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A first-order limit law for functionals of two independent fractional Brownian motions in the critical case

Junna Bi · Fangjun Xu

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Abstract We prove a first-order limit law for functionals of two independent d-dimensional fractional Brownian motions with the same Hurst index $H = 2/d$ ($d \geq 4$), using the method of moments and extending a result by LeGall in the case of Brownian motion.

Keywords Limit theorem · Fractional Brownian motion · Method of moments · Short range dependence

Mathematics Subject Classification (2010) Primary 60F17 · Secondary 60G15, 60G22

1 Introduction

Let $\left\{B_t^H = (B_t^1, \ldots, B_t^d), t \geq 0\right\}$ be a d-dimensional fractional Brownian motion (fBm) with Hurst index H in (0, 1). Let $B^{H,1}$ and $B^{H,2}$ be two independent copies of B^H with $H = 2/d \in (0, 1/2]$. If $Hd = 2$, then the intersection local time of $B^{H,1}$ and $B^{H,2}$ does not exist (see [8,9]). This is called the critical case. When $H = 1/2$ and $d = 4$, the following convergence in law was proved by LeGall (see [3]):

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J. Bi · F. Xu (✉)
School of Finance and Statistics, East China Normal University, Shanghai 200241, China
e-mail: fangjunxu@gmail.com

J. Bi
e-mail: jnbi@sfs.ecnu.edu.cn

$$\frac{1}{\log n}\int_0^n\int_0^n f(B_u^{\frac{1}{2},1}-B_v^{\frac{1}{2},2})\,\mathrm{d}u\,\mathrm{d}v \xrightarrow{\mathcal{L}} \left(\frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} f(x)\,\mathrm{d}x\right)N^2$$

as n tends to infinity, where f is a continuous function with compact support and N is the standard normal random variable.

We will generalize the above limit theorem to fBms. The next theorem is the main result of this paper.

Theorem 1.1 *Suppose $Hd = 2$, $H \le 1/2$, and f is a bounded measurable function on $\mathbb{R}^d$ with $\int_{\mathbb{R}^d} |f(x)||x|^\beta\,\mathrm{d}x < \infty$ for some $\beta > 0$. Then, for any t_1 and $t_2 > 0$,*

$$\frac{1}{n}\int_0^{e^{nt_1}}\int_0^{e^{nt_2}} f(B_u^{H,1}-B_v^{H,2})\,\mathrm{d}u\,\mathrm{d}v \xrightarrow{\mathcal{L}} C_{f,d}\,(t_1\wedge t_2)\,N^2 \tag{1.1}$$

as n tends to infinity, where

$$C_{f,d} = \frac{d}{4}\,B\left(\frac{d}{4},\frac{d}{4}\right)\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x \tag{1.2}$$

with $B(\cdot,\cdot)$ being the Beta function, and N is a real-valued standard normal random variable.

Remark 1.2 Since the function f is bounded, we can always assume $\beta \le 1$. Moreover, the assumption on f implies $f \in L^p(\mathbb{R}^d)$ for any $p \ge 1$.

Remark 1.3 We use a different normalization to make sure that the limiting distribution in (3.3) depending on times t_1 and t_2. Moreover, the above limit theorem can also be generalized to several independent fBms, see [1] for the Brownian motion case.

Limit theorems for functionals of two independent fBms in the case $Hd < 2$ have been studied in [6]. Here, we focus on the critical case $Hd = 2$ and just consider the first-order limit law. That is, $\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x$ is not required to be equal to 0. The second-order limit law would probably be considered in another paper. In [6], the first-order limit law follows immediately from the scaling property of fBm. However, this method fails in the critical case. For the recent development of limit theorems for functionals of one fBm, we refer to [4,5,10] and references therein.

As we all know, fBm with Hurst index not equal to 1/2 is neither a Markov process nor a semimartingale. Therefore, the methodology once applied for Brownian motion and Markov processes cannot be used directly to prove Theorem 1.1. We use method of moments to show our result and only consider the case $H \le 1/2$. So far, we still have no idea to deal with the case $d = 3$ and $H = 2/3$. The main reason for $H \le 1/2$ is that we need to use Lemma 2.4 in [10] when showing the convergence of moments (see Step 3 in the proof of Proposition 3.3), while Lemma 2.4 in [10] fails when $H > 1/2$. This methodology has been applied when considering limit laws for functionals of one fBm in the critical case $Hd = 1$, see [7,10]. We borrow some ideas from LeGall's paper [3] and extend his result for Brownian motion to fBms with $H \le 1/2$. However, this extension is not trivial mainly because the fBm with $H < 1/2$ does not have the independent increment property. New ideas are needed.

The paper is outlined in the following way. After some preliminaries in Section 2, Section 3 is devoted to the Proof of Theorem 1.1. Throughout this paper, if not mentioned otherwise, the letter c, with or without a subscript, denotes a generic positive finite constant whose exact value is independent of n and may change from line to line. Moreover, we use ι to denote $\sqrt{-1}$, $x \cdot y$ the usual inner product in $\mathbb{R}^d$ and $B(0, r)$ the ball in $\mathbb{R}^d$ centered at the origin with radius r.

2 Preliminaries

Let $\left\{B_t^H = (B_t^1, \dots, B_t^d), t \geq 0\right\}$ be a d-dimensional fBm with Hurst index H in (0, 1), defined on some probability space $(\Omega, \mathcal{F}, P)$. That is, the components of B^H are independent centered Gaussian processes with covariance function

$$\mathbb{E}\left(B_t^i B_s^i\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

We shall use the following property of fBm B^H.

Lemma 2.1 *Given $n \geq 1$, there exist two constants κ_H and β_H depending only on n, H and d, such that for any $0 = s_0 < s_1 < \cdots < s_n$ and $x_i \in \mathbb{R}^d$, $1 \leq i \leq n$, we have*

$$\begin{aligned}\kappa_H \sum_{i=1}^n |x_i|^2 (s_i - s_{i-1})^{2H} &\leq \operatorname{Var}\left(\sum_{i=1}^n x_i \cdot \left(B_{s_i}^H - B_{s_{i-1}}^H\right)\right)\\ &\leq \beta_H \sum_{i=1}^n |x_i|^2 (s_i - s_{i-1})^{2H}.\end{aligned}$$

The proof of Lemma 2.1 can be found in [4]. Moreover, inequalities in Lemma 2.1 can be rewritten as

$$\kappa_H \sum_{i=1}^n \Big|\sum_{j=i}^n x_j\Big|^2 (s_i - s_{i-1})^{2H} \leq \operatorname{Var}\left(\sum_{i=1}^n x_i \cdot B_{s_i}^H\right) \leq \beta_H \sum_{i=1}^n \Big|\sum_{j=i}^n x_j\Big|^2 (s_i - s_{i-1})^{2H}. \tag{2.1}$$

3 Proof of Theorem 1.1

In this section, we will show Theorem 1.1. For any positive real numbers t_1 and t_2, define

$$F_n(t_1, t_2) = \frac{1}{n} \int_0^{e^{nt_1}} \int_0^{e^{nt_2}} f(B_u^{H,1} - B_v^{H,2})\, \mathrm{d}u\, \mathrm{d}v.$$

We first show that the limiting distribution of $F_n(t_1, t_2)$ depends on $t_1 \wedge t_2$.

Lemma 3.1

$$\lim_{n\to\infty} \mathbb{E}\left[|F_n(t_1,t_2) - F_n(t_1\wedge t_2, t_1\wedge t_2)|\right] = 0.$$

Proof Without loss of generality, we assume $t_1 \le t_2$ and then obtain

$$\begin{aligned}
\mathbb{E}\left[|F_n(t_1,t_2) - F_n(t_1,t_1)|\right] &\le \frac{1}{n}\mathbb{E}\left[\int_0^{e^{nt_1}}\int_{e^{nt_1}}^{e^{nt_2}} |f(B_u^{H,1} - B_v^{H,2})|\,\mathrm{d}u\,\mathrm{d}v\right] \\
&\le \frac{1}{n}\int_0^{e^{nt_1}}\int_{e^{nt_1}}^{+\infty}\int_{\mathbb{R}^d} |f(x)|(u^{2H}+v^{2H})^{-\frac{d}{2}}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}x \\
&= \frac{1}{n}\int_0^{1}\int_{1}^{+\infty}\int_{\mathbb{R}^d} |f(x)|(s^{2H}+t^{2H})^{-\frac{d}{2}}\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x \\
&\le \frac{1}{n}\left(\int_{\mathbb{R}^d}|f(x)|\,\mathrm{d}x\right)\int_0^1\left(\int_1^{+\infty} s^{-2}\,\mathrm{d}s\right)\mathrm{d}t\,\mathrm{d}u \\
&= \frac{1}{n}\int_{\mathbb{R}^d}|f(x)|\,\mathrm{d}x,
\end{aligned}$$

where we use the fact that the probability density function $p_{u,v}(x)$ of $B_u^{H,1} - B_v^{H,2}$ is less than $(u^{2H}+v^{2H})^{-\frac{d}{2}}$ in the second inequality and make the change of variables $s = e^{nt_1}u$ and $t = e^{nt_1}v$ in the first equality. This gives the desired result. □

Now we only need to consider the limiting distribution of $F_n(t,t)$ for $t > 0$. For simplicity of notation, we write $F_n(t)$ for $F_n(t,t)$. Using Remark 1.2 and an identity on page 184 of [2], $F_n(t)$ can be rewritten as

$$F_n(t) = \frac{1}{(2\pi)^d n}\int_0^{e^{nt}}\int_0^{e^{nt}}\int_{\mathbb{R}^d} \widehat{f}(x)\,\exp\left(\iota x\cdot(B_u^{H,1} - B_v^{H,2})\right)\,\mathrm{d}x\,\mathrm{d}u\,\mathrm{d}v,$$

where $\widehat{f}(x) = \int_{\mathbb{R}^d} e^{-\iota x\cdot y} f(y)\,\mathrm{d}y$.

Let

$$G_n(t) = \frac{1}{(2\pi)^d n}\int_0^{e^{nt}}\int_0^{e^{nt}}\int_{|x|<1} \widehat{f}(0)\,\exp\left(\iota x\cdot(B_u^{H,1} - B_v^{H,2})\right)\,\mathrm{d}x\,\mathrm{d}u\,\mathrm{d}v. \quad (3.1)$$

We show that $F_n(t)$ and $G_n(t)$ have the same limiting distribution.

Lemma 3.2

$$\lim_{n\to\infty} \mathbb{E}\left[|F_n(t) - G_n(t)|\right] = 0.$$

Proof We first observe that

$$F_n(t) - G_n(t) = J_{n,1}(t) + J_{n,2}(t) + J_{n,3}(t) + J_{n,4}(t),$$

where

$$\begin{aligned}
J_{n,1}(t) &= \frac{1}{n}\int_{[0,e^{nt}]^2-[1,e^{nt}]^2} f\left(B_u^{H,1} - B_v^{H,2}\right) \mathrm{d}u\, \mathrm{d}v, \\
J_{n,2}(t) &= \frac{1}{(2\pi)^d n}\int_1^{e^{nt}}\int_1^{e^{nt}}\int_{|x|\geq 1} \widehat{f}(x) \exp\left(\iota x\cdot\left(B_u^{H,1} - B_v^{H,2}\right)\right) \mathrm{d}x\, \mathrm{d}u\, \mathrm{d}v, \\
J_{n,3}(t) &= \frac{1}{(2\pi)^d n}\int_1^{e^{nt}}\int_1^{e^{nt}}\int_{|x|<1} \left(\widehat{f}(x) - \widehat{f}(0)\right) \\
&\quad \times \exp\left(\iota x\cdot\left(B_u^{H,1} - B_v^{H,2}\right)\right) \mathrm{d}x\, \mathrm{d}u\, \mathrm{d}v, \\
J_{n,4}(t) &= -\frac{\widehat{f}(0)}{(2\pi)^d n}\int_{[0,e^{nt}]^2-[1,e^{nt}]^2}\int_{|x|<1} \exp\left(\iota x\cdot\left(B_u^{H,1} - B_v^{H,2}\right)\right) \mathrm{d}x\, \mathrm{d}u\, \mathrm{d}v.
\end{aligned}$$

Since the function f is bounded and integrable,

$$\begin{aligned}
\mathbb{E}\left[|J_{n,1}(t)|\right] &\leq \frac{1}{n}\|f\|_\infty \int_0^1\int_0^1 \mathrm{d}u\, \mathrm{d}v \\
&\quad + \frac{1}{n}\left(\int_{\mathbb{R}^d} |f(x)|\, \mathrm{d}x\right)\int_0^1\int_1^{e^{nt}} (u^{2H}+v^{2H})^{-\frac{d}{2}}\, \mathrm{d}u\, \mathrm{d}v \\
&\leq \frac{1}{n}\left(\|f\|_\infty + \int_{\mathbb{R}^d} |f(x)|\, \mathrm{d}x\right).
\end{aligned}$$

Now it suffices to show

$$\lim_{n\to\infty} \mathbb{E}\left[|J_{n,i}(t)|^2\right] = 0, \quad \text{for } i = 2, 3, 4.$$

When $i = 2$,

$$\begin{aligned}
\mathbb{E}\left[|J_{n,2}(t)|^2\right] &\leq \frac{1}{n^2}\int_{|x_1|\geq 1}\int_{|x_2|\geq 1} |\widehat{f}(x_1)\widehat{f}(x_2)| \\
&\quad \times \left(\int_{[1,e^{nt}]^2} \exp\left(-\frac{1}{2}\mathrm{Var}\left(x_2\cdot B_{u_2}^H + x_1\cdot B_{u_1}^H\right)\right) \mathrm{d}u\right)^2 \mathrm{d}x \\
&\leq \frac{4}{n^2}\int_{[1,e^{nt}]^4}\int_{|x_1|\geq 1}\int_{|x_2|\geq 1} |\widehat{f}(x_1)|\widehat{f}(x_2)| 1_{\{u_1\leq u_2, v_1\leq v_2\}} \\
&\quad \times \exp\left(-\frac{1}{2}\mathrm{Var}\left(x_2\cdot B_{u_2}^H + x_1\cdot B_{u_1}^H\right) - \frac{1}{2}\mathrm{Var}\left(x_2\cdot B_{v_2}^H + x_1\cdot B_{v_1}^H\right)\right) \mathrm{d}x\, \mathrm{d}u\, \mathrm{d}v,
\end{aligned}$$

where in the last inequality we used the Cauchy–Schwartz inequality.

Using Lemma 2.1 and the fact that $|\widehat{f}|$ is bounded,

$$\begin{aligned}\mathbb{E}\,[|J_{n,2}(t)|^2] &\le \frac{c_1}{n^2}\int_{[1,e^{nt}]^4}\int_{|x_1|\ge 1}\int_{|x_2|\ge 1}|\widehat{f}(x_2)| \\ &\quad\times \exp\left(-\frac{\kappa_H}{2}\left(|x_2|^2((u_2-u_1)^{2H}+(v_2-v_1)^{2H})\right)\right) \\ &\quad\times \exp\left(-\frac{\kappa_H}{2}\left(|x_1+x_2|^2\left(u_1^{2H}+v_1^{2H}\right)\right)\right)\,\mathrm{d}x\,\mathrm{d}u\,\mathrm{d}v \\ &\le \frac{c_2}{n^2}\left(\int_{|x_2|\ge 1}|\widehat{f}(x_2)||x_2|^{-d}\,\mathrm{d}x_2\right)\left(\int_{[1,e^{nt}]^2}(u_1^{2H}+v_1^{2H})^{-\frac{d}{2}}\,\mathrm{d}u_1\,\mathrm{d}v_1\right) \\ &\le \frac{c_3}{n},\end{aligned}$$

where the second inequality follows from integrating with respect to x_1, u_2 and v_2 and the last inequality holds because

$$\int_{|x_2|\ge 1}|\widehat{f}(x_2)||x_2|^{-d}\,\mathrm{d}x_2 \le \int_{|x_2|\ge 1}|\widehat{f}(x_2)|^2\,\mathrm{d}x_2 + \int_{|x_2|\ge 1}|x_2|^{-2d}\,\mathrm{d}x_2 < \infty$$

and $\int_{[1,e^{nt}]^2}\left(u_1^{2H}+v_1^{2H}\right)^{-\frac{d}{2}}\,\mathrm{d}u_1\,\mathrm{d}v_1$ is less than a constant multiple of nt.

When $i=3$, using inequalities $|\widehat{f}(x)-\widehat{f}(0)| < c_\beta |x|^\beta$ and (2.1), we can obtain

$$\begin{aligned}\mathbb{E}\,[|J_{n,3}(t)|^2] &\le \frac{c_4}{n^2}\int_{[1,e^{nt}]^4}\int_{|x_1|<1}\int_{|x_2|<1}|x_1|^\beta|x_2|^\beta 1_{\{u_1\le u_2, v_1\le v_2\}} \\ &\quad\times \exp\left(-\frac{1}{2}\mathrm{Var}\left(x_2\cdot\left(B^{H,1}_{u_2}-B^{H,2}_{v_2}\right)+x_1\cdot\left(B^{H,1}_{u_1}-B^{H,2}_{v_1}\right)\right)\right)\,\mathrm{d}x\,\mathrm{d}u\,\mathrm{d}v \\ &\le \frac{c_4}{n^2}\int_{[1,e^{nt}]^4}\int_{|x_1|<1}\int_{|x_2|<1}|x_2|^\beta \\ &\quad\times \exp\left(-\frac{\kappa_H}{2}\left(|x_2|^2((u_2-u_1)^{2H}+(v_2-v_1)^{2H})\right)\right) \\ &\quad\times \exp\left(-\frac{\kappa_H}{2}\left(|x_1+x_2|^2\left(u_1^{2H}+v_1^{2H}\right)\right)\right)\,\mathrm{d}x\,\mathrm{d}u\,\mathrm{d}v \\ &\le \frac{c_5}{n^2}\left(\int_{|x_2|<1}|x_2|^{\beta-d}\,\mathrm{d}x_2\right)\left(\int_{[1,e^{nt}]^2}\left(u_1^{2H}+v_1^{2H}\right)^{-\frac{d}{2}}\,\mathrm{d}u_1\,\mathrm{d}v_1\right) \le \frac{c_6}{n}.\end{aligned}$$

When $i=4$, using similar arguments as $i=2$ and $i=3$, we obtain

$$\begin{aligned}\mathbb{E}\,[|J_{n,4}(t)|^2] &\le \frac{c_7}{n^2}\int_{[0,1]^2\times[1,e^{nt}]^2}\int_{|x_1|<1}\int_{|x_2|<1}1_{\{u_1\le u_2, v_1\le v_2\}} \\ &\quad\times \exp\left(-\frac{1}{2}\mathrm{Var}\left(x_2\cdot\left(B^{H,1}_{u_2}-B^{H,2}_{v_2}\right)+x_1\cdot\left(B^{H,1}_{u_1}-B^{H,2}_{v_1}\right)\right)\right)\,\mathrm{d}x\,\mathrm{d}u\,\mathrm{d}v \\ &\le \frac{c_7}{n^2}\int_{[0,1]^2\times[1,e^{nt}]^2}\int_{|x_1|<1}\int_{|x_2|<1}\exp\left(-\frac{\kappa_H}{2}\left(|x_2|^2((u_2-u_1)^{2H}+(v_2-v_1)^{2H})\right)\right)\end{aligned}$$

$$\times \exp\left(-\frac{\kappa_H}{2}\left(|x_1+x_2|^2\left(u_1^{2H}+v_1^{2H}\right)\right)\right) \mathrm{d}x\, \mathrm{d}u\, \mathrm{d}v$$
$$\leq \frac{c_8}{n^2}\int_{|x|<1}\left(\int_0^{e^{nt}} \exp\left(-\frac{\kappa_H}{2}|x|^2 u^{2H}\right)\mathrm{d}u\right)^2 \mathrm{d}x$$
$$\leq \frac{c_9}{n},$$

where in the last third inequality we used (2.1).

Combing all these estimates gives the required result. □

For the simplicity of notation, we set

$$\overline{G}_n(t)=\frac{1}{n}\int_0^{e^{nt}}\int_0^{e^{nt}}\int_{B(0,1)} \exp\left(-\iota x\cdot\left(B_u^{H,1}-B_v^{H,2}\right)\right)\mathrm{d}x\,\mathrm{d}u\,\mathrm{d}v.$$

Note that

$$G_n(t)=\frac{\widehat{f}(0)}{(2\pi)^d}\,\overline{G}_n(t). \tag{3.2}$$

So the limiting distribution of $G_n(t)$ can be easily obtained from that of $\overline{G}_n(t)$.

We next give the limiting distribution of $\overline{G}_n(t)$.

Proposition 3.3 *Suppose $Hd=2$, $H\leq 1/2$, and f is a bounded measurable function on $\mathbb{R}^d$ with $\int_{\mathbb{R}^d}|f(x)||x|^\beta\,\mathrm{d}x<\infty$ for some $\beta>0$. Then, for any $t>0$,*

$$\overline{G}_n(t)\xrightarrow{\mathcal{L}}(2\pi)^{\frac{d}{2}}\frac{d}{4}B\left(\frac{d}{4},\frac{d}{4}\right)t\,N^2 \tag{3.3}$$

as n tends to infinity, where $B(\cdot,\cdot)$ is the Beta function and N is a real-valued standard normal random variable.

Proof The proof will be done in several steps.

Step 1. We first show tightness.

Let I_m^n be the m-th moment of $\overline{G}_n(t)$. Then

$$I_m^n=\frac{1}{n^m}\int_{B^m(0,1)}\left(\int_{[0,e^{nt}]^m}\exp\left(-\frac{1}{2}\mathrm{Var}\left(\sum_{i=1}^m x_i\cdot B_{u_i}^H\right)\right)\mathrm{d}u\right)^2\mathrm{d}x.$$

Define

$$I_n(x)=\int_{D_m}\exp\left(-\frac{1}{2}\mathrm{Var}\left(\sum_{i=1}^m x_i\cdot B_{u_i}^H\right)\right)\mathrm{d}u$$

and

$$I_n^\sigma(x)=\int_{D_m}\exp\left(-\frac{1}{2}\mathrm{Var}\left(\sum_{i=1}^m x_{\sigma(i)}\cdot B_{u_i}^H\right)\right)\mathrm{d}u$$

for any $\sigma \in \mathscr{P}_m$, where $\mathscr{P}_m$ is the set of all permutations of $\{1, 2, \ldots, m\}$ and

$$D_m = \left\{0 < u_1 < \cdots < u_m < e^{nt}\right\}.$$

Then

$$I_m^n = \frac{m!}{n^m} \sum_{\sigma \in \mathscr{P}_m} \int_{B^m(0,1)} I_n(x)\, I_n^\sigma(x)\, \mathrm{d}x.$$

Applying the Cauchy–Schwartz inequality,

$$\begin{aligned}
I_m^n &\le \frac{m!}{n^m} \sum_{\sigma \in \mathscr{P}_m} \left(\int_{B^m(0,1)} \left(I_n(x)\right)^2 \mathrm{d}x\right)^{1/2} \left(\int_{B^m(0,1)} \left(I_n^\sigma(x)\right)^2 \mathrm{d}x\right)^{1/2} \\
&= \frac{(m!)^2}{n^m} \int_{B^m(0,1)} (I_n(x))^2\, \mathrm{d}x \\
&\le \frac{(m!)^2}{n^m} \int_{B^m(0,1)} \left(\int_{D_m} \exp\left(-\frac{\kappa_H}{2}\left(\sum_{i=1}^m \Big|\sum_{j=i}^m x_j\Big|^2 (u_i - u_{i-1})^{2H}\right)\right) \mathrm{d}u\right)^2 \mathrm{d}x,
\end{aligned}$$

where in the last inequality we used Lemma 2.1.

For $i = 1, \ldots, m$, we make the change of variables

$$y_i = \sum_{j=i}^m x_j \quad \text{and} \quad w_i = u_i - u_{i-1} \tag{3.4}$$

with the convention $u_0 = 0$ and then obtain

$$\begin{aligned}
I_m^n &\le \frac{(m!)^2}{n^m} \int_{B^m(0,m)} \left(\int_{[0,e^{nt}]^m} \exp\left(-\frac{\kappa_H}{2}\left(\sum_{i=1}^m |y_i|^2 w_i^{2H}\right)\right) \mathrm{d}w\right)^2 \mathrm{d}y \\
&= (m!)^2 \left(\frac{1}{n} \int_{|y_1| < m e^{nHt}} \left(\int_0^1 \exp\left(-\frac{\kappa_H}{2} |y_1|^2 w_1^{2H}\right) \mathrm{d}w_1\right)^2 \mathrm{d}y_1\right)^m \\
&\le (m!)^2 \left(\frac{c_0}{n} \int_{|y_1| < m e^{nHt}} \left(1 \wedge |y_1|^{-2d}\right) \mathrm{d}y_1\right)^m \\
&\le c_{m,H,t},
\end{aligned} \tag{3.5}$$

where $c_{m,H,t}$ is a finite positive constant depending only on m, H and t.

Step 2. We show that I_m^n is asymptotically equal to $I_{m,\gamma}^n$ defined in (3.6) below.

For any positive constant $\gamma > 1$, let

$$I_{n,\gamma}(x) = \int_{D_{m,\gamma}} \exp\left(-\frac{1}{2} \mathrm{Var}\left(\sum_{i=1}^m x_i \cdot B_{u_i}^H\right)\right) \mathrm{d}u$$

and

$$I_{n,\gamma}^{\sigma}(x) = \int_{D_{m,\gamma}} \exp\Big(-\frac{1}{2}\mathrm{Var}\Big(\sum_{i=1}^{m} x_{\sigma(i)} \cdot B_{u_i}^H\Big)\Big)\,\mathrm{d}u,$$

where

$$D_{m,\gamma} = D_m - \cup_{1\le k\neq \ell\le m}\{\Delta u_\ell/\gamma < \Delta u_k < \gamma\Delta u_\ell\}$$

and $\Delta u_k = u_k - u_{k-1}$ with the convention $u_0 = 0$.

Set

$$I_{m,\gamma}^{n} = \frac{m!}{n^m}\sum_{\sigma\in\mathscr{P}_m}\int_{B^m(0,1)} I_{n,\gamma}(x)\, I_{n,\gamma}^{\sigma}(x)\,\mathrm{d}x. \tag{3.6}$$

Then

$$\begin{aligned} I_m^n - I_{m,\gamma}^n &= \frac{m!}{n^m}\sum_{\sigma\in\mathscr{P}_m}\int_{B^m(0,1)}\Big[\big(I_n(x) - I_{n,\gamma}(x)\big)\, I_n^{\sigma}(x) + \Big(I_n^{\sigma}(x) - I_{n,\gamma}^{\sigma}(x)\Big)\, I_{n,\gamma}(x)\Big]\,\mathrm{d}x \\ &\le \frac{2m!}{n^m}\sum_{\sigma\in\mathscr{P}_m}\int_{B^m(0,1)}\big[\big(I_n(x) - I_{n,\gamma}(x)\big)\, I_n^{\sigma}(x)\big]\,\mathrm{d}x. \end{aligned}$$

Using Cauchy–Schwartz inequality and then inequality (3.5),

$$I_m^n - I_{m,\gamma}^n \le c_1\bigg(\frac{1}{n^m}\int_{B^m(0,1)}\big(I_n(x) - I_{n,\gamma}(x)\big)^2\,\mathrm{d}x\bigg)^{1/2}. \tag{3.7}$$

Note that

$$\begin{aligned} &\int_{B^m(0,1)}\big(I_n(x) - I_{n,\gamma}(x)\big)^2\,\mathrm{d}x \\ &\quad = \int_{B^m(0,1)}\bigg(\int_{D_m - D_{m,\gamma}}\exp\Big(-\frac{1}{2}\mathrm{Var}\Big(\sum_{i=1}^{m}\big(\sum_{j=i}^{m} x_j\big)\cdot\big(B_{u_i}^H - B_{u_{i-1}}^H\big)\Big)\Big)\mathrm{d}u\bigg)^2\mathrm{d}x \\ &\quad \le \int_{B^m(0,m)}\bigg(\int_{D_m - D_{m,\gamma}}\exp\Big(-\frac{\kappa_H}{2}\sum_{i=1}^{m}|y_j|^2 w_i^{2H}\Big)\mathrm{d}w\bigg)^2\mathrm{d}y, \end{aligned} \tag{3.8}$$

where in the last inequality we used the change of variables in (3.4) and Lemma 2.1.

Recall the definitions of D_m and $D_{m,\gamma}$. We obtain

$$\begin{aligned}
&\int_{B^m(0,1)} \big(I_n(x)-I_{n,\gamma}(x)\big)^2\,\mathrm{d}x\\
&\le c_2 \sum_{1\le k\ne\ell\le m} n^{m-2}\int_{|y_k|<m}\int_{|y_\ell|<m}\left(\int_0^{e^{nt}}\int_{w_\ell/\gamma}^{\gamma w_\ell}\exp\left(-\frac{\kappa_H}{2}\left(|y_k|^2 w_k^{2H}\right.\right.\right.\\
&\quad \left.\left.\left.+|y_\ell|^2 w_\ell^{2H}\right)\right)\mathrm{d}w_k\,\mathrm{d}w_\ell\right)^2\mathrm{d}y_k\,\mathrm{d}y_\ell\\
&= c_2 \sum_{1\le k\ne\ell\le m} n^{m-2}\int_{|y_k|<m}\int_{|y_\ell|<m}\int_0^{e^{nt}}\int_0^{e^{nt}}\int_{w_\ell/\gamma}^{\gamma w_\ell}\int_{\tau_\ell/\gamma}^{\gamma\tau_\ell}\\
&\quad \times\exp\left(-\frac{\kappa_H}{2}\left(|y_k|^2\left(w_k^{2H}+\tau_k^{2H}\right)+|y_\ell|^2\left(w_\ell^{2H}+\tau_\ell^{2H}\right)\right)\right)\mathrm{d}w_k\,\mathrm{d}\tau_k\,\mathrm{d}w_\ell\,\mathrm{d}\tau_\ell\,\mathrm{d}y_k\,\mathrm{d}y_\ell\\
&\le c_3 \sum_{1\le k\ne\ell\le m} n^{m-2}\int_0^{e^{nt}}\int_0^{e^{nt}}\int_{w_\ell/\gamma}^{\gamma w_\ell}\int_{\tau_\ell/\gamma}^{\gamma\tau_\ell}\left(w_k^{2H}+\tau_k^{2H}\right)^{-\frac{d}{2}}\\
&\quad \times\left(1\wedge\left(w_\ell^{2H}+\tau_\ell^{2H}\right)^{-\frac{d}{2}}\right)\mathrm{d}w_k\,\mathrm{d}\tau_k\,\mathrm{d}w_\ell\,\mathrm{d}\tau_\ell\\
&\le c_4\sum_{\ell=1}^m(\ln\gamma)\,n^{m-2}\int_0^{e^{nt}}\int_0^{e^{nt}}\left(1\wedge\left(w_\ell^{2H}+\tau_\ell^{2H}\right)^{-\frac{d}{2}}\right)\mathrm{d}w_\ell\,\mathrm{d}\tau_\ell\\
&\le c_5\,(\ln\gamma)\,n^{m-1}.
\end{aligned}\tag{3.9}$$

Combining inequalites (3.7), (3.8) and (3.9) gives

$$0\le I_m^n - I_{m,\gamma}^n \le c_6\sqrt{\frac{\ln\gamma}{n}}. \tag{3.10}$$

Step 3. We obtain estimates for $I_{m,\gamma}^n$.

For any $a_1>0$, $a_2>0$, $b_1>0$ and $b_2>0$, define

$$\begin{aligned}
&J_m^n(a_1,a_2,b_1,b_2)\\
&\quad=\frac{m!}{n^m}\sum_{\sigma\in\mathscr{P}_m}\int_{B^m(0,a_1)}\int_{[0,a_2e^{nt}]^{2m}}\\
&\qquad\times\exp\left(-b_1\sum_{i=1}^m|y_i|^2u_i^{2H}-b_2\sum_{i=1}^m\Big|\sum_{j=i}^m y_{\sigma(j)}-y_{\sigma(j)+1}\Big|^2v_i^{2H}\right)\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y,\\
&J_{m,\gamma,1}^n(a_1,a_2,b_1,b_2)\\
&\quad=\frac{m!}{n^m}\sum_{\sigma\in\mathscr{P}_m}\int_{B^m(0,a_1)}\int_{[0,a_2e^{nt}]^{2m}-O_{m,\gamma}}\\
&\qquad\times\exp\left(-b_1\sum_{i=1}^m|y_i|^2u_i^{2H}-b_2\sum_{i=1}^m\Big|\sum_{j=i}^m y_{\sigma(j)}-y_{\sigma(j)+1}\Big|^2v_i^{2H}\right)\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y
\end{aligned}$$

and

$$\begin{aligned} & J^n_{m,\gamma,2}(a_1,a_2,b_1,b_2) \\ & \quad = \frac{m!}{n^m}\sum_{\sigma\in\mathscr{P}_m}\int_{B^m_\gamma(0,a_1)}\int_{[0,a_2e^{nt}]^{2m}} \\ & \qquad \times \exp\Bigg(-b_1\sum_{i=1}^m|y_i|^2u_i^{2H}-b_2\sum_{i=1}^m|\sum_{j=i}^m y_{\sigma(j)}-y_{\sigma(j)+1}|^2v_i^{2H}\Bigg)\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y, \end{aligned}$$

where $O_{m,\gamma}=\cup_{1\le k\neq\ell\le m}\{u_\ell/\gamma<u_k<\gamma u_\ell \text{ or } v_\ell/\gamma<v_k<\gamma v_\ell\}$ and

$$\begin{aligned} B^m_\gamma(0,a_1) &= \left\{y_i\in\mathbb{R}^d : |y_i|<a_1, i=1,2,\ldots,m\right\} \\ &\quad -\cup_{1\le i\neq j\le m}\left\{|y_j|/\gamma<|y_i|<\gamma|y_j|\right\}. \end{aligned}$$

Using similar arguments when we obtain (3.10),

$$0\le J^n_m(a_1,a_2,b_1,b_2)-J^n_{m,\gamma,1}(a_1,a_2,b_1,b_2)\le c_7\sqrt{\frac{\ln\gamma}{n}} \tag{3.11}$$

and

$$0\le J^n_m(a_1,a_2,b_1,b_2)-J^n_{m,\gamma,2}(a_1,a_2,b_1,b_2)\le c_8\sqrt{\frac{\ln\gamma}{n}}. \tag{3.12}$$

By Lemma 2.4 in [10], (3.11) and (3.12), we can obtain

$$\begin{aligned} I^n_{m,\gamma} &\le J^n_m\left(m,1,\frac{1}{2}-\frac{c_9}{2\gamma^H},\frac{1}{2}-\frac{c_9}{2\gamma^H}\right) \\ &\le c_{10}\sqrt{\frac{\ln\gamma}{n}}+J^n_{m,\gamma,2}\left(m,1,\frac{1}{2}-\frac{c_9}{2\gamma^H},\frac{1}{2}-\frac{c_9}{2\gamma^H}\right) \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} I^n_{m,\gamma} &\ge -c_{12}\sqrt{\frac{\ln\gamma}{n}}+J^n_{m,\gamma,1}\left(\frac{1}{m},\frac{1}{m},\frac{1}{2}+\frac{c_{11}}{2\gamma^H},\frac{1}{2}+\frac{c_{11}}{2\gamma^H}\right) \\ &\ge -c_{13}\sqrt{\frac{\ln\gamma}{n}}+J^n_m\left(\frac{1}{m},\frac{1}{m},\frac{1}{2}+\frac{c_{11}}{2\gamma^H},\frac{1}{2}+\frac{c_{11}}{2\gamma^H}\right). \end{aligned} \tag{3.14}$$

Step 4. We obtain estimates for I^n_m.

For any $a_1>0$, $a_2>0$, $b_1>0$ and $b_2>0$, define

$$\begin{aligned} & R^n_m(a_1,a_2,b_1,b_2) \\ & \quad = \frac{m!}{n^m}\sum_{\sigma\in\mathscr{P}_m}\int_{B^m(0,a_1)}\int_{[0,a_2e^{nt}]^{2m}} \end{aligned}$$

$$\times \exp\bigg(- b_1 \sum_{i=1}^{m} |y_i|^2 u_i^{2H} - b_2 \sum_{i=1}^{m} \sup_{j \in A_i^\sigma} |y_j|^2 v_i^{2H} \bigg) \mathrm{d}u\, \mathrm{d}v\, \mathrm{d}y \tag{3.15}$$

and

$$\begin{aligned} &R^n_{m,\gamma}(a_1, a_2, b_1, b_2) \\ &\quad = \frac{m!}{n^m} \sum_{\sigma \in \mathscr{P}_m} \int_{B^m_\gamma(0,a_1)} \int_{[0,a_2 e^{nt}]^{2m}} \\ &\qquad \times \exp\bigg(- b_1 \sum_{i=1}^{m} |y_i|^2 u_i^{2H} - b_2 \sum_{i=1}^{m} \sup_{j \in A_i^\sigma} |y_j|^2 v_i^{2H} \bigg) \mathrm{d}u\, \mathrm{d}v\, \mathrm{d}y, \end{aligned}$$

where

$$A_i^\sigma = \big\{\sigma(i), \ldots, \sigma(m)\big\} \Delta \big\{\sigma(i)+1, \ldots, \sigma(m)+1\big\}$$

with Δ being the symmetric difference operator for two sets.

Using similar arguments when we obtain (3.10),

$$R^n_m(a_1, a_2, b_1, b_2) - R^n_{m,\gamma}(a_1, a_2, b_1, b_2) \le c_{14} \sqrt{\frac{\ln \gamma}{n}}. \tag{3.16}$$

Thanks to (3.13) and (3.14) in **Step 3**, we obtain

$$\begin{aligned} I^n_m &\le c_{15} \sqrt{\frac{\ln \gamma}{n}} + J^n_{m,\gamma,2}\left(m, 1, \frac{1}{2} - \frac{c_9}{2\gamma^H}, \frac{1}{2} - \frac{c_9}{2\gamma^H}\right) \\ &\le c_{15} \sqrt{\frac{\ln \gamma}{n}} + R^n_m\left(m, 1, \frac{1}{2} - \frac{c_9}{2\gamma^H}, \frac{1}{2} - \frac{c_9}{2\gamma^H} - \frac{m}{\gamma}\right) \end{aligned}$$

and

$$\begin{aligned} I^n_m &\ge -c_{16} \sqrt{\frac{\ln \gamma}{n}} + J^n_{m,\gamma}\left(\frac{1}{m}, \frac{1}{m}, \frac{1}{2} + \frac{c_{11}}{2\gamma^H}, \frac{1}{2} + \frac{c_{11}}{2\gamma^H}\right) \\ &\ge -c_{16} \sqrt{\frac{\ln \gamma}{n}} + R^n_{m,\gamma}\left(\frac{1}{m}, \frac{1}{m}, \frac{1}{2} + \frac{c_{11}}{2\gamma^H}, \frac{1}{2} + \frac{c_{11}}{2\gamma^H} + \frac{m}{\gamma}\right) \\ &\ge -c_{17} \sqrt{\frac{\ln \gamma}{n}} + R^n_m\left(\frac{1}{m}, \frac{1}{m}, \frac{1}{2} + \frac{c_{11}}{2\gamma^H}, \frac{1}{2} + \frac{c_{11}}{2\gamma^H} + \frac{m}{\gamma}\right), \end{aligned}$$

where we used (3.16) in the last inequality.

Step 5. We obtain the limit of I^n_m and then show the convergence of corresponding moments.

By Lemma 4.1 in the Appendix,

$$\begin{aligned}\limsup_{n\to\infty} I_m^n &\le \limsup_{\gamma\to\infty}\limsup_{n\to\infty} R_m^n\left(m, 1, \frac{1}{2}-\frac{c_9}{2\gamma^H}, \frac{1}{2}-\frac{c_9}{2\gamma^H}-\frac{m}{\gamma}\right)\\ &= \left(2t\,(2\pi)^{\frac{d}{2}}\frac{\Gamma^2\left(\frac{d+4}{4}\right)}{\Gamma\left(\frac{d+2}{2}\right)}\right)^m (2m-1)!!\\ &= (2\pi)^{\frac{md}{2}}\left(\frac{d}{4}B\left(\frac{d}{4},\frac{d}{4}\right)\right)^m (2m-1)!!\,t^m\end{aligned}$$

and

$$\begin{aligned}\liminf_{n\to\infty} I_m^n &\ge \liminf_{\gamma\to\infty}\liminf_{n\to\infty} R_m^n\left(\frac{1}{m}, \frac{1}{m}, \frac{1}{2}+\frac{c_{11}}{2\gamma^H}, \frac{1}{2}+\frac{c_{11}}{2\gamma^H}+\frac{m}{\gamma}\right)\\ &= \left(2t\,(2\pi)^{\frac{d}{2}}\frac{\Gamma^2\left(\frac{d+4}{4}\right)}{\Gamma\left(\frac{d+2}{2}\right)}\right)^m (2m-1)!!\\ &= (2\pi)^{\frac{md}{2}}\left(\frac{d}{4}B\left(\frac{d}{4},\frac{d}{4}\right)\right)^m (2m-1)!!\,t^m.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} I_m^n = (2\pi)^{\frac{md}{2}}\left(\frac{d}{4}B\left(\frac{d}{4},\frac{d}{4}\right)\right)^m (2m-1)!!\,t^m.$$

This completes the proof. □

Proof of Theorem 1.1. This follows easily from Lemmas 3.1 and 3.2, (3.2) and Proposition 3.3.

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Appendix

In this section, we prove a lemma which is important in the Proof of Theorem 1.1.

Recall that $\mathscr{P}_m$ is the set of all permutations of $\{1, 2, \dots, m\}$. For any $(\tau,\sigma) \in \mathscr{P}_m \times \mathscr{P}_m$, define

$$\phi_{\tau,\sigma}(i) = \sup\left\{\tau(j) : j \in A_i^\sigma\right\},$$

where

$$A_i^\sigma = \left\{\sigma(i), \dots, \sigma(m)\right\}\Delta\left\{\sigma(i)+1, \dots, \sigma(m)+1\right\}.$$

Let Ω_m be the set of all $(\tau,\sigma) \in \mathscr{P}_m \times \mathscr{P}_m$ such that $\phi_{\tau,\sigma}$ is bijective. By Lemma 4 in [1], the number of elements in Ω_m is

$$\#\Omega_m = \prod_{i=1}^{m}(2i-1) = (2m-1)!!.$$

Recall the definition of $R_m^n(a_1, a_2, b_1, b_2)$ in (3.15). The following lemma shows that

$$\lim_{n\to\infty} R_m^n(a_1, a_2, b_1, b_2)$$

exists and does not depend on a_1 and a_2.

Lemma 3.4

$$\lim_{n\to\infty} R_m^n(a_1, a_2, b_1, b_2) = \left(\frac{2\pi^{\frac{d}{2}} t\, \Gamma^2\left(\frac{d+4}{4}\right)}{(b_1 b_2)^{\frac{d}{4}}\, \Gamma\left(\frac{d+2}{2}\right)}\right)^m (2m-1)!!. \tag{3.17}$$

Proof Recall that $Hd = 2$. Then

$$R_m^n(a_1, a_2, b_1, b_2) = \frac{m!}{n^m} \sum_{\sigma\in\mathscr{P}_m} \int_{|y_1|<a_1 e^{nHt}} \cdots \int_{|y_m|<a_1 e^{nHt}} \int_{[0,a_2]^{2m}} \times \exp\left(-b_1 \sum_{i=1}^m |y_i|^2 u_i^{2H} - b_2 \sum_{i=1}^m \sup_{j\in A_i^\sigma} |y_j|^2 v_i^{2H}\right) \mathrm{d}u\, \mathrm{d}v\, \mathrm{d}y.$$

It is easy to see that

$$\begin{aligned}
&\limsup_{n\to\infty} R_m^n(a_1, a_2, b_1, b_2) \\
&= \limsup_{n\to\infty} \frac{m!}{n^m} \sum_{\sigma\in\mathscr{P}_m} \int_{1<|y_1|<a_1 e^{nHt}} \cdots \int_{1<|y_m|<a_1 e^{nHt}} \int_{[0,a_2]^{2m}} \\
&\quad \times \exp\left(-b_1 \sum_{i=1}^m |y_i|^2 u_i^{2H} - b_2 \sum_{i=1}^m \sup_{j\in A_i^\sigma} |y_j|^2 v_i^{2H}\right) \mathrm{d}u\, \mathrm{d}v\, \mathrm{d}y \\
&\le \limsup_{n\to\infty} \frac{m!}{n^m} \int_{[0,+\infty)^{2m}} \exp\left(-b_1 \sum_{i=1}^m u_i^{2H} - b_2 \sum_{i=1}^m v_i^{2H}\right) \mathrm{d}u\, \mathrm{d}v \\
&\quad \times \sum_{\sigma\in\mathcal{P}_m} \int_{1<|y_1|<a_1 e^{nHt}} \cdots \int_{1<|y_m|<a_1 e^{nHt}} \left(\prod_{i=1}^m |y_i|^{-\frac{d}{2}}\right)\left(\prod_{i=1}^m (\sup_{j\in A_i^\sigma} |y_j|)^{-\frac{d}{2}}\right) \mathrm{d}y.
\end{aligned}$$

Making the change of variables $r_i = |y_i|$ for $i = 1, 2, \ldots, m$,

$$\limsup_{n\to\infty} R_m^n(a_1, a_2, b_1, b_2)$$
$$\le \left(\frac{\Gamma\left(\frac{d+4}{4}\right)}{b_1^{\frac{d}{4}}}\right)^m \left(\frac{\Gamma\left(\frac{d+4}{4}\right)}{b_2^{\frac{d}{4}}}\right)^m \left(\frac{d\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d+2}{2}\right)}\right)^m$$
$$\times \limsup_{n\to\infty} \frac{m!}{n^m} \sum_{\sigma\in\mathscr{P}_m} \int_{(1,a_1e^{nHt})^m} \Big(\prod_{i=1}^m r_i^{-\frac{d}{2}}\Big)\Big(\prod_{i=1}^m (\sup_{j\in A_i^\sigma} r_j)^{-\frac{d}{2}}\Big)\, \mathrm{d}r.$$

Making another change of variables $r_i = e^{n\alpha_i}$ for $i = 1, 2, \ldots, m$, the right-hand side of the above inequality is equal to

$$\left(\frac{d\pi^{\frac{d}{2}}\,\Gamma^2\left(\frac{d+4}{4}\right)}{(b_1b_2)^{\frac{d}{4}}\Gamma\left(\frac{d+2}{2}\right)}\right)^m \limsup_{n\to\infty} m! \sum_{\sigma\in\mathscr{P}_m} \int_{(0,\frac{\ln a_1}{n}+Ht)^m}$$
$$\exp\left(\frac{d}{2}n\Big(\sum_{i=1}^m \alpha_i - \sum_{i=1}^m \sup_{j\in A_i^\sigma} \alpha_j\Big)\right)\mathrm{d}\alpha$$
$$= \left(\frac{d\pi^{\frac{d}{2}}\,\Gamma^2\left(\frac{d+4}{4}\right)}{(b_1b_2)^{\frac{d}{4}}\Gamma\left(\frac{d+2}{2}\right)}\right)^m (Ht)^m\,\#\Omega_m,$$

where in the last equality we used the dominated convergence theorem and the definition of the set Ω_m.

Therefore,

$$\limsup_{n\to\infty} R_m^n(a_1, a_2, b_1, b_2) \le \left(\frac{2\pi^{\frac{d}{2}}t\,\Gamma^2\left(\frac{d+4}{4}\right)}{(b_1b_2)^{\frac{d}{4}}\,\Gamma\left(\frac{d+2}{2}\right)}\right)^m (2m-1)!!. \tag{3.18}$$

On the other hand,

$$\liminf_{n\to\infty} R_m^n(a_1, a_2, b_1, b_2)$$
$$= \liminf_{n\to\infty} \frac{m!}{n^m} \sum_{\sigma\in\mathscr{P}_m} \int_{K^H<|y_1|<a_1e^{nHt}} \cdots \int_{K^H<|y_m|<a_1e^{nHt}} \int_{[0,a_2]^{2m}}$$
$$\times \exp\left(-b_1\sum_{i=1}^m |y_i|^2u_i^{2H} - b_2\sum_{i=1}^m \sup_{j\in A_i^\sigma} |y_j|^2v_i^{2H}\right)\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y$$
$$\ge \liminf_{n\to\infty} \frac{m!}{n^m} \int_{[0,Ka_2]^{2m}} \exp\left(-b_1\sum_{i=1}^m u_i^{2H} - b_2\sum_{i=1}^m v_i^{2H}\right)\mathrm{d}u\,\mathrm{d}v$$
$$\times \sum_{\sigma\in\mathcal{P}_m} \int_{K^H<|y_1|<a_1e^{nHt}} \cdots \int_{K^H<|y_m|<a_1e^{nHt}} \Big(\prod_{i=1}^m |y_i|^{-\frac{d}{2}}\Big)\Big(\prod_{i=1}^m (\sup_{j\in A_i^\sigma} |y_j|)^{-\frac{d}{2}}\Big)\,\mathrm{d}y$$

$$\geq \int_{[0,Ka_2]^{2m}} \exp\Big(-b_1\sum_{i=1}^m u_i^{2H} - b_2\sum_{i=1}^m v_i^{2H}\Big)\mathrm{d}u\,\mathrm{d}v\left(\frac{d\pi^{\frac{d}{2}}}{\Gamma(\frac{d+2}{2})}\right)^m$$
$$\times \liminf_{n\to\infty}\frac{m!}{n^m}\sum_{\sigma\in\mathscr{P}_m}\int_{(K^H,a_1e^{nHt})^m}\Big(\prod_{i=1}^m r_i^{-\frac{d}{2}}\Big)\Big(\prod_{i=1}^m(\sup_{j\in A_i^\sigma} r_j)^{-\frac{d}{2}}\Big)\,dr$$
$$= \int_{[0,Ka_2]^{2m}} \exp\Big(-b_1\sum_{i=1}^m u_i^{2H} - b_2\sum_{i=1}^m v_i^{2H}\Big)\mathrm{d}u\,\mathrm{d}v\left(\frac{d\pi^{\frac{d}{2}}}{\Gamma(\frac{d+2}{2})}\right)^m$$
$$\times \liminf_{n\to\infty} m!\sum_{\sigma\in\mathscr{P}_m}\int_{(\frac{H\ln K}{n},\frac{\ln a_1}{n}+Ht)^m}\exp\left(\frac{d}{2}n\Big(\sum_{i=1}^m\alpha_i-\sum_{i=1}^m\sup_{j\in A_i^\sigma}\alpha_j\Big)\right)d\alpha$$
$$= \int_{[0,Ka_2]^{2m}} \exp\Big(-b_1\sum_{i=1}^m u_i^{2H} - b_2\sum_{i=1}^m v_i^{2H}\Big)\mathrm{d}u\,\mathrm{d}v\left(\frac{2\pi^{\frac{d}{2}}t}{\Gamma\left(\frac{d+2}{2}\right)}\right)^m(2m-1)!!.$$

Since the positive constant K can be arbitrarily large,

$$\liminf_{n\to\infty} R_m^n(a_1,a_2,b_1,b_2)$$
$$\geq \int_{[0,+\infty)^{2m}} \exp\Big(-b_1\sum_{i=1}^m u_i^{2H} - b_2\sum_{i=1}^m v_i^{2H}\Big)\mathrm{d}u\,\mathrm{d}v\left(\frac{2\pi^{\frac{d}{2}}t}{\Gamma\left(\frac{d+2}{2}\right)}\right)^m(2m-1)!!$$
$$= \left(\frac{2\pi^{\frac{d}{2}}t\,\Gamma^2\left(\frac{d+4}{4}\right)}{(b_1b_2)^{\frac{d}{4}}\,\Gamma\left(\frac{d+2}{2}\right)}\right)^m(2m-1)!!, \tag{3.19}$$

where the last equality follows from a change of variables, the definition of Gamma function and the fact $Hd=2$.

Combining (3.18) and (3.19) gives the desired result (3.17). □

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