + \frac{z}{N^{1/(2p)}}\right)\right) \\ &= \det\left(\mathbf{I} - \frac{\mathbf{J}}{\theta} - G'_\mu(\xi_n)\frac{z}{N^{1/(2p)}}(1 + o(1))\mathbf{J} - \frac{1}{\sqrt{N}}\mathbf{Z}_N\left(\xi_n + \frac{z}{N^{1/(2p)}}\right)\right) \end{aligned}$$

where

$$Z_N(z) := \sqrt{N}\mathbf{Q}^{-1}\mathbf{U}_{2r}^*\left((z - \mathbf{H}_N)^{-1} - G_\mu(z)\mathbf{I}\right)\mathbf{U}_{2r}\mathbf{QJ}.$$

Remind that by definition, $G_\mu(\xi_n) = \theta^{-1}$. From here, the reasoning to end the proof is the exact same than the one from [10, Lemma 5.1]. Nevertheless, we still have to prove that, for all θ and for all n, for all compact set K and for all $z \in K$,

$$\mathbf{Z}_N\left(\xi_n + \frac{z}{N^{1/(2p)}}\right) = \mathbf{Z}_N(\xi_n) + o(1), \qquad \text{and} \quad \mathbf{Z}_N(\xi_n) \text{ converges weakly.} \tag{24}$$

To do so, we write (thanks to 5.1),

$$\begin{aligned} \mathbf{Z}_N\left(\xi_n + \frac{z}{N^{1/(2p)}}\right) &= \sqrt{N}\mathbf{Q}^{-1}\mathbf{U}_{2r}^*\left((\xi_n - \mathbf{H}_N)^{-1} - \frac{1}{\theta}\right)\mathbf{U}_{2r}\mathbf{QJ} \\ &+ \sum_{k=1}^{p}\left(\frac{-z}{N^{1/(2p)}}\right)^k \sqrt{N}\mathbf{Q}^{-1}\mathbf{U}_{2r}^*\left((\xi_n - \mathbf{H}_N)^{-(k+1)} - \int \frac{\mu(\mathrm{d}x)}{(\xi_n - x)^{k+1}}\right)\mathbf{U}_{2r}\mathbf{QJ} \\ &+ \frac{1}{N^{1/(2p)}}\mathbf{QU}_{2r}^*\left(\xi_n - \mathbf{H}_N\right)^{-(k+1)}\left(\xi_n + \frac{z}{N^{1/(2p)}} - \mathbf{H}_N\right)^{-1}\mathbf{U}_{2r}\mathbf{QJ} + o(1). \end{aligned}$$

The last term is a $o(1)$ since $\mathrm{dist}(\xi_n, \mathrm{Spec}(\mathbf{H}_N)) > \varepsilon$ and one can conclude if (1), (2) are satisfied in Assumption 2.6. Otherwise, if it's $(0')$, $(1')$, $(2')$, we write

$$\begin{aligned} \mathbf{Z}_N\left(\xi_n + \frac{z}{N^{1/(2p)}}\right) &= \sqrt{N}\mathbf{Q}^{-1}\mathbf{U}_{2r}^*\left(\frac{1}{N}\operatorname{Tr}(\xi_n - \mathbf{H}_N)^{-1} - \frac{1}{\theta}\right)\mathbf{U}_{2r}\mathbf{QJ} \\ &\quad + \sqrt{N}\mathbf{Q}^{-1}\mathbf{U}_{2r}^*\left((\xi_n - \mathbf{H}_N)^{-1} - \frac{1}{N}\operatorname{Tr}(\xi_n - \mathbf{H}_N)^{-1}\right)\mathbf{U}_{2r}\mathbf{QJ} \\ &\quad + \sum_{k=1}^{p}\left(\frac{-z}{N^{1/(2p)}}\right)^k \sqrt{N}\mathbf{Q}^{-1}\mathbf{U}_{2r}^*\left((\xi_n - \mathbf{H}_N)^{-(k+1)} - \frac{1}{N}\right. \\ &\quad \left.\operatorname{Tr}(\xi_n - \mathbf{H}_N)^{-1}\right)\mathbf{U}_{2r}\mathbf{QJ} \end{aligned}$$

$$+\frac{1}{N^{1/(2p)}}\mathbf{Q}\mathbf{U}_{2r}^{*}\left(\xi_n-\mathbf{H}_N\right)^{-(p+1)}$$
$$\left(\xi_n+\frac{z}{N^{1/(2p)}}-\mathbf{H}_N\right)^{-1}\mathbf{U}_{2r}\mathbf{Q}\mathbf{J}+o\left(1\right).$$

5 Linear Algebra Lemmas

Lemma 5.1 *Let* $\mathbf{A}$ *be a matrix and* $\lambda\in\mathbb{C}$ *be such that both* $\mathbf{A}$ *and* $\mathbf{A}+\lambda\mathbf{I}$ *are non-singular. Then, for all* $p\geq 1$*,*

$$(\mathbf{A}+\lambda\mathbf{I})^{-1}=\sum_{k=1}^{p}(-\lambda)^{k-1}\mathbf{A}^{-k}+(-\lambda)^{p}\mathbf{A}^{-p}\left(\mathbf{A}+\lambda\mathbf{I}\right)^{-1}$$

Lemma 5.2 *(Schur's complement [22]) For any* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$*, one has, when it makes sense*

$$\begin{pmatrix}\mathbf{A}&\mathbf{B}\\\mathbf{C}&\mathbf{D}\end{pmatrix}^{-1}=\begin{pmatrix}\left(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right)^{-1}&-\left(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right)^{-1}\mathbf{B}\mathbf{D}^{-1}\\-\mathbf{D}^{-1}\mathbf{C}\left(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right)^{-1}&\mathbf{D}^{-1}+\mathbf{D}^{-1}\mathbf{C}\left(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right)^{-1}\mathbf{B}\mathbf{D}^{-1}\end{pmatrix}$$

5.1 Fluctuations of the Entries of UCI Random Matrices

We give here some results on the fluctuations of the entries of UCI matrices, which means matrices of the form $\mathbf{H}:=\mathbf{U}\mathbf{D}\mathbf{U}^{*}$ where $\mathbf{U}$ is Haar-distributed and $\mathbf{D}$ is a complex diagonal matrix.

Theorem 5.3 *(Fluctuations of the entries of UCI random matrices) Let* $\mathbf{T}$ *be an* $N\times N$ *diagonal matrix such that*

$$\operatorname{Tr}\mathbf{T}=0,\qquad\frac{1}{N}\operatorname{Tr}\mathbf{T}\mathbf{T}^{*}\to\sigma^{2},\qquad\frac{1}{N}\operatorname{Tr}\mathbf{T}^{2}\to\tau^{2},$$
$$\forall k\geq 1,\ \frac{1}{N}\operatorname{Tr}(\mathbf{T}\mathbf{T}^{*})^{k}=O\left(1\right). \tag{25}$$

Let $\mathbf{u}_{t_1},\ldots,\mathbf{u}_{t_p}$ *be* p *distinct columns of a Haar-distributed unitary matrix. Then,*

$$\left(\sqrt{N}\langle\mathbf{u}_{t_i},\ \mathbf{T}\mathbf{u}_{t_j}\rangle\right)_{i,j=1}^{p},$$

converges in distribution to a centered complex Gaussian vector $\left(\mathcal{G}_{i,j}\right)_{i,j=1}^{p}$ *with covariance*

$$\mathbb{E}\left[\mathcal{G}_{i,j}\mathcal{G}_{k,\ell}\right]=\delta_{i,\ell}\delta_{j,k}\tau^{2}\ ;\ \mathbb{E}\left[\mathcal{G}_{i,j}\overline{\mathcal{G}}_{k,\ell}\right]=\delta_{i,k}\delta_{j,\ell}\sigma^{2}$$

Remark 5.4 If $\mathbf{H}:=\mathbf{U}\mathbf{D}\mathbf{U}^{*}$ satisfies (1) and Assumption 2.1, then $\mathbf{T}:=\mathbf{D}-\frac{1}{N}\operatorname{Tr}\mathbf{D}$ satisfies (25).

Here comes a version of Theorem 5.3, with several matrices diagonal **T**. Due to the complex values of the diagonal matrices, the following theorem is not a simple consequence of Theorem 5.3 and Cramér–Wold theorem.

Theorem 5.5 *Let* $\mathbf{T}_1, \ldots, \mathbf{T}_q$ *be* $N \times N$ *diagonal matrices such that for all* $m, n \in \{1, \ldots, q\}$

$$\operatorname{Tr} \mathbf{T}_m = 0, \qquad \frac{1}{N} \operatorname{Tr} \mathbf{T}_m \mathbf{T}_n^* \to \sigma_{m,n}^2, \qquad \frac{1}{N} \operatorname{Tr} \mathbf{T}_m \mathbf{T}_n \to \tau_{m,n}^2,$$
$$\forall k \geq 1, \ \frac{1}{N} \operatorname{Tr}(\mathbf{T}_m \mathbf{T}_m^*)^k = O(1).$$

Let $\mathbf{u}_{t_1}, \ldots, \mathbf{u}_{t_p}$ *be* p *distinct columns of an Haar-distributed matrix. Then,*

$$\left(\sqrt{N}\langle \mathbf{u}_{t_i}, \ \mathbf{T}_m \mathbf{u}_{t_j} \rangle\right)_{\substack{1 \leq i \leq p \\ 1 \leq j \leq p \\ 1 \leq m \leq q}},$$

converges in distribution to a centered complex Gaussian vector $\left(\mathscr{G}_{i,j,m}\right)_{\substack{1 \leq i \leq p \\ 1 \leq j \leq p \\ 1 \leq m \leq q}}$ *with covariance*

$$\mathbb{E}\left[\mathscr{G}_{i,j,m}\mathscr{G}_{k,\ell,n}\right] = \delta_{i,\ell}\delta_{j,k}\tau_{m,n}^2 \ ; \ \mathbb{E}\left[\mathscr{G}_{i,j,m}\overline{\mathscr{G}}_{k,\ell,n}\right] = \delta_{i,k}\delta_{j,\ell}\sigma_{m,n}^2$$

Proof of Theorem 5.3 Without any loss of generality, due to the invariance by conjugation by a matrix of permutation, we can suppose that $t_1 = 1, t_2 = 2, \ldots, t_p = p$. Then, we just need to show that

$$X = \sqrt{N} \operatorname{Tr}\left(\mathbf{U}^* \mathbf{T} \mathbf{U} \mathbf{A}\right)$$

where **A** is a $N \times N$ deterministic matrix of the form

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_p & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{A}_p = \left(a_{i,j}\right)_{i,j=1}^p \in \mathscr{M}_p(\mathbb{C}),$$

is a asymptotically Gaussian. Before starting, we remind some definition. Let $(\mathbf{M}_1, \ldots, \mathbf{M}_q)$ be q matrices. For any permutation $\sigma \in S_q$, with cycle decomposition

$$\sigma = (i_{1,1} \cdots i_{1,k_1})(i_{2,1} \cdots i_{2,k_2}) \cdots (i_{r,1} \cdots i_{r,k_r}),$$

we denote by

$$\operatorname{Tr}_\sigma \left(\mathbf{M}_t\right)_{t=1}^q := \prod_{j=1}^r \operatorname{Tr}\left(\mathbf{M}_{t_{i_{j,1}}} \cdots \mathbf{M}_{t_{i_{j,r_j}}}\right). \tag{26}$$

For example, if $\sigma = (13)(256) \in S_6$, then

$$\mathrm{Tr}_\sigma \left(\mathbf{M}_t\right)_{t=1}^{6} = \mathrm{Tr}(\mathbf{M}_1\mathbf{M}_3)\,\mathrm{Tr}(\mathbf{M}_2\mathbf{M}_5\mathbf{M}_6)\,\mathrm{Tr}(\mathbf{M}_4).$$

Let $M(2n)$ be the set of all *perfect matching* on $\{1, \ldots, 2n\}$ which is a subset of S_{2n} of the permutation which are the product of n transpositions with disjoint support. For example,

$$M(4) = \{(12)(34), (13)(24), (14)(23)\}.$$

Then, if the following lemma is true, one can conclude the proof. □

Lemma 5.6 *Let* $\mathbf{T}_1, \ldots, \mathbf{T}_q$ *be* q *diagonal matrix such that for all* $i, j \in \{1, \ldots, q\}$,

$$\mathrm{Tr}\,\mathbf{T}_i = 0; \quad \frac{1}{N}\mathrm{Tr}\,\mathbf{T}_i\mathbf{T}_j \longrightarrow \tau_{i,j}; \quad \forall k \geq 1,\ \frac{1}{N}\mathrm{Tr}\left(\mathbf{T}_i\mathbf{T}_i^*\right)^k = O(1). \tag{27}$$

Let $\mathbf{A}_1, \ldots, \mathbf{A}_q$ *be* q *matrices of the form*

$$\mathbf{A}_i = \begin{pmatrix} \mathbf{A}_{0,i} & 0 \\ 0 & 0 \end{pmatrix}$$

where the $\mathbf{A}_{0,i}$*'s are* $K \times K$ *matrices independent from* N *where* K *is a fixed integer. Let* $\mathbf{U}$ *be a Haar-distributed matrix. Then, as* N *goes to infinity,*

$$\mathbb{E}\left[\prod_{t=1}^{q}\sqrt{N}\,\mathrm{Tr}\left(\mathbf{U}^*\mathbf{T}_t\mathbf{U}\mathbf{A}_t\right)\right] \longrightarrow \begin{cases} \displaystyle\sum_{\sigma\in M(q)} \mathrm{Tr}_\sigma\left(\mathbf{A}_i\right)_{i=1}^{q} \prod_{t=1}^{q/2} \tau_{\sigma(2t-1),\sigma(2t)} & \text{if } q \text{ is even.} \\ 0 & \text{if } q \text{ is odd.} \end{cases}$$

Indeed, once we suppose Lemma 5.6 satisfied, we need to compute for all p, q

$$\begin{aligned}&\mathbb{E}\left[\left[\sqrt{N}\,\mathrm{Tr}\,\mathbf{U}^*\mathbf{T}\mathbf{U}\mathbf{A}\right]^p\left[\overline{\sqrt{N}\,\mathrm{Tr}\,\mathbf{U}^*\mathbf{T}\mathbf{U}\mathbf{A}}\right]^q\right] \\ &\quad= \mathbb{E}\left[\left[\sqrt{N}\,\mathrm{Tr}\,\mathbf{U}^*\mathbf{T}\mathbf{U}\mathbf{A}\right]^p\left[\sqrt{N}\,\mathrm{Tr}\,\mathbf{U}^*\mathbf{T}^*\mathbf{U}\mathbf{A}^*\right]^q\right],\end{aligned}$$

in order to apply Lemma 5.7. According to Lemma 5.6, for $\mathbf{T}_t \equiv \mathbf{T}$ and $\mathbf{A}_t \equiv \mathbf{A}$, we have

$$\mathbb{E}\left[\sqrt{N}\,\mathrm{Tr}\,\mathbf{U}^*\mathbf{T}\mathbf{U}\mathbf{A}\right]^q \longrightarrow \begin{cases} \mathrm{Card}\,M(q)\,\mathrm{Tr}(\mathbf{A}^2)^{q/2}\tau^{q/2} & \text{if } q \text{ is even,} \\ 0 & \text{if } q \text{ is odd.} \end{cases}$$

(remind that $\mathrm{Card}\,M(q) = (q-1)(q-3)\cdots 3$) which means that the limit distribution of X already satisfies (30) and (31). Let $p \geq 1$ and $q \geq 2$ be two fixed integers such that $p+q$ is even; then, using notations from (26), we know thanks to Lemma 5.6 that

$$\mathbb{E}\left[\left[\sqrt{N}\operatorname{Tr}\mathbf{U}^*\mathbf{TUA}\right]^p\left[\sqrt{N}\operatorname{Tr}\mathbf{U}^*\mathbf{T}^*\mathbf{UA}^*\right]^q\right]$$
$$= \frac{1}{N^{\frac{p+q}{2}}}\sum_{\sigma\in M(p+q)}\operatorname{Tr}_\sigma\left(\mathbf{A}_t\right)_{t=1}^{p+q}\operatorname{Tr}_\sigma\left(\mathbf{T}_t\right)_{t=1}^{p+q}+o\left(1\right) \tag{28}$$

where

$$(\mathbf{T}_1,\ldots,\mathbf{T}_{p+q}) = (\underbrace{\mathbf{T},\ldots,\mathbf{T}}_{p},\underbrace{\mathbf{T}^*,\ldots,\mathbf{T}^*}_{q}) \quad\text{and}\quad (\mathbf{A}_1,\ldots,\mathbf{A}_{p+q})$$
$$= (\underbrace{\mathbf{A},\ldots,\mathbf{A}}_{p},\underbrace{\mathbf{A}^*,\ldots,\mathbf{A}^*}_{q}).$$

We rewrite the right side of (28) summing according to the value of $\sigma(1)$.

$$\sum_{\sigma\in M(p+q)}\operatorname{Tr}_\sigma\left(\mathbf{A}_t\right)_{t=1}^{p+q}\operatorname{Tr}_\sigma\left(\mathbf{T}_t\right)_{t=1}^{p+q} = \sum_{a=2}^{p+q}\sum_{\substack{\sigma\in M(p+q)\\ \sigma(1)=a}}\operatorname{Tr}_\sigma\left(\mathbf{A}_t\right)_{t=1}^{p+q}\operatorname{Tr}_\sigma\left(\mathbf{T}_t\right)_{t=1}^{p+q}$$
$$= \sum_{a=2}^{p+q}\operatorname{Tr}(\mathbf{A}_1\mathbf{A}_a)\operatorname{Tr}(\mathbf{T}_1\mathbf{T}_a)\sum_{\substack{\sigma\in M(p+q)\\ \sigma(1)=a}}\operatorname{Tr}_{\sigma\circ(1a)}\left(\mathbf{A}_t\right)_{t=1}^{p+q}\operatorname{Tr}_{\sigma\circ(1a)}\left(\mathbf{T}_t\right)_{t=1}^{p+q}$$
$$= \sum_{a=2}^{p}\operatorname{Tr}\mathbf{A}^2\operatorname{Tr}\mathbf{T}^2\sum_{\sigma\in M(p+q-2)}\operatorname{Tr}_\sigma\left(\widehat{\mathbf{A}}_t\right)_{t=1}^{p+q-2}\operatorname{Tr}_\sigma\left(\widehat{\mathbf{T}}_t\right)_{t=1}^{p+q-2}$$
$$+\sum_{a=p+1}^{p+q}\operatorname{Tr}\mathbf{AA}^*\operatorname{Tr}\mathbf{TT}^*\sum_{\sigma\in M(p+q-2)}\operatorname{Tr}_\sigma\left(\widetilde{\mathbf{A}}_t\right)_{t=1}^{p+q-2}\operatorname{Tr}_\sigma\left(\widetilde{\mathbf{T}}_t\right)_{t=1}^{p+q-2},$$

where

$$(\widehat{\mathbf{A}}_1,\ldots,\widehat{\mathbf{A}}_{p+q-2}) = (\underbrace{\mathbf{A},\ldots,\mathbf{A}}_{p-2},\underbrace{\mathbf{A}^*,\ldots,\mathbf{A}^*}_{q}) \quad\text{and}\quad (\widetilde{\mathbf{A}}_1,\ldots,\widetilde{\mathbf{A}}_{p+q-2})$$
$$= (\underbrace{\mathbf{A},\ldots,\mathbf{A}}_{p-1},\underbrace{\mathbf{A}^*,\ldots,\mathbf{A}^*}_{q-1}).$$

At last, one easily deduces that

$$\mathbb{E}\left[\left[\sqrt{N}\operatorname{Tr}\mathbf{U}^*\mathbf{TUA}\right]^p\left[\sqrt{N}\operatorname{Tr}\mathbf{U}^*\mathbf{T}^*\mathbf{UA}^*\right]^q\right]$$
$$= \frac{1}{N}\operatorname{Tr}\mathbf{T}^2\operatorname{Tr}\mathbf{A}^2(p-1)\,\mathbb{E}\left[\left[\sqrt{N}\operatorname{Tr}\mathbf{U}^*\mathbf{TUA}\right]^{p-2}\left[\sqrt{N}\operatorname{Tr}\mathbf{U}^*\mathbf{T}^*\mathbf{UA}^*\right]^q\right]$$
$$+\frac{1}{N}\operatorname{Tr}\mathbf{TT}^*\operatorname{Tr}\mathbf{AA}^*q\,\mathbb{E}\left[\left[\sqrt{N}\operatorname{Tr}\mathbf{U}^*\mathbf{TUA}\right]^{p-1}\left[\sqrt{N}\operatorname{Tr}\mathbf{U}^*\mathbf{T}^*\mathbf{UA}^*\right]^{q-1}\right]+o\left(1\right)$$

and so $\sqrt{N}\,\mathrm{Tr}(\mathbf{U}^*\mathbf{TAU})$ satisfies (32) which means according to Lemma 5.7 that its limit distribution is Gaussian.

At last, to compute to covariance of the $\left(\mathscr{G}_{i,j}\right)$'s, one can simply use [11, Lemma A.6].

Proof of Theorem 5.5 This time, we shall use Lemma 5.8 to show that for any $\mathbf{A}_1, \ldots, \mathbf{A}_r$, $N \times N$ deterministic matrix of the form

$$\mathbf{A}_m = \begin{pmatrix} \mathbf{A}_{m,p} & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{A}_{m,p} = \left(a_{i,j}^m\right)_{i,j=1}^p \in \mathscr{M}_p(\mathbb{C}),$$

the vector

$$\left(\frac{1}{N}\,\mathrm{Tr}\left(\mathbf{U}^*\mathbf{T}_1\mathbf{U}\mathbf{A}_1\right), \ldots, \frac{1}{N}\,\mathrm{Tr}\left(\mathbf{U}^*\mathbf{T}_r\mathbf{U}\mathbf{A}_r\right)\right)$$

converges weakly to a Gaussian multivariate. Thanks to Theorem 5.3, we know that for each m,

$$\frac{1}{N}\,\mathrm{Tr}\left(\mathbf{U}^*\mathbf{T}_m\mathbf{U}\mathbf{A}_m\right)$$

is asymptotically Gaussian. Then, we show that

$$\mathbb{E}\Bigg[\left(\frac{1}{N}\,\mathrm{Tr}\left(\mathbf{U}^*\mathbf{T}_1\mathbf{U}\mathbf{A}_1\right)\right)^{p_1}\left(\frac{1}{N}\,\mathrm{Tr}\left(\mathbf{U}^*\mathbf{T}_1^*\mathbf{U}\mathbf{A}_1^*\right)\right)^{q_1} \cdots \left(\frac{1}{N}\,\mathrm{Tr}\left(\mathbf{U}^*\mathbf{T}_r\mathbf{U}\mathbf{A}_r\right)\right)^{p_r}\left(\frac{1}{N}\,\mathrm{Tr}\left(\mathbf{U}^*\mathbf{T}_r^*\mathbf{U}\mathbf{A}_r^*\right)\right)^{q_r}\Bigg]$$

satisfies (35) and (36) using Lemma 5.6. □

Proof of Lemma 5.6 We know from [26, Proposition3.4]

$$\mathbb{E}\left[\prod_{t=1}^{q}\sqrt{N}\,\mathrm{Tr}\left(\mathbf{U}^*\mathbf{T}_t\mathbf{U}\mathbf{A}_t\right)\right] = N^{q/2}\sum_{\sigma,\tau\in S_q}\mathrm{Wg}(\tau\circ\sigma^{-1})\,\mathrm{Tr}_\tau\left(\mathbf{A}_i\right)_{i=1}^q\,\mathrm{Tr}_\sigma\left(\mathbf{T}_i\right)_{i=1}^q$$

where Wg is a function called the *Weingarten function*. Moreover, for $\sigma \in S_q$, the asymptotical behavior of $\mathrm{Wg}(\sigma)$ is at most given by

$$\mathrm{Wg}(\sigma) = O\left(N^{-q}\right). \tag{29}$$

First, one should notice that if σ has one invariant point (which means a cycle of size one in its cycle decomposition), then

$$\mathrm{Tr}_\sigma\left(\mathbf{T}_i\right)_{i=1}^q = 0,$$

also, if σ has r cycles in its cycle decomposition, then, by the Holder inequality,

$$\mathrm{Tr}_\sigma\left(\mathbf{T}_i\right)_{i=1}^q = O\left(N^r\right).$$

Actually, the maximum of cycles in its decomposition that can have σ without any 1-sized cycle is $\left\lfloor \frac{q}{2} \right\rfloor$ so that, using (29)

$$N^{q/2}\,\mathrm{Wg}(\sigma\circ\tau^{-1})\,\mathrm{Tr}_\tau\left(\mathbf{A}_i\right)_{i=1}^q\,\mathrm{Tr}_\sigma\left(\mathbf{T}_i\right)_{i=1}^q = O\left(N^{\lfloor \frac{q}{2}\rfloor-\frac{q}{2}}\right),$$

so that first, if q is odd

$$\mathbb{E}\left[\prod_{t=1}^q \sqrt{N}\,\mathrm{Tr}\left(\mathbf{U}^*\mathbf{T}_t\mathbf{U}\mathbf{A}_t\right)\right] = o\left(1\right).$$

Moreover, if $q = 2r$, then the only way to have

$$N^{q/2}\,\mathrm{Wg}(\sigma\circ\tau^{-1})\,\mathrm{Tr}_\tau\left(\mathbf{A}_i\right)_{i=1}^q\,\mathrm{Tr}_\sigma\left(\mathbf{T}_i\right)_{i=1}^q \neq o\left(1\right)$$

is to have

- $\tau = \sigma$,
- σ is a product of $\frac{q}{2} = r$ transpositions with disjoint support.

One easily conclude. □

5.2 Moments of a Complex Gaussian Variable

The following lemma allows to prove that a random variable is Gaussian if and only if its moments satisfy an induction relation.

Lemma 5.7 *Let Z be a complex Gaussian variable such that*

$$\mathbb{E}\left[Z\right] = 0, \quad \mathbb{E}\left[Z^2\right] = \tau^2, \quad \mathbb{E}\left[|Z|^2\right] = \sigma^2. \tag{30}$$

Then, for all $p \geq 1$

$$\mathbb{E}\left[Z^{2p}\right] = p!!\tau^{2p} \;\; and \;\; \mathbb{E}\left[Z^{2p+1}\right] = 0, \qquad \left(where\ p!! := \frac{(2p)!}{2^p p!}\right) \tag{31}$$

also, for all $p, q \geq 0$,

$$\begin{aligned}\mathbb{E}\left[Z^{p+2}\overline{Z}^{q+2}\right] &= \sigma^2(q+2)\,\mathbb{E}\left[Z^{p+1}\overline{Z}^{q+1}\right] + \tau^2(p+1)\,\mathbb{E}\left[Z^p\overline{Z}^{q+2}\right]\\ &= \sigma^2(p+2)\,\mathbb{E}\left[Z^{p+1}\overline{Z}^{q+1}\right] + \overline{\tau}^2(q+1)\,\mathbb{E}\left[Z^{p+2}\overline{Z}^{q}\right].\end{aligned} \tag{32}$$

Conversely, any complex random variable Z satisfying (30)*,* (31)*, and* (32) *is a complex Gaussian variable.*

Proof First, recall that if $Z = X_1 + \mathrm{i}\, X_2$ is a complex random Gaussian such that

$$\mathbb{E}\,[Z] \;=\; 0, \quad \mathbb{E}\left[Z^2\right] \;=\; \tau^2, \quad \mathbb{E}\left[|Z|^2\right] \;=\; \sigma^2,$$

, then its Fourier transform is given, for $t = t_1 + \mathrm{i}\, t_2 \in \mathbb{C}$, by

$$\begin{aligned}\Phi_Z(t) &:= \mathbb{E} \exp\left(i(X_1 t_1 + X_2 t_2)\right)\\ &= \exp\left(-\frac{1}{4}\big((t_1^2+t_2^2)\sigma^2 + (t_1^2-t_2^2)\operatorname{Re}(\tau^2) + 2t_1t_2\operatorname{Im}(\tau^2)\big)\right)\end{aligned}$$

We define the differential operators

$$\partial_t \;:=\; \partial_1 + \mathrm{i}\,\partial_2 \;;\; \partial_{\bar t} \;:=\; \partial_1 - \mathrm{i}\,\partial_2 \tag{33}$$

so that

$$\mathbb{E}\left[Z^p \overline{Z}^q\right] = (-\,\mathrm{i})^{p+q}\,\partial_t^p\,\partial_{\bar t}^q\,\Phi(t)\big|_{t=0}. \tag{34}$$

One can easily compute

$$\partial_t \Phi(t) \;=\; -\frac{1}{2}\left(t\sigma^2 + \bar t\tau^2\right)\Phi(t); \quad \partial_{\bar t}\Phi(t) \;=\; -\frac{1}{2}\left(\bar t\sigma^2 + t\bar\tau^2\right)\Phi(t);$$

therefore, for any $p \geq 0$, $q \geq 0$,

$$\begin{aligned}\partial_t^{p+2}\Phi(t) &= \partial_t^{p+1}\left(-\frac{1}{2}\left(t\sigma^2 + \bar t\tau^2\right)\Phi(t)\right)\\ &= -\frac{1}{2}t\sigma^2\partial_t^{p+1}\Phi(t) - \frac{1}{2}\tau^2\partial_t^{p+1}\left(\bar t\Phi(t)\right)\\ &= -\frac{1}{2}t\sigma^2\partial_t^{p+1}\Phi(t) - \frac{1}{2}\tau^2\bar t\partial_t^{p+1}\Phi(t) - \tau^2(p+1)\partial_t^p\Phi(t)\end{aligned}$$

and

$$\begin{aligned}\partial_t^{p+2}\partial_{\bar t}^{q+2}\Phi(t) &= \partial_t^{p+1}\partial_{\bar t}^{q+2}\left(-\frac{1}{2}\left(t\sigma^2 + \bar t\tau^2\right)\Phi(t)\right)\\ &= -\frac{1}{2}\sigma^2\partial_{\bar t}^{q+2}\left(t\partial_t^{p+1}\Phi(t)\right) - \frac{1}{2}\tau^2\partial_t^{p+1}\left(\bar t\partial_{\bar t}^{q+2}\Phi(t)\right)\\ &= -\sigma^2\left(\frac{t}{2}\partial_t^{p+1}\partial_{\bar t}^{q+2}\Phi(t) + (q+2)\partial_t^{p+1}\partial_{\bar t}^{q+1}\Phi(t)\right)\\ &\quad - \tau^2\left(\frac{\bar t}{2}\partial_t^{p+1}\partial_{\bar t}^{q+2}\Phi(t) + (p+1)\partial_t^p\partial_{\bar t}^{q+2}\Phi(t)\right),\end{aligned}$$

hence,

$$\mathbb{E}\left[Z^{p+2}\right] = \tau^2(p+1)\,\mathbb{E}\left[Z^p\right],$$
$$\mathbb{E}\left[Z^{p+2}\overline{Z}^{q+2}\right] = \sigma^2(q+2)\,\mathbb{E}\left[Z^{p+1}\overline{Z}^{q+1}\right] + \tau^2(p+1)\,\mathbb{E}\left[Z^p\overline{Z}^{q+2}\right]$$

and the same way,

$$\mathbb{E}\left[\overline{Z}^{p+2}\right] = \overline{\tau}^2(p+1)\,\mathbb{E}\left[\overline{Z}^p\right],$$
$$\mathbb{E}\left[Z^{p+2}\overline{Z}^{q+2}\right] = \sigma^2(p+2)\,\mathbb{E}\left[Z^{p+1}\overline{Z}^{q+1}\right] + \overline{\tau}^2(q+1)\,\mathbb{E}\left[Z^{p+2}\overline{Z}^q\right].$$

Conversely, one can easily prove by induction that any complex random variable Z satisfying (30), (31), and (32) has all its moments uniquely determined, and since the complex Gaussian variable also satisfies (30), (31), and (32), one can conclude. □

More generally, one can show the following lemma

Lemma 5.8 *Let $(X_1, \ldots, X_r)$ be a centered complex Gaussian vector. Then, for all nonnegative integers $p_1, q_1, \ldots, p_r, q_r$, for all $i \in \{1, \ldots, r\}$*

$$\begin{aligned}
&\textit{if } p_i \geq 1, \quad \mathbb{E}\left[X_1^{p_1}\overline{X_1}^{q_1}\cdots X_r^{p_r}\overline{X_r}^{q_r}\right] \\
&= (p_i - 1)\,\mathbb{E}\left[X_i^2\right]\mathbb{E}\left[X_1^{p_1}\overline{X_1}^{q_1}\cdots X_i^{p_i-2}\overline{X_i}^{q_i}\cdots X_r^{p_r}\overline{X_r}^{q_r}\right] \\
&\quad + \sum_{\substack{j=1\\ j\neq i}}^{r} p_j\,\mathbb{E}\left[X_iX_j\right]\mathbb{E}\left[X_1^{p_1}\overline{X_1}^{q_1}\cdots X_i^{p_i-1}\cdots X_j^{p_j-1}\cdots X_r^{p_r}\overline{X_r}^{q_r}\right] \\
&\quad + \sum_{j=1}^{r} q_j\,\mathbb{E}\left[X_i\overline{X_j}\right]\mathbb{E}\left[X_1^{p_1} \quad l{X_1}^{q_1}\cdots X_i^{p_i-1}\cdots \overline{X_j^{q_j-1}}\cdots X_r^{p_r}\overline{X_r}^{q_r}\right] \qquad (35)\\
&\textit{if } q_i \geq 1, \quad \mathbb{E}\left[X_1^{p_1}\overline{X_1}^{q_1}\cdots X_r^{p_r}\overline{X_r}^{q_r}\right] \\
&= (q_i - 1)\,\mathbb{E}\left[\overline{X_i^2}\right]\mathbb{E}\left[X_1^{p_1}\overline{X_1}^{q_1}\cdots X_i^{p_i}\overline{X_i}^{q_i-2}\cdots X_r^{p_r}\overline{X_r}^{q_r}\right] \\
&\quad + \sum_{j=1}^{r} p_j\,\mathbb{E}\left[\overline{X_i}X_j\right]\mathbb{E}\left[X_1^{p_1}\overline{X_1}^{q_1}\cdots \overline{X_i^{q_i-1}}\cdots X_j^{p_j-1}\cdots X_r^{p_r}\overline{X_r}^{q_r}\right] \\
&\quad + \sum_{\substack{j=1\\ j\neq i}}^{r} q_j\,\mathbb{E}\left[\overline{X_iX_j}\right]\mathbb{E}\left[X_1^{p_1}\overline{X_1}^{q_1}\cdots \overline{X_i^{q_i-1}}\cdots \overline{X_j^{q_j-1}}\cdots X_r^{p_r}\overline{X_r}^{q_r}\right] \qquad (36)
\end{aligned}$$

with the convention that $X^{-1} = 0$.

Conversely, if $X_1, \ldots, X_r$ are r centered Gaussian variables satisfying (35) and (36), then $(X_1, \ldots, X_r)$ is a centered complex Gaussian vector.

Proof In the same spirit as the proof of Lemma 5.7, we obtain (35) and (36) by derivating the Fourier transform. The converse is proved by induction. □

References

1. Anderson, G., Guionnet, A., Zeitouni, O.: An Introduction to Random Matrices. Cambridge Studies in Advanced Mathematics, vol. 118. Cambridge University Press (2009)
2. Bai, Z.D., Silverstein, J.W.: Spectral Analysis of Large Dimensional Random Matrices, 2nd edn. Springer, New York (2009)
3. Baik, J., Ben Arous, G., Péché, S.: Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices. Ann. Probab. **33**(5), 1643–1697 (2005)
4. Beardon, A.: Complex Analysis: The Winding Number Principle in Analysis and Topology. Wiley, New York (1979)
5. Belinschi, S.T., Bercovici, H., Capitaine, M., Février, M.: Outliers in the spectrum of large deformed unitarily invariant models. arXiv:1207.5443v1, (2012)
6. Benaych-Georges, F., Guionnet, A., Maida, M.: Fluctuations of the extreme eigenvalues of finite rank deformations of random matrices. Electron. J. Probab. **16**(60), 1621–1662 (2011)
7. Benaych-Georges, F., Guionnet, A., Maida, M.: Large deviations of the extreme eigenvalues of random deformations of matrices. Probab. Theory Relat. Fields **154**(3), 703–751 (2012)
8. Benaych-Georges, F., Nadakuditi, R.R.: The eigenvalues and eigenvectors of finite, low rank perturbations of large random matrices. Adv. Math. **227**(1), 494–521 (2011)
9. Benaych-Georges, F., Nadakuditi, R.R.: The singular values and vectors of low rank perturbations of large rectangular random matrices. J. Multivar. Anal. **111**, 120–135 (2012)
10. Benaych-Georges, F., Rochet, J.: Outliers in the single ring theorem. Probab. Theory Relat. Fields (2015). doi:10.1007/s00440-015-0632-x
11. Benaych-Georges, F., Rochet, J.: Fluctuations for analytic test functions in the single ring theorem. arXiv preprint arXiv:1504.05106, (2015)
12. Bordenave, C., Capitaine, M.: Outlier eigenvalues for deformed i.i.d. random matrices. arXiv:1403.6001
13. Capitaine, M., Donati-Martin, C., Féral, D.: The largest eigenvalue of finite rank deformation of large Wigner matrices: convergence and non universality of the fluctuations. Ann. Probab. **37**(1), 1–47 (2009)
14. Capitaine, M., Donati-Martin, C., Féral, D.: Central limit theorems for eigenvalues of deformations of Wigner matrices. Ann. I.H.P. Probab. Stat. **48**(1), 107–133 (2012)
15. Capitaine, M., Donati-Martin, C., Féral, D., Février, M.: Free convolution with a semi-circular distribution and eigenvalues of spiked deformations of Wigner matrices. Electron. J. Probab. **16**(64), 1750–1792 (2011)
16. Erdös, L., Yau, H.T., Yin, J.: Bulk universality for generalized Wigner matrices. arXiv preprint arXiv:1001.3453
17. Féral, D., Péché, S.: The largest eigenvalue of rank one deformation of large Wigner matrices. Commun. Math. Phys. **272**, 185–228 (2007)
18. Fyodorov, Y.V., Sommers, H.-J.: Statistics of S-matrix poles in few-channel chaotic scattering: crossover from isolated to overlapping resonances. JETP Lett. **63**, 1026–1030 (1996)
19. Fyodorov, Y.V., Sommers, H.-J.: Statistics of resonance poles, phase shifts and time delays in quantum chaotic scattering: random matrix approach for systems with broken time reversal invariance. J. Math. Phys. **38**, 1918–1981 (1997)
20. Fyodorov, Y.V., Khoruzhenko, B.A.: Systematic analytical approach to correlation functions of resonances in quantum chaotic scattering. Phys. Rev. Lett. **83**, 65–68 (1999)
21. Fyodorov, Y.V., Sommers, H.-J.: Random matrices close to Hermitian or unitary: overview of methods and results. J. Phys. A Math. Gen. **36**, 3303–3347 (2003)
22. Horn, R.A., Johnson, C.R.: Matrix Analysis, Cambridge University Press, ISBN 978-0-521-38632-6 (1985)
23. Johnstone, I.M.: On the distribution of the largest eigenvalue in principal components analysis. Ann. Stat. **29**(2), 295–327 (2001)
24. Knowles, A., Yin, J.: The isotropic semicircle law and deformation of Wigner matrices. Commun. Pure Appl. Math. **66**(11), 1663–1750 (2013)
25. Knowles, A., Yin, J.: The outliers of a deformed Wigner matrix. Ann. Probab. **42**(5), 1980–2031 (2014)

26. Mingo, J.A., Śniady, P., Speicher, R.: Second order freeness and fluctuations of random matrices: II. Unitary random matrices. Adv. Math. **209**(1), 212–240 (2007)
27. Rajagopalan, A.B.: Outlier eigenvalue fluctuations of perturbed iid matrices. arXiv preprint arXiv:1507.01441
28. O'Rourke, S., Renfrew, D.: Low rank perturbations of large elliptic random matrices, arXiv
29. Péché, S.: The largest eigenvalue of small rank perturbations of Hermitian random matrices. Prob. Theory Relat. Fields **134**, 127–173 (2006)
30. Pizzo, A., Renfrew, D., Soshnikov, A.: On finite rank deformations of Wigner matrices. Ann. Inst. H. Poincaré Probab. Stat. **49**(1), 64–94 (2013)
31. Pizzo, A., Renfrew, D., Soshnikov, A.: Fluctuations of matrix entries of regular functions of Wigner matrices. J. Stat. Phys. **146**(3), 550–591 (2012)
32. Tao, T.: Outliers in the spectrum of iid matrices with bounded rank perturbations. Probab. Theory Relat. Fields **155**, 231–263 (2013)
33. Sinai, Y., Soshnikov, A.: Central limit theorem for traces of large random symmetric matrics with independent matrix elements. Bull. Braz. Math. Soc. **29**(1), 1–24 (1998)