

Two Non-closure Properties on the Class of Subexponential Densities

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Abstract Relations between subexponential densities and locally subexponential distributions are discussed. It is shown that the class of subexponential densities is neither closed under convolution roots nor closed under asymptotic equivalence. A remark is given on the closure under convolution roots for the class of convolution equivalent distributions.

Keywords Subexponential densities · Local subexponentiality · Convolution roots · Asymptotic equivalence

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1 Introduction and Main Results

In what follows, we denote by \mathbb{R} the real line and by \mathbb{R}_+ the half line $[0, \infty)$. Let \mathbb{N} be the totality of positive integers. The symbol $\delta_a(dx)$ stands for the delta measure at $a \in \mathbb{R}$. Let η and ρ be probability measures on \mathbb{R} . We denote the convolution of η and ρ by $\eta * \rho$ and denote n -th convolution power of ρ by ρ^{n*} . Let $f(x)$ and $g(x)$ be integrable functions on \mathbb{R} . We denote by $f^{n\otimes}(x)$ n -th convolution power of $f(x)$ and by $f \otimes g(x)$ the convolution of $f(x)$ and $g(x)$. For positive functions $f_1(x)$ and $g_1(x)$ on $[a, \infty)$ for some $a \in \mathbb{R}$, we define the relation $f_1(x) \sim g_1(x)$ by

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$\lim_{x \rightarrow \infty} f_1(x)/g_1(x) = 1$. We also define the relation $a_n \sim b_n$ for positive sequences $\{a_n\}_{n=A}^{\infty}$ and $\{b_n\}_{n=A}^{\infty}$ with $A \in \mathbb{N}$ by $\lim_{n \rightarrow \infty} a_n/b_n = 1$. We define the class \mathcal{P}_+ as the totality of probability distributions on \mathbb{R}_+ . In this paper, we prove that the class of subexponential densities is not closed under two important closure properties. We say that a measurable function $g(x)$ on \mathbb{R} is a density function if $\int_{-\infty}^{\infty} g(x)dx = 1$ and $g(x) \geq 0$ for all $x \in \mathbb{R}$.

- Definition 1.1** (i) A nonnegative measurable function $g(x)$ on \mathbb{R} belongs to the class \mathbf{L} if $g(x) > 0$ for all sufficiently large $x > 0$ and if $g(x+a) \sim g(x)$ for any $a \in \mathbb{R}$.
- (ii) A measurable function $g(x)$ on \mathbb{R} belongs to the class \mathcal{L}_d if $g(x)$ is a density function and $g(x) \in \mathbf{L}$.
- (iii) A measurable function $g(x)$ on \mathbb{R} belongs to the class \mathcal{S}_d if $g(x) \in \mathcal{L}_d$ and $g \otimes g(x) \sim 2g(x)$.
- (iv) A distribution ρ on \mathbb{R} belongs to the class \mathcal{L}_{ac} if there is $g(x) \in \mathcal{L}_d$ such that $\rho(dx) = g(x)dx$.
- (v) A distribution ρ on \mathbb{R} belongs to the class \mathcal{S}_{ac} if there is $g(x) \in \mathcal{S}_d$ such that $\rho(dx) = g(x)dx$.

Densities in the class \mathcal{S}_d are called *subexponential densities* and those in the class \mathcal{L}_d are called *long-tailed densities*. The study on the class \mathcal{S}_d goes back to Chover et al. [2]. Let ρ be a distribution on \mathbb{R} . Note that $c^{-1}\rho((x-c, x])$ is a density function on \mathbb{R} for every $c > 0$.

- Definition 1.2** (i) Let $\Delta := (0, c]$ with $c > 0$. A distribution ρ on \mathbb{R} belongs to the class \mathcal{L}_{Δ} if $\rho((x, x+c]) \in \mathbf{L}$.
- (ii) Let $\Delta := (0, c]$ with $c > 0$. A distribution ρ on \mathbb{R} belongs to the class \mathcal{S}_{Δ} if $\rho \in \mathcal{L}_{\Delta}$ and $\rho * \rho((x, x+c]) \sim 2\rho((x, x+c])$.
- (iii) A distribution ρ on \mathbb{R} belongs to the class \mathcal{L}_{loc} if $\rho \in \mathcal{L}_{\Delta}$ for each $\Delta := (0, c]$ with $c > 0$.
- (iv) A distribution ρ on \mathbb{R} belongs to the class \mathcal{S}_{loc} if $\rho \in \mathcal{S}_{\Delta}$ for each $\Delta := (0, c]$ with $c > 0$.
- (v) A distribution $\rho \in \mathcal{L}_{loc}$ belongs to the class \mathcal{UL}_{loc} if there exists $p(x) \in \mathcal{L}_d$ such that $c^{-1}\rho((x-c, x]) \sim p(x)$ uniformly in $c \in (0, 1]$.
- (vi) A distribution $\rho \in \mathcal{S}_{loc}$ belongs to the class \mathcal{US}_{loc} if there exists $p(x) \in \mathcal{S}_d$ such that $c^{-1}\rho((x-c, x]) \sim p(x)$ uniformly in $c \in (0, 1]$.

Distributions in the class \mathcal{S}_{loc} are called *locally subexponential*; those in the class \mathcal{US}_{loc} are called *uniformly locally subexponential*. The class \mathcal{S}_{Δ} was introduced by Asmussen et al. [1] and the class \mathcal{S}_{loc} was by Watanabe and Yamamuro [14]. Detailed accounts of the classes \mathcal{S}_d and \mathcal{S}_{loc} are found in the book of Foss et al. [6]. First, we present some interesting results on the classes \mathcal{S}_d and \mathcal{S}_{loc} .

Proposition 1.1 *We have the following.*

- (i) Let $\Delta := (0, c]$ with $c > 0$ and let $p(x) := c^{-1}\mu((x-c, x])$ for a distribution μ on \mathbb{R}_+ . Then $\mu \in \mathcal{S}_{\Delta}$ if and only if $p(x) \in \mathcal{S}_d$. Moreover, $\mu \in \mathcal{S}_{loc} \cap \mathcal{P}_+$ if and only if there exists a density function $q(x)$ on \mathbb{R}_+ such that $q(x) \in \mathcal{S}_d$ and $c^{-1}\mu((x-c, x]) \sim q(x)$ for every $c > 0$.

- (ii) Let $\rho_1(dx) := q_1(x)dx$ be a distribution on \mathbb{R}_+ . If $q_1(x)$ is continuous with compact support and if $\rho_2 \in \mathcal{S}_{loc} \cap \mathcal{P}_+$, then $\rho_1 * \rho_2(dx) = \left(\int_{0-}^{x+} q_1(x-u)\rho_2(du)\right) dx$ and $\int_{0-}^{x+} q_1(x-u)\rho_2(du) \in \mathcal{S}_d$.
- (iii) Let μ be a distribution on \mathbb{R}_+ . If there exist distributions ρ_c for $c > 0$ such that, for every $c > 0$, the support of ρ_c is included in $[0, c]$ and $\rho_c * \mu \in \mathcal{S}_{loc}$, then $\mu \in \mathcal{S}_{loc}$.

Definition 1.3 (i) We say that a class \mathcal{C} of probability distributions on \mathbb{R} is closed under convolution roots if $\mu^{n*} \in \mathcal{C}$ for some $n \in \mathbb{N}$ implies that $\mu \in \mathcal{C}$.

- (ii) Let $p_1(x)$ and $p_2(x)$ be density functions on \mathbb{R} . We say that a class \mathcal{C} of density functions is closed under asymptotic equivalence if $p_1(x) \in \mathcal{C}$ and $p_2(x) \sim cp_1(x)$ with $c > 0$ implies that $p_2(x) \in \mathcal{C}$.

The class \mathcal{S}_{ac} is a proper subclass of the class \mathcal{US}_{loc} because a distribution in \mathcal{US}_{loc} can have a point mass. Moreover, the class \mathcal{US}_{loc} is a proper subclass of the class \mathcal{S}_{loc} as the following theorem shows.

Theorem 1.1 *There exists a distribution $\mu \in \mathcal{S}_{loc} \setminus \mathcal{US}_{loc}$ such that $\mu^{2*} \in \mathcal{S}_{ac}$.*

Corollary 1.1 *We have the following.*

- (i) *The class \mathcal{S}_{ac} is not closed under convolution roots.*
- (ii) *The class \mathcal{US}_{loc} is not closed under convolution roots.*
- (iii) *The class \mathcal{L}_{ac} is not closed under convolution roots.*
- (iv) *The class \mathcal{UL}_{loc} is not closed under convolution roots.*

The class \mathcal{S}_d is closed under asymptotic equivalence in the one-sided case. See (ii) of Lemma 2.1 below. However, Foss et al. [6] suggest the possibility of non-closure under asymptotic equivalence for the class \mathcal{S}_d in the two-sided case. We exactly prove it as follows.

Theorem 1.2 *The class \mathcal{S}_d is not closed under asymptotic equivalence; that is, there exist $p_1(x) \in \mathcal{S}_d$ and $p_2(x) \notin \mathcal{S}_d$ such that $p_2(x) \sim cp_1(x)$ with $c > 0$.*

In Sect. 2, we prove Proposition 1.1. In Sect. 3, we prove Theorems 1.1 and 1.2. In Sect. 4, we give a remark on the closure under convolution roots.

2 Proof of Proposition 1.1

We present two lemmas for the proofs of main results and then prove Proposition 1.1.

Lemma 2.1 *Let $f(x)$ and $g(x)$ be density functions on \mathbb{R}_+ .*

- (i) *If $f(x) \in \mathcal{L}_d$, then $f^{n\otimes}(x) \in \mathcal{L}_d$ for every $n \in \mathbb{N}$.*
- (ii) *If $f(x) \in \mathcal{S}_d$ and $g(x) \sim cf(x)$ with $c > 0$, then $g(x) \in \mathcal{S}_d$.*
- (iii) *Assume that $f(x) \in \mathcal{L}_d$. Then, $f(x) \in \mathcal{S}_d$ if and only if*

$$\lim_{A \rightarrow \infty} \limsup_{x \rightarrow \infty} \frac{1}{f(x)} \int_A^{x-A} f(x-u)f(u)du = 0.$$

Proof Proof of assertion (i) is due to Theorem 4.3 of [6]. Proofs of assertions (ii) and (iii) are due to Theorems 4.8 and 4.7 of [6], respectively. \square

Lemma 2.2 (i) *Let $\Delta := (0, c]$ with $c > 0$. Assume that $\rho \in \mathcal{L}_\Delta \cap \mathcal{P}_+$. Then, $\rho \in \mathcal{S}_\Delta$ if and only if*

$$\lim_{A \rightarrow \infty} \limsup_{x \rightarrow \infty} \frac{1}{\rho((x, x + c])} \int_{A+}^{(x-A)-} \rho((x - u, x + c - u]) \rho(du) = 0.$$

- (ii) *Assume that $\rho \in \mathcal{L}_{loc} \cap \mathcal{P}_+$. Then, $\rho^{n*} \in \mathcal{L}_{loc}$ for every $n \in \mathbb{N}$. Moreover, $\rho((x - c, x]) \sim c\rho((x - 1, x])$ for every $c > 0$.*
- (iii) *Let $\rho_2 \in \mathcal{P}_+$. If $\rho_1 \in \mathcal{S}_{loc} \cap \mathcal{P}_+$ and $\rho_2((x - c, x]) \sim c_1\rho_1((x - c, x])$ with $c_1 > 0$ for every $c > 0$, then $\rho_2 \in \mathcal{S}_{loc} \cap \mathcal{P}_+$.*

Proof Proof of assertion (i) is due to Theorem 4.21 of [6]. First assertion of (ii) is due to Corollary 4.19 of [6]. Second one is proved as (2.6) in Theorem 2.1 of [14]. Proof of assertion (iii) is due to Theorem 4.22 of [6]. \square

Proof of (i) of Proposition 1.1 Let $\rho(dx) := c^{-1}1_{[0,c)}(x)dx$. First, we prove that if $\mu \in \mathcal{S}_{loc} \cap \mathcal{P}_+$, then $\rho * \mu \in \mathcal{S}_{ac}$. We can assume that $c = 1$. Suppose that $\mu \in \mathcal{S}_{loc}$. Let $p(x) := \mu((x - 1, x])$. We have $\rho * \mu(dx) = \mu((x - 1, x])dx$ and hence $p(x) \in \mathcal{L}_d$. Let A be a positive integer and let X, Y be independent random variables with the same distribution μ . Then, we have for $x > 2A + 2$

$$\begin{aligned} & \int_A^{x-A} p(x - u)p(u)du \\ &= 2 \int_A^{x/2} p(x - u)p(u)du \\ &= 2 \int_A^{x/2} P(x - u - 1 < X \leq x - u, u - 1 < Y \leq u)du \\ &\leq 2 \int_A^{x/2} P(X > A, Y > A, x - 2 < X + Y \leq x, u - 1 < Y \leq u)du \\ &\leq 2 \sum_{n=A}^\infty \int_n^{n+1} P(X > A, Y > A, x - 2 < X + Y \leq x, n - 1 < Y \leq n + 1)du \\ &\leq 4P(X > A, Y > A, x - 2 < X + Y \leq x) \\ &\leq 4 \int_{A+}^{(x-A)-} \mu((x - 2 - u, x - u])\mu(du). \end{aligned}$$

Since $\mu \in \mathcal{S}_{loc}$, we obtain from (i) of Lemma 2.2 that

$$\lim_{A \rightarrow \infty} \limsup_{x \rightarrow \infty} \frac{\int_A^{x-A} p(x - u)p(u)du}{p(x)} = 0.$$

Thus, we see from (iii) of Lemma 2.1 that $p(x) \in \mathcal{S}_d$.

Conversely, suppose that $p(x) \in \mathcal{S}_d$. Then, we have $\mu \in \mathcal{L}_\Delta$. Let $[y]$ be the largest integer not exceeding a real number y . Choose sufficiently large integer $A > 0$. Note that there are positive constants c_j for $1 \leq j \leq 4$ such that

$$c_1 p(x - n) \leq p(x - u) \leq c_2 p(x - n) \text{ and } c_3 p(n) \leq p(u) \leq c_4 p(n)$$

for $n \leq u \leq n + 1, A \leq n \leq [x + 1 - A]$, and $x > 2A + 2$. Thus, we find that

$$\begin{aligned} &P(A < X, A < Y, x < X + Y \leq x + 1) \\ &\leq \sum_{n=A}^{[x+1-A]} \int_n^{n+1} \mu((x - u, x + 1 - u]) \mu(du) \\ &= \sum_{n=A}^{[x+1-A]} \int_n^{n+1} p(x - u + 1) \mu(du) \\ &\leq c_2 \sum_{n=A}^{[x+1-A]} p(x - n + 1) p(n + 1) \\ &\leq \frac{c_2}{c_1 c_3} \sum_{n=A}^{[x+1-A]} \int_n^{n+1} p(x - u + 1) p(u + 1) du \\ &\leq \frac{c_2}{c_1 c_3} \int_A^{x+2-A} p(x - u + 1) p(u + 1) du \end{aligned}$$

Since $p(x) \in \mathcal{S}_d$, we establish from (iii) of Lemma 2.1 that

$$\lim_{A \rightarrow \infty} \limsup_{x \rightarrow \infty} \frac{P(A < X, A < Y, x < X + Y \leq x + 1)}{P(x < X \leq x + 1)} = 0.$$

Thus, $\mu \in \mathcal{S}_\Delta$ by (i) of Lemma 2.2. Note from (ii) of Lemma 2.2 that if $\mu \in \mathcal{S}_{loc}$, then $c^{-1} \mu((x - c, x]) \sim \mu((x - 1, x])$ for every $c > 0$. Thus, the second assertion is true. □

Proof of (ii) of Proposition 1.1 Suppose that $\rho_1(dx) := q_1(x)dx$ be a distribution on \mathbb{R}_+ such that $q_1(x)$ is continuous with compact support in $[0, N]$. Let $q(x) := \int_{0-}^{x+} q_1(x - u) \rho_2(du)$. For $M \in \mathbb{N}$, there are $\delta(M) > 0$ and $a_n = a_n(M) \geq 0$ for $n \in \mathbb{N}$ such that $\lim_{M \rightarrow \infty} \delta(M) = 0$ and $a_n \leq q_1(x) \leq a_n + \delta(M)$ for $M^{-1}(n - 1) < x \leq M^{-1}n$ and $1 \leq n \leq MN$. Define $J(M; x)$ as

$$J(M; x) := \sum_{n=1}^{MN} a_n \rho_2((x - M^{-1}n, x - M^{-1}(n - 1)]).$$

Then, we have

$$J(M; x) \sim \rho_2((x - 1, x]) \sum_{n=1}^{MN} a_n M^{-1} \tag{2.1}$$

and for $x > N$

$$J(M; x) \leq q(x) \leq J(M; x) + \delta(M)\rho_2((x - N, x]).$$

Since $\lim_{M \rightarrow \infty} \delta(M) = 0$ and

$$\lim_{M \rightarrow \infty} \sum_{n=1}^{MN} a_n M^{-1} = \int_0^N q_1(x) dx = 1,$$

we obtain from (2.1) that

$$q(x) \sim \rho_2((x - 1, x]).$$

Since $\rho_2 \in \mathcal{S}_{loc}$, we conclude from (i) of Proposition 1.1 that $q(x) \in \mathcal{S}_d$. \square

Proof of (iii) of Proposition 1.1 Suppose that the support of ρ_c is included in $[0, c]$ and $\rho_c * \mu \in \mathcal{S}_{loc}$ for every $c > 0$. Let X and Y be independent random variables with the same distribution μ , and let X_c and Y_c be independent random variables with the same distribution ρ_c . Define $J_1(c; c_1; a; x)$ and $J_2(c; c_1; a; x)$ for $a \in \mathbb{R}$ and $c_1 > 0$ as

$$\begin{aligned} J_1(c; c_1; a; x) &:= \frac{P(x + a < X + X_c \leq x + c_1 + a)}{P(x < X + X_c \leq x + c_1 + c)}, \\ J_2(c; c_1; a; x) &:= \frac{P(x + a < X + X_c \leq x + c_1 + c + a)}{P(x < X + X_c \leq x + c_1)}. \end{aligned}$$

We see that

$$J_1(c; c_1; a; x) \leq \frac{P(x + a < X \leq x + c_1 + a)}{P(x < X \leq x + c_1)} \leq J_2(c; c_1; a; x). \quad (2.2)$$

Since $\rho_c * \mu \in \mathcal{L}_{loc}$, we obtain that

$$\lim_{x \rightarrow \infty} J_1(c; c_1; a; x) = \frac{c_1}{c_1 + c}$$

and

$$\lim_{x \rightarrow \infty} J_2(c; c_1; a; x) = \frac{c_1 + c}{c_1}.$$

Thus, as $c \rightarrow 0$ we have by (2.2)

$$\lim_{x \rightarrow \infty} \frac{P(x + a < X \leq x + c_1 + a)}{P(x < X \leq x + c_1)} = 1,$$

and hence $\mu \in \mathcal{L}_{loc}$. We find from $\rho_c * \mu \in \mathcal{S}_{loc}$ and (i) of Lemma 2.2 that

$$\begin{aligned} & \lim_{A \rightarrow \infty} \limsup_{x \rightarrow \infty} \frac{P(X > A, Y > A, x < X + Y \leq x + c_1)}{P(x < X \leq x + c_1)} \\ & \leq \lim_{A \rightarrow \infty} \limsup_{x \rightarrow \infty} \frac{P(X > A, Y > A, x < X + X_c + Y + Y_c \leq x + c_1 + 2c)}{P(x < X + X_c \leq x + c_1)} = 0. \end{aligned}$$

Thus, we see from (i) of Lemma 2.2 that $\mu \in \mathcal{S}_{loc}$. □

3 Proofs of Theorems 1.1 and 1.2

For the proofs of the theorems, we introduce a distribution μ as follows. Let $1 < x_0 < b$ and choose $\delta \in (0, 1)$ satisfying $\delta < (x_0 - 1) \wedge (b - x_0)$. We take a continuous periodic function $h(x)$ on \mathbb{R} with period $\log b$ such that $h(\log x) > 0$ for $x \in [1, x_0) \cup (x_0, b]$ and

$$h(\log x) = \begin{cases} 0 & \text{for } x = x_0, \\ \frac{-1}{\log |x - x_0|} & \text{for each } x \text{ with } 0 < |x - x_0| < \delta. \end{cases}$$

Let

$$\phi(x) := x^{-\alpha-1} h(\log x) 1_{[1, \infty)}(x)$$

with $\alpha > 0$. Here, the symbol $1_{[1, \infty)}(x)$ stands for the indicator function of the set $[1, \infty)$. Define a distribution μ as

$$\mu(dx) := M^{-1} \phi(x) dx,$$

where $M := \int_1^\infty x^{-1-\alpha} h(\log x) dx$.

Lemma 3.1 *We have $\mu \in \mathcal{L}_{loc}$.*

Proof Let $\{y_n\}$ be a sequence such that $1 \leq y_n \leq b$ and $\lim_{n \rightarrow \infty} y_n = y$ for some $y \in [1, b]$. Then, we put $x_n = b^{m_n} y_n$, where m_n is a positive integer and $\lim_{n \rightarrow \infty} x_n = \infty$. In what follows, $c > 0$ and $c_1 \geq 0$.

Case 1. Suppose that $y \neq x_0$. Let $x_n + c_1 \leq u \leq x_n + c_1 + c$. Then, we have

$$y_n + b^{-m_n} c_1 \leq b^{-m_n} u \leq y_n + b^{-m_n} (c_1 + c), \tag{3.1}$$

and thereby $\lim_{n \rightarrow \infty} b^{-m_n} u = y$. This yields that

$$h(\log u) = h(\log(b^{-m_n} u)) \sim h(\log y).$$

Hence, we obtain that

$$\int_{x_n+c_1}^{x_n+c_1+c} \phi(u) du = \int_{x_n+c_1}^{x_n+c_1+c} u^{-1-\alpha} h(\log u) du$$

$$\sim x_n^{-1-\alpha} \int_{x_n+c_1}^{x_n+c_1+c} h(\log u)du \sim cx_n^{-1-\alpha} h(\log y),$$

so that

$$\int_{x_n}^{x_n+c} \phi(u)du \sim \int_{x_n+c_1}^{x_n+c_1+c} \phi(u)du \tag{3.2}$$

Case 2. Suppose that $y = x_0$. Let $x_n + c_1 \leq u \leq x_n + c_1 + c$ and put

$$E_n := \{u : |b^{-m_n}u - x_0| \leq \epsilon b^{-m_n}\},$$

where $\epsilon > 0$. For sufficiently large n , we have for $u \in E_n$

$$-\log |b^{-m_n}u - x_0| \geq -\log \epsilon b^{-m_n} \geq \frac{1}{2}m_n \log b \tag{3.3}$$

Set $\lambda_n := |y_n - x_0|b^{m_n}$. It suffices that we consider the case where there exists a limit of λ_n as $n \rightarrow \infty$, so we may put $\lambda := \lim_{n \rightarrow \infty} \lambda_n$. This limit permits infinity. We divide λ in the two cases where $\lambda < \infty$ and $\lambda = \infty$.

Case 2-1. Suppose that $0 \leq \lambda < \infty$. Now, we have

$$\begin{aligned} & \int_{x_n+c_1}^{x_n+c_1+c} h(\log u)du \\ &= \int_{[x_n+c_1, x_n+c_1+c] \setminus E_n} h(\log u)du + \int_{[x_n+c_1, x_n+c_1+c] \cap E_n} h(\log u)du. \end{aligned}$$

Let $u \in [x_n + c_1, x_n + c_1 + c] \setminus E_n$. For sufficiently large n , we have by (3.1)

$$\begin{aligned} \epsilon b^{-m_n} &\leq |b^{-m_n}u - x_0| \leq |b^{-m_n}u - y_n| + |y_n - x_0| \\ &\leq b^{-m_n}(c + c_1) + b^{-m_n}\lambda_n \leq b^{-m_n}(c + c_1 + \lambda + 1). \end{aligned}$$

This implies that

$$-\log |b^{-m_n}u - x_0| \sim m_n \log b.$$

For sufficiently large n , it follows that

$$\begin{aligned} \int_{[x_n+c_1, x_n+c_1+c] \setminus E_n} h(\log u)du &= \int_{[x_n+c_1, x_n+c_1+c] \setminus E_n} h(\log b^{-m_n}u)du \\ &= \int_{[x_n+c_1, x_n+c_1+c] \setminus E_n} \frac{-1}{\log |b^{-m_n}u - x_0|} du \\ &\sim \int_{[x_n+c_1, x_n+c_1+c] \setminus E_n} \frac{1}{m_n \log b} du \end{aligned}$$

As we have

$$c \geq \int_{[x_n+c_1, x_n+c_1+c] \setminus E_n} du \geq \int_{[x_n+c_1, x_n+c_1+c]} du - \int_{E_n} du \geq c - 2\epsilon,$$

it follows that

$$(1 - \epsilon) \cdot \frac{c - 2\epsilon}{m_n \log b} \leq \int_{[x_n+c_1, x_n+c_1+c] \setminus E_n} h(\log u) du \leq (1 + \epsilon) \cdot \frac{c}{m_n \log b}$$

for sufficiently large n . Furthermore, we see from (3.3) that

$$\begin{aligned} \int_{[x_n+c_1, x_n+c_1+c] \cap E_n} h(\log u) du &= \int_{[x_n+c_1, x_n+c_1+c] \cap E_n} \frac{-1}{\log |b^{-m_n} u - x_0|} du \\ &\leq \frac{2}{m_n \log b} \int_{E_n} du \leq \frac{4\epsilon}{m_n \log b}. \end{aligned}$$

Hence, we obtain that

$$\begin{aligned} \int_{x_n+c_1}^{x_n+c_1+c} \phi(u) du &\sim x_n^{-1-\alpha} \int_{x_n+c_1}^{x_n+c_1+c} h(\log u) du \\ &\sim x_n^{-1-\alpha} \frac{c}{m_n \log b}, \end{aligned}$$

so that (3.2) holds.

Case 2-2. Suppose that $\lambda = \infty$. For u with $x_n + c_1 + \leq u \leq x_n + c_1 + c$, we see from (3.1) that

$$|y_n - x_0| - (c + c_1)b^{-m_n} \leq |b^{-m_n} u - x_0| \leq |y_n - x_0| + (c + c_1)b^{-m_n},$$

that is,

$$(1 - (c + c_1)\lambda_n^{-1})|y_n - x_0| \leq |b^{-m_n} u - x_0| \leq (1 + (c + c_1)\lambda_n^{-1})|y_n - x_0|.$$

This implies that

$$\begin{aligned} \int_{x_n+c_1}^{x_n+c_1+c} \phi(u) du &\sim x_n^{-1-\alpha} \int_{x_n+c_1}^{x_n+c_1+c} \frac{-1}{\log |b^{-m_n} u - x_0|} du \\ &\sim x_n^{-1-\alpha} \frac{-c}{\log |y_n - x_0|}, \end{aligned}$$

so we get (3.2). The lemma has been proved. □

Lemma 3.2 *We have*

$$\phi \otimes \phi(x) \sim 2M \int_x^{x+1} \phi(u) du = 2M^2 \mu((x, x + 1]).$$

Proof Let $\{y_n\}$ be a sequence such that $1 \leq y_n \leq b$ and $\lim_{n \rightarrow \infty} y_n = y$ for some $y \in [1, b]$. We put $x_n = b^{m_n} y_n$, where m_n is a positive integer and $\lim_{n \rightarrow \infty} x_n = \infty$. Now, we have

$$\begin{aligned} \phi \otimes \phi(x_n) &= \int_1^{x_n-1} \phi(x_n - u)\phi(u)du \\ &= 2 \int_1^{2^{-1}x_n} \phi(x_n - u)\phi(u)du \\ &= 2 \left(\int_1^{(\log x_n)^\beta} + \int_{(\log x_n)^\beta}^{2^{-1}x_n} \right) \phi(x_n - u)\phi(u)du =: 2(J_1 + J_2). \end{aligned}$$

Here, we took β satisfying $\alpha\beta > 1$. Put $K := \sup\{h(\log x) : 1 \leq x \leq b\}$. Then, we have

$$J_2 \leq K^2 \int_1^{2^{-1}x_n} \frac{du}{(\log x_n)^\beta u^{1+\alpha}(x_n - u)^{1+\alpha}} \leq K^2 \left(\frac{2}{x_n}\right)^{1+\alpha} \cdot \alpha^{-1}(\log x_n)^{-\alpha\beta}.$$

We consider the two cases where $y \neq x_0$ and $y = x_0$.

Case 1. Suppose that $y \neq x_0$. If $1 \leq u \leq (\log x_n)^\beta$, then

$$h(\log(x_n - u)) = h(\log(y_n - b^{-m_n}u)) \sim h(\log y).$$

Hence, we obtain that

$$\begin{aligned} J_1 &= \int_1^{(\log x_n)^\beta} (x_n - u)^{-1-\alpha} u^{-1-\alpha} h(\log(x_n - u))h(\log u)du \\ &\sim x_n^{-1-\alpha} \int_1^{(\log x_n)^\beta} u^{-1-\alpha} h(\log(x_n - u))h(\log u)du \\ &\sim Mx_n^{-1-\alpha} h(\log y), \end{aligned}$$

so that

$$\phi \otimes \phi(x_n) = 2(J_1 + J_2) \sim 2J_1 \sim 2Mx_n^{-1-\alpha} h(\log y).$$

Case 2. Suppose that $y = x_0$. Put $\gamma_n := b^{m_n}|y_n - x_0|(\log x_n)^{-\beta}$ and

$$E'_n := \{u : |y_n - x_0 - b^{-m_n}u| \leq \epsilon b^{-m_n}\},$$

where $0 < \epsilon < 1$. It suffices that we consider the case where there exists a limit of γ_n , so we may put $\gamma := \lim_{n \rightarrow \infty} \gamma_n$. This limit permits infinity. Furthermore, we divide γ in the two cases where $\gamma < \infty$ and $\gamma = \infty$.

Case 2-1. Suppose that $0 \leq \gamma < \infty$. Take sufficiently large n . Set

$$J'_{11} := \int_{[1, (\log x_n)^\beta] \setminus E'_n} u^{-1-\alpha} h(\log(x_n - u)) h(\log u) du,$$

$$J'_{12} := \int_{[1, (\log x_n)^\beta] \cap E'_n} u^{-1-\alpha} h(\log(x_n - u)) h(\log u) du.$$

Let $u \in [1, (\log x_n)^\beta] \setminus E'_n$. We have

$$\begin{aligned} \epsilon b^{-m_n} &\leq |y_n - x_0 - b^{-m_n} u| \leq |y_n - x_0| + b^{-m_n} u \\ &\leq (\gamma + 2) b^{-m_n} (\log x_n)^\beta. \end{aligned}$$

This implies that

$$-\log |y_n - x_0 - b^{-m_n} u| \sim m_n \log b.$$

It follows that

$$\begin{aligned} J'_{11} &= \int_{[1, (\log x_n)^\beta] \setminus E'_n} u^{-1-\alpha} h(\log(y_n - b^{-m_n} u)) h(\log u) du \\ &= \int_{[1, (\log x_n)^\beta] \setminus E'_n} u^{-1-\alpha} h(\log u) \frac{-1}{\log |y_n - x_0 - b^{-m_n} u|} du \\ &\sim \frac{1}{m_n \log b} \int_{[1, (\log x_n)^\beta] \setminus E'_n} u^{-1-\alpha} h(\log u) du. \end{aligned}$$

Here, we see that, for sufficiently large n ,

$$M - \epsilon - 2\epsilon K \leq \int_{[1, (\log x_n)^\beta] \setminus E'_n} u^{-1-\alpha} h(\log u) du \leq M,$$

and thereby

$$(1 - \epsilon) \frac{M - \epsilon - 2\epsilon K}{m_n \log b} \leq J'_{11} \leq (1 + \epsilon) \frac{M}{m_n \log b}.$$

Let $u \in E'_n$. Then, we have

$$\begin{aligned} h(\log(x_n - u)) &= h(\log(y_n - b^{-m_n} u)) \\ &= \frac{-1}{\log |y_n - x_0 - b^{-m_n} u|} \leq \frac{2}{m_n \log b}. \end{aligned}$$

Hence, we see that

$$J'_{12} \leq \frac{2}{m_n \log b} \int_{[1, (\log x_n)^\beta] \cap E'_n} u^{-\alpha-1} h(\log u) du \leq \frac{4K\epsilon}{m_n \log b}.$$

We consequently obtain that

$$J_1 \sim x_n^{-1-\alpha}(J'_{11} + J'_{12}) \sim \frac{Mx_n^{-1-\alpha}}{m_n \log b},$$

so that

$$\phi \otimes \phi(x_n) = 2(J_1 + J_2) \sim 2J_1 \sim \frac{2Mx_n^{-1-\alpha}}{m_n \log b}.$$

Case 2-2. Suppose that $\gamma = \infty$. Note that $[1, (\log x_n)^\beta] \cap E'_n$ is empty for sufficiently large n . Let $1 \leq u \leq (\log x_n)^\beta$. Since

$$|y_n - x_0|(1 - \gamma_n^{-1}) \leq |y_n - x_0 - b^{-m_n u}| \leq |y_n - x_0|(1 + \gamma_n^{-1}),$$

we see that

$$\log |y_n - x_0 - b^{-m_n u}| \sim \log |y_n - x_0|.$$

This yields that

$$\begin{aligned} J_1 &\sim x_n^{-1-\alpha} \int_{[1, (\log x_n)^\beta]} u^{-1-\alpha} h(\log u) \cdot \frac{-1}{\log |y_n - x_0 - b^{-m_n u}|} du \\ &\sim \frac{-M}{\log |y_n - x_0|} x_n^{-1-\alpha}. \end{aligned}$$

For sufficiently large n , we have

$$\begin{aligned} J_2 \times x_n^{1+\alpha} (-\log |y_n - x_0|) &\leq \frac{2^{1+\alpha} K^2}{\alpha} \cdot \frac{-\log |y_n - x_0|}{(\log x_n)^{\alpha\beta}} \\ &= \frac{2^{1+\alpha} K^2}{\alpha} \cdot \frac{-\log \gamma_n + m_n \log b - \log(\log x_n)^\beta}{(\log x_n)^{\alpha\beta}} \\ &\leq \frac{2^{1+\alpha} K^2}{\alpha} \cdot \frac{m_n \log b}{(\log x_n)^{\alpha\beta}}, \end{aligned}$$

so that $\lim_{n \rightarrow \infty} J_2/J_1 = 0$. We consequently obtain that

$$\phi \otimes \phi(x_n) = 2(J_1 + J_2) \sim 2J_1 \sim 2x_n^{-1-\alpha} \frac{-M}{\log |y_n - x_0|}.$$

Combining the above calculations with the proof of Lemma 3.1, we reach the following: If $y \neq x_0$, then

$$\phi \otimes \phi(x_n) \sim 2Mx_n^{-1-\alpha} h(\log y) \sim 2M \int_{x_n}^{x_n+1} \phi(u) du.$$

Suppose that $y = x_0$. Recall λ in the proof of Lemma 3.1. If $0 \leq \gamma < \infty$ and $\lambda = \infty$, then we have $-\log |y_n - x_0| \sim m_n \log b$. Hence,

$$\begin{aligned} \phi \otimes \phi(x_n) &\sim 2M \frac{x_n^{-1-\alpha}}{m_n \log b} \\ &\sim 2M \frac{-x_n^{-1-\alpha}}{\log |y_n - x_0|} \sim 2M \int_{x_n}^{x_n+1} \phi(u) du. \end{aligned}$$

If $0 \leq \gamma < \infty$ and $0 \leq \lambda < \infty$, then

$$\phi \otimes \phi(x_n) \sim 2M \frac{x_n^{-1-\alpha}}{m_n \log b} \sim 2M \int_{x_n}^{x_n+1} \phi(u) du.$$

If $\gamma = \infty$, then $\lambda = \infty$ and

$$\phi \otimes \phi(x_n) \sim 2M \frac{-x_n^{-1-\alpha}}{\log |y_n - x_0|} \sim 2M \int_{x_n}^{x_n+1} \phi(u) du.$$

The lemma has been proved. □

Proof of Theorem 1.1 We have $\mu \in \mathcal{L}_{loc}$ by Lemma 3.1. It follows from Lemma 3.2 that

$$\begin{aligned} \mu * \mu((x, x + 1]) &= M^{-2} \int_x^{x+1} \phi \otimes \phi(u) du \\ &\sim 2 \int_x^{x+1} \mu((u, u + 1]) du \sim 2\mu((x, x + 1]). \end{aligned}$$

Let $c > 0$. Furthermore, we see from $\mu \in \mathcal{L}_{loc}$ and (ii) of Lemma 2.2 that

$$\mu * \mu((x, x + c]) \sim c\mu * \mu((x, x + 1]) \quad \text{and} \quad \mu((x, x + c]) \sim c\mu((x, x + 1]).$$

Hence, we get

$$\mu * \mu((x, x + c]) \sim 2\mu((x, x + c]),$$

and thereby $\mu \in \mathcal{S}_{loc}$. Thus, $\mu((x - 1, x]) \in \mathcal{S}_d$ by (i) of Proposition 1.1. Since we see that

$$\phi \otimes \phi(x) \sim 2M \int_x^{x+1} \phi(u) du = 2M^2 \mu((x, x + 1]),$$

we have $\mu^{2*} \in \mathcal{S}_{ac}$ by (ii) of Lemma 2.1. However, we have $\mu \notin \mathcal{UL}_{loc}$ because, for $c = b^{-m(n)}$ with $m(n) \in \mathbb{N}$, we see that as $n \rightarrow \infty$

$$c^{-1} \int_{b^n x_0}^{b^{n+1} x_0} M^{-1} \phi(u) du \sim \frac{M^{-1} b^{-(\alpha+1)n} x_0^{-\alpha-1}}{(m(n) + n) \log b}.$$

The above relation implies that the convergence of the definition of the class \mathcal{UL}_{loc} fails to satisfy uniformity. Since $\mathcal{US}_{loc} \subset \mathcal{UL}_{loc}$, the theorem has been proved. \square

Proof of Corollary 1.1 Proofs of assertions (i) and (ii) are clear from Theorem 1.1. We find from the proof of Theorem 1.1 that $\mu \notin \mathcal{UL}_{loc}$ but $\mu^{2*} \in \mathcal{S}_{ac}$. Since $\mathcal{S}_{ac} \subset \mathcal{L}_{ac} \subset \mathcal{UL}_{loc}$, assertions (iii) and (iv) are true. \square

Choose x_1 and x_2 satisfying that $1 < x_0 < x_0 + x_1 < x_0 + x_2 < b$. Let $\{n_k\}_{k=1}^\infty$ be an increasing sequence of positive integers satisfying $\sum_{k=1}^\infty 1/\sqrt{n_k} = 1$. Let $B_k := (-b^{n_k}x_2, -b^{n_k}x_1]$ and $D_k := (b^{n_k}x_0, b^{n_k}x_0 + 1]$ for $k \in \mathbb{N}$. Choose a distribution μ_1 satisfying that $\mu_1(B_k) = 1/\sqrt{n_k}$ for all $k \in \mathbb{N}$ and $\mu_1((\cup_{k=1}^\infty B_k)^c) = 0$.

Lemma 3.3 *We have, for $c \in \mathbb{R}$,*

$$\lim_{k \rightarrow \infty} \frac{\mu * \mu_1(D_k + c)}{\mu(D_k)} = \infty.$$

Proof We have, uniformly in $v \in [x_1, x_2]$,

$$\mu((b^n(x_0 + v), b^n(x_0 + v) + 1]) \sim M^{-1}b^{-(\alpha+1)n}(x_0 + v)^{-\alpha-1}h(\log(x_0 + v))$$

and

$$\mu((b^n x_0, b^n x_0 + 1]) \sim M^{-1} \frac{b^{-(\alpha+1)n} x_0^{-\alpha-1}}{n \log b}.$$

Thus, there exists $c_1 > 0$ such that c_1 does not depend on $v \in [x_1, x_2]$ and that

$$\liminf_{n \rightarrow \infty} \frac{\mu((b^n(x_0 + v), b^n(x_0 + v) + 1])}{n\mu((b^n x_0, b^n x_0 + 1])} \geq c_1.$$

Hence, we obtain from Lemma 3.1 that

$$\begin{aligned} \liminf_{k \rightarrow \infty} \frac{\mu * \mu_1(D_k + c)}{\mu(D_k)} &\geq \liminf_{k \rightarrow \infty} \int_{B_k} \frac{\mu(D_k - u + c)}{\mu(D_k)} \mu_1(du) \\ &= \liminf_{k \rightarrow \infty} \int_{B_k} \frac{\mu(D_k - u)}{\mu(D_k)} \mu_1(du) \\ &\geq c_1 \liminf_{k \rightarrow \infty} \frac{n_k}{\sqrt{n_k}} = \infty. \end{aligned}$$

Thus, we have proved the lemma. \square

Proof of Theorem 1.2 Define distributions ρ_1 and ρ_2 as

$$\rho_1(dx) := 2^{-1}\delta_0(dx) + 2^{-1}\mu(dx), \quad \rho_2(dx) := 2^{-1}\mu_1(dx) + 2^{-1}\mu(dx).$$

Thus, $\rho_1 \in \mathcal{S}_{loc}$ by Theorem 1.1 and (iii) of Lemma 2.2. Let $\rho(dx) := f(x)dx$, where $f(x)$ is continuous with compact support in $[0, 1]$. Define distributions $p_1(x)dx$ and $p_2(x)dx$ as

$$p_1(x)dx := \rho * \rho_1(dx) = 2^{-1} f(x)dx + 2^{-1} \rho * \mu(dx)$$

and

$$p_2(x)dx := \rho * \rho_2(dx) = 2^{-1} \rho * \mu_1(dx) + 2^{-1} \rho * \mu(dx).$$

Then, we find that $p_1(x) = p_2(x)$ for all sufficiently large $x > 0$ and $p_1(x) \in \mathcal{S}_d$ by (ii) of Proposition 1.1. We establish from Lemma 3.3 and Fatou’s lemma that

$$\begin{aligned} & \liminf_{k \rightarrow \infty} \frac{\int_{D_k} p_2 \otimes p_2(x)dx}{\int_{D_k} p_2(x)dx} \\ & \geq \liminf_{k \rightarrow \infty} \frac{\int_0^2 \mu * \mu_1(D_k - u) f^{2\otimes}(u)du}{\int_0^1 \mu(D_k - u) f(u)du} \\ & \geq \int_0^2 \liminf_{k \rightarrow \infty} \frac{\mu * \mu_1(D_k - u)}{\mu(D_k)} f^{2\otimes}(u)du = \infty. \end{aligned}$$

Thus, we conclude that $p_2(x) \notin \mathcal{S}_d$. □

4 A Remark on the Closure Under Convolution Roots

The tail of a measure ξ on \mathbb{R} is denoted by $\bar{\xi}(x)$, that is, $\bar{\xi}(x) := \xi((x, \infty))$ for $x \in \mathbb{R}$. Let $\gamma \in \mathbb{R}$. The γ -exponential moment of ξ is denoted by $\widehat{\xi}(\gamma)$, namely $\widehat{\xi}(\gamma) := \int_{-\infty}^{\infty} e^{\gamma x} \xi(dx)$.

Definition 4.1 Let $\gamma \geq 0$.

- (i) A distribution ρ on \mathbb{R} is said to belong to the class $\mathcal{L}(\gamma)$ if $\bar{\rho}(x) > 0$ for every $x \in \mathbb{R}$ and if

$$\bar{\rho}(x + a) \sim e^{-\gamma a} \bar{\rho}(x) \quad \text{for every } a \in \mathbb{R}.$$

- (ii) A distribution ρ on \mathbb{R} belongs to the class $\mathcal{S}(\gamma)$ if $\rho \in \mathcal{L}(\gamma)$ with $\widehat{\rho}(\gamma) < \infty$ and if

$$\overline{\rho * \rho}(x) \sim 2\widehat{\rho}(\gamma)\bar{\rho}(x).$$

- (iii) Let $\gamma_1 \in \mathbb{R}$. A distribution ρ on \mathbb{R} belongs to the class $\mathcal{M}(\gamma_1)$ if $\widehat{\rho}(\gamma_1) < \infty$.

The convolution closure problem on the class $\mathcal{S}(\gamma)$ with $\gamma \geq 0$ is negatively solved by Leslie [9] for $\gamma = 0$ and by Klüppelberg and Villasenor [8] for $\gamma > 0$. The same problem on the class \mathcal{S}_d is also negatively solved by Klüppelberg and Villasenor [8].

On the other hand, the fact that the class $\mathcal{S}(0)$ of subexponential distributions is closed under convolution roots is proved by Embrechts et al. [5] in the one-sided case and by Watanabe [13] in the two-sided case. Embrechts and Goldie conjecture that $\mathcal{L}(\gamma)$ with $\gamma \geq 0$ and $\mathcal{S}(\gamma)$ with $\gamma > 0$ are closed under convolution roots in [3,4], respectively. They also prove in [4] that if $\mathcal{L}(\gamma) \cap \mathcal{P}_+$ with $\gamma > 0$ is closed under convolution roots, then $\mathcal{S}(\gamma) \cap \mathcal{P}_+$ with $\gamma > 0$ is closed under convolution roots. However, Shimura and Watanabe [12] prove that the class $\mathcal{L}(\gamma)$ with $\gamma \geq 0$ is not closed under convolution roots, and we find that Xu et al. [16] show the same conclusion in the case $\gamma = 0$. Pakes [10] and Watanabe [13] show that $\mathcal{S}(\gamma)$ with $\gamma > 0$ is closed under convolution roots in the class of infinitely divisible distributions on \mathbb{R} . It is still open whether the class $\mathcal{S}(\gamma)$ with $\gamma > 0$ is closed under convolution roots. Shimura and Watanabe [11] show that the class \mathcal{OS} is not closed under convolution roots. Watanabe and Yamamuro [15] pointed out that \mathcal{OS} is closed under convolution roots in the class of infinitely divisible distributions.

Let $\gamma \in \mathbb{R}$. For $\mu \in \mathcal{M}(\gamma)$, we define the *exponential tilt* $\mu_{(\gamma)}$ of μ as

$$\mu_{(\gamma)}(\mathrm{d}x) := \frac{1}{\widehat{\mu}(\gamma)} e^{\gamma x} \mu(\mathrm{d}x).$$

Exponential tilts preserve convolutions, that is, $(\mu * \rho)_{(\gamma)} = \mu_{(\gamma)} * \rho_{(\gamma)}$ for distributions $\mu, \rho \in \mathcal{M}(\gamma)$. Let \mathcal{C} be a distribution class. For a class $\mathcal{C} \subset \mathcal{M}(\gamma)$, we define the class $\mathfrak{E}_\gamma(\mathcal{C})$ by

$$\mathfrak{E}_\gamma(\mathcal{C}) := \{\mu_{(\gamma)} : \mu \in \mathcal{C}\}.$$

It is obvious that $\mathfrak{E}_\gamma(\mathcal{M}(\gamma)) = \mathcal{M}(-\gamma)$ and that $(\mu_{(\gamma)})_{(-\gamma)} = \mu$ for $\mu \in \mathcal{M}(\gamma)$. The class $\mathfrak{E}_\gamma(\mathcal{S}(\gamma))$ is determined by Watanabe and Yamamuro as follows. Analogous result is found in Theorem 2.1 of Klüppelberg [7].

Lemma 4.1 (Theorem 2.1 of [14]) *Let $\gamma > 0$.*

- (i) *We have $\mathfrak{E}_\gamma(\mathcal{L}(\gamma) \cap \mathcal{M}(\gamma)) = \mathcal{L}_{loc} \cap \mathcal{M}(-\gamma)$ and hence $\mathfrak{E}_\gamma(\mathcal{L}(\gamma) \cap \mathcal{M}(\gamma) \cap \mathcal{P}_+) = \mathcal{L}_{loc} \cap \mathcal{P}_+$. Moreover, if $\rho \in \mathcal{L}(\gamma) \cap \mathcal{M}(\gamma)$, then we have*

$$\rho_{(\gamma)}((x, x + c]) \sim \frac{c\gamma}{\widehat{\rho}(\gamma)} e^{\gamma x} \bar{\rho}(x) \text{ for all } c > 0.$$

- (ii) *We have $\mathfrak{E}_\gamma(\mathcal{S}(\gamma)) = \mathcal{S}_{loc} \cap \mathcal{M}(-\gamma)$ and thereby $\mathfrak{E}_\gamma(\mathcal{S}(\gamma) \cap \mathcal{P}_+) = \mathcal{S}_{loc} \cap \mathcal{P}_+$.*

Finally, we present a remark on the closure under convolution roots for the three classes $\mathcal{S}(\gamma) \cap \mathcal{P}_+$, $\mathcal{S}_{loc} \cap \mathcal{P}_+$, and $\mathcal{S}_{ac} \cap \mathcal{P}_+$.

Proposition 4.1 *The following are equivalent:*

- (1) *The class $\mathcal{S}(\gamma) \cap \mathcal{P}_+$ with $\gamma > 0$ is closed under convolution roots.*
- (2) *The class $\mathcal{S}_{loc} \cap \mathcal{P}_+$ is closed under convolution roots.*
- (3) *Let μ be a distribution on \mathbb{R}_+ and let $p_c(x) := c^{-1}\mu((x - c, x])$ for $c > 0$. Then, $\{p_c^{n \otimes}(x) : c > 0\} \subset \mathcal{S}_d$ for some $n \in \mathbb{N}$ implies $\{p_c(x) : c > 0\} \subset \mathcal{S}_d$.*

Proof Proof of the equivalence between (1) and (2) is due to Lemma 4.1. Let $n \geq 2$. Suppose that (2) holds and, for some n , $p_c^{n \otimes}(x) \in \mathcal{S}_d$ for every $c > 0$. Let $f_c(x) = c^{-1}1_{(0,c)}(x)$. We have $p_c^{n \otimes}(x)dx = ((f_c(x)dx) * \mu)^{n*} \in \mathcal{S}_{loc}$. We see from assertion (2) that $(f_c(x)dx) * \mu \in \mathcal{S}_{loc}$ and hence, by (iii) of Proposition 1.1, we have $\mu \in \mathcal{S}_{loc}$, that is, $p_c(x) \in \mathcal{S}_d$ for every $c > 0$ by (i) of Proposition 1.1. Conversely, suppose that (3) holds and $\mu^{n*} \in \mathcal{S}_{loc}$. Note that $f_c^{n \otimes}(x)$ is continuous with compact support in \mathbb{R}_+ . Thus, we see from (ii) of Proposition 1.1 that $p_c^{n \otimes}(x) = \int_{0-}^{x+} f_c^{n \otimes}(x-u)\mu^{n*}(du) \in \mathcal{S}_d$ for every $c > 0$. We obtain from assertion (3) that $p_c(x) \in \mathcal{S}_d$ for every $c > 0$, that is, $\mu \in \mathcal{S}_{loc}$ by (i) of Proposition 1.1. \square

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