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A Large Deviation Principle for Symmetric Markov Processes with Feynman–Kac Functional

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Abstract We establish a large deviation principle for the occupation distribution of a symmetric Markov process with Feynman–Kac functional. As an application, we show the L^p-independence of the spectral bounds of a Feynman–Kac semigroup. In particular, we consider one-dimensional diffusion processes and show that if no boundaries are natural in Feller's boundary classification, the L^p-independence holds, and if one of the boundaries is natural, the L^p-independence holds if and only if the L^2-spectral bound is non-positive.

Keywords Large deviation · Feynman–Kac semigroup · Spectral bound · Dirichlet form

Mathematics Subject Classification (2000) 60J45 · 60J40 · 35J10

1 Introduction

In this paper we study Donsker–Varadhan type large deviations for symmetric Markov processes with Feynman–Kac functional; in particular, we prove the uniform upper bound for each closed set and we apply it to show the L^p-independence of spectral bounds of Feynman–Kac semigroups.

Let $\mathbb{M} = (\Omega, X_t, \mathbb{P}_x, \zeta)$ be an m-symmetric Markov process on a locally compact separable metric space X. Here m is a positive Radon measure with full support and ζ is the lifetime. We impose on the Markov process $\mathbb{M}$ the assumptions (I), (II) and (III) below. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be the Dirichlet form on $L^2(X; m)$ generated by $\mathbb{M}$. Let μ

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be a Green-tight Kato measure (in notation, $\mu \in \mathcal{K}_\infty$) and A_t^μ the positive continuous additive functional in the Revuz correspondence to μ. We then define the Feynman–Kac semigroup $\{p_t^\mu\}_{t\geq 0}$ by

$$p_t^\mu f(x) = \mathbb{E}_x\big(e^{A_t^\mu} f(X_t); t < \zeta\big)$$

for a bounded Borel function f on X. We may regard $\{p_t^\mu\}_{t\geq 0}$ as the semigroup generated by the Schrödinger form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}))$:

$$\mathcal{E}^\mu(u,v) = \mathcal{E}(u,v) - \int_X u(x)v(x)\,d\mu(x), \quad u,v \in \mathcal{D}(\mathcal{E}). \tag{1.1}$$

Let $\mathcal{P}$ be the set of probability measures on X equipped with the weak topology. We define the function I^μ on $\mathcal{P}$ by

$$I^\mu(\nu) = \begin{cases} \mathcal{E}^\mu(\sqrt{f}, \sqrt{f}) & \text{if } \nu = f \cdot m, \sqrt{f} \in \mathcal{D}(\mathcal{E}), \\ \infty & \text{otherwise.} \end{cases} \tag{1.2}$$

Given $\omega \in \Omega$ with $0 < t < \zeta(\omega)$, we define the occupation distribution $L_t(\omega) \in \mathcal{P}$ by

$$L_t(\omega)(A) = \frac{1}{t}\int_0^t 1_A\big(X_s(\omega)\big)\,ds$$

for a Borel set A of X, where 1_A is the indicator function of the set A. Then we will establish the main theorem:

Theorem 1.1 *Assume* (I), (II) *and* (III) *below. Let μ be a measure in $\mathcal{K}_\infty$.*

(i) *For each open set $G \subset \mathcal{P}$,*

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x\big(e^{A_t^\mu}; L_t \in G, t < \zeta\big) \geq -\inf_{\nu\in G} I^\mu(\nu).$$

(ii) *For each closed set $K \subset \mathcal{P}$,*

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in X} \mathbb{E}_x\big(e^{A_t^\mu}; L_t \in K, t < \zeta\big) \leq -\inf_{\nu\in K} I^\mu(\nu).$$

The infimum of $I^\mu(\nu)$ attains at the normalized ground state of the generalized Schrödinger operator, the generator of the semigroup $\{p_t^\mu\}$ (see Remark 4.1). In this sense, Theorem 1.1 says a large deviation from the ground state, not from the invariant measure. The essential idea of the proof for Theorem 1.1 lies in Donsker–Varadhan [9], where the one-dimensional Brownian motion was treated; however, since A_t^μ is not generally regarded as a function of L_t, we need to extend the Donsker–Varadhan's argument to Markov processes with Feynman–Kac functional.

The lower bound (i) was proved in [21]. An important fact for the proof is that any irreducible symmetric Markov process can be transformed to a symmetric ergodic process by a certain supermartingale multiplicative functional (Theorem 3.1). For the proof of the upper bound (ii), we will first introduce a new rate function which is regarded as a version of so-called *I-function* introduced in Donsker and Varadhan [10]; suppose that $\mu \in \mathcal{K}_\infty$ is *gaugeable*, that is,

$$\sup_{x\in X} \mathbb{E}_x\left(e^{A^\mu_\zeta}\right) < \infty$$

and let $h(x) = \mathbb{E}_x(\exp(A^\mu_\zeta))$. After consideration of the Feynman–Kac functional, we define the modified I-function by

$$I(\nu) = -\inf_{\substack{\phi\in\mathcal{D}_+(\mathcal{H}^\mu)\\ \epsilon>0}} \int_X \frac{\mathcal{H}^\mu\phi}{\phi+\epsilon h}\,d\nu, \qquad \nu \in \mathcal{P}, \tag{1.3}$$

where $\mathcal{H}^\mu$ is the generalized Schrödinger operator and $\mathcal{D}_+(\mathcal{H}^\mu)$ is its suitable domain. The operator $\mathcal{H}^\mu$ is formally written as $\mathcal{H}^\mu = \mathcal{L} + \mu$, where $\mathcal{L}$ is the generator of the Markov process $\mathbb{M}$. Next we will show the upper bound with this modified I-function I and finally identify the function I with I^μ by the similar argument as in [10]. The function h is said to be a *gauge function* and some necessary and sufficient conditions for the measure μ being gaugeable are known (cf. [3, 6]). For an analytic condition for the gaugeability, see Theorem 2.1 below.

In [20, 21], we dealt with the large deviation principle for symmetric Markov processes with finite lifetime or Feynman–Kac functional. Theorem 1.1 can be regarded as a final result in the sense that it says the *full* large deviation principle for symmetric Markov processes with Feynman–Kac functional; in [20] we proved Theorem 1.1 for symmetric Markov processes without Feynman–Kac functional. We there used the identity function 1 for the gauge function h to define the I-function. Noting that the identity function is harmonic for the Markov generator $\mathcal{L}$ and the gauge function h is harmonic for the Schrödinger operator $\mathcal{H}^\mu$, we can regard the function I as an extension of the I-function in [20]. In [21] we proved the upper bound (ii) for each compact set of $\mathcal{P}$ without assuming (III). We there did not need to add ϵh in (1.3) because the Markov process was supposed to be conservative and the I-function was defined by taking the infimum over uniformly positive functions in a domain of $\mathcal{H}^\mu$. We can show that the function I is independent of h if the function h is uniformly positive and bounded, that is, I is identical to the Schrödinger form (1.2). This is an extension of the known fact due to Donsker and Varadhan that if a Markov process is symmetric, then the associated I-function is identified with its Dirichlet form. We would like to emphasize that the definition (1.3) of the rate function I is a key point for the proof of the upper bound, Theorem 1.1(ii).

A technically important remark on the proof of Theorem 1.1 is that it suffices to prove it for the β-subprocess of $\mathbb{M}$, the killed process by $\exp(-\beta t)$, $\beta > 0$. Owing to this, we may assume that $\mathbb{M}$ is transient. Moreover, since every Green-tight measure becomes gaugeable with respect to the β-subprocess of $\mathbb{M}$ if β is large enough (Lemma 4.4), we may also assume that μ is gaugeable. The β-subprocess is a useful tool in studying the original process. This tool becomes available by extending the large deviation to symmetric Markov processes with finite lifetime.

When $G = K = \mathcal{P}$, Theorem 1.1 tells us that

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x\big(e^{A_t^\mu}; t < \zeta\big) \\
&\quad = \lim_{t\to\infty} \frac{1}{t} \log \sup_{x\in X} \mathbb{E}_x\big(e^{A_t^\mu}; t < \zeta\big) \\
&\quad = -\inf\left\{\mathcal{E}^\mu(u,u) : u \in \mathcal{D}(\mathcal{E}), \int_X u^2\, dm = 1\right\}. \qquad (1.4)
\end{aligned}$$

Equation (1.4) says that the spectral bound of the semigroup p_t^μ on $L^p(X;m)$ is independent of p; indeed, let $\|p_t^\mu\|_{p,p}$ be the operator norm of p_t^μ from $L^p(X;m)$ to $L^p(X;m)$ and define the L^p-spectral bound of p_t^μ by

$$\rho_p(\mu) = -\lim_{t\to\infty} \frac{1}{t} \log \|p_t^\mu\|_{p,p}, \quad 1 \le p \le \infty. \qquad (1.5)$$

Since

$$\sup_{x\in X} \mathbb{E}_x\big(e^{A_t^\mu}; t < \zeta\big) = \sup_{x\in X} p_t^\mu 1(x) = \|p_t^\mu\|_{\infty,\infty},$$

(1.4) implies that $\rho_\infty(\mu) = \rho_2(\mu)$ and consequently $\rho_p(\mu)$ is independent of p by the Riesz–Thorin interpolation theorem ([7, 1.1.5]).

We gave in [24] an alternate proof of the L^p-independence for a different class of symmetric Markov processes whose semigroup is conservative, transforms $C_\infty(X)$ to itself and does not always satisfy (III). Here $C_\infty(X)$ is the set of continuous functions vanishing the infinity Δ. Our method in [24] is as follows: we first note that if the state space is compact, only the Feller property is necessary to verify the upper bound. We thus extend the Markov process $\mathbb{M}$ to the one-point compactification X_Δ by making the infinity Δ a trap, and prove the upper bound for this extended Markov process. Then the rate function becomes a function on the set of probability measures on X_Δ, not on X; the adjoined point Δ makes a contribution to the rate function. We showed in [24] that the infimum of the rate function on the set of probability measures on X_Δ is equal to the infimum of the original rate function on the set of probability measures on X, if and only if the L^2-spectral bound is non-positive. Consequently we obtained a necessary and sufficient condition for the L^p-independence. For non-local Feynman–Kac semigroups, see [25, 27].

The uniform upper bound (ii) is crucial for the proof of L^p-independence, and so is the assumption (III). In Sect. 5, we will consider one-dimensional diffusion processes and show that if no boundaries are natural in Feller's boundary classification, the assumption (III) is fulfilled. As a result, the L^p-independence holds if no boundaries are natural. We see by exactly the same argument as in [24] that if one of boundaries is natural, then the L^p-independence holds if and only if the L^2-spectral bound is non-positive. The case treated in [24] is corresponding to when the both boundaries are natural. For example, consider the one-dimensional diffusion process with generator $(1/2)\Delta + k \cdot d/dx$ on $(-\infty, \infty)$. Here k is a constant. Then the both boundaries are natural and $\rho_2(\mu)$ equals $k^2/2$; however, $\rho_\infty(\mu) = 0$ because of the conservativeness.

Consequently, Theorem 1.1 does not hold when G and K are the whole space $\mathcal{P}$. This example was given in [11]. Next consider the Ornstein–Uhlenbeck process, the diffusion process generated by $(1/2)\Delta - x \cdot d/dx$ on $(-\infty, \infty)$. Then both boundaries are natural and $\rho_2(\mu)$ and $\rho_\infty(\mu)$ are zero, consequently the L^p-independence follows; however, Theorem 1.1 is not applicable because the uniform upper bound (ii) is not known, while the locally uniform upper bound was shown in [11]. In this sense, we can say that the L^p-independence of the Ornstein–Uhlenbeck operator holds for the different reason from Theorem 1.1 below.

The Gärtner–Ellis theorem is a useful theorem for the proof of the large deviation principle (cf. [8]). To prove the large deviation principle of A_t^μ/t by employing the Gärtner–Ellis theorem, we need to prove the existence of the logarithmic moment generating function of A_t^μ; that is, for each $\theta \in \mathbb{R}$, the limit

$$C(\theta) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x\left(e^{\theta A_t^\mu}; t < \zeta\right)$$

exists. Equation (1.4) implies that $C(\theta)$ exists and equals to $-\rho_2(\theta\mu)$. We will discuss the large deviation principle for additive functionals of one-dimensional diffusion processes (Theorem 5.2).

2 Notations and Some Facts

Let X be a locally compact separable metric space and m a positive Radon measure on X with full support. Let $\mathbb{M} = (\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, \mathbb{P}_x, X_t, \zeta)$ be an m-symmetric on X. Here $\{\mathcal{F}_t\}_{t\ge 0}$ is the minimal (augmented) admissible filtration, θ_t, $t \ge 0$, are the shift operators satisfying $X_s(\theta_t) = X_{s+t}$ identically for $s, t \ge 0$. Let $X_\Delta = X \cup \{\Delta\}$ be the one-point compactification of X, and ζ be the lifetime of $\mathbb{M}$, $\zeta = \inf\{t \ge 0 : X_t = \Delta\}$. Let $\{p_t\}_{t\ge 0}$ be the semigroup and $\{R_\alpha\}_{\alpha\ge 0}$ the resolvent:

$$p_t f(x) = \mathbb{E}_x\big(f(X_t)\big), \qquad R_\alpha f(x) = \int_0^\infty e^{-\alpha t} p_t f(x)\, dt.$$

We then impose three assumptions on $\mathbb{M}$.

(I) (Irreducibility) If a Borel set A is p_t-invariant, i.e., $p_t(1_A f)(x) = 1_A p_t f(x)$ m-a.e. for any $f \in L^2(X; m) \cap \mathcal{B}_b(X)$ and $t > 0$, then A satisfies either $m(A) = 0$ or $m(X \setminus A) = 0$. Here $\mathcal{B}_b(X)$ is the space of bounded Borel functions on X.

(II) (Strong Feller Property) For each t, $p_t(\mathcal{B}_b(X)) \subset C_b(X)$, where $C_b(X)$ is the space of bounded continuous functions on X.

(III) For any $\epsilon > 0$, there exists a compact set K such that

$$\sup_{x\in X} R_1 1_{K^c}(x) \le \epsilon.$$

Here 1_{K^c} is the indicator function of the complement of the compact set K.

Remark 2.1 The assumption (II) implies that the transition probability kernel $p_t(x, dy)$ is absolutely continuous with respect to m, $p_t(x, dy) = p_t(x, y)\, dm(y)$. The next assumption is a resolvent version of (II):

(II′) (Resolvent Strong Feller Property) For each $\alpha > 0$, $R_\alpha(\mathcal{B}_b(X)) \subset C_b(X)$, where $C_b(X)$ is the space of bounded continuous functions.

Under the assumption (II′) the resolvent kernel $R_\alpha(x, dy)$ is absolutely continuous with respect to m, $R_\alpha(x, dy) = R_\alpha(x, y)\,dm(y)$, and so is the transition probability $p_t(x, dy)$ by [13, Theorem 4.2.4]. The assumption (II′) is weaker than the assumption (II) and can be checked more easily for time-changed processes (see Appendix).

Remark 2.2 We know from the resolvent equation that $\|R_1 1_{K^c}\|_\infty \le \alpha \|R_\alpha 1_{K^c}\|_\infty$ for $\alpha > 1$ and $\|R_1 1_{K^c}\|_\infty \le (1/\alpha)\|R_\alpha 1_{K^c}\|_\infty$ for $\alpha < 1$. Hence the α-resolvent R_α satisfies the assumption (III) for all $\alpha > 0$.

Remark 2.3 (i) If $m(X) < \infty$ and $\|R_1\|_{\infty,1} < \infty$, then

$$\sup_{x\in X} R_1 1_{K^c}(x) = \|R_1\|_{\infty,1} \cdot m\big(K^c\big)$$

and the assumption (III) is fulfilled. Here, $\|R_1\|_{\infty,1}$ is the operator norm of R_1 from $L^1(X; m)$ to $L^\infty(X; m)$.

(ii) If $R_1 1 \in C_\infty(X)$, then the assumption (III) is fulfilled. Indeed, since by the strong Markov property

$$\begin{aligned} R_1 1_{K^c}(x) &= \mathbb{E}_x\left(\int_0^\infty e^{-t} 1_{K^c}(X_t)\,dt\right) = \mathbb{E}_x\left(\int_{\sigma_{K^c}}^\infty e^{-t} 1_{K^c}(X_t)\,dt\right) \\ &= \mathbb{E}_x\big(e^{-\sigma_{K^c}} R_1 1_{K^c}(X_{\sigma_{K^c}})\big) \end{aligned}$$

($\sigma_{K^c} = \inf\{t > 0; X_t \in K^c\}$), we have

$$\sup_{x\in X} R_1 1_{K^c}(x) = \sup_{x\in K^c} R_1 1_{K^c}(x) \le \sup_{x\in K^c} R_1 1(x).$$

Remark 2.4 The assumption (III) is equivalent to the statement that the measure m is Green-tight in Definition 2.1(ii) below.

We further assume that $\mathbb{M}$ is transient; however, this assumption is not necessary to prove Theorem 1.1 because it is enough to do it for the β-subprocess of $\mathbb{M}$. Note that the β-subprocess also satisfies the assumptions (I), (II) and (III). We denote by $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ the Dirichlet form generated by $\mathbb{M}$ ([13, p. 29]). Every function u in $\mathcal{D}(\mathcal{E})$ admits a quasi-continuous version $\tilde{u}$ (see [13, Theorem 2.1.3]). In the sequel we always assume that every function $u \in \mathcal{D}(\mathcal{E})$ is represented by its quasi-continuous version. A positive Borel measure μ on X is said to be *smooth*, if there exists a positive continuous additive functional (PCAF in abbreviation) A_t^μ of $\mathbb{M}$ such that for any non-negative Borel function $f \in \mathcal{B}_+(X)$ and γ-excessive function h,

$$\lim_{t\to 0} \frac{1}{t}\mathbb{E}_{hm}\left(\int_0^t f(X_s)\,dA_s^\mu\right) = \int_X f(x)h(x)\,d\mu(x) \tag{2.1}$$

(see [13, p. 188]). The measure μ is called the *Revuz measure* corresponding to A_t^μ. We introduce classes of smooth measures.

Definition 2.1 (i) A positive smooth Radon measure μ on X is said to be in the *Kato class* (in notation, $\mu \in \mathcal{K}$), if

$$\lim_{t \downarrow 0} \sup_{x \in X} \mathbb{E}_x\left(A_t^\mu\right) = 0. \tag{2.2}$$

(ii) A measure $\mu \in \mathcal{K}$ is said to be *Green-tight* (in notation, $\mu \in \mathcal{K}_\infty$), if for any $\epsilon > 0$, there exists a compact set $K \subset X$ such that

$$\sup_{x \in X} \int_{K^c} R(x, y)\, d\mu(y) < \epsilon, \tag{2.3}$$

where $R(x, y)$ denotes the 0-resolvent density $R_0(x, y)$.

Remark 2.5 We suppose that every measure in $\mathcal{K}$ is a Radon measure. As a result, the associated PCAF is of no exceptional set, that is, a classical one ([13, Theorem 5.1.7], [2, Proposition 3.8]). We know from [2, Theorem 3.9] that for all $x \in X$,

$$\mathbb{E}_x\left(A_t^\mu\right) = \int_0^t \int_X p_t(x, y)\, d\mu(y), \qquad \mathbb{E}_x\left(A_\zeta^\mu\right) = \int_X R(x, y)\, d\mu(y). \tag{2.4}$$

Remark 2.6 The definition of $\mathcal{K}_\infty$ is different from that of Z.-Q. Chen [3, Definition 2.2], where he assumes in addition that there exists a positive constant δ such that for all measurable sets $B \subset K$ with $\mu(B) < \delta$,

$$\sup_{x \in X} \int_B R(x, y)\, d\mu(y) < \epsilon. \tag{2.5}$$

Chen in [3] showed that if a measure $\mu \in \mathcal{K}_\infty$ satisfies (2.5), two statements in Proposition 2.1 below are equivalent. We only need the sufficient part for the proof of Theorem 1.1. For this reason, we remove the condition (2.5) in the definition of $\mathcal{K}_\infty$.

We see from [2, Lemma 3.5] that a measure μ in $\mathcal{K}$ is β-potential-bounded, $\sup_{x \in X} R_\beta \mu(x) < \infty$, for any $\beta > 0$. Here

$$R_\beta \mu(x) = \int_X R_\beta(x, y)\, d\mu(y).$$

Equation (2.2) in Definition 2.1(i) is equivalent to

$$\lim_{\beta \to \infty} \|R_\beta \mu\|_\infty = 0 \tag{2.6}$$

(e.g. [1]). Moreover, it is known from [19, Theorem 3.1] that

$$\int_X u^2 d\mu \leq \|R_\beta \mu\|_\infty \cdot \mathcal{E}_\beta(u, u). \tag{2.7}$$

We define the Feynman–Kac semigroup $\{p_t^\mu\}_{t \geq 0}$ by

$$p_t^\mu f(x) = \mathbb{E}_x\left(e^{A_t^\mu} f(X_t); t < \zeta\right).$$

Then the semigroup $\{p_t^\mu\}_{t \geq 0}$ possesses the following properties:

Theorem 2.1 *Let $\mu = \mu^+ - \mu^- \in \mathcal{K} - \mathcal{K}$.*

(i) *There exist constants c and β such that*

$$\|p_t^\mu\|_{p,p} \le ce^{\beta t}, \quad 1 \le p \le \infty, t > 0.$$

Here $\| \ \|_{p,p}$ means the operator norm on $L^p(X;m)$;

(ii) *$\{p_t^\mu\}_{t\ge 0}$ is a strongly continuous symmetric semigroup on $L^2(X;m)$ and the closed form generated by p_t^μ is identical to $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}))$;*

(iii) *For each $f \in \mathcal{B}_b(X)$, $p_t^\mu f \in C_b(X)$.*

Proof (i) This assertion is a consequence of [2, Proposition 5.2, Theorem 6.1(i)].

(ii) By (2.7), the Dirichlet space $\mathcal{D}(\mathcal{E})$ is contained in $L^2(X;\mu)$. Thus this assertion follows from [2, Theorem 6.1(ii)].

(iii) See [6, Proposition 3.12]. □

For a measure μ in $\mathcal{K}$, define

$$\lambda(\mu) = \inf\left\{\mathcal{E}(u,u) : u \in \mathcal{D}(\mathcal{E}), \int_X u^2(x)\mu(dx) = 1\right\}. \tag{2.8}$$

On account of Lemma 3.1 in [22], we see that $\lambda(\mu)$ is the principal eigenvalue of the time-changed process of $\mathbb{M}$ by A_t^μ. If a measure $\mu \in \mathcal{K}$ satisfies

$$\sup_{x\in X} \mathbb{E}_x\left(e^{A_\zeta^\mu}\right) < \infty,$$

then μ is said to be *gaugeable*.

Proposition 2.1 *Let $\mu \in \mathcal{K}_\infty$. Then*

$$\lambda(\mu) > 1 \quad \Longrightarrow \quad \sup_{x\in X} \mathbb{E}_x\left(e^{A_\zeta^\mu}\right) < \infty.$$

For the proof of Proposition 2.1 see Appendix.

The super-gauge theorem follows from Proposition 2.1 ([3, Theorem 5.4]).

Corollary 2.1 *If $\mu \in \mathcal{K}_\infty$ satisfies $\lambda(\mu) > 1$, then there exists a positive constant ϵ such that $(1+\epsilon)\mu$ is gaugeable.*

Proof There exists a positive constant ϵ such that $\lambda(\mu) > 1+\epsilon$. By the definition of $\lambda(\mu)$,

$$\lambda\big((1+\epsilon)\mu\big) = \frac{1}{1+\epsilon}\lambda(\mu) > 1.$$

□

3 Transform of Symmetric Markov Processes

In [4], we studied a class of supermartingale multiplicative functionals which transform each symmetric Markov process to an ergodic one. For the proof of Theorem 1.1, the transformation played a crucial role.

Let $\mu \in \mathcal{K}_\infty$ and $\kappa(\mu)$ the constant in Theorem 2.1(i). If $\alpha > \kappa(\mu)$ and $f \in \mathcal{B}_b(X)$, we define the resolvent R_α^μ by

$$R_\alpha^\mu f(x) = \mathbb{E}_x\left(\int_0^\infty e^{-\alpha t + A_t^\mu} f(X_t)\,dt\right).$$

We set

$$\mathcal{D}_+\big(\mathcal{H}^\mu\big) = \big\{R_\alpha^\mu f : \alpha > \kappa(\mu),\ f \in L^2(X;m) \cap C_b(X),\ f \geq 0 \text{ and } f \not\equiv 0\big\},$$

and define the generator $\mathcal{H}^\mu$ by

$$\mathcal{H}^\mu u = \alpha u - f, \quad u = R_\alpha^\mu f \in \mathcal{D}_+\big(\mathcal{H}^\mu\big). \tag{3.1}$$

Here $\kappa(\mu)$ is defined by

$$\kappa(\mu) = \lim_{t\to\infty} \frac{1}{t} \log \big\| p_t^\mu \big\|_{\infty,\infty}.$$

Theorem 2.1(i) says that $\kappa(\mu)$ is finite and the function $R_\alpha^\mu f$ is finite for $\alpha > \kappa(\mu)$ and $f \in L^2(X;m) \cap C_b(X)$. Each function $\phi = R_\alpha^\mu f \in \mathcal{D}_+(\mathcal{H}^\mu)$ is strictly positive because $P_x(\sigma_O < \zeta) > 0$ for any $x \in X$ by the assumption (I). Here O is a non-empty open set $\{x \in X : f(x) > 0\}$ and $\sigma_O = \inf\{t > 0 : X_t \in O\}$.

Suppose that a measure $\mu \in \mathcal{K}_\infty$ satisfies $\lambda(\mu) > 1$. Let

$$h(x) = \mathbb{E}_x\big(e^{A_\zeta^\mu}\big), \tag{3.2}$$

which is said to be the *gauge function* of μ. Then Proposition 2.1 yields

$$1 \leq h(x) \leq C_h\Big(:= \sup_{x\in X} h(x)\Big) < \infty.$$

Remark 3.1 If a measure μ in $\mathcal{K}_\infty$ satisfies $\lambda(\mu) > 1$, then

$$\big\| p_t^\mu \big\|_{\infty,\infty} \leq \sup_{x\in X} \mathbb{E}_x\big(e^{A_\zeta^\mu}\big) < \infty,$$

and thus

$$\lim_{t\to\infty} \frac{1}{t} \log \big\| p_t^\mu \big\|_{\infty,\infty} \leq 0.$$

Hence we can take any $\alpha > 0$ in the definition of $\mathcal{D}_+(\mathcal{H}^\mu)$.

Lemma 3.1 *The function h defined in (3.2) is bounded continuous.*

Proof Since by the Markov property

$$e^{A_t^\mu} h(X_t)1_{\{t<\zeta\}} = e^{A_t^\mu}\mathbb{E}_{X_t}\big(e^{A_\zeta^\mu}\big)1_{\{t<\zeta\}} = \mathbb{E}_x\big(e^{A_t^\mu + A_{\zeta(\theta_t)}^\mu(\theta_t)} 1_{\{t<\zeta\}}|\mathcal{F}_t\big)$$

$$= \mathbb{E}_x\big(e^{A_\zeta^\mu} 1_{\{t<\zeta\}}|\mathcal{F}_t\big),$$

we have

$$p_t^\mu h(x) = \mathbb{E}_x\big(e^{A_\zeta^\mu}; t<\zeta\big), \tag{3.3}$$

and so by Hölder's inequality

$$h(x) - p_t^\mu h(x) = \mathbb{E}_x\big(e^{A_\zeta^\mu}; t \geq \zeta\big)$$

$$\leq \big(\mathbb{E}_x\big(e^{(1+\epsilon)A_\zeta^\mu}\big)\big)^{1/(\epsilon+1)} \cdot \mathbb{P}_x(t\geq\zeta)^{\epsilon/(\epsilon+1)}.$$

Since by Corollary 2.1

$$\sup_{x\in X} \mathbb{E}_x\big(e^{(1+\epsilon)A_\zeta^\mu}\big) < \infty,$$

and $\mathbb{P}_x(t\geq\zeta) = 1 - p_t 1(x)$ converges to 0 locally uniformly as $t \downarrow 0$, we see that $p_t^\mu h$ converges to h locally uniformly as $t\downarrow 0$. The function $p_t^\mu h$ is continuous by [5] and so is h. □

Lemma 3.2 *Let h be the function defined in* (3.2). *Put $h(\Delta)=1$ and define*

$$M_t^h = e^{A_t^\mu} h(X_t) - h(X_0).$$

Then M_t^h is a martingale with respect to $(\mathbb{P}_x, \{\mathcal{F}_t\})$.

Proof Noting $h(\Delta)=1$, we have

$$\mathbb{E}_x\big(e^{A_t^\mu} h(X_t)\big) = \mathbb{E}_x\big(e^{A_t^\mu} h(X_t); t<\zeta\big) + \mathbb{E}_x\big(e^{A_\zeta^\mu}; t\geq\zeta\big).$$

The first term on the right-hand side equals $\mathbb{E}_x(e^{A_\zeta^\mu}; t<\zeta)$ by (3.3) and thus $\mathbb{E}_x(e^{A_t^\mu} h(X_t)) = h(x)$, that is, $\mathbb{E}_x(M_t^h) = 0$. Since

$$M_{s+t}^h = M_s^h + e^{A_s^\mu} M_t^h(\theta_s),$$

we have

$$\mathbb{E}_x\big(M_{s+t}^h|\mathcal{F}_s\big) = M_s^h + e^{A_s^\mu}\mathbb{E}_{X_s}\big(M_t^h\big) = M_s^h.$$ □

Lemma 3.3 *For $\phi = R_\alpha^\mu f \in \mathcal{D}_+(\mathcal{H}^\mu)$, let*

$$M_t^{\mu,\phi} = e^{A_t^\mu}\phi(X_t) - \phi(X_0) - \int_0^t e^{A_s^\mu}\mathcal{H}^\mu\phi(X_s)\,ds.$$

Then $M_t^{\mu,\phi}$ is a martingale with respect to $(\mathbb{P}_x, \{\mathcal{F}_t\})$.

Proof First we show that $\mathbb{E}_x(M_t^{\mu,\phi})=0$. By the Markov property

$$\begin{aligned}
&\alpha\cdot\mathbb{E}_x\left(\int_0^t e^{A_s^\mu}R_\alpha^\mu f(X_s)\,ds\right)\\
&=\alpha\cdot\mathbb{E}_x\left(\int_0^t e^{A_s^\mu}\mathbb{E}_{X_s}\left(\int_0^\infty e^{-\alpha u+A_u^\mu}f(X_u)\,du\right)ds\right)\\
&=\alpha\cdot\mathbb{E}_x\left(\int_0^t\left(\int_0^\infty \mathbb{E}_x\left(e^{-\alpha u+A_s^\mu+A_u^\mu(\theta_s)}f(X_{u+s})|\mathcal{F}_s\right)du\right)ds\right)\\
&=\alpha\cdot\mathbb{E}_x\left(\int_0^t e^{\alpha s}\left(\int_s^\infty e^{-\alpha u+A_u^\mu}f(X_u)\,du\right)ds\right),
\end{aligned}\tag{3.4}$$

and the right-hand side equals

$$\mathbb{E}_x\left(\int_0^t\left(e^{\alpha u}-1\right)e^{-\alpha u+A_u^\mu}f(X_u)\,du\right)+\mathbb{E}_x\left(\int_t^\infty\left(e^{\alpha t}-1\right)e^{-\alpha u+A_u^\mu}f(X_u)\,du\right)$$

by interchanging the order of integration. The first term equals

$$\mathbb{E}_x\left(\int_0^t e^{A_u^\mu}f(X_u)\,du\right)-\mathbb{E}_x\left(\int_0^t e^{-\alpha u+A_u^\mu}f(X_u)\,du\right)$$

and the second term equals

$$\mathbb{E}_x\left(e^{A_t^\mu}\mathbb{E}_{X_t}\left(\int_0^\infty e^{-\alpha s+A_s^\mu}f(X_s)\,ds\right)\right)-\mathbb{E}_x\left(\int_t^\infty e^{-\alpha u+A_u^\mu}f(X_u)\,du\right).$$

Hence the left-hand side of (3.4) equals

$$\mathbb{E}_x\left(\int_0^t e^{A_u^\mu}f(X_u)\,du\right)+\mathbb{E}_x\left(e^{A_t^\mu}R_\alpha^\mu f(X_t)\right)-R_\alpha^\mu f(x),$$

which implies $\mathbb{E}_x(M_t^{\mu,\phi})=0$. Since $M_{s+t}^{\mu,\phi}=M_s^{\mu,\phi}+e^{A_s^\mu}M_t^{\mu,\phi}(\theta_s)$, we have the lemma for the same reason as in Lemma 3.2. □

For $\phi=R_\alpha^\mu g\in\mathcal{D}_+(\mathcal{H}^\mu)$ and $\epsilon>0$, let $\phi_\epsilon=\phi+\epsilon h$ and put

$$M_t^{\mu,\phi_\epsilon}=e^{A_t^\mu}\phi_\epsilon(X_t)-\phi_\epsilon(X_0)-\int_0^t e^{A_s^\mu}\mathcal{H}^\mu\phi(X_s)\,ds.$$

Then, by Lemmas 3.2 and 3.3, M_t^{μ,ϕ_ϵ} is a martingale with respect to $\mathbb{P}_x$. Let $M_t^{[\phi_\epsilon]}$ be the martingale part of the semimartingale $\phi_\epsilon(X_t)-\phi_\epsilon(X_0)=M_t^{[\phi_\epsilon]}+N_t^{[\phi_\epsilon]}$. Then M_t^{μ,ϕ_ϵ} is also written as

$$M_t^{\mu,\phi_\epsilon}=\int_0^t e^{A_s^\mu}\,dM_s^{[\phi_\epsilon]},\quad \mathbb{P}_x\text{-a.e. } x\in X.\tag{3.5}$$

Indeed, applying Itô's formula to $F(x,y)=xy$, we see that

$$e^{A_t^\mu}\phi_\epsilon(X_t)-\phi_\epsilon(X_0)=F\big(e^{A_t^\mu},\phi_\epsilon(X_t)\big)-F\big(e^{A_0^\mu},\phi_\epsilon(X_0)\big)$$
$$=\int_0^t e^{A_s^\mu}\,dM_s^{[\phi_\epsilon]}+\int_0^t e^{A_s^\mu}\,dN_s^{[\phi_\epsilon]}+\int_0^t \phi_\epsilon(X_s)e^{A_s^\mu}\,dA_s^\mu,$$

and thus the martingale part of $e^{A_t^\mu}\phi_\epsilon(X_t)-\phi_\epsilon(X_0)$ equals the right-hand side of (3.5).

Let us define the multiplicative functional (MF in abbreviation) $L_t^{\phi_\epsilon}$ by

$$L_t^{\phi_\epsilon}=e^{A_t^\mu}\cdot\frac{\phi_\epsilon(X_t)}{\phi_\epsilon(X_0)}\exp\bigg(-\int_0^t\frac{\mathcal{H}^\mu\phi}{\phi_\epsilon}(X_s)\,ds\bigg)1_{\{t<\zeta\}}. \tag{3.6}$$

When $\epsilon=0$, we write L_t^ϕ for $L_t^{\phi_0}$.

Lemma 3.4 *For* $\epsilon>0$,

$$L_t^{\phi_\epsilon}-1=\int_0^t\frac{1}{\phi_\epsilon(X_0)}\exp\bigg(-\int_0^s\frac{\mathcal{H}^\mu\phi}{\phi_\epsilon}(X_u)\,du\bigg)\,dM_s^{\mu,\phi_\epsilon}. \tag{3.7}$$

Proof The right-hand side of (3.7) is equal to

$$\frac{1}{\phi_\epsilon(X_0)}\int_0^t\exp\bigg(-\int_0^s\frac{\mathcal{H}^\mu\phi}{\phi_\epsilon}(X_u)\,du\bigg)\Big(d\big(e^{A_s^\mu}\phi_\epsilon(X_s)\big)-e^{A_s^\mu}\mathcal{H}^\mu\phi(X_s)\,ds\Big).$$

Noting that

$$d\bigg(e^{A_s^\mu}\phi_\epsilon(X_s)\exp\bigg(-\int_0^s\frac{\mathcal{H}^\mu\phi}{\phi_\epsilon}(X_u)\,du\bigg)\bigg)$$
$$=\exp\bigg(-\int_0^s\frac{\mathcal{H}^\mu\phi}{\phi_\epsilon}(X_u)\,du\bigg)\Big(d\big(e^{A_s^\mu}\phi_\epsilon(X_s)\big)-e^{A_s^\mu}\mathcal{H}^\mu\phi(X_s)\,ds\Big),$$

we have the lemma. □

For $\epsilon=0$, define the sequence of open sets $\{G_n\}_{n=1}^\infty$ by $G_n=\{x\in X:\phi(x)>\frac{1}{n}\}$. As remarked in the second paragraph of Sect. 3, the function ϕ is strictly positive continuous, and thus $G_n\uparrow X$. Let τ_n be the first leaving time from G_n, $\tau_n=\inf\{t>0:X_t\notin G_n\}$. We have the next lemma in the same way as in Lemma 3.4.

Lemma 3.5 *For each* n,

$$L_{t\wedge\tau_n}^\phi-1=\int_0^{t\wedge\tau_n}\frac{1}{\phi(X_0)}\exp\bigg(-\int_0^s\frac{\mathcal{H}^\mu\phi}{\phi}(X_u)\,du\bigg)\,dM_s^{\mu,\phi}. \tag{3.8}$$

We see from Lemmas 3.4 and 3.5 that for each $\epsilon \geq 0$

$$\mathbb{E}_x\big(L_t^{\phi_\epsilon}\big) \leq \liminf_{n\to\infty} E_x\big(L_{t\wedge\tau_n}^{\phi_\epsilon}\big) \leq 1, \quad x \in X,$$

and $L_t^{\phi_\epsilon}$ is a supermartingale MF. Denote by $\mathbb{M}^{\phi_\epsilon} = (\Omega, X_t, P_x^{\phi_\epsilon}, \zeta)$ the transformed process of $\mathbb{M}$ by $L_t^{\phi_\epsilon}$. We see from Lemma 3.4 that L_t^{ϕ} satisfies the Doléans–Dade equation

$$L_t^{\phi} = 1 + \int_0^t L_{t-}^{\phi} \frac{1}{\phi(X_{s-})} dM_s^{[\phi]}.$$

We note that the function ϕ belongs to $\mathcal{D}(\mathcal{E})$ by [2, Proposition 5.2, Theorem 6.1]. The transformation by L_t^{ϕ}, $\phi \in \mathcal{D}(\mathcal{E})$, was thoroughly studied in [4]. For example, the next theorem is a consequence of [4, Theorem 2.6, Theorem 2.8].

Theorem 3.1 *$\mathbb{M}^{\phi}$ is a $\phi^2 m$-symmetric ergodic process.*

Remark 3.2 For $\phi = R_\alpha^\mu g > 0$, $g \in \mathcal{B}_b^+(X)$, the operator $\mathcal{H}^\mu\phi$ is defined in the same way as (3.1). Since Lemma 3.3 holds for this ϕ, the MF L_t^{ϕ} defined by (3.6) satisfies that $L_0^{\phi} = 1$ and $\mathbb{E}_x(L_t^{\phi}) \leq 1$.

4 A Large Deviation Principle

In this section we will prove the main theorem. As mentioned in the Introduction, Theorem 1.1(i) was proved in [12, 21]. For the completeness, we will give a sketch of the proof. For the proof of Theorem 1.1(ii), a new definition of I-function is essential. After the definition we can prove it by the similar argument as in [10, 20].

Let $\mathcal{P}$ the set of probability measures on X equipped with the weak topology. Define the function I^μ on $\mathcal{P}$ by

$$I^\mu(\nu) = \begin{cases} \mathcal{E}^\mu(\sqrt{f}, \sqrt{f}) & \text{if } \nu = f \cdot m, \sqrt{f} \in \mathcal{D}(\mathcal{E}), \\ \infty & \text{otherwise.} \end{cases}$$

For $t < \zeta(\omega)$, let

$$L_t(\omega)(A) = \frac{1}{t} \int_0^t 1_A\big(X_s(\omega)\big)\, ds, \quad A \in \mathcal{B}(X).$$

Proposition 4.1 *Let $\mu \in \mathcal{K}_\infty$. Then for each open set $G \in \mathcal{P}$*

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x\big(e^{A_t^\mu};\ L_t \in G, t < \zeta\big) \geq -\inf_{\nu\in G} I^\mu(\nu). \tag{4.1}$$

Proof Let $\phi = R_\alpha^\mu f \in \mathcal{D}_+(\mathcal{H}^\mu)$ and $\phi^2 \cdot m \in G$. Let $L_t^\phi := L_t^{\phi_0}$ be the MF defined by (3.6) and denote by $\mathbb{M}^\phi = (\Omega, X_t, P_x^\phi)$ the transformed process of the Hunt process $\mathbb{M}$ by L^ϕ. We have this proposition by exactly the same argument as in [21].

Set

$$\Omega_1 = \left\{\omega \in \Omega : \lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{\mathcal{H}^\mu\phi}{\phi}\big(X_s(\omega)\big)\,ds = \int_X \phi\mathcal{H}^\mu\phi\,dm\right\},$$

$$\Omega_2 = \big\{\omega \in \Omega : L_t(\omega) \text{ converges to } \phi^2 m\big\}.$$

We then know from Theorem 3.1 that for $i = 1, 2$, $\mathbb{P}_x^\phi(\Omega_i) = 1$ $\phi^2 m$-a.s., so that $\mathbb{P}_x^\phi(\Omega_i) = 1$ for any $x \in X$ on account of the shift invariance of Ω_i and the absolute continuity of the transition probability of $\mathbb{M}^\phi$. Hence,

$$\mathbb{P}_x^\phi\big(S(t,\epsilon)\big) \longrightarrow 1 \quad t \to \infty \text{ for } \forall x \in X,$$

where

$$S(t,\epsilon) = \left\{\omega \in \Omega : \left|\int_X \frac{\mathcal{H}^\mu\phi}{\phi}(x)L_t(\omega, dx) - \int_X \phi\mathcal{H}^\mu\phi\,dm\right| < \epsilon, L_t(\omega) \in G\right\}.$$

Since

$$\begin{aligned}
&\mathbb{E}_x\big(e^{A_t^\mu}; L_t \in G, t < \zeta\big)\\
&= \mathbb{E}_x^\phi\big(L_t^{\phi^{-1}} e^{A_t^\mu}; L_t \in G, t < \zeta\big)\\
&\geq \exp\left(t\left(\int_X \phi\mathcal{H}^\mu\phi\,dm - \epsilon\right)\right)\mathbb{E}_x^\phi\left(\frac{\phi(X_0)}{\phi(X_t)}; S(t,\epsilon)\right)\\
&\geq \exp\left(t\left(\int_X \phi\mathcal{H}^\mu\phi\,dm - \epsilon\right)\right)\frac{\phi(x)}{\|\phi\|_\infty}\big(1 - \mathbb{P}_x^\phi(\Omega - S(t,\epsilon)\big),
\end{aligned}$$

we have

$$\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}_x\big(e^{A_t^\mu}; L_t \in G, t < \zeta\big) \geq \int_X \phi\mathcal{H}^\mu\phi\,dm - \epsilon.$$

Noting that the set $\{\phi \in \mathcal{D}_+(\mathcal{H}^\mu) : \|\phi\|_2 = 1\}$ is dense in the set $\{\phi \in \mathcal{D}(\mathcal{E}) : \phi \geq 0, \|\phi\|_2 = 1\}$ with respect to $\mathcal{E}_{\alpha_0}^\mu$ $(\alpha_0 > \kappa(\mu))$, we arrive at the theorem. □

Lemma 4.1 *Let $\mu \in \mathcal{K}_\infty$. If $\lambda(\mu) > 1$, then R_1^μ also satisfies the assumption* (III); *for any $\epsilon > 0$ there exists a compact set K_ϵ such that* $\sup_{x\in X} R_1^\mu 1_{K_\epsilon^c}(x) \leq \epsilon$.

Proof By Corollary 2.1, $\sup_{x\in X}\mathbb{E}_x(e^{(1+\epsilon)A_\zeta^\mu}) < \infty$ for small ϵ. Since

$$R_1^\mu 1_{K^c}(x) = \int_0^\infty e^{-t}\mathbb{E}_x\big(e^{A_t^\mu}1_{K^c}(X_t)\big)\,dt$$

$$\leq \left(\int_0^\infty e^{-t} \mathbb{E}_x \left(e^{(1+\epsilon)A_t^\mu} \right) dt \right)^{1/(1+\epsilon)} \cdot R_1 1_{K^c}(x)^{\epsilon/(1+\epsilon)}$$

$$\leq \left(\sup_{x \in X} \mathbb{E}_x \left(e^{(1+\epsilon)A_\zeta^\mu} \right) \right)^{1/(1+\epsilon)} \cdot R_1 1_{K^c}(x)^{\epsilon/(1+\epsilon)},$$

the proof of this lemma is completed. □

For $\mu \in \mathcal{K}_\infty$ with $\lambda(\mu) > 1$, let h be the gauge function of μ and put

$$\phi_\epsilon = \phi + \epsilon h, \quad \phi \in \mathcal{D}_+(\mathcal{H}^\mu), \epsilon > 0.$$

We define the function on $\mathcal{P}$ by

$$I(\nu) = - \inf_{\substack{\phi \in \mathcal{D}_+(\mathcal{H}^\mu) \\ \epsilon > 0}} \int_X \frac{\mathcal{H}^\mu \phi}{\phi_\epsilon} \, d\nu. \tag{4.2}$$

Proposition 4.2 *Let $\mu \in \mathcal{K}_\infty$ with $\lambda(\mu) > 1$. Then, for each closed subset K of $\mathcal{P}$*

$$\limsup_{t \to \infty} \frac{1}{t} \log \sup_{x \in X} \mathbb{E}_x \left(e^{A_t^\mu}; L_t \in K, t < \zeta \right) \leq - \inf_{\nu \in K} I(\nu). \tag{4.3}$$

Proof For $\phi \in \mathcal{D}_+(\mathcal{H}^\mu)$, let $L_t^{\phi_\epsilon}$ be the MF defined in (3.6). Then, since $L_t^{\phi_\epsilon}$ is a local martingale with $L_0^{\phi_\epsilon} = 1$,

$$\mathbb{E}_x \left(e^{A_t^\mu} \cdot \frac{\phi_\epsilon(X_t)}{\phi_\epsilon(X_0)} \exp\left(- \int_0^t \frac{\mathcal{H}^\mu \phi}{\phi_\epsilon}(X_s) \, ds \right); t < \zeta \right) \leq 1, \tag{4.4}$$

and thus

$$\sup_{x \in X} \mathbb{E}_x \left(\exp\left(A_t^\mu - \int_0^t \frac{\mathcal{H}^\mu \phi}{\phi_\epsilon}(X_s) \, ds; t < \zeta \right) \right) \leq \frac{\|\phi\|_\infty + \epsilon \|h\|_\infty}{\epsilon}.$$

Furthermore, for any Borel set C of $\mathcal{P}$

$$\mathbb{E}_x \left(\exp\left(A_t^\mu - \int_0^t \frac{\mathcal{H}^\mu \phi}{\phi_\epsilon}(X_s) \, ds \right); L_t \in C, t < \zeta \right)$$

$$= \mathbb{E}_x \left(e^{A_t^\mu} \cdot \exp\left(-t \int_X \frac{\mathcal{H}^\mu \phi}{\phi_\epsilon}(x) L_t(dx) \right); L_t \in C, t < \zeta \right)$$

$$\geq \mathbb{E}_x \left(e^{A_t^\mu}; L_t \in C, t < \zeta \right) \cdot \exp\left(-t \cdot \sup_{\nu \in C} \int_X \frac{\mathcal{H}^\mu \phi}{\phi_\epsilon}(x) \, d\nu(dx) \right).$$

Hence

$$\sup_{x \in X} E_x \left(e^{A_t^\mu}; L_t \in C, t < \zeta \right)$$

$$\leq \left(\frac{\|\phi\|_\infty + \epsilon \|h\|_\infty}{\epsilon} \right) \exp\left(t \cdot \sup_{\nu \in C} \int_X \frac{\mathcal{H}^\mu \phi}{\phi_\epsilon}(x)\, d\nu(dx) \right)$$

and thus

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{x \in X} \mathbb{E}_x \left(e^{A_t^\mu}; L_t \in C, t < \zeta \right) \leq \inf_{\substack{\phi \in \mathcal{D}_+(\mathcal{H}^\mu) \\ \epsilon > 0}} \sup_{\nu \in C} \int_X \frac{\mathcal{H}^\mu \phi}{\phi_\epsilon}\, d\nu. \tag{4.5}$$

To derive (4.3) from (4.5), we have only to imitate the argument in [9]. Indeed, let K be a compact set of $\mathcal{P}$ and set

$$\ell = \sup_{\nu \in K} \inf_{\substack{\phi \in \mathcal{D}_+(\mathcal{H}^\mu) \\ \epsilon > 0}} \int_X \frac{\mathcal{H}^\mu \phi}{\phi_\epsilon}\, d\nu.$$

Then, given $\delta > 0$, for every $\nu \in K$ there exist $\phi_\nu \in \mathcal{D}_+(\mathcal{H}^\mu)$ and $\epsilon_\nu > 0$ such that

$$\int_X \frac{\mathcal{H}^\mu \phi_\nu}{\phi_\nu + \epsilon_\nu h}\, d\nu \leq \ell + \delta.$$

The function $\frac{\mathcal{H}^\mu \phi_\nu}{\phi_\nu + \epsilon_\nu h}$ is bounded and continuous on X, so that there exists a neighborhood $N(\nu)$ of ν such that

$$\int_X \frac{\mathcal{H}^\mu \phi_\nu}{\phi_\nu + \epsilon_\nu h} d\lambda \leq \ell + 2\delta \quad \text{for } \lambda \in N(\nu).$$

Since $\{N(\nu)\}_{\nu \in K}$ is an open covering of K, there exist $\nu_1, \ldots, \nu_k$ in K such that $K \subset \bigcup_{j=1}^k N(\nu_j)$. Put $N_j = N(\nu_j)$. We then have for $1 \leq j \leq k$

$$\sup_{\nu \in N_j} \int_X \frac{\mathcal{H}^\mu \phi_{\nu_j}}{\phi_{\nu_j} + \epsilon_{\nu_j} h}\, d\nu \leq \ell + 2\delta,$$

and thus

$$\max_{1 \leq j \leq k} \inf_{\substack{\phi \in \mathcal{D}_+(\mathcal{H}^\mu) \\ \epsilon > 0}} \sup_{\nu \in N_j} \int_X \frac{\mathcal{H}^\mu \phi}{\phi_\epsilon}\, d\nu \leq \ell + 2\delta.$$

Therefore, by (4.5)

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x \left(e^{A_t^\mu}; L_t \in K, t < \zeta \right) \\ &\quad \leq \max_{1 \leq j \leq k} \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x \left(e^{A_t^\mu}; L_t \in N_j, t < \zeta \right) \\ &\quad \leq \max_{1 \leq j \leq k} \inf_{\substack{\phi \in \mathcal{D}_+(\mathcal{H}^\mu) \\ \epsilon > 0}} \sup_{\mu \in N_j} \int_X \frac{\mathcal{H}^\mu \phi}{\phi_\epsilon}\, d\nu \leq \ell + 2\delta. \end{aligned} \tag{4.6}$$

Since δ is arbitrary, (4.3) holds for each compact set.

To prove (4.3) for each closed set, we follow the argument in [11, Lemma 7.1]. We know from Lemma 4.1 that for $\epsilon > 0$, there exists a compact set K_ϵ such that $\sup_{x\in X} R_1^\mu 1_{K_\epsilon^c}(x) \le \epsilon$. Put

$$V_\epsilon(x) = -\frac{\mathcal{H}^\mu R_1^\mu 1_{K_\epsilon^c}(x)}{R_1^\mu 1_{K_\epsilon^c}(x) + \epsilon h(x)} = \frac{1_{K_\epsilon^c}(x) - R_1^\mu 1_{K_\epsilon^c}(x)}{R_1^\mu 1_{K_\epsilon^c}(x) + \epsilon h(x)}$$

and define the measure $Q_{x,t}$ on $\mathcal{P}$ by

$$Q_{x,t}(C) = \mathbb{E}_x\big(e^{A_t^\mu}; L_t \in C, t < \zeta\big), \quad C \in \mathcal{B}(\mathcal{P}).$$

We then see from Remark 3.2 that

$$\int_{\mathcal{P}} \exp\left(t \int_X V_\epsilon(x)\nu(dx)\right) Q_{x,t} \le \frac{R_1^\mu 1_{K_\epsilon^c} + \epsilon h}{\epsilon} \le \frac{\epsilon + C_h\epsilon}{\epsilon} = 1 + C_h, \tag{4.7}$$

where $C_h = \sup_{x\in X} h(x)$. If $0 < \epsilon \le 1/(2 + C_h)$, then for $x \in K_\epsilon$ the function $V_\epsilon(x)$ is negative and for $x \in K_\epsilon^c$

$$V_\epsilon(x) \ge \frac{1-\epsilon}{\epsilon + C_h\epsilon} \ge \frac{1 - 1/(2 + C_h)}{\epsilon + C_h\epsilon} = \frac{1}{(2 + C_h)\epsilon}.$$

Hence the set K_ϵ^c is written as

$$K_\epsilon^c = \left\{x \in X : \ V_\epsilon(x) \ge \frac{1}{(2 + C_h)\epsilon}\right\}.$$

Since $V_\epsilon(x) > -1$, we have

$$\begin{aligned} &\int_{\mathcal{P}} \exp\left(t \int_X V_\epsilon(x)\nu(dx)\right) dQ_{x,t} \\ &= \int_{\mathcal{P}} \exp\left(t \int_{K_\epsilon^c} V_\epsilon(x)\nu(dx) + t \int_{K_\epsilon} V_\epsilon(x)\nu(dx)\right) dQ_{x,t} \\ &\ge \int_{\mathcal{P}} \exp\left(\frac{t}{(2 + C_h)\epsilon} \nu(K_\epsilon^c) - t\right) dQ_{x,t}. \end{aligned} \tag{4.8}$$

Let

$$\mathcal{M}_\epsilon^\delta = \big\{\nu \in \mathcal{P} : \nu\big(K_\epsilon^c\big) > \delta\big\}.$$

Then it follows from (4.7) and (4.8) that for $0 < \epsilon \le 1/(2 + C_h)$

$$Q_{x,t}\big(\mathcal{M}_\epsilon^\delta\big) \le (1 + C_h) \cdot \exp\left(t - \frac{t\delta}{(2 + C_h)\epsilon}\right).$$

For any $\lambda > 2 + C_h$, set $J_\lambda = \bigcup_{n=1}^{\infty} \mathcal{M}^{\frac{2+C_h}{n}}_{\frac{1}{\lambda n^2}}$. Then

$$Q_{x,t}(J_\lambda) \le \sum_{n=1}^{\infty} Q_{x,t}\Big(\mathcal{M}^{\frac{2+C_h}{n}}_{\frac{1}{\lambda n^2}}\Big) = \sum_{n=1}^{\infty}(1+C_h)e^{(t-t\lambda n)}$$

$$= (1+C_h)\cdot \frac{e^{(1-\lambda)t}}{1-e^{-\lambda t}},$$

and thus

$$\limsup_{t\to\infty}\frac{1}{t}\log\sup_{x\in X} Q_{x,t}(J_\lambda) \le 1-\lambda.$$

We see by definition that the set J_λ^c is tight and closed with respect to the weak topology, that is, a compact subset of $\mathcal{P}$. Hence for each closed subset K

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\log\sup_{x\in X} Q_{x,t}(K) \\
&\quad\le \Big(\limsup_{t\to\infty}\frac{1}{t}\log\sup_{x\in X} Q_{x,t}\big(K\cap J_\lambda^c\big)\Big) \vee \Big(\limsup_{t\to\infty}\frac{1}{t}\log\sup_{x\in X} Q_{x,t}(K\cap J_\lambda)\Big) \\
&\quad\le \Big(-\inf_{\nu\in K\cap J_\lambda^c} I^\mu(\nu)\Big)\vee(1-\lambda) \le \Big(-\inf_{\nu\in K} I^\mu(\nu)\Big)\vee(1-\lambda).
\end{aligned}$$

The proof is completed by letting λ to ∞. □

Denote by $\mathcal{B}_b^+(X)$ the set of non-negative bounded Borel functions on X. Let us define a function on $\mathcal{P}$ by

$$I_\alpha(\nu) = -\inf_{\substack{u\in\mathcal{B}_b^+(X)\\ \epsilon>0}} \int_X \log\bigg(\frac{\alpha R_\alpha^\mu u + \epsilon h}{u+\epsilon h}\bigg)\,d\mu. \tag{4.9}$$

Lemma 4.2 *It holds that*

$$I_\alpha(\nu) \le \frac{I(\nu)}{\alpha}, \quad \nu\in\mathcal{P}.$$

Proof For $u = R_\alpha^\mu f \in \mathcal{D}_+(\mathcal{H}^\mu)$ and $\epsilon > 0$, set

$$\phi(\alpha) = -\int_X \log\bigg(\frac{\alpha R_\alpha^\mu u + \epsilon h}{u+\epsilon h}\bigg)\,d\nu.$$

Then, noting that $\frac{d}{d\alpha}(R_\alpha^\mu u) = -(R_\alpha^\mu)^2 u$, we have

$$\frac{d\phi}{d\alpha}(\alpha) = -\int_X \frac{R_\alpha^\mu u - \alpha(R_\alpha^\mu)^2 u}{\alpha R_\alpha^\mu u + \epsilon h}\,d\nu = \int_X \frac{\mathcal{H}^\mu (R_\alpha^\mu)^2 u}{\alpha R_\alpha^\mu u + \epsilon h}\,d\nu. \tag{4.10}$$

Since

$$\big(\alpha\big(R^\mu_\alpha\big)^2 u - R^\mu_\alpha u\big)\big(\alpha^2\big(R^\mu_\alpha\big)^2 u + \epsilon h\big) - \big(\alpha\big(R^\mu_\alpha\big)^2 u - R^\mu_\alpha u\big)\big(\alpha R^\mu_\alpha u + \epsilon h\big)$$

equals $\alpha(\alpha(R^\mu_\alpha)^2 u - R^\mu_\alpha u)^2 \geq 0$, we have

$$\frac{\alpha(R^\mu_\alpha)^2 u - R^\mu_\alpha u}{\alpha R^\mu_\alpha u + \epsilon h} \geq \frac{\alpha(R^\mu_\alpha)^2 u - R^\mu_\alpha u}{\alpha^2(R^\mu_\alpha)^2 u + \epsilon h},$$

and thus

$$\int_X \frac{\mathcal{H}^\mu (R^\mu_\alpha)^2 u}{\alpha R^\mu_\alpha u + \epsilon h}\, d\nu \geq \int_X \frac{\mathcal{H}^\mu (R^\mu_\alpha)^2 u}{\alpha^2(R^\mu_\alpha)^2 u + \epsilon h}\, d\nu$$

$$= -\frac{1}{\alpha^2}\left(-\int_X \frac{\mathcal{H}^\mu (R^\mu_\alpha)^2 u}{(R^\mu_\alpha)^2 u + \frac{\epsilon}{\alpha^2} h}\, d\nu\right) \geq -\frac{1}{\alpha^2} I(\nu).$$

Therefore

$$\phi(\infty) - \phi(\alpha) = \int_X \log\left(\frac{\alpha R^\mu_\alpha u + \epsilon h}{u + \epsilon h}\right) d\nu \geq -\frac{I(\nu)}{\alpha},$$

which implies

$$-\inf_{\substack{u \in \mathcal{D}_+(\mathcal{H}^\mu)\\ \epsilon > 0}} \int_X \log\left(\frac{\alpha R^\mu_\alpha u + \epsilon h}{u + \epsilon h}\right) d\nu \leq \frac{I(\nu)}{\alpha}.$$

Since by Theorem 2.1(i) and Remark 3.1, $\|\beta R^\mu_\beta f\|_\infty \leq C\|f\|_\infty$, $\beta > 0$, and $\beta R^\mu_\beta f(x) \to f(x)$ as $\beta \to \infty$,

$$\int_X \log\left(\frac{\alpha R^\mu_\alpha(\beta R^\mu_\beta f) + \epsilon h}{\beta R^\mu_\beta f + \epsilon h}\right) d\mu \overset{\beta\to\infty}{\longrightarrow} \int_X \log\left(\frac{\alpha R^\mu_\alpha f + \epsilon h}{f + \epsilon h}\right) d\nu. \tag{4.11}$$

Define the measure ν_α by

$$\nu_\alpha(A) = \int_X \alpha R^\mu_\alpha(x, A)\, d\nu(x) \quad A \in \mathcal{B}(X).$$

Given $v \in \mathcal{B}^+_b(X)$, take a sequence $\{g_n\}_{n=1}^\infty \subset C^+_b(X) \cap L^2(X; m)$ such that

$$\int_X |v - g_n|\, d(\nu_\alpha + \nu) \longrightarrow 0 \quad \text{as } n \to \infty.$$

We then have

$$\int_X \big|\alpha R^\mu_\alpha v - \alpha R^\mu_\alpha g_n\big|\, d\nu \leq \int_X \alpha R^\mu_\alpha\big(|v - g_n|\big)\, d\nu = \int_X |v - g_n|\, d\nu_\alpha \longrightarrow 0$$

as $n \longrightarrow \infty$, and so

$$\int_X \log\left(\frac{\alpha R^\mu_\alpha g_n + \epsilon h}{g_n + \epsilon h}\right) d\nu \overset{n\to\infty}{\longrightarrow} \int_X \log\left(\frac{\alpha R^\mu_\alpha v + \epsilon h}{v + \epsilon h}\right) d\nu. \tag{4.12}$$

Hence, combining (4.11) and (4.12) we have

$$\inf_{u\in\mathcal{D}_+(\mathcal{H}^\mu)}\int_X \log\left(\frac{\alpha R_\alpha^\mu u+\epsilon h}{u+\epsilon h}\right)d\nu = \inf_{u\in\mathcal{B}_b^+}\int_X \log\left(\frac{\alpha R_\alpha^\mu u+\epsilon h}{u+\epsilon h}\right)d\nu,$$

which implies the lemma. □

Lemma 4.3 *If $I(\nu)<\infty$, then ν is absolutely continuous with respect to m.*

Proof By the similar argument in the proof of [10, Lemma 4.1], we obtain this lemma. Indeed, for $a>0$ and $A\in\mathcal{B}(X)$, set $u(x)=a1_A(x)+1\in\mathcal{B}_b^+(X)$, where 1_A is the indicator function the set A. Then

$$\int_X \log\left(\frac{\alpha R_\alpha^\mu u+\epsilon h}{u+\epsilon h}\right)d\nu = \int_X \log\left(\frac{a\alpha R_\alpha^\mu(x,A)+\alpha R_\alpha^\mu(x,X)+\epsilon h}{a1_A(x)+1+\epsilon h}\right)d\nu.$$

Define the measure ν_α as in the proof of Lemma 4.2. Put

$$c_\alpha=\int_X \alpha R_\alpha^\mu(x,X)\,d\nu(x)\big(=\nu_\alpha(X)\big),\quad k=\int_X h\,d\nu.$$

Noting that $h\ge 1$, we see from Lemma 4.2 and Jensen's inequality that

$$\log\big(a\nu_\alpha(A)+c_\alpha+k\epsilon\big)\ge \nu(A)\log(a+1+\epsilon)+\nu\big(A^c\big)\log(1+\epsilon)-I(\nu)/\alpha,$$

and by letting $\epsilon\to 0$

$$\log\big(a\nu_\alpha(A)+c_\alpha\big)\ge \nu(A)\log(a+1)-I(\nu)/\alpha.$$

Since $\log x\le x-1$ for $x>0$, we have

$$a\nu_\alpha(A)+c_\alpha-1\ge \nu(A)\log(a+1)-I(\nu)/\alpha,$$

and so

$$\nu_\alpha(A)-\nu(A)\ge \frac{-I(\nu)/\alpha+\nu(A)(\log(a+1)-a)+1-c_\alpha}{a}.$$

Noting that $\log(a+1)-a<0$, we have

$$\nu_\alpha(A)-\nu(A)\ge \frac{-I(\nu)/\alpha+(\log(a+1)-a)+1-c_\alpha}{a}$$

for all $A\in\mathcal{B}(X)$ and

$$\begin{aligned}\nu(A)-\nu_\alpha(A)&=1-c_\alpha+\big(\nu_\alpha\big(A^c\big)-\nu\big(A^c\big)\big)\\ &\ge \frac{-I(\nu)/\alpha+(\log(a+1)-a)+(1-c_\alpha)(a+1)}{a}\end{aligned}$$

for all $A \in \mathcal{B}(X)$. Therefore we can conclude that

$$\sup_{A\in\mathcal{B}(X)} \left|\nu(A) - \nu_\alpha(A)\right| \le \frac{a - \log(a+1) + I(\nu)/\alpha + (1-c_\alpha)(a+1)}{a}.$$

Note that $c_\alpha \to 1$ as $\alpha \to \infty$. Then since

$$\limsup_{\alpha\to\infty} \sup_{A\in\mathcal{B}(X)} \left|\nu(A) - \nu_\alpha(A)\right| \le \frac{a - \log(a+1)}{a}$$

and the right-hand side converges to 0 as $a \to 0$, the lemma follows from Remark 2.1. □

Proposition 4.3 *It holds that for $\nu \in \mathcal{P}$*

$$I(\nu) = I^\mu(\nu).$$

Proof We follow the argument in the proof of [10, Theorem 5]. Suppose that $I(\nu) = \ell < \infty$. By Lemma 4.4, ν is absolutely continuous with respect to m. Let us denote by f its density and let $f^n = \sqrt{f} \wedge n$. Since $\log(1-x) \le -x$ for $-\infty < x < 1$ and

$$-\infty < \frac{f^n - \alpha R_\alpha^\mu f^n}{f^n + \epsilon h} < 1,$$

$$\begin{aligned}\int_X \log\left(\frac{\alpha R_\alpha^\mu f^n + \epsilon h}{f^n + \epsilon h}\right) f\,dm &= \int_X \log\left(1 - \frac{f^n - \alpha R_\alpha^\mu f^n}{f^n + \epsilon h}\right) f\,dm \\ &\le -\int_X \frac{f^n - \alpha R_\alpha^\mu f^n}{f^n + \epsilon h} f\,dm,\end{aligned}$$

so

$$\int_X \frac{f^n - \alpha R_\alpha^\mu f^n}{f^n + \epsilon h} f\,dm \le I_\alpha(f\cdot m).$$

By letting $n \to \infty$ and $\epsilon \to 0$,

$$\int_X \sqrt{f}\big(\sqrt{f} - \alpha R_\alpha^\mu \sqrt{f}\big)\,dm \le I_\alpha(f\cdot m) \le \frac{I(f\cdot m)}{\alpha},$$

which implies that $\sqrt{f} \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}^\mu(\sqrt{f}, \sqrt{f}) \le I(f\cdot m)$.

Let $\phi \in \mathcal{D}_+(\mathcal{H}^\mu)$ and define the semigroup P_t^ϕ by

$$P_t^\phi f(x) = \mathbb{E}_x\left(e^{A_t^\mu} \cdot \frac{\phi_\epsilon(X_t)}{\phi_\epsilon(X_0)} \exp\left(-\int_0^t \frac{\mathcal{H}^\mu \phi}{\phi_\epsilon}(X_s)\,ds\right) f(X_t)\right).$$

Then, P_t^ϕ is $(\phi + \epsilon h)^2 m$-symmetric and satisfies $P_t^\phi 1 \le 1$ by virtue of (4.2). Given $\nu = f\cdot m \in \mathcal{F}_1$ with $\sqrt{f} \in \mathcal{D}(\mathcal{E})$, set

$$S_t^\phi \sqrt{f}(x) = \mathbb{E}_x\left(e^{A_t^\mu} \cdot \exp\left(-\int_0^t \frac{\mathcal{H}^\mu \phi}{\phi_\epsilon}(X_s)\,ds\right)\sqrt{f}(X_t)\right).$$

Then

$$\begin{aligned}\int_X (S_t^\phi \sqrt{f})^2\, dm &= \int_X \phi_\epsilon^2 \left(P_t^\phi \left(\frac{\sqrt{f}}{\phi_\epsilon} \right) \right)^2 dm \\ &\le \int_X \phi_\epsilon^2 P_t^\phi \left(\left(\frac{\sqrt{f}}{\phi_\epsilon} \right)^2 \right) dm \\ &\le \int_X \phi_\epsilon^2 \left(\frac{\sqrt{f}}{\phi_\epsilon} \right)^2 dm = \int_X f\, dm.\end{aligned}$$

Hence

$$0 \le \lim_{t\to 0} \frac{1}{t} \big(\sqrt{f} - S_t^\phi \sqrt{f}, \sqrt{f}\big)_m = \mathcal{E}^\mu \big(\sqrt{f}, \sqrt{f}\big) + \int_X \frac{\mathcal{H}^\mu \phi}{\phi_\epsilon} f\, dm,$$

and thus $\mathcal{E}^\mu(\sqrt{f}, \sqrt{f}) \ge I(f \cdot m)$. □

We now remark that by considering the β-subprocess of $\mathbb{M}$, we can assume without loss of the generality that $\mathbb{M}$ is transient and μ is gaugeable. Here the β-subprocess of $\mathbb{M}$ is the m-symmetric Markov process with transition probability $e^{-\beta t} p_t(x, y) m(dy)$, $\beta > 0$. Let us denote by $\mathbb{M}^\beta = (\Omega, \mathbb{P}_x^\beta, X_t, \zeta)$ the subprocess. Then clearly the subprocess $\mathbb{M}^\beta$ also fulfills the assumptions (I), II) and (III). The Dirichlet form generated by $\mathbb{M}^\beta$ is identical to $\mathcal{E}_\beta$ $(:= \mathcal{E} + \beta(\, , \,)_m)$. Let $\mathcal{K}_\infty^\beta$ be the set of Green-tight measures defined by using the β-resolvent density $R_\beta(x, y)$ in place of $R(x, y)$. According to the resolvent equation, the space $\mathcal{K}_\infty^\beta$ is independent of $\beta > 0$.

Lemma 4.4 *If $\mu \in \mathcal{K}_\infty^1$, then for large β*

$$\lambda_\beta(\mu) := \inf \left\{ \mathcal{E}_\beta(u, u) : \int_X u^2\, d\mu = 1 \right\} > 1.$$

Proof By (2.7) and (2.6),

$$\int_X u^2\, d\mu \le \|R_\beta \mu\|_\infty \cdot \mathcal{E}_\beta(u, u)$$

and $\lim_{\beta\to\infty} \|R_\beta \mu\|_\infty = 0$. Hence $\lambda_\beta(\mu) > 1$ for β large enough, and thus this lemma follows form Proposition 2.1. □

Combining Lemma 4.4 with Proposition 2.1, we see that each measure $\mu \in \mathcal{K}_\infty^1$ becomes gaugeable with respect to the β-subprocess for large β. We define $I^{\mu,\beta}$ in the same manner as I^μ by using $\mathcal{E}_\beta$. By applying Propositions 4.1 and 4.2 to the β-subprocess, we can prove Theorem 1.1 for the subprocess. Since

$$\inf_{\nu\in G} I^{\mu,\beta}(\nu) = \inf_{\nu\in G} I^\mu(\nu) - \beta$$

and

$$\mathbb{E}_x^{\beta}\left(e^{A_t^{\mu}}; L_t \in G, t < \zeta\right) = e^{-\beta t} \cdot \mathbb{E}_x\left(e^{A_t^{\mu}}; L_t \in G, t < \zeta\right),$$

Theorem 1.1 for the subprocess yields that for the original Markov process.

Remark 4.1 We see that the generalized Schrödinger operator $\mathcal{H}^{\mu}$ admits the ground state. Indeed, put

$$\rho_2(\mu) = \inf\left\{\mathcal{E}^{\mu}(u,u) : u \in \mathcal{D}(\mathcal{E}), \int_X u^2\, d\mu = 1\right\},$$

and let $\{u_n\}$ be a minimizing sequence of $\mathcal{D}(\mathcal{E})$, i.e., $\int_X u_n^2\, d\mu = 1$ and $\rho_2(\mu) = \lim_{n\to\infty} \mathcal{E}^{\mu}(u_n, u_n)$. By (2.7),

$$\int_X u_n^2\, d\mu \le \|R_\alpha \mu\|_\infty \cdot \left(\mathcal{E}(u_n,u_n) + \alpha\right)$$

and $\|R_\alpha \mu\|_\infty < 1$ for large α because $\mu \in \mathcal{K}$. Hence

$$\mathcal{E}(u_n,u_n) \le \frac{\sup_n \mathcal{E}^{\mu}(u_n,u_n) + \alpha \|R_\alpha \mu\|_\infty}{1 - \|R_\alpha \mu\|_\infty} < \infty.$$

We then see from the assumption (III) that for any $\epsilon > 0$ there exists a compact set K such that

$$\sup_n \int_{K^c} u_n^2\, dm \le \|R_1 I_{K^c}\|_\infty \cdot \left(\sup_n \mathcal{E}(u_n,u_n) + \alpha\right) < \epsilon,$$

that is, the subset $\{u_n^2 \cdot m\}$ of $\mathcal{P}(X)$ is tight. Hence a subsequence $\{u_{n_k}^2 \cdot m\}$ weakly converges to a probability measure ν. Moreover, it follows from Proposition 4.3 that the function I^{μ} is lower semi-continuous with respect to the weak topology. Hence

$$I^{\mu}(\nu) \le \liminf_{k\to\infty} I^{\mu}\left(u_{n_k}^2 \cdot m\right) = \liminf_{k\to\infty} \mathcal{E}^{\mu}(u_{n_k}, u_{n_k}) < \infty$$

and the probability measure ν is expressed by $\nu = u_0^2 \cdot m$, $u_0 \in \mathcal{D}(\mathcal{E})$. We now conclude that u_0 is the ground state, $\lambda_2(\mu) = \mathcal{E}^{\mu}(u_0, u_0)$. The uniqueness of the ground state is derived from the irreducibility (I).

Remark 4.2 Let μ be a signed Radon measure whose positive part μ^+ is in $\mathcal{K}_\infty$ and negative part μ^- is in $\mathcal{K}$. Let $\mathbb{M}^{\mu^-}$ be the subprocess by the MF $\exp(-A_t^{\mu^-})$. Then Theorem 2.1(iii) says that the process $\mathbb{M}^{\mu^-}$ satisfies (I), (II) and (III). Applying the results above to $\mathbb{M}^{\mu^-}$, we establish Theorem 1.1 for $\mu = \mu^+ - \mu^- \in \mathcal{K}_\infty - \mathcal{K}$.

The next corollary is a consequence of Theorem 1.1 with $G = K = \mathcal{P}$.

Corollary 4.1 *For* $\mu \in \mathcal{K}_\infty^1$

$$-\lim_{t\to\infty} \frac{1}{t} \log \sup_{x\in X} \mathbb{E}_x\left(e^{A_t^{\mu}}; t < \zeta\right)$$

$$= \inf\left\{\mathcal{E}^\mu(u,u) : u \in \mathcal{D}(\mathcal{E}), \int_X u^2\,dm = 1\right\}. \tag{4.13}$$

Let $\|p_t^\mu\|_{p,p}$ be the operator norm of p_t^μ from $L^p(X;m)$ to $L^p(X;m)$ and define the L^p-spectral bound by

$$\rho_p(\mu) = -\lim_{t\to\infty} \frac{1}{t}\log \|p_t^\mu\|_{p,p}, \quad 1 \le p \le \infty.$$

Note that $\|p_t^\mu\|_{\infty,\infty} = \sup_{x\in X} \mathbb{E}_x(\exp(A_t^\mu); t < \zeta)$ and $\rho_2(\mu)$ equals the right-hand side of (4.13) by the spectral theorem. Then Corollary 4.1 yields

$$\rho_\infty(\mu) = \rho_2(\mu). \tag{4.14}$$

By the symmetry and positivity of p_t^μ,

$$\|p_t^\mu\|_{2,2} \le \|p_t^\mu\|_{p,p} \le \|p_t^\mu\|_{\infty,\infty}, \quad 1 < p < \infty. \tag{4.15}$$

Hence the next theorem is an immediate consequence of (4.14) and the Riesz–Thorin interpolation theorem.

Theorem 4.1 *Let $\mu \in \mathcal{K}_\infty^1$. Then under the assumptions* (I), (II) *and* (III), *the spectral bound $\rho_p(\mu)$, $1 \le p \le \infty$, is independent of p.*

Remark 4.3 The inequality (4.15) says that $\rho_2(\mu) \ge \rho_\infty(\mu)$. Hence the uniform upper bound in Theorem 1.1(ii) with $K = \mathcal{P}$ is essential for the proof of the L^p-independence.

5 One-Dimensional Diffusion Processes

In order to illustrate the power of our main Theorem 1.1, we consider one-dimensional diffusion process and obtain a necessary and sufficient condition for L^p-independence of their diffusion semigroups in terms of speed measures and scale functions. To this end we need to check the assumption (III). Let $I = (r_1, r_2)$, $-\infty \le r_1 < 0 < r_2 \le \infty$. Let s be strictly increasing continuous function on I and m a strictly increasing function on I. We define

$$D_m u(x) = \lim_{h\to 0} \frac{u(x+h) - u(x)}{m(x+h) - m(x)}, \qquad D_s^+ u(x) = \lim_{h\downarrow 0} \frac{u(x+h) - u(x)}{s(x+h) - s(x)},$$

if the limits exist. Let us recall Feller's boundary classification. Put

$$\rho = \int_0^{r_i}\left(\int_0^y dm(x)\right) ds(y), \qquad \sigma = \int_0^{r_i}\left(\int_0^y ds(x)\right) dm(y)$$

($r_i = r_1$ or r_2). By the Feller's boundary classification, we call

$$r_i \text{ a regular boundary if } \rho < \infty, \sigma < \infty,$$

$$r_i \text{ an exit boundary if } \rho < \infty, \sigma = \infty,$$

$$r_i \text{ an entrance boundary if } \rho = \infty, \sigma < \infty,$$

$$r_i \text{ a natural boundary if } \rho = \infty, \sigma = \infty.$$

We denote $\mathbb{M} = (\mathbb{P}_x, X_t, \zeta)$ the minimal diffusion process generated by $D_m D_s^+$, that is, the Dirichlet boundary condition is imposed if r_i is a regular or exit boundary. The Dirichlet form generated by $\mathbb{M}$ is written as

$$\mathcal{E}(u, v) = -\int_{r_1}^{r_2} D_m D_s^+ u \cdot v \, dm = \int_{r_1}^{r_2} D_s^+ u(x) \cdot D_s^+ v(x) \, ds. \tag{5.1}$$

Let $u_1(x)$ (resp. $u_2(x)$) be a positive increasing (resp. decreasing) solution of the equation $(1 - D_m D_s^+)u = 0$ and W the Wronskian. We may assume that $W = 1$. Then $R_1(x, y)$ is written by

$$R_1(x, y) = \begin{cases} u_1(x)u_2(y), & r_1 < x \le y < r_2, \\ u_2(x)u_1(y), & r_1 < y \le x < r_2 \end{cases}$$

(e.g. [14, 5.14]).

Lemma 5.1 *Suppose that r_2 is regular, exit, or entrance. Then for any $\epsilon > 0$ there exists $0 < r < r_2$ such that*

$$\sup_{x \in I} R_1 1_{(r, r_2)}(x) < \epsilon.$$

Proof We know from [14, Theorem 5.14.1] that if r_2 is regular or exit, then

$$\lim_{x \to r_2} R_1 1(x) = 0.$$

Hence for any $\epsilon > 0$ there exists a constant $r > 0$ such that

$$\sup_{r_1 < x < r_2} R_1 1_{(r, r_2)}(x) = \sup_{r < x < r_2} R_1 1_{(r, r_2)}(x) \le \sup_{r < x < r_2} R_1 1(x) < \epsilon.$$

If r_2 is an entrance boundary, we see from [14, Theorem 5.14.1] that for a bounded Borel function g on I

$$\lim_{x \uparrow r_2} R_1 g(x) = u_2(r_2) \int_{r_1}^{r_2} g(x) u_1(x) \, dm(x),$$

where $u_2(r_2) = \lim_{x \uparrow r_2} u_2(x) < \infty$. Hence noting that the function $R_1 1_{(r, r_2)}(x)$ is increasing in x, we have for $0 < r < r_2$

$$\sup_{x \in I} R_1 1_{(r, r_2)}(x) = \lim_{x \uparrow r_2} R_1 1_{(r, r_2)}(x) = u_2(r_2) \int_{r+}^{r_2} u_1(x) \, dm(x)$$

and the left-hand side converges to 0 as $r \to r_2$. □

The next corollary follows from Theorem 1.1.

Corollary 5.1 *Assume that no boundaries are natural. Then for* $\mu \in \mathcal{K}^1_\infty$, $\rho_p(\mu)$ *is independent of* p.

Suppose that the boundary, say r_2, is natural. Then we can show by the same argument as in [24] that $\rho_p(\mu)$ is independent of p if and only if $\rho_2(\mu) \le 0$, while we supposed in [24] that the symmetric Markov process is conservative. Indeed, we extend the diffusion to $(r_1, r_2]$ by making the adjoined point r_2 a trap, that is, the transition probability $\bar{p}_t(x, dy)$ on $(r_1, r_2]$ defined by

$$\bar{p}_t(x, E) = p_t\big(x, E \setminus \{r_2\}\big), \quad x \in (r_1, r_2), E \in \mathcal{B}\big((r_1, r_2]\big)$$

and

$$\bar{p}_t(r_2, E) = \begin{cases} 1 & r_2 \in E, \\ 0 & r_2 \notin E. \end{cases}$$

We first suppose that r_1 is regular or exit. Let $\bar{\mathbb{M}} = (\bar{P}_x, X_t, \zeta)$ be the diffusion process on $(r_1, r_2]$ with transition probability $\bar{p}_t(x, dy)$. We regard r_1 as the infinity Δ of $\bar{\mathbb{M}}$. Furthermore, we take β large enough so that $\mu \in \mathcal{K}^1_\infty$ is gaugeable with respect to the β-subprocess of $\mathbb{M}$, and denote by $\bar{\mathbb{M}}^\beta = (\bar{P}^\beta_x, X_t)$ the β-subprocess of $\bar{\mathbb{M}}$. We will apply the facts shown in the previous section to the β-subprocess $\bar{\mathbb{M}}^\beta$. Let $\bar{p}^\mu_t$ and $\bar{R}^\mu_\beta$ be the semigroup and the resolvent of $\bar{\mathbb{M}}^\beta$: for $f \in \mathcal{B}_b((r_1, r_2])$

$$\bar{p}^\mu_t f(x) = \bar{\mathbb{E}}_x\big(e^{A^\mu_t} f(X_t); t < \zeta\big), \qquad \bar{R}^\mu_\beta f(x) = \int_0^\infty e^{-\beta t} \bar{p}^\mu_t f(x)\, dt.$$

Lemma 5.2 *Suppose that* r_2 *is a natural boundary. Then for a bounded continuous function* f *on* $(r_1, r_2]$,

$$\lim_{x \uparrow r_2} p^\mu_t f(x) = f(r_2).$$

Proof We see from [14, Theorem 5.14.1] that for $f \in C_b((r_1, r_2])$

$$\lim_{x \uparrow r_2} R_\beta f(x) = \frac{f(r_2)}{\beta}. \tag{5.2}$$

Let f be a strictly positive function in $C_\infty(I)$. For $r_1 < r < x < r_2$

$$\mathbb{P}_x(\sigma_r \le t) \le \frac{e^{\beta t}}{R_\beta f(r)} \mathbb{E}_x\big(e^{-\beta \sigma_r} R_\beta f(X_{\sigma_r})\big)$$

and

$$R_\beta f(x) \ge \mathbb{E}_x\left(\int_{\sigma_r}^\infty e^{-\beta t} f(X_t)\, dt\right) = \mathbb{E}_x\big(e^{-\beta \sigma_r} R_\beta f(X_{\sigma_r})\big),$$

where σ_r is the first hitting time at r, $\sigma_r = \inf\{t > 0 : X_t = r\}$. Hence we have

$$\mathbb{P}_x(\sigma_r \le t) \le \frac{e^{\beta t}}{R_\beta f(r)} \cdot R_\beta f(x) \longrightarrow 0, \quad x \uparrow r_2,$$

which implies that for any $f \in C_\infty(I)$

$$\lim_{x \uparrow r_2} p_t f(x) = 0,$$

and so

$$\mathbb{E}_x\bigl(e^{A_t^\mu} f(X_t); t < \zeta\bigr) \le \mathbb{E}_x\bigl(e^{2A_t^\mu}\bigr)^{1/2} \cdot p_t\bigl(f^2\bigr)(x)^{1/2} \longrightarrow 0, \quad x \uparrow r_2.$$

Since $p^\mu f(x) = p^\mu(f - f(r_2))(x) + f(r_2)p^\mu 1(x)$, it is enough to show that $\lim_{x\uparrow r_2} p_t^\mu 1(x) = \lim_{x\uparrow r_2} \mathbb{E}_x(\exp(A_t^\mu); t < \zeta) = 1$.

Let $K \subset I$ be a compact set and denote by μ_{K^c} the restriction of the measure μ on the complement of K, $\mu_{K^c}(\cdot) = \mu(K^c \cap \cdot)$. Since

$$R_\beta \mu_{K^c}(x) \ge \mathbb{E}_x\left(\int_0^t e^{-\beta s} 1_{K^c}(X_s)\, dA_s^\mu\right) \ge e^{-\beta t} \cdot \mathbb{E}_x\bigl(A_t^{\mu_{K^c}}\bigr),$$

we have

$$\sup_{x\in I} \mathbb{E}_x\bigl(A_t^{\mu_{K^c}}\bigr) \le e^{\beta t} \cdot \sup_{x\in I} R_\beta \mu_{K^c}(x) \longrightarrow 0, \quad K \uparrow I$$

by the definition of $\mathcal{K}_\infty^1$. By Khasminskii's lemma,

$$\sup_{x\in I} \mathbb{E}_x\bigl(e^{A_t^{\mu_{K^c}}}\bigr) \le \frac{1}{1 - \sup_{x\in I} \mathbb{E}_x(A_t^{\mu_{K^c}})}$$

and thus

$$\lim_{K\uparrow I} \sup_{x\in I} \mathbb{E}_x\bigl(e^{A_t^{\mu_{K^c}}}\bigr) = 1.$$

Hence we have $\lim_{x\uparrow r_2} \mathbb{E}_x(\exp(A_t^\mu); t < \zeta) = 1$ by the same argument as in [24, Theorem 2.1(iv)]. □

Set

$$\mathcal{D}_+\bigl(\bar{\mathcal{H}}^{\mu,\beta}\bigr) = \bigl\{\phi = \bar{R}_{\alpha+\beta}^\mu g : \alpha > 0, g \in C_b\bigl((r_1, r_2]\bigr) \text{ with } g \ge \exists\epsilon > 0\bigr\}.$$

On account of Remark 3.1 and Lemma 5.2 we see that $\phi \in \mathcal{D}_+(\bar{\mathcal{H}}^{\mu,\beta})$ is a bounded continuous function on $(r_1, r_2]$. Let

$$\eta = \inf\{t > 0 : X_{t-} = r_1\}, \qquad \rho = \inf\bigl\{t > 0 : X_{t-} \in (r_1, r_2), X_t = r_1\bigr\}.$$

Then $\zeta = \eta \wedge \rho$, that is, η is the predictable part of ζ and ρ is the inaccessible part of ζ. Let

$$h(x) = \mathbb{E}_x^\beta\bigl(e^{A_\zeta^\mu}; \eta = \zeta\bigr), \qquad h(r_1) = 1.$$

Then the function h satisfies that

$$h(x) - p_t^\mu h(x) = \mathbb{E}_x^\beta\big(e^{A_\zeta^\mu}; \eta = \zeta, t < \zeta\big)$$

and the argument in Lemma 3.1 leads us to the continuity of h on I. Moreover,

$$h(r_2) = \lim_{x\uparrow r_2} h(x) = 0 \tag{5.3}$$

because $\lim_{x\uparrow r_2} \mathbb{P}_x^\beta(\eta = \zeta) = 0$ and

$$\mathbb{E}_x^\beta\big(e^{A_\zeta^\mu}; \eta = \zeta\big) \le \mathbb{E}_x^\beta\big(e^{(1+\epsilon)A_\zeta^\mu}\big)^{1/(1+\epsilon)} \cdot \mathbb{P}_x^\beta(\eta = \zeta)^{\epsilon/(1+\epsilon)}.$$

Denote by $\mathcal{P}$ (resp. $\bar{\mathcal{P}}$) the set of probability measures on (r_1, r_2) (resp. $(r_1, r_2]$). Let us define the function on $\bar{\mathcal{P}}$ by

$$\bar{I}^\beta(\nu) = -\inf_{\substack{\phi\in\mathcal{D}_+(\bar{\mathcal{H}}^{\mu,\beta})\\ \epsilon>0}} \int_{(r_1,r_2]} \frac{\bar{\mathcal{H}}^{\mu,\beta}\phi}{\phi + \epsilon h}\, d\nu, \quad \nu \in \bar{\mathcal{P}}, \tag{5.4}$$

where $\bar{\mathcal{H}}^{\mu,\beta}\phi = \alpha \bar{R}^\mu_{\alpha+\beta} g - g$ for $\phi = \bar{R}^\mu_{\alpha+\beta} g \in \mathcal{D}_+(\bar{\mathcal{H}}^{\mu,\beta})$.

Lemma 5.3 *For $\nu \in \bar{\mathcal{P}}$ with $\nu((r_1, r_2)) > 0$, put*

$$\hat{\nu} = \hat{\nu}(\bullet) = \nu(\bullet)/\nu\big((r_1, r_2)\big) \in \mathcal{P}.$$

Then

$$\bar{I}^\beta(\nu) = \nu\big((r_1, r_2)\big) \cdot I^\beta(\hat{\nu}) + \nu\big(\{r_2\}\big) \cdot \beta, \quad \nu \in \bar{\mathcal{P}}.$$

Proof For $\phi = \bar{R}^\mu_{\alpha+\beta} g \in \mathcal{D}_+(\bar{\mathcal{H}}^{\mu,\beta})$,

$$\lim_{x\uparrow r_2} \phi(x) = \frac{1}{\alpha+\beta} g(r_2),$$

$$\lim_{x\uparrow r_2} \bar{\mathcal{H}}^{\mu,\beta}\phi(x) = \lim_{x\uparrow r_2}\big(\alpha \bar{R}^\mu_{\alpha+\beta} g(x) - g(x)\big) = -\frac{\beta}{\alpha+\beta} g(r_2)$$

by (5.2). In addition, $\bar{R}_\beta f(x) = R_\beta f(x)$ on $x \in (r_1, r_2)$ and so $\bar{\mathcal{H}}^{\mu,\beta}\phi(x) = \mathcal{H}^{\mu,\beta}\phi(x)$ on $x \in (r_1, r_2)$. Hence we have by (5.3)

$$\frac{\bar{\mathcal{H}}^{\mu,\beta}\phi(r_2)}{\phi(r_2) + \epsilon h(r_2)} = \frac{-\frac{\beta}{\alpha+\beta} \cdot g(r_2)}{\frac{1}{\alpha+\beta} \cdot g(r_2) + \epsilon h(r_2)} = -\beta,$$

and for $\nu \in \bar{\mathcal{P}}$

$$\bar{I}^\beta(\nu) = -\inf_{\substack{\phi\in\mathcal{D}_+(\bar{\mathcal{H}}^{\mu,\beta})\\ \epsilon>0}} \int_{(r_1,r_2]} \frac{\bar{\mathcal{H}}^{\mu,\beta}\phi}{\phi + \epsilon h}\, d\nu$$

$$= - \inf_{\substack{\phi \in \mathcal{D}_+(\mathcal{H}^{\mu,\beta}) \\ \epsilon > 0}} \int_{(r_1,r_2)} \frac{\mathcal{H}^{\mu,\beta}\phi}{\phi + \epsilon h} \, d\nu + \beta \cdot \nu\big(\{r_2\}\big)$$

$$= \nu\big((r_1, r_2)\big) \cdot I^{\beta}(\hat{\nu}) + \nu\big(\{r_2\}\big) \cdot \beta. \qquad \square$$

Proposition 5.1 *Let $\mu \in \mathcal{K}^1_\infty$. Suppose that r_2 is a natural boundary and r_1 is a regular or exit boundary. Then, for each closed set $K \subset \bar{\mathcal{P}}$*

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{x \in I} \mathbb{E}^{\beta}_x \big(e^{A^{\mu}_t}; L_t \in K, t < \zeta\big) \le - \inf_{\nu \in K} \bar{I}^{\beta}(\nu). \tag{5.5}$$

Proof Since for $x \in (r_1, r_2)$

$$\mathbb{E}^{\beta}_x \big(e^{A^{\mu}_t}; L_t \in K, t < \zeta\big) = \bar{\mathbb{E}}^{\beta}_x \big(e^{A^{\mu}_t}; L_t \in K, t < \zeta\big),$$

we can prove this proposition by exactly the same argument as in Proposition 4.2. $\square$

The set $\bar{\mathcal{P}} \setminus \{\delta_{r_2}\}$ is in one-to-one correspondence to $(0, 1] \times \mathcal{P}$ through the map:

$$\nu \in \bar{\mathcal{P}} \setminus \{\delta_{r_2}\} \mapsto \big(\nu\big((r_1, r_2)\big), \hat{\nu}(\bullet) = \nu(\bullet)/\nu\big((r_1, r_2)\big) \in (0, 1] \times \mathcal{P}. \tag{5.6}$$

Then Lemma 5.3 says that

$$\inf_{\nu \in \bar{\mathcal{P}}} \bar{I}^{\beta}(\nu) = \Big(\inf_{\nu \in \bar{\mathcal{P}} \setminus \{\delta_{r_2}\}} \bar{I}^{\beta}(\nu)\Big) \wedge \bar{I}^{\beta}(\delta_{r_2}) = \Big(\inf_{\nu \in \bar{\mathcal{P}} \setminus \{\delta_{r_2}\}} \bar{I}^{\beta}(\nu)\Big) \wedge \beta$$

$$= \Big(\inf_{0<\gamma\le 1, \nu \in \mathcal{P}} \big\{\gamma I^{\beta}(\nu) + (1-\gamma)\beta\big\}\Big) \wedge \beta$$

$$= \inf_{0 \le \gamma \le 1} \big\{\gamma\big(\rho_2(\mu) + \beta\big) + (1-\gamma)\beta\big\}. \tag{5.7}$$

Hence if $\rho_2(\mu) \le 0$, then the right-hand side equals $\rho_2(\mu) + \beta$. Moreover, Proposition 5.1 implies that $\rho_\infty(\mu) + \beta \ge \rho_2(\mu) + \beta$. As a result, we have $\rho_\infty(\mu) = \rho_2(\mu)$ on account of Remark 4.3. On the other hand, if $\rho_2(\mu) > 0$, then the right-hand side of (5.7) equals β, and thus $\rho_\infty(\mu) + \beta \ge \beta$. In addition, $\lim_{x \uparrow r_2} p^{\mu}_1 1(x) = 1$ by Lemma 5.2. Hence $\|p^{\mu}_t\|_{\infty,\infty} \ge 1$ and so $\rho_\infty(\mu) \le 0$. Therefore we can conclude that if $\rho_2(\mu) > 0$, then $\rho_\infty(\mu) = 0$.

If r_1 is entrance or natural, we need not add ϵh in the definition of I-function because the diffusion process is conservative and so each function ϕ in $\mathcal{D}_+(\bar{\mathcal{H}}^{\mu,\beta})$ is strictly positive, that is, there exists a positive constant $\delta > 0$ such that $\phi(x) \ge \delta$ on (r_1, r_2). If r_1 is entrance, we extend $\mathbb{M}$ to $(r_1, r_2]$ by making r_2 a trap. Then we can show by the same arguments as above that the L^p-independence holds if and only if $\rho_2(\mu) \le 0$. If r_1 is natural, we extend $\mathbb{M}$ to $[r_1, r_2]$ by making both r_1 and r_2 traps. Then $\bar{I}^{\beta}(\nu)$ is written as for $\nu \in \bar{\mathcal{P}}(:= \mathcal{P}([r_1, r_2]))$:

$$\bar{I}^{\beta}(\nu) = \nu\big((r_1, r_2)\big) \cdot I^{\beta}(\hat{\nu}) + \nu\big(\{r_1\}\big) \cdot \beta + \nu\big(\{r_2\}\big) \cdot \beta,$$

and

$$\inf_{\nu\in\bar{\mathcal{P}}} \bar{I}^{\beta}(\nu) = \Bigl(\inf_{\nu\in\bar{\mathcal{P}}:\nu((r_1,r_2))>0} \bar{I}^{\beta}(\nu)\Bigr) \wedge \beta$$
$$= \inf_{0\le\gamma\le 1}\bigl\{\gamma\bigl(\rho_2(\mu)+\beta\bigr) + (1-\gamma)\beta\bigr\}.$$

Therefore the same conclusion follows. We now sum up the facts above:

Theorem 5.1 *Let $\mu \in \mathcal{K}_\infty^1$. If no boundaries are natural, then $\rho_p(\mu)$, $1 \le p \le \infty$, is independent of p. If one of the boundaries is natural, then $\rho_p(\mu)$ is independent of p if and only if $\rho_2(\mu) \le 0$.*

Finally we consider a large deviation principle for the additive functional A_t^μ of a one-dimensional process. To establish the large deviation principle by applying the Gärtner–Ellis theorem, we need the existence of the *logarithmic moment generating function* ([8, Assumption 2.3.2]). Theorem 5.1 and Remark 4.1 lead us to the next corollary.

Corollary 5.2 *Let $\mu \in \mathcal{K}_\infty^1$ and assume that no boundaries are natural. Then for $\theta \in \mathbb{R}$*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x\bigl(e^{\theta A_t^\mu}; t<\zeta\bigr) = -\rho_2(\theta\mu).$$

For a positive bounded function v, the measure $v\cdot m$ belongs to $\mathcal{K}$ because

$$\limsup_{t\downarrow 0}\sup_{x\in I} \mathbb{E}_x\left(\int_0^t v(X_s)\,ds\right) \le \lim_{t\downarrow 0}\|v\|_\infty t = 0.$$

Note that the assumption (III) is equivalent to that $m \in \mathcal{K}_\infty^1$, and thus $v\cdot m$ belongs to $\mathcal{K}_\infty^1$ for $v \in \mathcal{B}_b^+$. Therefore we assume that no boundaries are natural, the limit in Corollary 5.2 exists for $\int_0^t v(X_s)\,ds$. Moreover, if no boundaries are natural, the resolvent R_1 of the diffusion is compact [16, Theorem 3.1], and so is the resolvent R_1^v of the Feynman–Kac semigroup because R_1^v is written by

$$R_1^v f(x) = R_1 f(x) + R_1\bigl(vR_1^v f\bigr)(x).$$

Consequently, $\rho_2(\theta(v\cdot m))$ is differentiable in θ by the analytic perturbation theorem [15, Chap. VII]. Therefore, employing the Gärtner–Ellis theorem, we have:

Theorem 5.2 *Assume that no boundaries are natural. Then for a bounded positive Borel function v, $\int_0^t v(X_s)\,ds/t$ obeys the large deviation principle with rate function $I(\lambda) = \sup\{\lambda\theta - C(\theta) : \theta \in \mathbb{R}\}$:*

(i) *For each closed set $K \in \mathbb{R}$,*

$$\limsup_{t\to\infty} \frac{1}{t}\log \mathbb{P}_x\left(\frac{1}{t}\int_0^t v(X_s)\,ds \in K; t<\zeta\right) \le -\inf_{\lambda\in K} I(\lambda).$$

(ii) *For each open set* $G \subset \mathbb{R}$,

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}_x\left(\frac{1}{t}\int_0^t v(X_s)\,ds \in G; t < \zeta\right) \geq -\inf_{\lambda\in G} I(\lambda).$$

Appendix

In this section we will prove Proposition 2.1 if a symmetric Markov process satisfies (I) and (II). For $\mu \in \mathcal{K}$ its fine support is defined by

$$f\text{-supp}[\mu] = \{x \in X : \mathbb{P}_x(\tau = 0) = 1\}, \quad \tau = \inf\{t > 0 : A_t^\mu > 0\}.$$

Lemma 6.1 *Let* $\mathbb{M}$ *be a symmetric Markov process satisfying* (I), (II). *Let* μ *be a measure in* $\mathcal{K}_\infty^\beta$ *whose fine support is identical to the topological support*, supp$[\mu]$. *Then the time-changed process of the* β*-subprocess by* A_t^μ *satisfies* (I), (II$'$) *and* (III).

Proof Denote by $\mathbb{M}^{\beta,\mu} = (\mathbb{P}_x^{\beta,\mu}, X_t, \zeta)$ the time-changed process of the β-subprocess by A_t^μ. $\mathbb{M}^{\beta,\mu}$ satisfies (I) because the irreducibility is stable under time-changed transform ([18, Theorems 8.2, 8.5]).

Let $R_\alpha^{\beta,\mu}$ be the α-resolvent of $\mathbb{M}^{\beta,\mu}$. Let $\tau_t = \inf\{s > 0 : A_s^\mu > t\}$. Then for $f \in \mathcal{B}_b(X)$

$$R_\alpha^{\beta,\mu} f(x) = \mathbb{E}_x^\beta\left(\int_0^\infty e^{-\alpha t} f(X_{\tau_t})\,dt\right) = \mathbb{E}_x^\beta\left(\int_0^\infty e^{-\alpha A_t^\mu} f(X_t)\,dA_t^\mu\right)$$
$$= \mathbb{E}_x\left(\int_0^\infty e^{-\beta t - \alpha A_t^\mu} f(X_t)\,dA_t^\mu\right).$$

Note that by Theorem 2.1(iii),

$$\mathbb{E}_x\left(\int_s^\infty e^{-\beta t-\alpha A_t^\mu} f(X_t)\,dA_t^\mu\right)$$
$$= \mathbb{E}_x\left(e^{-\beta s-\alpha A_s^\mu}\mathbb{E}_{X_s}\left(\int_0^\infty e^{-\beta t-\alpha A_t^\mu} f(X_t)\,dA_t^\mu\right)\right)$$
$$= \mathbb{E}_x\left(e^{-\beta s-\alpha A_s^\mu} R_\alpha^{\beta,\mu} f(X_s)\right) = e^{-\beta s} p_s^{-\alpha\mu} R_\alpha^{\beta,\mu} f(x) \in C_b(X).$$

Then since

$$\sup_{x\in X}\left|R_\alpha^{\beta,\mu} f(x) - e^{-\beta s} p_s^{-\alpha\mu} R_\alpha^{\beta,\mu} f(x)\right|$$
$$= \sup_{x\in X}\mathbb{E}_x\left(\int_0^s e^{-\beta t-\alpha A_t^\mu} f(X_t)\,dA_t^\mu\right) \leq \|f\|_\infty \sup_{x\in X}\mathbb{E}_x(A_s^\mu)\downarrow 0, \quad s\downarrow 0,$$

by the definition of the Kato class, we see that $R_\alpha^{\beta,\mu} f \in C_b(X)$, that is, $\mathbb{M}^{\beta,\mu}$ satisfies (II$'$).

Finally, since

$$R_\alpha^{\beta,\mu} 1_{K^c}(x) \le \mathbb{E}_x\left(\int_0^\infty e^{-\beta t} 1_{K^c}(X_t)\, dA_t^\mu\right) = R_\beta(1_{K^c}\mu),$$

the property (III) follows from the definition of $\mu \in \mathcal{K}_\infty^\beta$. □

Lemma 6.2 *Let g be a strictly positive function in $C_\infty(X)$, the set of continuous functions vanishing the infinity Δ. Then the measure $g \cdot m$ belongs to $\mathcal{K}_\infty^\beta$.*

Proof Since

$$R_\beta(1_{K^c} \cdot g)(x) \le R_\beta\Big(\sup_{x\in K^c} g(x) \cdot 1_{K^c}\Big)(x) \le \sup_{x\in K^c} g(x) \cdot R_\beta 1(x),$$

we have

$$\sup_{x\in X} R_\beta(1_{K^c} \cdot g)(x) \le \frac{1}{\beta} \cdot \sup_{x\in K^c} g(x) \longrightarrow 0, \quad K \uparrow X.$$ □

Proposition 6.1 *It holds that for $\mu \in \mathcal{K}_\infty^\beta$*

$$\lambda^\beta(\mu) > 1 \quad \Longrightarrow \quad \sup_{x\in X} \mathbb{E}_x^\beta\big(e^{A_\zeta^\mu}\big) < \infty.$$

Proof Let g be a function in Lemma 6.2 and $\mathbb{M}^{\beta,g} = (\mathbb{P}_x^{\beta,g}, X_t, \zeta)$ be the subprocess of $\mathbb{M}^\beta$ by $\exp(-\int_0^t g(X_s)\,ds)$. Then $\mathbb{M}^{\beta,g}$ satisfies (I) and (II). Since the fine support of $\mu + g \cdot m$ equals the whole space X, the time-changed process of $\mathbb{M}^{\beta,g}$ by PCAF $A_t^\mu + \int_0^t g(X_t)\,dt$ satisfies (I), (II′) and (III). Then the assertion that

$$\lambda^{\beta,g}(\mu) > 1 \quad \Longleftrightarrow \quad \sup_{x\in X} \mathbb{E}_x^{\beta,g}\left(\exp\left(A_\zeta^\mu + \int_0^\zeta g(X_t)\,dt\right)\right) < \infty \tag{6.1}$$

is a consequence of [23, Corollary 4.9], where

$$\lambda^{\beta,g}(\mu) = \inf\left\{\mathcal{E}_\beta(u,u) + \int_X u^2 g\,dm : u \in \mathcal{D}(\mathcal{E}), \int_X u^2(x)(d\mu + g\,dm) = 1\right\}.$$

Put

$$A_t^+ = A_t^\mu + \int_0^t g(X_s)\,ds, \qquad A_t^- = \int_0^t g(X_s)\,ds.$$

Then we see from [17, (62.13)] and [3, (2.17), (2.19)] that the expectation on the right-hand side of (6.1) equals

$$\begin{aligned}\mathbb{E}_x^\beta\left(\int_0^\zeta e^{A_t^+}\, d\big(-e^{-A_t^-}\big) + e^{A_\zeta^+} e^{-A_\zeta^-}\right) &= 1 + \mathbb{E}_x^\beta\left(\int_0^\zeta e^{-A_t^-}\, d(e^{A_t^+})\right) \\ &= \mathbb{E}_x^\beta\left(e^{A_\zeta^\mu} + \int_0^\zeta e^{A_t^\mu}\, dA_t^-\right) \ge \mathbb{E}_x^\beta\big(e^{A_\zeta^\mu}\big).\end{aligned}$$

Moreover, we show in the same way as in [26, Lemma 3.1] that the left-hand side of (6.1) is equivalent with $\lambda^\beta(\mu) > 1$. The proof is completed. □

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