J Theor Probab (2011) 24: 577–594
DOI 10.1007/s10959-010-0303-x

One-Sided Cauchy–Stieltjes Kernel Families

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Received: 30 June 2009 / Revised: 16 May 2010 / Published online: 20 August 2010

Abstract This paper continues the study of a kernel family which uses the Cauchy–Stieltjes kernel $1/(1-\theta x)$ in place of the celebrated exponential kernel $\exp(\theta x)$ of the exponential families theory. We extend the theory to cover generating measures with support that is unbounded on one side. We illustrate the need for such an extension by showing that cubic pseudo-variance functions correspond to free-infinitely divisible laws without the first moment. We also determine the domain of means, advancing the understanding of Cauchy–Stieltjes kernel families also for compactly supported generating measures.

Keywords Exponential families · Cauchy kernel · Free stable law

Mathematics Subject Classification (2000) 60E10 · 46L54

1 Introduction and Definition

According to Wesołowski [16], the kernel family generated by a kernel $k(x,\theta)$ with generating measure ν is the set of probability measures

$$\left\{k(x,\theta)/L(\theta)\nu(dx)\colon \theta\in\Theta\right\},$$

Research partially supported by NSF grant DMS-0904720 and by Taft Research Seminar 2008/09.

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where $L(\theta) = \int k(x,\theta)\nu(dx)$ is the normalizing constant, and ν is the generating measure. The theory of exponential families is based on the kernel $k(x,\theta) = e^{\theta x}$; see, e.g., [9, 10], or [8, Sect. 2.3]. The author of [6] initiated the study of the Cauchy–Stieltjes kernel

$$k(x,\theta) = \frac{1}{1-\theta x},$$

for compactly supported generating measures ν. In the neighborhood of $\theta = 0$, such families can be parameterized by the mean, and under this parametrization the family (and measure ν) is uniquely determined by the variance function $V(m)$ and a real number m_0, which is the mean of ν. The Cauchy–Stieltjes Kernel (CSK) family has some properties analogous to classical exponential families, with convolution of measures replaced in some results by the free additive convolution.

For measures with unbounded support, one can still define the CSK family if the support of the generating measure is bounded above or below. In such situation, the family is parameterized by a "one-sided" range of θ of a fixed sign. In this note, we consider generating measures with support bounded from above and our CSK families are parameterized by $\theta > 0$. Part of our motivation is to admit the free additive $1/2$-stable law [4, p. 1054] as a generating measure; for exponential families, the celebrated inverse Gaussian law is $1/2$-stable and corresponds to cubic variance function.

Throughout the paper,

$$B = B(\nu) = \max\{0, \sup \operatorname{supp}(\nu)\} \tag{1.1}$$

denotes the (sometimes strict) non-negative upper bound for the support of ν. Occasionally we will want to explain how the formulas change for the "dual case" of measures with support bounded from below. Then we will use $A = A(\nu) = \min\{0, \inf \operatorname{supp}(\nu)\}$. Of course, $\operatorname{supp}(\nu) \subset [A, B]$ when both expressions are finite.

Definition 1.1 Suppose ν is a non-degenerate (i.e., not a point mass) probability measure with support bounded from above. For $0 \le \theta B(\nu) < 1$, let

$$M(\theta) = \int \frac{1}{1-\theta x}\nu(dx).$$

The (one-sided) Cauchy–Stieltjes kernel family generated by ν is the family of probability measures

$$\mathcal{K}_+(\nu) = \left\{ P_\theta(dx) = \frac{1}{M(\theta)(1-\theta x)}\nu(dx)\colon\ 0 < \theta < \theta_+ \right\}, \tag{1.2}$$

where $\theta_+ = 1/B(\nu)$ (here $1/0$ is interpreted as ∞).

To simplify the statements such as reparameterization (2.2) of $\mathcal{K}_+(\nu)$, we chose to exclude $\theta = 0$ so that $\nu \notin \mathcal{K}_+(\nu)$. Alternatively, one could include ν as P_0 in the family $\mathcal{K}_+(\nu)$; then one would need to write (2.2) with the left-closed domain of means $[m_0, m_+)$ and one would need to allow also the extended value $m_0 = -\infty$.

Similarly, one may define the one-sided CSK family for a generating measure ν with support bounded from below. Then the one-sided CSK family $\mathcal{K}_-(\nu)$ is defined for $\theta_- < \theta < 0$, where θ_- is either $1/A(\nu)$ or $-\infty$. Finally, if ν has compact support, then the natural domain for the parameter θ of the two-sided CSK family $\mathcal{K}(\nu) = \mathcal{K}_-(\nu) \cup \mathcal{K}_+(\nu) \cup \{\nu\}$ is $\theta_- < \theta < \theta_+$. For definiteness, we concentrate on the case of measures bounded from above, and we explicitly allow generating measures with unbounded support, and perhaps without moments.

2 Parameterizations by the Mean

Since $\theta > 0$, the mean $m(\theta) = \int x P_\theta(dx)$ exists for all measures in (1.2). A calculation gives

$$m(\theta) = \frac{M(\theta) - 1}{\theta M(\theta)}. \tag{2.1}$$

Indeed,

$$m(\theta) = \int_{(-\infty,b]} \frac{1 - (1 - \theta x)}{\theta M(\theta)(1 - \theta x)} \nu(dx) = \frac{1}{\theta} - \frac{1}{\theta M(\theta)}.$$

To verify that $\theta \mapsto M(\theta)$ is differentiable on $(0, \theta_+)$, we check that one can differentiate under the integral sign. For this, we first observe that since $\mathrm{supp}(\nu) \subset (-\infty, B]$, for $\theta \in (0, \theta_+)$ the expression $1 - \theta x$ is positive for all x from the support of ν. So

$$\begin{aligned} \int \frac{|x|}{|1 - \theta x|^2} \nu(dx) &\le \frac{1}{\theta} \int \frac{|\theta x - 1| + 1}{|1 - \theta x|^2} \nu(dx) \\ &= \frac{1}{\theta} \int \frac{(1 - \theta x) + 1}{(1 - \theta x)^2} \nu(dx) \\ &\le \frac{M(\theta)}{\theta} + \frac{M(\theta)}{\theta(1 - \theta B)} < \infty. \end{aligned}$$

Now fix $0 < \alpha < \beta < \theta_+$. Then, since for x in the support of ν the map $\theta \mapsto \frac{\partial}{\partial \theta}(\frac{1}{1-\theta x}) = \frac{x}{(1-\theta x)^2}$ is increasing on $(0, \theta_+)$, we have

$$\frac{x}{(1 - \alpha x)^2} \le \frac{x}{(1 - \theta x)^2} \le \frac{x}{(1 - \beta x)^2},$$

for all $\theta \in [\alpha, \beta]$.

For x in the support of ν, define

$$g(x) = \frac{|x|}{(1 - \alpha x)^2} + \frac{|x|}{(1 - \beta x)^2}.$$

Then we have that $g \ge 0$, g is ν-integrable because α and β are in $(0, \theta_+)$, and $|\frac{\partial}{\partial \theta}(\frac{1}{1-\theta x})| = \frac{|x|}{(1-\theta x)^2} \le g(x)$, for all $\theta \in [\alpha, \beta]$. With this, we have that $\theta \mapsto M(\theta)$

is differentiable on $(0, \theta_+)$, and we can differentiate under the integral sign. The reasoning from [6] can now be repeated to show that $m(\theta)$ is increasing. Indeed, differentiating (2.1) we get

$$m'(\theta) = \frac{M(\theta) + \theta M'(\theta) - (M(\theta))^2}{\theta^2 (M(\theta))^2}.$$

Since ν is non-degenerate,

$$\begin{aligned} &M(\theta) + \theta M'(\theta) - \big(M(\theta)\big)^2 \\ &\quad = \int \frac{1}{(1-\theta x)^2} \nu(dx) - \left(\int \frac{1}{1-\theta x} \nu(dx) \right)^2 > 0 \end{aligned}$$

for all $0 < \theta < \theta_+$. Thus the function $\theta \mapsto m(\theta)$ is increasing on $(0, \theta_+)$. Denoting by ψ the inverse function, we are thus lead to parametrization of $\mathcal{K}_+(\nu)$ by the mean,

$$\mathcal{K}_+(\nu) = \big\{ Q_m(dx) = P_{\psi(m)}(dx) \colon m \in (m_0, m_+) \big\}, \tag{2.2}$$

where the so-called domain of means (m_0, m_+) is the image under $\theta \mapsto m(\theta)$ of the interval $(0, \theta_+)$. It is clear that $m_+ \le \sup \operatorname{supp}(\nu)$ and

$$m_0 = \lim_{\theta \searrow 0} m(\theta) = \int x \nu(dx) \ge -\infty.$$

We will give alternative representations for m_0 and m_+ later in the paper, see Propositions 3.8(iii) and 3.4.

3 The Pseudo-Variance Function

The variance function

$$V(m) = \int (x-m)^2 Q_m(dx) \tag{3.1}$$

is the fundamental concept of the theory of exponential families, and also of the theory of CSK families as presented in [6]. Unfortunately, if ν does not have the first moment (which is the case for free 1/2-stable laws), all measures in the CSK family generated by ν have infinite variance. We therefore introduce the following substitute which coincides with the variance function when the mean of ν is zero.

Definition 3.1 For $m \in (m_0, m_+)$, the *pseudo-variance function* is defined as

$$\mathbb{V}(m) = m \left(\frac{1}{\psi(m)} - m \right). \tag{3.2}$$

Note that the pseudo-variance function may take negative values: $\mathbb{V}(m)$ is negative for all $m_0 < m < 0$, as in this case $1/\psi(m) - m > 0$.

Since (3.2) is difficult to interpret, we will clarify in Proposition 3.2 how the pseudo-variance is related to the variance function when $\int |x|\nu(dx) < \infty$. To derive this relation, we need first to point out how the pseudo-variance function enters into the representation of a CSK family as a free-exponential or q-exponential family; compare with [7].

Proposition 3.1 *Suppose $\mathbb{V}$ is a pseudo-variance of the CSK family $\mathcal{K}(\nu)$ generated by a probability measure ν with support in $(-\infty, b]$ for some $b \in \mathbb{R}$. The explicit parametrization of $\mathcal{K}_+(\nu)$ by the mean (2.2) is as follows*:

$$Q_m(dx) = \frac{\mathbb{V}(m)}{\mathbb{V}(m) + m(m-x)}\nu(dx). \tag{3.3}$$

Proof For completeness, we include a version of the argument implicit in [6]: If $\theta = \psi(m) = m/(m^2 + \mathbb{V}(m))$, then from (3.2) we get

$$\frac{1}{M(\theta)(1-\theta x)} = \frac{m^2 + \mathbb{V}(m)}{M(\theta)(\mathbb{V}(m) + m(m-x))}.$$

From (2.1), $1/M(\theta) = 1 - \psi(m)m = \mathbb{V}(m)/(m^2 + \mathbb{V}(m))$, proving (3.3). □

The following result shows that the pseudo-variance functions are closely related to the variance functions.

Proposition 3.2 *If ν is non-degenerate with support bounded from above, and with finite first moment $m_0 = \int x\nu(dx)$ then the variance function V as defined by (3.1) exists, and*

$$\mathbb{V}(m) = \frac{m}{m - m_0}V(m)$$

for $m \in (m_0, m_+)$. In particular, if $m_0 = 0$ then $\mathbb{V}(m) = V(m)$ on $(0, m_+)$.

Proof For completeness, we include a version of the argument presented in [6]. A different argument based on (3.4) is presented in [7].

If ν has the first moment, then $m_0 = \int x\nu(dx)$ is a real number and $x(x-m)$ is integrable with respect to measures $Q_m \in \mathcal{K}(\nu)$ for all $m \in (m_0, m_+)$. The following calculation gives the answer:

$$\begin{aligned}
V(m) &= \int x(x-m)Q_m(dx) = \int \frac{x(x-m)\mathbb{V}(m)}{\mathbb{V}(m) + m(m-x)}\nu(dx) \\
&= \frac{\mathbb{V}(m)}{m}\int \frac{xm(x-m)}{\mathbb{V}(m) + m(m-x)}\nu(dx) \\
&= \frac{\mathbb{V}(m)}{m}\int \frac{x\mathbb{V}(m) - x(\mathbb{V}(m) + m(m-x))}{\mathbb{V}(m) + m(m-x)}\nu(dx) \\
&= \frac{\mathbb{V}(m)}{m}\left(\int xQ_m(dx) - \int x\nu(dx)\right) = \frac{\mathbb{V}(m)}{m}(m - m_0).
\end{aligned}$$ □

Remark 3.1 If ν has also finite second moment and $m_0 \neq 0$, then $\mathbb{V}(m)$ has a simple pole at $m = m_0$. In Corollary 3.9, we will verify that $m/\mathbb{V}(m)$ is positive, continuous, and increasing on (m_0, m_+), so this must be the only pole.

Remark 3.2 Using the finite difference operator

$$(\Delta_m f)(m) := \frac{f(m) - f(0)}{m},$$

it is straightforward to verify that when $0 \in (m_0, m_+)$ the density $g(m,x) := \frac{\mathbb{V}(m)}{\mathbb{V}(m)+m(m-x)}$ satisfies the equation

$$\Delta_m g(m,x) = \frac{x-m}{\mathbb{V}(m)} g(m,x). \tag{3.4}$$

This is a finite-difference analog of the differential equation satisfied by the density of an exponential family [15]. In [7], this equation is used to verify directly from (3.3) that m is the mean and that $\mathbb{V}(m)$ is the variance function when $\int x\nu(dx) = 0$.

In principle, the generating measure ν can be determined from the pseudo-variance function $\mathbb{V}$ by the following method. Given $\mathbb{V}(m)$, solve (3.2) for ψ, find the inverse function $m(\theta)$, and solve (2.1) for $M(\theta)$. This effectively determines the distribution via Stieltjes inversion formula (3.8). To ensure that this procedure indeed works we need several technical results.

Proposition 3.3 *Suppose $\mathbb{V}$ is the pseudo-variance function of a CSK family generated by a probability measure with $B = B(\nu) < \infty$ (recall (1.1)). Let*

$$z = z(m) = m + \frac{\mathbb{V}(m)}{m}. \tag{3.5}$$

Then $m \mapsto z(m)$ is continuous, strictly decreasing on (m_0, m_+), $z(m) > 0$ on (m_0, m_+), $z(m) \nearrow \infty$ as $m \searrow m_0$, and $z(m) \searrow B$ as $m \nearrow m_+$.

Proof Rewriting (3.2), we see that $z(m) = 1/\psi(m)$ is strictly decreasing, positive and increases without bound as $m \searrow m_0$. Clearly, as an inverse of the differentiable function $\theta \mapsto m(\theta)$, function $\psi(m)$ is continuous. □

3.1 Cauchy Transform

The Cauchy transform of a probability measure ν on Borel sets of $\mathbb{R}$ is an analytic mapping G from the upper complex half-plane $\mathbb{C}^+$ into the lower half-plane $\mathbb{C}^-$ given by

$$G_\nu(z) = \int \frac{1}{z-x} \nu(dx). \tag{3.6}$$

(We will drop subscript ν when the measure is clear from the context.) It is known (see [1] or [5, Proposition 5.1]) that an analytic function $G : \mathbb{C}^+ \to \mathbb{C}^-$ is a Cauchy transform if and only if

$$\lim_{t\to\infty} itG(it) = 1 \tag{3.7}$$

and that the corresponding probability measure ν is determined uniquely from the Stieltjes inversion formula

$$\nu(dx) = \lim_{\varepsilon\searrow 0} \frac{-\Im G(x+i\varepsilon)}{\pi} dx, \tag{3.8}$$

where the limit is in the sense of weak convergence of measures (see e.g., [1, p. 125]).

For a probability measure ν with support in $(-\infty, b]$, G_ν is analytic on the slit complex plane $\mathbb{C} \setminus (-\infty, b]$. This shows that a probability measure ν with support bounded from above is determined uniquely by $G_\nu(z)$ on $z \in (b, \infty)$ for some b.

Furthermore, $\lim_{z\to\infty} zG(z) = 1$, and since G is non-negative and decreasing on (b, ∞), the limit $\lim_{z\searrow b} G_\nu(z)$ exists as an extended number in $(0, \infty]$. It will be convenient to write $1/G(b) \in [0, \infty)$ for the limit $\lim_{z\searrow b} 1/G_\nu(z)$ even if $G(z)$ is undefined at $z = b$.

The following shows how the upper end of the domain of means is related directly to Cauchy transform.

Proposition 3.4 *If ν has support bounded from above with $B = B(\nu) < \infty$ given by* (1.1), *then $m_+ = B - 1/G(B)$. (Here $1/G(B) := \lim_{z\searrow B} 1/G(z)$ can be zero.)*

Proof Since $M(\theta) = \frac{1}{\theta} G(\frac{1}{\theta})$, from (2.1) we get $m_+ = \lim_{\theta\nearrow 1/B} m(\theta) = \lim_{\theta\nearrow 1/B}(1/\theta - 1/G(1/\theta)) = B - 1/G(B)$. □

Remark 3.3 (Precise Domain of Means) If ν is compactly supported and $A = \min\{0, \inf \mathrm{supp}(\nu)\}$ and $B = B(\nu)$, then the domain of means for the two sided CSK family generated by ν is the interval (m_-, m_+) with $m_- = A - 1/G(A)$, $m_+ = B - 1/G(B)$. This gives a more precise information about the domain of means considered in [6, Theorem 3.1].

From the fact that measures (3.3) integrate to 1, we get the following (see [6, Theorem 3.1]).

Proposition 3.5 *Suppose $\mathbb{V}$ is the pseudo-variance function of the CSK family $\mathcal{K}_+(\nu)$ generated by a probability measure ν with support bounded from above. For z given by* (3.5), *the Cauchy–Stieltjes transform of ν satisfies*

$$G(z) = \frac{m}{\mathbb{V}(m)}. \tag{3.9}$$

In particular, $0 < \frac{m}{\mathbb{V}(m)} < G(B(\nu))$. The generating measure ν is determined uniquely by the pseudo-variance function $\mathbb{V}(\cdot)$.

Proof Integrating each measure in (3.3), we get (3.9). Since G is positive and decreasing on $(B(\nu),\infty)$, from (3.9) we see that $0<m/\mathbb{V}(m)<G(B(\nu))$ on (m_0,m_+). By Proposition 3.3, this determines $G(z)$ on an interval. Thus by the uniqueness of analytic extension, this determines $G(z)$ for all $z\in\mathbb{C}^+$, and hence it determines ν via (3.8). □

Corollary 3.6 *If $\mathbb{V}$ is a pseudo-variance function of a CSK family generated by probability measure with support bounded from above, then $\frac{m}{\mathbb{V}(m)}\to 0$ and $\frac{m^2}{\mathbb{V}(m)}\to 0$ as $m\searrow m_0$.*

Proof By Proposition 3.3, $z(m)\to\infty$. So (3.9) and the properties of the Cauchy transform imply that $G(z(m))=m/\mathbb{V}(m)\to 0$ and $zG(z)=1+m^2/\mathbb{V}(m)\to 1$. □

Remark 3.4 Proposition 3.5 shows that pseudo-variance function $m\mapsto\mathbb{V}(m)$ contains more information than the variance function $m\mapsto V(m)$: $\mathbb{V}$ is defined for ν without moments, and it determines the generating measure ν without the need to supply its mean. In particular, for a two-sided family with the domain of means (m_-,m_+) as in Remark 3.3, its pseudo-variance function determines also $m_0=\int x\nu(dx)$.

The following technical result is needed later on to verify that the R-transform is strictly increasing on an interval. (The inequality gives the lower bound for the difference quotient of $F=1/G$ on (b,∞); compare [12, Proposition 2.1].)

Proposition 3.7 *If ν is a non-degenerate probability measure with the support bounded above by $b\in\mathbb{R}$, then for $b<z_1<z_2$,*

$$G(z_1)-G(z_2)>(z_2-z_1)G(z_1)G(z_2). \tag{3.10}$$

Proof We have

$$G(z_1)-G(z_2)=(z_2-z_1)\int_{(-\infty,b]}\frac{1}{(z_1-x)(z_2-x)}\nu(dx). \tag{3.11}$$

Note that for $x,y\le b$,

$$\left(\frac{1}{z_1-x}-\frac{1}{z_1-y}\right)\left(\frac{1}{z_2-x}-\frac{1}{z_2-y}\right)$$
$$=\frac{(y-x)^2}{(z_1-x)(z_1-y)(z_2-x)(z_2-y)}\ge 0.$$

Choose $a<b$ such that $\nu((a,b])>0$. Since ν is non-degenerate,

$$\iint_{(a,b]\times(a,b]}(y-x)^2\nu(dx)\nu(dy)>0.$$

Therefore,

$$\begin{aligned}
&2\int_{(-\infty,b]} \frac{1}{(z_1-x)(z_2-x)}\nu(dx) - 2G(z_1)G(z_2)\\
&\quad= \iint_{(-\infty,b]\times(-\infty,b]} \left(\frac{1}{z_1-x}-\frac{1}{z_1-y}\right)\left(\frac{1}{z_2-x}-\frac{1}{z_2-y}\right)\nu(dx)\nu(dy)\\
&\quad= \iint_{(-\infty,b]\times(-\infty,b]} \frac{(y-x)^2}{(z_1-x)(z_1-y)(z_2-x)(z_2-y)}\nu(dx)\nu(dy)\\
&\quad\geq \frac{1}{(z_2-a)^4}\iint_{(a,b]\times(a,b]} (y-x)^2\nu(dx)\nu(dy) > 0.
\end{aligned}$$

Since $z_2 > z_1$, this together with (3.11) ends the proof. □

3.2 Free Convolution

It is known (see [5]) that if G is a Cauchy transform of a probability measure, then there exist $b > 0$ such that G is univalent in the domain

$$\Gamma_b^+ = \{z \in \mathbb{C}^+ : \Re z > b,\ \Im z < \Re z\}. \tag{3.12}$$

Since $G(\bar{z}) = \overline{G(z)}$, G is also univalent in $\Gamma_b^- = \overline{\Gamma_b^+}$. For measures with support bounded from above by $b > 0$, G is also one-to-one on (b, ∞). So increasing b if necessary, we can extend the region of univalence to

$$\Gamma_b = \{z \in \mathbb{C} : \Re z > b,\ |\Im z| < \Re z\}. \tag{3.13}$$

Then the composition-inverse function $K_\nu(z) = G_\nu^{\langle -1\rangle}(z)$ exists and is analytic for z in the domain $G(\Gamma_b)$. The R-transform is an analytic function in the same region, and is defined by

$$\mathcal{R}_\nu(z) = K_\nu(z) - 1/z\,. \tag{3.14}$$

(A warning is in place: some authors use $R(z) = z\mathcal{R}(z)$ as the R-transform!) As an analytic function, $\mathcal{R}_\nu$ is determined uniquely by its values on the interval $\mathbb{R} \cap G(\Gamma_b) = (0, G(b))$. In fact, on the real line $\mathcal{R}_\nu$ is defined on a potentially larger interval $(0, G(B(\nu)))$.

Our interest in the R-transform stems from its relation to free convolution: a free convolution $\mu \boxplus \nu$ of probability measures μ, ν on Borel sets of the real line is a uniquely defined probability measure $\mu \boxplus \nu$ such that

$$\mathcal{R}_{\mu\boxplus\nu}(z) = \mathcal{R}_\mu(z) + \mathcal{R}_\nu(z)$$

for all z in an appropriate domain (see [5, Sect. 5] for details; the exact form of this domain is not relevant for us, as we will be working only with the intervals in $\mathbb{R}$ and then appeal to the uniqueness of analytic extension.)

For $\alpha > 0$ we denote by $\nu^{\boxplus\alpha}$ the free convolution power of a probability measure ν, which is defined by

$$\mathcal{R}_{\nu^{\boxplus\alpha}}(z) = \alpha\mathcal{R}_\nu(z). \tag{3.15}$$

Convolution power of order $\alpha \in [1, \infty)$ exists by [3, Sect. 2]. Convolution power of order $\alpha > 0$ exists for $\boxplus$-infinitely divisible laws. The following result lists properties of R-transform that we need.

Proposition 3.8 *Suppose $\mathbb{V}$ is a pseudo-variance function of the CSK family $\mathcal{K}_+(\nu)$ generated by a probability measure ν with $b = \sup supp(\nu) < \infty$. Then*

(i) *$\mathcal{R}_\nu$ is strictly increasing on* $(0, G(b))$.
(ii) *For $m \in (m_0, m_+)$,*

$$\mathcal{R}_\nu\left(\frac{m}{\mathbb{V}(m)}\right) = m. \tag{3.16}$$

(iii) $\lim_{z\searrow 0} \mathcal{R}_\nu(z) = m_0 \geq -\infty$.
(iv) $\lim_{z\searrow 0} z\mathcal{R}_\nu(z) = 0$.

(*Of course, the only new contribution of* (iv) *is the case $m_0 = -\infty$.*)

Proof

(i) Choose $0 < x_1 < x_2 < G(b)$. Then $\mathcal{R}_\nu(x_j) = K_\nu(x_j) - 1/x_j$ are well defined. Let $u_j = K_\nu(x_j) = \mathcal{R}_\nu(x_j) + 1/x_j$ so that $x_j = G_\nu(u_j)$. Clearly, $u_1 > u_2 > b$. Then (3.10) says $1/G(u_1) - 1/G(u_2) > u_1 - u_2$ so $1/x_1 - 1/x_2 > K_\nu(x_1) - K_\nu(x_2)$, i.e., $\mathcal{R}_\nu(x_1) < \mathcal{R}_\nu(x_2)$.
(ii) This is the same as (3.9).
(iii) Since the limit exists by part (i), this is a consequence of (3.16).
(iv) By (3.16) with $z = m/\mathbb{V}(m)$, we have $z\mathcal{R}_\nu(z) = m^2/\mathbb{V}(m) \to 0$ as $m \to m_0$ by Corollary 3.6. Since $\mathcal{R}_\nu$ is increasing on $(0, G(b))$, this ends the proof. □

Corollary 3.9 *$m \mapsto m/\mathbb{V}(m)$ is strictly increasing and smooth function on* (m_0, m_+).

Proof We rewrite (3.16) as $m/\mathbb{V}(m) = \mathcal{R}_\nu^{\langle -1\rangle}(m)$, and use the fact that $\mathcal{R}$ is smooth and strictly increasing. □

It is worth mentioning here that one can use the R-transform to determine the domain of means and the pseudo-variance function of a CSK family, and even to define them. In fact, $(m_0, m_+) = \mathcal{R}_\nu((0, G(B(\nu))))$, and $\mathbb{V}$ is nothing but the function which gives $\mathcal{R}_\nu(z)/z$ as a function of $\mathcal{R}_\nu(z)$. For example, if ν is the inverse semicircle law with $\mathcal{R}_\nu(z) = -p/\sqrt{z}$, we have that $\mathcal{R}_\nu(z)/z = \frac{(\mathcal{R}_\nu(z))^3}{p^2}$ so that the pseudo-variance function of the generated CSK family is equal to $\frac{m^3}{p^2}$. This is the analogue of the fact that, for the classical exponential families, the variance function is the function which gives the second derivative of the cumulant function in terms of the first derivative.

Proposition 3.10 *Let $\mathbb{V}_\nu$ be the pseudo-variance function of the one sided CSK family generated by a probability measure ν with support bounded from above and with the mean $-\infty \le m_0 < \infty$. Then for $\alpha > 0$ such that $\nu^{\boxplus\alpha}$ is defined, the support of $\nu^{\boxplus\alpha}$ is bounded from above and for $m > \alpha m_0$ close enough to αm_0,*

$$\mathbb{V}_{\nu^{\boxplus\alpha}}(m) = \alpha \mathbb{V}_\nu(m/\alpha). \tag{3.17}$$

Proof We first show that the support of $\nu^{\boxplus\alpha}$ is bounded from above. For functions with support bounded from above, $\mathcal{R}_\nu$ is univalent in a domain that contains some open interval $(0, \delta)$. Therefore, $\mathcal{R}_{\nu^{\boxplus\alpha}}$ is univalent in the same domain. This shows that $G_{\nu^{\boxplus\alpha}}$ is analytic on a domain that contains (c, ∞), where $c = K_{\nu^{\boxplus\alpha}}(\delta)$. So the support of $\nu^{\boxplus\alpha}$ is bounded from above by c, see [5, Proposition 6.1].

From Proposition 3.8(iii), we see that the domain of means for one sided CSK generated by $\nu^{\boxplus\alpha}$ starts at $\lim_{z\searrow 0} \mathcal{R}_{\nu^{\boxplus\alpha}}(z) = \alpha m_0$. So for $m > \alpha m_0$ close enough to αm_0 so that $m/\alpha \in (m_0, m_+)$ and $m/\mathbb{V}_{\nu^{\boxplus\alpha}}(m) \in (0, G(B(\nu)))$ (recall Corollary 3.6) we can apply (3.16) and (3.15) to see that

$$\mathcal{R}_\nu\left(\frac{m}{\mathbb{V}_{\nu^{\boxplus\alpha}}(m)}\right) = \frac{1}{\alpha}\mathcal{R}_{\nu^{\boxplus\alpha}}\left(\frac{m}{\mathbb{V}_{\nu^{\boxplus\alpha}}(m)}\right) = m/\alpha = \mathcal{R}_\nu\left(\frac{m/\alpha}{\mathbb{V}_\nu(m/\alpha)}\right).$$

From Proposition 3.8(i), we know that $\mathcal{R}_\nu$ is one-to-one on $(0, G(B(\nu)))$, so

$$\frac{m}{\mathbb{V}_{\nu^{\boxplus\alpha}}(m)} = \frac{m/\alpha}{\mathbb{V}_\nu(m/\alpha)},$$

and formula (3.17) follows. □

We remark that the restriction of (3.17) to m "close enough" to αm_0 cannot be easily avoided, as we do not have a general formula for the upper end of the domain of means for $\nu^{\alpha\boxplus}$. (For the freely r-stable laws the upper end of the domain of means is $\alpha^r m_+$, so we do not expect a simple general formula.)

3.3 Affine Transformations

Here we collect the formulas that describe the effects of applying an affine transformation to the generating measure.

For $\delta \neq 0$ and $\gamma \in \mathbb{R}$, let $\varphi(\nu)$ be the image of ν under the affine map $x \mapsto \frac{x-\gamma}{\delta}$. In other words, if X is a random variable with law ν then $\varphi(\nu)$ is the law of $(X-\gamma)/\delta$, or $\varphi(\nu) = D_{1/\delta}(\nu \star \delta_{-\gamma})$, where $D_r(\mu)$ denotes the dilation of measure μ by a number $r \neq 0$, i.e., $D_r(\mu)(U) = \mu(U/r)$.

It is well known that $G_{\varphi(\nu)}(z) = \delta G_\nu(\delta z + \gamma)$ and $\mathcal{R}_{\varphi(\nu)}(z) = 1/\delta \mathcal{R}_\nu(z/\delta) - \gamma/\delta$. The effects of the affine transformation on the corresponding CSK family are as follows:

- Point m_0 is transformed to $(m_0 - \gamma)/\delta$. In particular, if $\delta < 0$ then $\varphi(\nu)$ has support bounded from below and then it generates the left-sided $\mathcal{K}_-(\varphi(\nu))$.

- For m close enough to $(m_0-\gamma)/\delta$ the pseudo-variance function is

$$\mathbb{V}_{\varphi(\nu)}(m)=\frac{m}{\delta(m\delta+\gamma)}\mathbb{V}_\nu(\delta m+\gamma). \tag{3.18}$$

In particular, if the variance function exists, then $V_{\varphi(\nu)}(m)=\frac{1}{\delta^2}V_\nu(\delta m+\gamma)$.

A special case worth noting is the reflection $\varphi(x)=-x$. If ν has support bounded from above and its right-sided CSK family $\mathcal{K}_+(\nu)$ has domain of means (m_0,m_+) and pseudo-variance function $\mathbb{V}_\nu(m)$, then $\varphi(\nu)$ generates the left-sided CSK family $\mathcal{K}_-(\varphi(\nu))$ with the domain of means $(-m_+,-m_0)$ and the pseudo-variance function $\mathbb{V}_{\varphi(\nu)}(m)=\mathbb{V}_\nu(-m)$.

3.4 Reproductive Property

Proposition 3.11 *If $\mathbb{V}$ is a pseudo-variance function of a CSK family generated by a probability measure ν with support bounded from above, then for $\lambda\ge 1$ measure*

$$\nu_\lambda := D_{1/\lambda}\big(\nu^{\boxplus\lambda}\big)$$

has also support bounded from above and there is $\delta>0$ such that the pseudo-variance function of the one sided CSK family generated by ν_λ is $\mathbb{V}(m)/\lambda$ for $m\in(m_0,m_0+\delta)$.

If ν is free-infinitely divisible, then the above holds for every $\lambda>0$. Conversely, if for every $\lambda>0$, there is $\delta=\delta(\lambda)>0$ such that $\mathbb{V}(m)/\lambda$ is a pseudo-variance function of some CSK family on $(m_0,m_0+\delta)$, then ν is free-infinitely divisible.

Proof This is closely related to Proposition 3.10 and is similar to [7, Proposition 4.3], see also [6]; the details are omitted. □

4 Quadratic and Cubic Pseudo-Variance Functions

In this section, we review the description of CSK families with quadratic variance functions, adding the precise domain of means, then we analyze certain cubic variance functions and point out the reciprocity relation between these two cases.

4.1 CSK Families with Quadratic Variance Functions

The generating measures of CSK families with quadratic variance functions $V(m)=a-bm+cm^2$ with $a>0$, i.e., with the pseudo-variance functions of the form

$$\mathbb{V}(m)=\frac{m(a-bm+cm^2)}{m-m_0} \tag{4.1}$$

were determined in [7]. (Up to affine transformations, it is enough to consider $m_0=0$ and $a=1$.) These are the so-called free Meixner laws ([2, 14]). Since free Meixner laws are compactly supported, they generate two-sided CSK families. Remark 3.3

can be used to determine the precise domain of means which was not previously available, except for an ad-hoc technique for the semi-circle law in [6, Example 4.1].

Unsurprisingly, the domain of means ends at the rightmost atom of ν when there is one; but may fall strictly inside the support of ν when there are no atoms in $(0,\infty)$. When $m_0 = 0$, a calculation gives

$$m_+ = \begin{cases} \frac{b-\sqrt{b^2-4ac}}{2c} & \text{if either } c>0,\ b>2\sqrt{ac} \text{ or } -1\le c<0; \\ a/b & \text{if } c=0 \text{ and } b>\sqrt{a}\ ; \\ \sqrt{a/(1+c)} & \text{when } \nu \text{ has no atoms in } (0,\infty). \end{cases} \tag{4.2}$$

(We recommend [14] for the treatment of atoms.)

4.2 A Class of Families with Cubic Pseudo-Variance Function

Next, we describe the class of Cauchy–Stieltjes kernel families with pseudo-variance functions of the form

$$\mathbb{V}(m) = m\big(am^2 + bm + c\big), \tag{4.3}$$

with $a > 0$. This class is important because it is related to the quadratic class by a relation of reciprocity which will be introduced in the next section.

Suppose that (4.3) is the pseudo-variance function generated by a distribution ν. Then (3.5) is a quadratic equation for m, so we can use (3.9) to express G as a function of real z large enough. By uniqueness of the analytic extension, we get

$$G_\nu(z) = \frac{b+1+2az-\sqrt{(b+1)^2+4a(z-c)}}{2(c+bz+az^2)}$$

for all z in the upper half plane $\mathbb{C}^+$. The Stieltjes inversion formula (3.8) gives

$$\begin{aligned} \nu(dx) = {} & \frac{\sqrt{4ac-(b+1)^2-4ax}}{2\pi(c+bx+ax^2)} 1_{(-\infty,c-(b+1)^2/(4a))}(x)\,dx \\ & + p(a,b,c)\delta_{-(b+\sqrt{b^2-4ac})/(2a)}, \end{aligned} \tag{4.4}$$

where the weight of the atom $p(a,b,c) = 1 - 1/\sqrt{b^2-4ac}$ if $b^2 > 4ac+1$, and is 0 otherwise. (In particular for $c = 0$, there is an atom at $-b/a$ if $b > 1$, or an atom at 0 if $b < -1$.)

From Proposition 3.4, we see that the domain of means is $(-\infty, m_+)$ with $m_+ = B(\nu) - 1/G(B(\nu))$. A calculation that goes over the cases when the support contains positive numbers shows that

$$m_+ = \begin{cases} -\frac{1+b}{2a} & \text{if } c>0,\ -\sqrt{1+4ac}\le b\le 2\sqrt{ac}-1; \\ -\frac{(b+\sqrt{b^2-4ac})}{2a} & \text{if } c>0 \text{ and } b\le -\sqrt{1+4ac}; \\ -\frac{b+1+\sqrt{(b+1)^2-4ac}}{2a} & \text{if either } c\le 0, \text{ or } b>2\sqrt{ac}-1. \end{cases} \tag{4.5}$$

The most interesting example in this class is the inverse semicircle law with the pseudo-variance function $\mathbb{V}(m) = m^3/p^2$ which corresponds to the case $a = 1/p^2$, $b = 0$ and $c = 0$. We have that

$$G_\nu(z) = \frac{p^2 + 2z - p\sqrt{4z + p^2}}{2z^2},$$

and the density of ν is

$$f(x) = \frac{p\sqrt{-p^2 - 4x}}{2\pi x^2} \tag{4.6}$$

on $(-\infty, -p^2/4)$. This is a free 1/2-stable density, see [4, p. 1054], see also [13].

The domain of means is $(m_0, m_+) = (-\infty, -p^2)$; this can be read out either from $\psi(m) = \frac{p^2}{m(m+p^2)}$, or from Proposition 3.4 where the last case of (4.5) is relevant here.

Similarly, one can use (4.3), (4.4) and (4.5) to get the free analogous of the five other members of the Letac–Mora class with variance function of degree 3. Keeping the names given in [11], we have

(i) Free Abel (or Free Borel–Tanner)

$$\nu(dx) = \frac{1}{\pi(1-x)\sqrt{-x}} 1_{(-\infty,0)}(x)\,dx,$$

with pseudo-variance function $\mathbb{V}(m) = m^2(m-1)$ and domain of the means $(-\infty, 0)$.

(ii) Free Ressel (or Free Kendall)

$$\nu(dx) = \frac{-1}{\pi x\sqrt{-1-x}} 1_{(-\infty,-1)}(x)\,dx,$$

with pseudo-variance function $\mathbb{V}(m) = m^2(m+1)$ and domain of the means $(-\infty, -2)$.

(iii) Free strict arcsine

$$\nu(dx) = \frac{\sqrt{3-4x}}{2\pi(1+x^2)} 1_{(-\infty,3/4)}(x)\,dx,$$

with pseudo-variance function $\mathbb{V}(m) = m(1+m^2)$ and domain of the means $(-\infty, -1/2)$.

(iv) Free large arcsine

$$\nu(dx) = \frac{r\sqrt{4 - 5r^2 - 4(1+r^2)x}}{2\pi(x^2 + r^2(1+x)^2)} 1_S(x)\,dx,$$

where $r > 0$ and $S = (-\infty, \frac{4-5r^2}{4(1+r^2)})$. The pseudo-variance function is

$$\mathbb{V}(m) = m\left(1 + 2m + \frac{1+r^2}{r^2} m^2\right),$$

and domain of the means is $(-\infty, -\frac{3r^2}{2(1+r^2)})$, if $r^2 \leq 4/5$, and $(-\infty, -\frac{3r^2+r\sqrt{5r^2-4}}{2(1+r^2)})$, if $r^2 > 4/5$.

(v) Free Takács

$$\nu(dx) = \frac{\sqrt{-5r^2 - 2r - 1 - 4r(1+r)x}}{2\pi r(1+x)(1+(1+1/r)x)} 1_S(x)\,dx + (1-r)^+\delta_{-1},$$

where $r > 0$ and $S = (-\infty, 1 - \frac{(1+3r)^2}{4r(1+r)})$. The pseudo-variance function is

$$\mathbb{V}(m) = m(1+m)\left(1 + \frac{1+r}{r} m\right),$$

and domain of the means is $(-\infty, -\frac{1+3r+\sqrt{5r^2+2r+1}}{2(1+r)})$.

4.3 Reciprocity

The notion of reciprocity between two natural exponential families is defined by a symmetric relation between the cumulant functions of two generating measures (see [11, Sect. 5]). Similarly, we can define the reciprocity between two Cauchy–Stieltjes Kernel Families by a relation between the R-transforms of the generating distributions.

Definition 4.1 Suppose $\widetilde{\nu}, \nu$ are probability measures with support bounded from above. We say that the corresponding one-sided Cauchy–Stieltjes kernel families $\mathcal{K}_+(\widetilde{\nu})$ and $\mathcal{K}_+(\nu)$ are reciprocal if $m_0 := \int x\nu(dx)$ and $\widetilde{m}_0 := \int x\widetilde{\nu}(dx)$ are of opposite signs and there is $\delta > 0$ such that

$$\mathcal{R}_{\widetilde{\nu}}\big(z\big|\mathcal{R}_\nu(z)\big|\big) = -\frac{1}{\mathcal{R}_\nu(z)} \tag{4.7}$$

for all z in $(0, \delta)$.

In this case, we also say that the distributions $\widetilde{\nu}$ and ν are reciprocal.

We note that $\mathcal{R}_\nu$ is defined for $z > 0$ small enough so, by Proposition 3.8(iv), both sides of the expression (4.7) are well defined for all $z \in (0, \delta)$ when $\delta > 0$ is small enough. We also remark that with $m_0 := \int x\nu(dx) \in [-\infty, \infty)$, in (4.7) we actually have

$$\big|\mathcal{R}_\nu(z)\big| = \begin{cases} \mathcal{R}_\nu(z) & \text{if } m_0 \geq 0; \\ -\mathcal{R}_\nu(z) & \text{if } m_0 < 0 \end{cases}$$

for $z > 0$ close enough to 0.

Note that (4.7) is equivalent to

$$\mathcal{R}_\nu\big(z'\big|\mathcal{R}_{\widetilde{\nu}}(z')\big|\big) = -\frac{1}{\mathcal{R}_{\widetilde{\nu}}(z')} \tag{4.8}$$

for all $z' > 0$ small enough, so reciprocity is a symmetric relation. Indeed, we first note that (4.7) implies $m_0 = -1/\widetilde{m}_0$ even if $m_0 = 0$ or $\widetilde{m}_0 = 0$. We consider separately the cases $m_0 \geq 0$ and $m_0 < 0$.

If $m_0 \geq 0$, we set $z' = z\mathcal{R}_\nu(z)$. From Proposition 3.8(iv), we see that $\mathcal{R}_\nu(z')$ is well defined for small enough $z > 0$. Then (4.7) is equivalent to

$$\mathcal{R}_{\widetilde{\nu}}(z') = -\frac{1}{\mathcal{R}_\nu(z)} \quad \text{and} \quad z = -z'\mathcal{R}_{\widetilde{\nu}}(z'). \tag{4.9}$$

Hence

$$\mathcal{R}_\nu\left(-z'\mathcal{R}_{\widetilde{\nu}}(z')\right) = -\frac{1}{\mathcal{R}_{\widetilde{\nu}}(z')}. \tag{4.10}$$

Since $\mathcal{R}_{\widetilde{\nu}}(z') < 0$, because $\widetilde{m}_0 < 0$, this is nothing but (4.8).

If $m_0 < 0$, we use the same reasoning using $z' = -z\mathcal{R}_\nu(z)$.

The reciprocity between $\mathcal{K}_+(\widetilde{\nu})$ and $\mathcal{K}_+(\nu)$ may also be expressed using the variance functions. More precisely, we have:

Theorem 4.1 *Let $\mathbb{V}_{\widetilde{\nu}}$ and $\mathbb{V}_\nu$ be the pseudo-variance functions of the right-sided Cauchy–Stieltjes kernel families generated by $\widetilde{\nu}$ and ν, with means $\widetilde{m}_0$ and m_0, respectively. Then $\mathcal{K}_+(\widetilde{\nu})$ and $\mathcal{K}_+(\nu)$ are reciprocal if and only if $m_0 = -1/\widetilde{m}_0$ (it is understood that $-1/0 = -\infty$), and*

$$\mathbb{V}_{\widetilde{\nu}}(m) = -|m|^3\mathbb{V}_\nu\left(-\frac{1}{m}\right) \tag{4.11}$$

for all $m > \widetilde{m}_0$ close enough to $\widetilde{m}_0$.

Proof Suppose $m_0 = -1/\widetilde{m}_0$ and (4.11) holds for all $\widetilde{m}_0 < m < M$. Decreasing M if necessary, we may ensure that $1/m \in (m_0, m_+)$. Choose $z' > 0$ such that $z' < M/\mathbb{V}_{\widetilde{\nu}}(M)$. Since $m \mapsto m/\mathbb{V}_{\widetilde{\nu}}(m)$ is a continuous function, we can find $m' > \widetilde{m}_0$ such that $z' = m'/\mathbb{V}_{\widetilde{\nu}}(m')$. Let $m = -1/m'$ and $z := m/\mathbb{V}_\nu(m)$. For our choice of m, m', from (3.16) we get

$$\mathcal{R}_{\widetilde{\nu}}(z') = -\frac{1}{\mathcal{R}_\nu(z)}, \tag{4.12}$$

and to deduce (4.7) we only need to note that $z' = z|\mathcal{R}_\nu(z)|$. The latter is a consequence of (4.11) and (3.16).

To prove the converse implication, suppose that (4.7) holds. Then taking the limit as $z \searrow 0$ we deduce that $m_0 = -1/\widetilde{m}_0$. Therefore, for all $m > m_0$ close enough to m_0 so that $z := m/\mathbb{V}_\nu(m)$ and $z' := z|\mathcal{R}_\nu(z)|$ are within the domain of (4.7), we deduce that (4.12) holds. As previously, for m close enough to m_0, z' is close enough to 0 so that we can find $m' > \widetilde{m}_0$ such that $z' = m'/\mathbb{V}_{\widetilde{\nu}}(m')$. Then (4.12) says that $mm' = -1$ (here we use (3.16) again), so the identities $z = m/\mathbb{V}_\nu(m)$ and $z' = m'/\mathbb{V}_{\widetilde{\nu}}(m')$ imply (4.11). $\square$

Remark 4.1 In particular, if $m_0 \geq 0$ then (4.11) says that for $m < 0$ close enough to m_0 we have $\mathbb{V}_{\widetilde{\nu}}(m) = m^3 \mathbb{V}_\nu(-1/m)$.

Of course, one can combine reciprocity with affine action $\varphi(x) = -x$. Correspondingly, one can extend the definition of reciprocity to pairs $\mathcal{K}_\pm(\nu)$ and $\mathcal{K}_\pm(\widetilde{\nu})$.

4.3.1 Example

As mentioned above, we have been interested in the class of the Cauchy–Stieltjes kernel families with pseudo-variance functions of the form (4.3) because it is the class the Cauchy–Stieltjes kernel families with pseudo-variance functions of degree three which are obtained by reciprocity from the families with quadratic variance functions. In fact, the CSK family generated with pseudo-variance (4.3) is reciprocal with the right-sided part of the quadratic CSK family with (pseudo)-variance (4.1) for $m_0 = 0$. In particular, the semicircle family with variance function equal to $\frac{1}{p^2}$, $m_+ = 1/p$ and the inverse semicircle family with pseudo-variance function equal to $\frac{m^3}{p^2}$, $m_+ = -p^2$ are reciprocal. For $z > 0$, their R-transforms $\mathcal{R}_{\widetilde{\nu}}(z) = z/p^2$ and $\mathcal{R}_\nu(z) = -p/\sqrt{z}$ are related by formula (4.7).

Comparing (4.2) and (4.5), we see that for reciprocal families the upper ends of the domain of means do not satisfy a simple relation.

We remark that for $c < 0$ the reciprocal of the free-infinitely divisible cubic family is free-binomial law which is not free-infinitely divisible.

Acknowledgements The authors thank M. Bożejko for information about free 1/2-stable distribution and J. Wesołowski for a copy of [16]. We also thank two anonymous referees for their detailed reports that helped to improve the paper.

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