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Free Jacobi Process

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Abstract In this paper, we define and study two parameters dependent free processes (λ, θ) called *free Jacobi*, obtained as the limit of its matrix counterpart when the size of the matrix goes to infinity. The main result we derive is a free SDE analogous to that satisfied in the matrix setting, derived under injectivity assumptions. Once we did, we examine a particular case for which the spectral measure is explicit and does not depend on time (stationary). This allows us to determine easily the parameters range ensuring our injectivity requirements so that our result applies. Then, we show that under an additional condition of invertibility at time $t = 0$, this range extends to the general setting. To proceed, we set a recurrence formula for the moments of the process via free stochastic calculus.

Keywords Unitary Brownian matrix · Free Jacobi process · Free additive Brownian motion · Free multiplicative Brownian motion · Polar decomposition

1 Introduction

The classification of classical diffusions relies on three central and interrelated processes: Brownian motion, squared Bessel and Jacobi processes. The two latters can be defined as (see [22]) the unique strong solutions of

$$dR_t = 2\sqrt{Z_t}dW_t + \delta\, dt,$$
$$dJ_t = 2\sqrt{J_t(1-J_t)}dB_t + (p - (p+q)J_t)\, dt$$

respectively, where δ, p, q are positive and W, B are two standard BMs. Except for the BM, these names are referring to Laguerre and Jacobi polynomials which are

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Table 1 Stochastic processes and their free analogs

Matrix size	Hermite	Laguerre	Jacobi
$d = 1$	Br. motion	Squared Bessel	Jacobi
$d > 1$	Hermitian Br. matrix	Wishart/Laguerre	matrix Jacobi
$d = \infty$	Free Br. motion	Free Wishart	?

eigenfunctions of the corresponding generators (see [3, 25]). A similar statement holds for BMs with Hermite polynomials. Then, their matrix extensions were developed through several works by Dyson [17] for Hermitian Brownian matrices, Bru [10] and others for Wishart and Laguerre processes and Doumerc for real and complex matrix Jacobi processes [15]. A parallel interpretation using multivariate orthogonal polynomials can be found in [4] and [20]. Then, it was quite natural to have an insight into the infinite dimensional case, that is when the size of the matrix goes to infinity. This started with Voiculescu for independent large random matrices in the so-called *Gaussian unitary ensemble* [24]. In this way, several results were derived for unitary matrices and in particular unitary processes [6, 18]. Few years later, free Wishart processes appeared in [12]. They are one parameter-dependent processes defined as a limit of their matrix analogs, Laguerre processes. Authors extend well-known results from matrix theory to this context via free stochastic calculus. For instance, a free SDE of squared Bessel type was derived. All what we said can be summarized in the array drawn in Table 1.

Our task consists in filling the remaining empty box. Our approach follows the one in [12] however, as we will see and as always, the Jacobi setting is more sophisticated and needs more computations. Here, we do recall some definitions and fix some notations that will be frequently used throughout the paper.

2 Definitions and Notations

2.1 Matrix Jacobi Process

We refer to [15] for facts on real matrix Jacobi processes. In the sequel, we are interested in its complex analog. Let $Y(d)$ be a $d \times d$ unitary Brownian matrix, that is a unitary matrix-valued process such that:

- $Y_0(d) = I_d$.
- $(Y_{t_i}(d)Y_{t_{i-1}}^{-1}(d), 1 \le i \le n)$ are independent for any collection $0 < t_1 < \cdots < t_n$.
- $Y_t(d)Y_s^{-1}(d)$, $s < t$ has the same distribution as $Y_{t-s}(d)$ [6].

Let $1 \le m, p \le d$ and denote by X the $m \times p$ upper left corner of $Y(d)$:

$$X \oplus 0 = P_m Y_t(d) Q_p := \begin{pmatrix} I_m & \\ & 0 \end{pmatrix} Y_t(d) \begin{pmatrix} I_p & \\ & 0 \end{pmatrix}.$$

Then $J(m) := XX^\star$ is a $m \times m$ complex matrix Jacobi process of parameters $(p, d-p)$ such that $0 \le J_t(m) \le I_m$. If X_t is the $m \times p$ left corner of $\tilde{Z}Y_t(d)$ where

$\tilde{Z}$ is a $d \times d$ unitary random matrix independent of Y, then $XX^\star$ is a $m \times m$ complex matrix Jacobi process starting from $X_0X_0^\star$. As for the real matrix case [15], $I_m - J$ is still a complex matrix Jacobi process of parameters $(d-p, p)$.

2.2 Free Probability

Recall that a noncommutative probability space (NCPS) is given by a unital algebra $\mathcal{A}$ with a linear functional $\Phi : \mathcal{A} \to \mathbb{C}$. An element in $(\mathcal{A}, \Phi)$ is called a random variable. The subalgebras $(\mathcal{A}_i)_{i\in I}$ are said to be free if for all $a_i \in \mathcal{A}_{j_i}$ such that $\Phi(a_i) = 0$ one has

$$\Phi\Big(\prod a_i\Big) = 0, \quad j_i \in I,\ j_i \neq j_{i+1}.$$

$a_1, \dots, a_n \in \mathcal{A}$ are free if the subalgebras $\mathcal{A}$ generated by $\{\mathbf{1}, a_i\}$ are to be ($\mathbf{1}$ denotes the unit of $\mathcal{A}$). The distribution of a random variable $a \in \mathcal{A}$ is given by its moments $\Phi(a^r)$, $r \geq 0$. Similarly, the distribution of $a_1, \dots, a_n$ is given by $\Phi(L(a_1, \dots, a_n))$ for all noncommutative polynomial $L \in \mathbb{C}[a_1, \dots, a_n]$. When this family is free, this factorizes into products of moments of a_i so that it is entirely determined by a_i's distributions. A famous realization of random variables is illustrated by random matrices of all order finite moments: the algebra is

$$\mathcal{A}_d := \bigcap_{p>0} L^p(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}) \otimes \mathcal{M}_d(\mathbb{C})$$

where $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is a filtered probability space and $\mathcal{M}_d(\mathbb{C})$ stands for the set of $d \times d$ complex matrices, equipped with the normalized trace expectation $\mathbb{E} \otimes \mathrm{tr}_d$. We say that the family of $d \times d$ random matrices $(A_s(d))_{s\in S}$ converges in distribution to the family of random variables $(a_s)_{s\in S}$ in some NCPS $(\mathcal{A}, \Phi)$ if and only if:

$$\lim_{d\to\infty} \mathbb{E}[\mathrm{tr}_d(A_{s_1}(d) \cdots A_{s_r}(d))] = \Phi(a_{s_1} \cdots a_{s_r}), \quad s_1, \dots, s_r \in S,$$

which implies that

$$\lim_{d\to\infty} \mathbb{E}[\mathrm{tr}_d(A_s^k(d))] = \Phi(a_s^k), \quad k \geq 1.$$

$(A_s(d), s \in S)$ is said to be *asymptotically free* if $(a_s)_{s\in S}$ form a free family. As stated before, independent random matrices enjoying some invariance properties are shown to be asymptotically free random variables in some NCPS. The starting point was with Voiculescu for independent $d \times d$ matrices belonging to the GUE with variance $1/d$ [24]. This is used to show that the normalized Hermitian BM converges in distribution to the *free additive Brownian motion*: it is a collection of self-adjoint random variables indexed by time, say $(a_t)_{t\geq 0}$ (or process) with free increments $(a_t - a_s,\ s < t)$ and such that $a_t - a_s$ has the same law as a_{t-s} given by the semicircle law σ_{t-s} (free additive Lévy process, [12]) where

$$\sigma_t(dx) = \frac{1}{2\pi t}\sqrt{4t - x^2}\,\mathbf{1}_{[-2\sqrt{t}, 2\sqrt{t}]}(x)dx.$$

A similar result we will use later is due to Biane [6]: the unitary Brownian matrix converges in distribution to the free multiplicative Brownian motion Y in some NCPS $(\mathscr{A}, \Phi)$. Recall that Y is unitary, $Y_0 = \mathbf{1}$, has free left increments, that is, for a collection of times $0 < t_1 < t_2 < \cdots < t_n$, $Y_{t_n}Y_{t_{n-1}}^{-1}, \dots, Y_{t_2}Y_{t_1}^{-1}$ are free and the law of Y_t, say ν_t, is given by the so-called Σ-transform (see [6]):

$$\Sigma_{\nu_t}(z) = e^{\frac{t}{2}\frac{1+z}{1-z}}, \qquad \nu_{t+s} = \nu_t \boxtimes \nu_s$$

where $\boxtimes$ denotes the free multiplicative convolution (free multiplicative Lévy process, see [5, 6]). For our purposes, we shall consider a von Neumann algebra $\mathscr{A}$ endowed with a faithful tracial state Φ (see [16] for details). This is known as a $W^\star$ NCPS. The L^q-norm is given by $\|a\|_{L^q} := \Phi[(aa^\star)^{q/2}]^{1/q}$ for $1 \le q < \infty$. The L^∞-norm or the algebra-norm is defined as the limit of the L^q-norm as q tends to infinity. It will be denoted by $\|\cdot\|_{L^\infty}$ or by $\|\cdot\|$ if there is no confusion.

3 Free Jacobi Process

Let $Y_t(d(m))$ be a $d(m) \times d(m)$ unitary Brownian matrix with $m \times p(m)$ upper left corner X_t such that:

$$\lim_{m\to\infty} \frac{m}{p(m)} = \lambda > 0, \qquad \lim_{m\to\infty} \frac{p(m)}{d(m)} = \theta \in]0, 1] \quad \text{so that} \lim_{m\to\infty} \frac{m}{d(m)} = \lambda\theta.$$

Let $Q_m := Q_{p(m)}$ with Q_p defined in Sect. 2.1. Then, $J_t(m) = X_t X_t^\star$ and:

$$A_t(m) := J_t(m) \oplus 0_{d(m)-m} = P_m Y_t(d(m)) Q_m Y_t^\star(d(m)) P_m.$$

It follows that:

$$\lim_{m\to\infty} \mathrm{tr}_{d(m)}(P_m) = \lambda\theta,$$
$$\lim_{m\to\infty} \mathrm{tr}_{d(m)}(Q_m) = \theta, \quad \theta \in]0, 1],$$

$$\lim_{m\to\infty} \mathbb{E}\{\mathrm{tr}_m[J_{t_1}(m)J_{t_2}(m)\cdots J_{t_n}(m)]\}$$
$$= \lim_{m\to\infty} \frac{d(m)}{m} \mathbb{E}\{\mathrm{tr}_{d(m)}[A_{t_1}(m)A_{t_2}(m)\dots A_{t_n}(m)]\}$$

for any collection $t_1, \dots, t_n$. Next, we make use of the following result ([18], p. 157):

Theorem 3.1 *Let $(U_s(m))_s$ be a family of independent $m \times m$ unitary random matrices such that the distribution of $U_s(m)$ is equal to that of $VU_s(m)V^\star$ for any unitary matrix V (unitary invariant) and such that $U_s(m)$ converges in distribution. Let $(D_t(m))_t$ be a family of $m \times m$ constant matrices converging in distribution and such that* $\sup_m \|D_t(m)\| < \infty$. *Then the families*

$$\{U_s(m), U_s^\star(m)\}_s, \{D_t(m), D_t^\star(m), t \ge 0\}$$

are asymptotically free as $m \to \infty$.

Note that $\{Y_t(d(m))Y_s^{-1}(d(m))\}_{0\le s<t}$ is a unitary invariant family since $(Y_t)_{t\ge 0}$ is right-left invariant. By the freeness of increments of Y mentioned above, Theorem 3.1 claims that:

$$\{(Y_t(d(m)))_{t\ge 0}, (Y_t^\star(d(m)))_{t\ge 0}\}, \qquad \{P_m, Q_m\}$$

are asymptotically free. Thus, its limiting distribution in $(\mathcal{A}_{d(m)}, \mathbb{E}\otimes \mathrm{tr}_{d(m)})$ as m goes to infinity is the distribution of $\{(Y_t)_{t\ge 0}, (Y_t^\star)_{t\ge 0}\}, \{P, Q\}$ in $(\mathcal{A}, \Phi)$ such that

- Y is a free multiplicative Brownian motion in $(\mathcal{A}, \Phi)$.
- P is a projection with $\Phi(P) = \lambda\theta \le 1$, $\theta \in]0, 1]$.
- Q is a projection with $\Phi(Q) = \theta$.
- $QP = PQ = \begin{cases} P & \text{if } \lambda \le 1, \\ Q & \text{if } \lambda > 1. \end{cases}$
- $\{(Y_t)_{t\ge 0}, (Y_t^\star)_{t\ge 0}\}$ and $\{P, Q\}$ are free.

Hence, we deduce that the limiting distribution of the complex matrix Jacobi process $(J_t)_{t\ge 0}$ in $(\mathcal{A}_m, \mathbb{E}\otimes \mathrm{tr}_m)$ is the distribution of $(PY_tQY_t^\star P)_{t\ge 0}$ in $P\mathcal{A}P$ equipped with the state

$$\tilde{\Phi} = \frac{1}{\Phi(P)}\Phi_{|P\mathcal{A}P} = \frac{1}{\lambda\theta}\Phi_{|P\mathcal{A}P}.$$

$(P\mathcal{A}P, \tilde{\Phi})$ is called the *compressed* NCPS. This suggests to define the free Jacobi process as follows:

Definition 3.1 *Let $(\mathcal{A}, \Phi)$ be a $W^\star$ NCPS. Let $\theta \in]0, 1]$ and $\lambda > 0$ such that $\lambda\theta \le 1$. Let P and Q be two projections such that*

$$\Phi(Q) = \theta, \qquad \Phi(P) = \lambda\theta, \quad \textit{and} \quad PQ = QP = \begin{cases} P & \textit{if } \lambda \le 1, \\ Q & \textit{if } \lambda > 1. \end{cases}$$

Let Y be a free multiplicative Brownian motion such that $\{(Y_t)_{t\ge 0}, (Y_t^\star)_{t\ge 0}\}$ and $\{P, Q\}$ is a free family in $(\mathcal{A}, \Phi)$. We will say that a process J in a $W^\star$ NCPS (B, Ψ) is a free Jacobi process with parameters (λ, θ), denoted by $FJP(\lambda, \theta)$, if its distribution in (B, Ψ) is equal to the distribution of the process $(PY_tQY_t^\star P)_{t\ge 0}$ in $(P\mathcal{A}P, (1/\Phi(P))\Phi_{|P\mathcal{A}P})$. This process starts from $J_0 = P$ if $\lambda \le 1$ and $J_0 = Q$ if $\lambda > 1$.

Equivalently, the law of J is the limiting distribution of a complex matrix Jacobi process when $\frac{m}{p(m)} \underset{m\to\infty}{\longrightarrow} \lambda$ and $\frac{m}{d(m)} \underset{m\to\infty}{\longrightarrow} \lambda\theta$.

We also define the free Jacobi process starting from J_0:

Definition 3.2 *Let Y be a free multiplicative Brownian motion and Z a unitary operator free with Y. Then, the process defined by $\tilde{Y} = YZ$ is a free multiplicative Brownian motion starting at $\tilde{Y}_0 = Z$. Moreover, if Z is free with $\{P, Q\}$, then the process $\tilde{J}$ defined by*:

$$\tilde{J}_t := P\tilde{Y}_tQ\tilde{Y}_t^\star P$$

is called a free Jacobi process with parameters (λ, θ) and starting from $\tilde{J}_0 = PZQZ^\star P$.

Since $P - J = PY_t(\mathbf{1} - Q)Y_t^\star P$ and $\mathbf{1} - Q$ is a projection, then:

Corollary 3.1 *If J is a $FJP(\lambda, \theta)$ with λ, θ as above and starting from J_0, then $P - J$ is still a $FJP(\lambda\theta/(1-\theta), 1-\theta)$ starting from $P - J_0$.*

For the sake of simplicity, we will write Y for a free multiplicative Brownian motion starting from Y_0 and J for a free Jacobi process ($FJP(\lambda, \theta)$) starting at J_0.

4 Free Jacobi Process and Free Stochastic Calculus

We refer to [6] and [7] for free stochastic calculus and notations. Let $(\mathcal{A}_t)_{t\geq 0}$ be an increasing family of unital, weakly closed $\star$-subalgebras of the von Neumann algebra $\mathcal{A}$. Then, $(\mathcal{A}, (\mathcal{A}_t), \Phi)$ is called a filtered $W^\star$ NCPS. Since Φ is tracial and faithful, there exists a unique conditional expectation denoted by $\Phi(\cdot|\mathcal{A}_t)$. Let $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ be the von Neumann tensor product algebra equipped with the tracial state $\Phi \otimes \Phi^{\mathrm{op}}$. A bi-process U is an element in $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ and is written as $(U_t = \sum_i A_t^i \otimes B_t^i)$. It is adapted if $U_t \in \mathcal{A}_t \otimes \mathcal{A}_t$ for all $t \geq 0$. The prefix "bi" and the superscript "op" refer to the fact that the integrator can be multiplied both to the left and to the right due to the noncommutativity. Furthermore, adapted bi-processes form a complex vector space that we endow with the norm:

$$\|U\|_\infty = \left(\int_0^\infty \|U_s\|^2_{L^\infty(\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}})} ds\right)^{1/2}$$

where the $L^\infty(\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}})$ norm is defined by:

$$\|\cdot\|_{L^\infty(\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}})} := \lim_{p\to\infty} \|\cdot\|_{L^p(\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}})}.$$

The completion of this space is denoted by $\mathcal{B}_\infty^a$. Recall also that, for fixed $t > 0$ and $U \in \mathcal{B}_\infty^a$, we have:

$$\int_0^t U_s \sharp dX_s = \int_0^\infty U_s \mathbf{1}_{[0,t]}(s) \sharp dX_s$$

where X is a free additive Brownian motion and

$$U_s \sharp dX_s := \sum_i A_s^i dX_s B_s^i.$$

For any two adapted bi-processes N and M belonging to $\mathcal{B}_\infty^a$:

$$\Phi\left\{\int_0^t N_s \sharp dX_s \int_0^t M_s \sharp dX_s\right\} = \int_0^t \langle N_s, M_s^\star\rangle\, ds, \tag{1}$$

where $\langle\ \rangle$ is the inner product in $L^2(\mathcal{A}, \Phi) \otimes L^2(\mathcal{A}, \Phi)$ (namely, if $N = a \otimes a'$ and $M = b \otimes b'$, then $M^\star = (b')^\star \otimes b^\star$ and $\langle N, M^\star\rangle = \Phi(ab')\Phi(a'b)$). Consider the process $(J_t := PY_tQY_t^\star P)_{t\geq 0}$, where P, Q are two projections as in the definition above, Y is a free multiplicative Brownian motion in $(\mathcal{A}, \Phi)$. Recall that Y satisfies the free SDE (see [6]):

$$dY_t = i\, dX_t\, Y_t - \frac{1}{2} Y_t dt, \qquad Y_0 \in \mathcal{A}$$

where $(X_t)_{t\geq 0}$ is a free additive Brownian motion in $(\mathcal{A}, \Phi)$. By free Itô's formula [6, 7, 19], we get:

$$\begin{aligned} d(Y_t Q Y_t^\star) &= (dY_t) Q Y_t^\star + Y_t Q (dY_t^\star) + \Phi(Y_t Q Y_t^\star) dt \\ &= i dX_t Y_t Q Y_t^\star - i Y_t Q Y_t^\star dX_t - Y_t Q Y_t^\star dt + \theta dt \end{aligned}$$

since X_t is self-adjoint. Thus, the free Jacobi process satisfies:

$$\begin{aligned} dJ_t &= P d(Y_t\, Q\, Y_t^\star) P = i P dX_t Y_t Q Y_t^\star P - i P Y_t Q Y_t^\star dX_t P - J_t\, dt + \theta P dt \\ &= i P Y_t (Y_t^\star dX_t Y_t) Q Y_t^\star P - i P Y_t Q (Y_t^\star dX_t Y_t) Y_t^\star P - J_t\, dt + \theta P dt, \end{aligned} \tag{2}$$

since Φ is tracial and Y_t is unitary (by definition). The next step consists of characterizing the process $(Y_t^\star dX_t Y_t)_{t\geq 0}$. This needs the following characterization of the free additive Brownian motion [8, 12] which is the free analogue of the Lévy characterization:

Theorem 4.1 *Let $(\mathcal{A}_s, s \in [0,1])$ be an increasing family of von Neumann subalgebras in a noncommutative probability space $(\mathcal{A}, \Phi)$, and let $(Z_s = (Z_s^1, \dots, Z_s^m); s \in [0,1])$ be an m-tuple of self-adjoint $(\mathcal{A}_s)$-adapted processes such that*:

- *Z is bounded and $Z_0 = 0$.*
- *$\Phi(Z_t^i | \mathcal{A}_s) = Z_s^i$ for all $1 \leq i \leq m$.*
- *$\Phi(|Z_t^i - Z_s^i|^4) \leq K(t-s)^2$ for some constant K and for all $1 \leq i \leq m$.*
- *For any $l, p \in \{1, \dots, m\}$ and all $A, B \in \mathcal{A}_s$, one has*:

$$\Phi(A(Z_t^p - Z_s^p) B (Z_t^l - Z_s^l)) = \mathbf{1}_{\{p=l\}} \Phi(A)\Phi(B)(t-s) + o(t-s),$$

then Z is a m-dimensional free Brownian motion.

It follows that:

Lemma 4.1 *The process $(S_t)_{t\geq 0} := (\int_0^t Y_s^\star dX_s Y_s)_{t\geq 0}$ is an $\mathcal{A}_t$-free Brownian motion.*

Proof One has to check the four conditions mentioned above are satisfied. Note that for all $T > 0$, $Y_t^\star \otimes Y_t \mathbf{1}_{[0,T]} \in \mathcal{B}_\infty^a$ since $\|Y_t\| = \|Y_t^\star\| = 1$. Take $A, B \in \mathcal{A}_s$, then (using (1) in the second line):

$$\begin{aligned} \Phi(A(S_t - S_s)B(S_t - S_s)) &= \Phi\left\{ \int_s^t A Y_r^\star dX_r Y_r \int_s^t B Y_r^\star dX_r Y_r \right\} \\ &= \Phi\left\{ \int_s^t (A Y_r^\star \otimes Y_r) \sharp dX_r \int_s^t (B Y_r^\star \otimes Y_r) \sharp dX_r \right\} \end{aligned}$$

$$= \int_s^t \Phi(Y_r A Y_r^\star)\Phi(Y_r B Y_r^\star)dr$$
$$= \Phi(A)\Phi(B)(t-s),$$

since Φ is tracial and Y_t is unitary. Hence, the fourth condition is fulfilled. For the third, we follow in the same way and use again the fact that Y_t is unitary to get:

$$\Phi(|S_t - S_s|^4) \le (t-s)^2.$$

The second condition results from the fact $(\int_0^t Y_s^\star dX_s Y_s)_{t\ge 0}$ defines an $\mathcal{A}_t$-martingale. Finally, it is easily seen from the end of the proof of Theorem 3.2.1 in [7] that:

$$\left\|\int_0^t Y_s^\star dX_s Y_s\right\| := \left\|\int_0^t Y_s^\star dX_s Y_s\right\|_{L^\infty(\mathcal{A})}$$
$$\le 2\sqrt{2}\left(\int_0^t \|Y_s^\star \otimes Y_s\|^2_{L^\infty(\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}})} ds\right)^{1/2} = 2\sqrt{2t}$$

since the integrand's norm is equal to 1 from the unitarity of Y_s. □

Thus, (2) transforms to:

$$dJ_t = iPY_t\, dS_t QY_t^\star P - iPY_t Q\, dS_t\, Y_t^\star P - J_t\, dt + \theta P dt$$
$$= iPY_t(1-Q)\, dS_t\, QY_t^\star P - iPY_t Q\, dS_t\, (1-Q)Y_t^\star P - J_t\, dt + \theta P dt.$$

In order to use the polar decomposition of $P - J_t$, we write:

$$P - J_t = (PY_t - PY_t Q)(Y_t^\star P - QY_t^\star P) := C^\star C$$

since Y_t is unitary, $Q^2 = Q$ and $P^2 = P$. Hence

$$QY_t^\star P = R_t\sqrt{J_t}, \qquad C = (1-Q)Y_t^\star P = V_t\sqrt{P - J_t},$$

which gives:

$$dJ_t = \sqrt{P-J_t}(iV_t^\star\, dS_t R_t)\sqrt{J_t} + \sqrt{J_t}((iR_t)^\star\, dS_t\, V_t)\sqrt{P-J_t} + (\theta P - J_t)dt. \tag{3}$$

Remark An elementary and needed relation is:

$$\sqrt{J_t} R_t^\star V_t \sqrt{P - J_t} = PY_t Q(1-Q)Y_t^\star P = 0 \tag{4}$$

since $Q = Q^2$.

Proposition 4.1 *Suppose that J_t and $P - J_t$ are injective operators in $P\mathcal{A}P$. Then, the following holds*:

- *$R_t P = R_t$ and $V_t P = V_t$.*

- $R_t^\star R_t = P$ *and* $V_t^\star V_t = P$.
- $PR_t^\star V_t P = PV_t^\star R_t P = 0$.

Proof Recall first that if T is an operator in $\mathcal{A}$, then the support E of T is the orthogonal projection on $(\ker T)^\perp = \overline{\operatorname{Im} T^\star}$ and satisfies $TE = T$ (see A. III in [16]). Furthermore, if we consider the polar decomposition of T, namely $T = A|T| = A(T^\star T)^{1/2}$, then E is also the support of A and the latter is partially isometric, that is $A^\star A = E$ and $AA^\star = F$ where F is the support of $T^\star$. Thus, the two first assertions follow if we prove that P is the support of both J_t and $P - J_t$. Indeed, the injectivity of J_t in $P\mathcal{A}P$ implies that $\ker J_t = \ker P$. Thus, we claim that P is the support of J_t ($(\ker P)^\perp = \operatorname{Im} P$) and the same result holds for $P - J_t$. The third is obvious when J_t and $P - J_t$ are invertible. Else, (4) is written in $P\mathcal{A}P$:

$$0 = \sqrt{J_t} R_t^\star V_t \sqrt{P - J_t} = \sqrt{J_t}(PR_t^\star V_t P)\sqrt{P - J_t}.$$

Since both J_t and $P - J_t$ are injective operators in $P\mathcal{A}P$, then:

$$(PR_t^\star V_t P)\sqrt{P - J_t} = 0 \Rightarrow \sqrt{P - J_t}(PV_t^\star R_t P) = 0 \Rightarrow (PV_t^\star R_t P) = 0. \qquad \square$$

Corollary 4.1 *Under the same assumption of Proposition 4.1, the process* $(W_t)_{t\geq 0}$ *defined by* $W_t := (i/\sqrt{\Phi(P)})\int_0^t (PV_s^\star \otimes R_s P)\sharp dS_s$ *is a* $P\mathcal{A}_t P$*-complex free Brownian motion.*

Proof Let us first recall that a process $Z : \mathbb{R}_+ \to \mathcal{A}$ is a complex $(\mathcal{A}_t)$-Brownian motion if it can be written $Z = (X^1 + \sqrt{-1}X^2)/\sqrt{2}$, where (X^1, X^2) is a two-dimensional $(\mathcal{A}_t)$-free Brownian motion. Note also that $(iZ_t)_{t\geq 0}$ is still a complex $(\mathcal{A}_t)$-Brownian motion since $(-X^2)$ is an $(\mathcal{A}_t)$-free Brownian motion. So, we shall show that:

$$\left(\tilde{W}_t := (1/\sqrt{\Phi(P)})\int_0^t (PV_s^\star \otimes R_s P)\sharp dS_s\right)_{t\geq 0}$$

is a $P\mathcal{A}_t P$-complex free Brownian motion. To proceed, it suffices to show that:

$$X_t^1 = \frac{\tilde{W}_t + \tilde{W}_t^\star}{\sqrt{2}} = \frac{1}{\sqrt{2\Phi(P)}}\left(\int_0^t (PV_s^\star \otimes R_s P)\sharp dS_s + \int_0^t (PR_s^\star \otimes V_s P)\sharp dS_s\right),$$
$$X_t^2 = \frac{\tilde{W}_t - \tilde{W}_t^\star}{\sqrt{2}i} = \frac{1}{\sqrt{2\Phi(P)}i}\left(\int_0^t (PV_s^\star \otimes R_s P)\sharp dS_s - \int_0^t (PR_s^\star \otimes V_s P)\sharp dS_s\right)$$

define two free $(\mathcal{A}_t)$-free Brownian motions using again the characterization given in Theorem 4.1. We will do this for X_1. Note that, since R_t and V_t are partially isometric, then $(PV_t^\star \otimes R_t P\mathbf{1}_{[0,T]})_{t\geq 0}$ and $(PR_t^\star \otimes V_t P\mathbf{1}_{[0,T]})_{t\geq 0} \in \mathcal{B}_\infty^a \forall T > 0$. Hence, the first condition follows since $(\int_0^t (PV_s^\star \otimes R_s P)dS_s)_{t\geq 0}$ and $(\int_0^t (PR_s^\star \otimes V_s P)dS_s)_{t\geq 0}$ are $P\mathcal{A}_t P$-martingales. For $A, B \in \mathcal{A}_s$ and using (1), one has:

$$\begin{aligned}
&\tilde{\Phi}(PAP(X_t^1 - X_s^1)PBP(X_t^1 - X_s^1)) \\
&\quad = \frac{1}{2\Phi(P)}\tilde{\Phi}\left(\int_s^t (PAPV_u^\star \otimes R_u P)\sharp dS_u + \int_s^t (PAPR_u^\star \otimes V_u P)\sharp dS_u\right) \\
&\qquad + \left(\int_s^t (PBPV_u^\star \otimes R_u P)\sharp dS_u + \int_s^t (PBPR_u^\star \otimes V_u P)\sharp dS_u\right) \\
&\quad = \frac{1}{2\Phi^2(P)}\Phi\left(\int_s^t (PAPV_u^\star \otimes R_u P)\sharp dS_u + \int_s^t (PAPR_u^\star \otimes V_u P)\sharp dS_u\right) \\
&\qquad + \left(\int_s^t (PBPV_u^\star \otimes R_u P)\sharp dS_u + \int_s^t (PBPR_u^\star \otimes V_u P)\sharp dS_u\right) \\
&\quad = \frac{1}{2}\int_s^t [\tilde{\Phi}(PAPV_u^\star V_u P)\tilde{\Phi}(R_u PBPR_u^\star) + \tilde{\Phi}(PAPR_u^\star R_u P)\tilde{\Phi}(V_u PBPV_u^\star)]du \\
&\qquad + \frac{1}{2}\int_s^t [\tilde{\Phi}(PAPV_u^\star R_u P)\tilde{\Phi}(R_u PBPV_u^\star) \\
&\qquad + \tilde{\Phi}(PAPR_u^\star V_u P)\tilde{\Phi}(V_u PBPR_u^\star)]du \\
&\quad = \frac{1}{2}\int_s^t [\tilde{\Phi}(PAP)\tilde{\Phi}(PBP) + \tilde{\Phi}(PAP)\tilde{\Phi}(PBP)]du \\
&\qquad + \frac{1}{2}\int_s^t [\tilde{\Phi}(APV_u^\star R_u P)\tilde{\Phi}(BPV_u^\star R_u P) + \tilde{\Phi}(APR_u^\star V_u P)\tilde{\Phi}(BPR_u^\star V_u P)]du \\
&\quad = \tilde{\Phi}(PAP)\tilde{\Phi}(PBP)(t-s),
\end{aligned}$$

since $PV_u^\star R_u P = PR_u^\star V_u P = 0$ and since R_u and V_u are partially isometric (Proposition 4.1). Similarly, the same result holds for X^2. Furthermore,

$$\begin{aligned}
&\tilde{\Phi}(PAP(X_t^1 - X_s^1)PBP(X_t^2 - X_s^2)) \\
&\quad = \frac{1}{2\Phi(P)}\tilde{\Phi}\left(\int_s^t (PAPV_u^\star \otimes R_u P)\sharp dS_u + \int_s^t (PAPR_u^\star \otimes V_u P)\sharp dS_u\right) \\
&\qquad + \left(\int_s^t (PBPV_u^\star \otimes R_u P)\sharp dS_u - \int_s^t (PBPR_u^\star \otimes V_u P)\sharp dS_u\right) \\
&\quad = \frac{1}{2}\int_s^t [-\tilde{\Phi}(PAPV_u^\star V_u P)\tilde{\Phi}(R_u PBPR_u^\star) \\
&\qquad + \tilde{\Phi}(PAPR_u^\star R_u P)\tilde{\Phi}(V_u PBPV_u^\star)]du \\
&\qquad + \frac{1}{2}\int_s^t [\tilde{\Phi}(PAPV_u^\star R_u P)\tilde{\Phi}(R_u PBPV_u^\star) \\
&\qquad - \tilde{\Phi}(PAPR_u^\star V_u P)\tilde{\Phi}(V_u PBPR_u^\star)]du \\
&\quad = \frac{1}{2}\int_s^t [-\tilde{\Phi}(PAP)\tilde{\Phi}(PBP) + \tilde{\Phi}(PAP)\tilde{\Phi}(PBP)]du
\end{aligned}$$

$$+\frac{1}{2}\int_s^t [\tilde{\Phi}(APV_u^\star R_u P)\tilde{\Phi}(BPV_u^\star R_u P)$$
$$-\tilde{\Phi}(APR_u^\star V_u P)\tilde{\Phi}(BPR_u^\star V_u P)]du = 0$$

which finishes the proof. □

Substituting R_t and V_t by $R_t P$ and $V_t P$ in (2) and using Corollary 4.1, we proved:

Theorem 4.2 *Given J_0 such that J_0 and $P - J_0$ are injective operators in $P\mathcal{A}P$, let $T := \inf\{s, \ker(J_s) \neq \ker P$ or $\ker(P - J_s) \neq \ker P\} > 0$ by continuity of the trajectories. Then, for all $t < T$,*

$$\begin{cases} dJ_t = \sqrt{\lambda\theta}\sqrt{P - J_t}dW_t\sqrt{J_t} + \sqrt{\lambda\theta}\sqrt{J_t}dW_t^\star\sqrt{P - J_t} + (\theta P - J_t)dt, \\ J_0 = PY_0QY_0^\star P \end{cases} \tag{5}$$

where $(W_t)_{t\geq 0}$ is a $P\mathcal{A}P$-complex free Brownian motion.

In the remainder of this paper, we will try to find the range of (λ, θ) ensuring the injectivity of both J_t and $P - J_t$. This is equivalent to find (λ, θ) for which the spectral measure of both J_t and $P - J_t$ has no atoms in 0. We first investigate the stationary case then deal with the general setting.

5 Free Jacobi Process: the Stationary Case

In this section, we will give some interest in the particular case when Y_0 is Haar distributed, that is $\Phi(Y_0^k) = \delta_{k0}$. Then Y_t remains Haar distributed for all $t > 0$. Thus, the law of J_t does not depend on time and such a process is called a stationary free Jacobi process. Its law has already been computed by both Capitaine and Casalis using the so-called *generalized free cumulants* [12] and Collins ($P = Q$, [14]). Here we will use Nica and Speicher's result on compression by free projections. More precisely, authors considered PaP for any operator $a \in \mathcal{A}$ free with P (cf. [21, 23]). This condition is fulfilled for $a = Y_t QY_t^\star$ since Y_t is Haar unitary. In fact, the following classical result holds (see [18]):

Lemma 5.1 *If U is Haar unitary and $\mathcal{B}$ is a sub-algebra which is free with U, then, $\forall A, B \in \mathcal{B}$, A and $UBU^\star$ are free.*

From [23], the law of J_t in $(P\mathcal{A}P, \tilde{\Phi})$ writes:

$$\mu_{J_t} = \boxplus^r \mu_{\lambda\theta a}$$

where $\Phi(P) = \lambda\theta = 1/r$ and $\boxplus$ denotes the free additive convolution. Since Φ is tracial and Q is a projection, then $\Phi(a^k) = \Phi(Q) = \theta$ for all $k \geq 1$. Thus,

$$\mu_a = (1-\theta)\delta_0 + \theta\delta_1.$$

Furthermore, the Cauchy transform of a writes

$$G_a(z) := \frac{1}{z} + \sum_{k\geq 1} \frac{\Phi(a^k)}{z^{k+1}} = \frac{z+\theta-1}{z(z-1)}.$$

Its inverse is then written:

$$K_a(z) = \frac{z+1+\sqrt{(z-1)^2+4\theta z}}{2z},$$

and finally,

$$R_a(z) := K_a(z) - \frac{1}{z} = \frac{z-1+\sqrt{(z-1)^2+4\theta z}}{2z}.$$

R is known as the R-transform. It plays the role of the log-Laplace transform in classical probability since it linearizes the free additive convolution. This means that if a and b are free, then $R_{a+b} = R_a + R_b$. Hence

$$R_{J_t}(z) = r R_{\lambda\theta a}(z) = R_a(\lambda\theta z)$$

where the last equality follows from the expression of the R-transform in terms of free cumulants and the multilinearity of these latters [23]. It follows that

$$R_{J_t}(z) = \frac{\lambda\theta z - 1 + \sqrt{(\lambda\theta z-1)^2 + 4\lambda\theta^2 z}}{2\alpha z} = \frac{z-r+\sqrt{(z-r)^2+4z/\lambda}}{2z},$$

which implies that:

$$K_{J_t}(z) = R_{J_t} + \frac{1}{z} = \frac{z+(2-r)+\sqrt{(z-r)^2+4z/\lambda}}{2z}.$$

which inverse is:

$$G_{J_t}(z) = \frac{(2-r)z + (1/\lambda - 1) + \sqrt{Az^2 - Bz + C}}{2z(z-1)},$$

where $A = r^2 = 1/(\lambda\theta)^2$, $B = 2(r + (r-2)/\lambda)$ and $C = (1-1/\lambda)^2$. Since J_t is self-adjoint and $0 \leq J_t \leq P$ then its spectrum lies in $[0, 1]$. Thus $z \in \mathbb{C} \setminus [0, 1]$ and is constrained to $\Im[G(z)] < 0$ when $\Im(z) > 0$ which determines the square root. The law of J_t takes the form:

$$\mu_{J_t}(dx) = a_0\delta_0(dx) + a_1\delta_1(dx) + g(x)dx,$$

where

$$a_0 = \lim_{y\to 0^+} -y\Im[G(iy)], \qquad a_1 = \lim_{y\to 0^+} -y\Im[G(1+iy)],$$

$$g(x) = \lim_{y\to 0^+} -\frac{1}{\pi}\Im[G(x+iy)] \quad \text{for some } x \in (0, 1).$$

Remark The last equality holds whenever $\lim_{z\in D\to x}\Im(G(z))=\Im(G(x))$ where D is the upper half-plane and $x\in\mathbb{R}$ (cf. [13]). In fact, from

$$G_F(z)=\int\frac{dF(\zeta)}{z-\zeta}$$

for some distribution function F, the following inversion formula holds (cf. [18, 23]):

$$F(b)-F(a)=\lim_{y\to 0^+}-\frac{1}{\pi}\int_a^b\Im(G(x+iy))dx$$

for any two continuity points a,b of F (weak convergence). Silverstein and Choi showed that if the limit above exists, then F is differentiable and dF has the density function with respect to the Lebesgue measure given by $F'(x)=\lim_{y\to 0^+}-(1/\pi)\Im(G(x+iy))$. See [13] for more details.

G_{J_t} was already obtained in [11], as a result:

$$\mu_{J_t}(dx)=\max(0,1-l)\delta_0(dx)+\max(0,1-k)\delta_1(dx)+g(x)\mathbf{1}_{[x_-,x_+]}dx$$

where $l=1/\lambda$, $k=(1-\theta)/(\lambda\theta)$ and

$$x_\pm=(\sqrt{\theta(1-\lambda\theta)}\pm\sqrt{\lambda\theta(1-\theta)})^2,$$

$$g(x)=\frac{\sqrt{(x-x_-)(x_+-x)}}{2\lambda\theta\pi x(1-x)}.$$

Note that $\lambda\in]0,1]$, $1/\theta\geq\lambda+1\Leftrightarrow l\geq 1$, $k\geq 1$ so that $a_0=b_0=0$ and $\lambda=1\Rightarrow x_-=0$, $\theta=1/\lambda+1\Rightarrow x_+=1$. As a byproduct:

Proposition 5.1 $\forall\lambda\in]0,1]$, $1/\theta\geq\lambda+1(\theta\in]0,1/2]$ *for instance*), J_t *and* $P-J_t$ *are injective operators in the compressed space* $P\mathcal{A}P$. *For* $\lambda\in]0,1[$ *and* $1/\theta>\lambda+1$, *they are invertible in* $P\mathcal{A}P$. *As a result*, $(J_t)_{t\geq 0}$ *is a solution of* (5).

Remark In [15], Doumerc derived for $p(m)\geq m+1$ and $q(m)\geq m+1$ where $q(m)=d(m)-p(m)$, the following SDE for the real matrix Jacobi process:

$$dJ_t=\sqrt{I_m-J_t}dB_t\sqrt{J_t}+\sqrt{J_t}dB_t^T\sqrt{I_m-J_t}+(p(m)I_m-(p(m)+q(m))J_t)dt$$

where $(B_t)_{t\geq 0}$ is a real $m\times m$ Brownian matrix. If both J_0 and I_m-J_0 are injective, then this SDE has a unique strong solution. The complex version satisfies:

$$dJ_t=\sqrt{I_m-J_t}dB_t\sqrt{J_t}+\sqrt{J_t}dB_t^\star\sqrt{I_m-J_t}+(p(m)I_m-(p(m)+q(m))J_t)dt$$

where $(B_t)_{t\geq 0}$ is a complex $m\times m$ Brownian matrix. A similar uniqueness result holds for $p(m),q(m)\geq m$. A Heuristically, if we consider the ratio $dJ_t/(d(m))$ and let m go to infinity, then this SDE converges weakly (up to a constant) to its free counterpart, since normalized complex Brownian matrix converges in distribution to the free complex Brownian motion. It is also worth noting that conditions $p(m)\geq m$ and $q(m)\geq m$ are in agreement with $\lambda\in[0,1]$ and $1/\theta\geq\lambda+1$.

6 Free Jacobi Process: the General Case

In this section, we will suppose that $\lambda \le 1$ and $1/\theta \ge \lambda + 1$. Let $Y_0 \in \mathcal{A}$ such that $0 < J_0 := PY_0QY_0^\star P < P$, that is J_0 and $P - J_0$ are invertible in $P\mathcal{A}P$. By continuity of paths, the result of Theorem 4.2 holds for $t < T$:

$$\begin{cases} dJ_t = U_t \sharp dX_t + V_t \sharp dY_t + (\theta P - J_t)dt, \\ J_0 = PY_0QY_0^\star P, \quad 0 < J_0 < P \end{cases}$$

where

$$W_t = \frac{X_t + \sqrt{-1}Y_t}{\sqrt{2}},$$

$$U_t = \sqrt{\frac{\lambda\theta}{2}}(\sqrt{P - J_t} \otimes \sqrt{J_t} + \sqrt{J_t} \otimes \sqrt{P - J_t}) = \sum_{i=1}^{2} A_t^i \otimes B_t^i,$$

$$V_t = i\sqrt{\frac{\lambda\theta}{2}}(\sqrt{P - J_t} \otimes \sqrt{J_t} - \sqrt{J_t} \otimes \sqrt{P - J_t}) = \sum_{i=1}^{2} C_t^i \otimes D_t^i$$

and X and Y are two free $P\mathcal{A}_t P$-free-Brownian motions. Now, let us recall that for any operator $Z \in P\mathcal{A}P$, we set (see [7]):

$$\partial Z^n = \sum_{k=0}^{n-1} Z^k \otimes Z^{n-k-1},$$

$$\Delta_U(Z^n) = 2\sum_{i,j} \sum_{\substack{k,l \ge 0 \\ k+l \le n-2}} Z^k A_t^i B_t^j Z^{n-k-l-2} \tilde{\Phi}(B_t^i Z^l A_t^j)$$

where $U = \sum_i A^i \otimes B^i$ is an adapted bi-process.

Proposition 6.1 *Let X, Y be two free free-Brownian motions, U, V be two adapted integrable bi-processes and K an adapted process in $P\mathcal{A}P$. Let*

$$dM_t = U_t \sharp dX_t + V_t \sharp dY_t + K_t dt$$

then, for every polynomial R, we have:

$$dR(M_t) = \partial R(M_t) \sharp (U_t \sharp dX_t) + \partial R(M_t) \sharp (V_t \sharp dY_t) + \partial R(M_t) \sharp K_t dt \\ + \frac{1}{2}(\Delta_U R(M_t) + \Delta_V R(M_t))dt.$$

Proof When $V = 0 \otimes 0$, this is the free Itô's formula stated in [7]. By linearity, it suffices to prove the formula for monomials. To do this, we shall proceed by induction. Hence, assume that:

$$dM_t^n = \partial M_t^n \sharp (U_t \sharp dX_t) + \partial M_t^n \sharp (V_t \sharp dY_t) + \partial M_t^n \sharp K_t dt \\ + \frac{1}{2}(\Delta_U M_t^n + \Delta_V M_t^n)dt$$

By free integration by parts formula (see [6]), we have:

$$\begin{aligned} dM_t^{n+1} &= d(M_t M_t^n) = dM_t M_t^n + M_t dM_t^n + (dM_t)(dM_t^n) \\ &= (1\otimes M_t^n + M_t\partial M_t^n)\sharp(U_t\sharp dX_t) + (1\otimes M_t^n + M_t\partial M_t^n)\sharp(V_t\sharp dY_t) \\ &\quad + (1\otimes M_t + M_t\partial M_t^n)\sharp K_t dt + \frac{1}{2}M_t(\Delta_U M_t^n + \Delta_V M_t^n)dt \\ &\quad + (dM_t)(dM_t^n). \end{aligned}$$

On the other hand, we can easily see that:

$$1\otimes M_t^n + M_t\partial M_t^n = 1\otimes M_t^n + \sum_{k=1}^{n} M_t^k \otimes M_t^{n-k} = \partial M_t^{n+1}.$$

Then, using the fact that $(dX)(dY) = 0$ by the freeness of X and Y [19], we get:

$$\begin{aligned} (dM_t)(dM_t^n) &= \sum_{i,j}\sum_{l=0}^{n-1} A_t^i B_t^j M_t^{n-l-1}\tilde{\Phi}(B_t^i M_t^l A_t^j) \\ &\quad + \sum_{i,j}\sum_{l=0}^{n-1} C_t^i D_t^j M_t^{n-l-1}\tilde{\Phi}(D_t^i M_t^l C_t^j). \end{aligned}$$

Moreover,

$$\begin{aligned} M_t\Delta_U(M_t^n) &= 2\sum_{i,j}\sum_{\substack{k,l\geq 0\\ k+l\leq n-2}} M_t^{k+1} A_t^i B_t^j M_t^{n-k-l-2}\tilde{\Phi}(B_t^i M_t^l A_t^j) \\ &= 2\sum_{i,j}\sum_{l=0}^{n-2}\sum_{k=1}^{n-l-1} M_t^k A_t^i B_t^j M_t^{n-k-l-1}\tilde{\Phi}(B_t^i M_t^l A_t^j) \end{aligned}$$

and the same holds for

$$M_t\Delta_V(M_t^n) = 2\sum_{i,j}\sum_{l=0}^{n-2}\sum_{k=1}^{n-l-1} M_t^k C_t^i D_t^j M_t^{n-k-l-1}\tilde{\Phi}(C_t^i M_t^l D_t^j).$$

Consequently, we get:

$$\frac{1}{2}M_t(\Delta_V(M_t^n) + \Delta_U(M_t^n)) + (dM_t)(dM_t^n) = \frac{1}{2}(\Delta_V(M_t^{n+1}) + \Delta_U(M_t^{n+1})). \quad \square$$

6.1 A Recurrence Formula for Free Jacobi Moments

Corollary 6.1 *Let $m_n(t) := \tilde{\Phi}(J_t^n)$ for $n \geq 2$ and $t < T$. Then, we have the following recurrence relation*:

$$\begin{aligned} m_n(t) &= m_n(0) - n\int_0^t m_n(s)ds + n\theta\int_0^t m_{n-1}(s)ds \\ &\quad + \lambda\theta n\sum_{k=0}^{n-2}\int_0^t m_{n-k-1}(s)(m_k(s) - m_{k+1}(s))ds \end{aligned}$$

or equivalently,

$$\frac{dm_n(t)}{dt} = -nm_n(t) + n\theta m_{n-1}(t) + \lambda\theta n \sum_{k=0}^{n-2} m_{n-k-1}(t)(m_k(t) - m_{k+1}(t)).$$

Proof Using Proposition 6.1, we get:

$$dJ_t^n = \text{martingale} + \sum_{k=0}^{n-1} J_t^k(\theta P - J_t)J_t^{n-k-1}dt + \frac{1}{2}(\Delta_U(J_t^n) + \Delta_V(J_t^n))dt.$$

Next, we compute

$$\begin{aligned}
\Delta_U(J_t^n) &= 2\sum_{i,j=1}^{2} \sum_{\substack{k,l\geq 0\\ k+l\leq n-2}} J_t^k A_t^i B_t^j J_t^{n-k-l-2}\tilde{\Phi}(B_t^i J_t^l A_t^j) \\
&\overset{(1)}{=} 2\sum_{i,j=1}^{2} \sum_{\substack{k,l\geq 0\\ k+l\leq n-2}} A_t^i B_t^j J_t^{n-l-2}\tilde{\Phi}(B_t^i J_t^l A_t^j) \\
&= 2\sum_{i,j=1}^{2}\sum_{l=0}^{n-2}(n-l-1)A_t^i B_t^j J_t^{n-l-2}\tilde{\Phi}(B_t^i J_t^l A_t^j) \\
&= 2\sum_{i,j=1}^{2}\sum_{l=0}^{n-2}(l+1)A_t^i B_t^j J_t^{l}\tilde{\Phi}(B_t^i J_t^{n-l-2} A_t^j) \\
&\overset{(2)}{=} 2\sum_{l=0}^{n-2}(l+1)\Big[\frac{\lambda\theta}{2}(P-J_t)J_t^l\tilde{\Phi}(J_t^{n-l-1}) + \frac{\lambda\theta}{2}J_t^{l+1}\tilde{\Phi}(J_t^{n-l-2}(P-J_t)) \\
&\quad + \lambda\theta\sqrt{P-J_t}\sqrt{J_t}J_t^l\tilde{\Phi}(J_t^{n-l-2}\sqrt{J_t}\sqrt{P-J_t})\Big]
\end{aligned}$$

where in both (1) and (2), we used the fact that A_t^i, B_t^j and J_t commute $\forall i, j \in \{1, 2\}$. Similarly,

$$\begin{aligned}
\Delta_V(J_t^n) = 2\sum_{l=0}^{n-2}(l+1)\Big[&\frac{\lambda\theta}{2}(P-J_t)J_t^l\tilde{\Phi}(J_t^{n-l-1}) + \frac{\lambda\theta}{2}J_t^{l+1}\tilde{\Phi}(J_t^{n-l-2}(P-J_t)) \\
&- \lambda\theta\sqrt{P-J_t}\sqrt{J_t}J_t^l\tilde{\Phi}(J_t^{n-l-2}\sqrt{J_t}\sqrt{P-J_t})\Big].
\end{aligned}$$

Thus, we have:

$$\begin{aligned}
&\frac{1}{2}(\Delta_U(J_t^n) + \Delta_V(J_t^n)) \\
&\quad = \lambda\theta\sum_{l=0}^{n-2}(l+1)[(P-J_t)J_t^l\tilde{\Phi}(J_t^{n-l-1}) + J_t^{l+1}\tilde{\Phi}(J_t^{n-l-2}(P-J_t))].
\end{aligned}$$

Taking the expectation, it yields

$$\begin{aligned}
&\frac{1}{2}\tilde{\Phi}(\Delta_U(J_t^n)+\Delta_V(J_t^n))\\
&=\lambda\theta\left(\sum_{l=0}^{n-2}(l+1)[\tilde{\Phi}((P-J_t)J_t^l)\tilde{\Phi}(J_t^{n-l-1})]\right.\\
&\qquad\left.+\sum_{l=0}^{n-2}(l+1)[\tilde{\Phi}(J_t^{l+1})\tilde{\Phi}(J_t^{n-l-2}(P-J_t))]\right)\\
&=\lambda\theta\left(\sum_{l=0}^{n-2}(l+1)[\tilde{\Phi}((P-J_t)J_t^l)\tilde{\Phi}(J_t^{n-l-1})]\right.\\
&\qquad\left.+\sum_{l=0}^{n-2}(n-l-1)[\tilde{\Phi}(J_t^{n-l-1})\tilde{\Phi}(J_t^l(P-J_t))]\right)\\
&=n\lambda\theta\sum_{l=0}^{n-2}[\tilde{\Phi}((P-J_t)J_t^l)\tilde{\Phi}(J_t^{n-l-1})]\\
&=n\lambda\theta\sum_{l=0}^{n-2}[m_{n-l-1}(t)(m_l(t)-m_{l+1}(t)].
\end{aligned}$$

Furthermore, since $P-J_t$ and J_t commute, we can easily see that:

$$\tilde{\Phi}\left(\sum_{k=0}^{n-1}J_t^k(\theta P-J_t)J_t^{n-k-1}\right)=n\tilde{\Phi}(J_t^{n-1}(\theta P-J_t)).$$ □

Proposition 6.2 *If J_0 and $P-J_0$ are invertible in $P\mathcal{A}P$, then for all $\lambda\in]0,1]$, $1/\theta\geq 1+\lambda$ and $t\geq 0$, $P-J_t$ and J_t are injective operators in $P\mathcal{A}P$.*

Proof Our inspiration comes from the matrix case in which we make use of the semi-martingale $\log\det(J_t(m))$. It is known that for a self-adjoint operator $a\in P\mathcal{A}P$ such that $0<a<P$,

$$\log(P-a)=-\sum_{n=1}^{\infty}\frac{a^n}{n}.$$

Since $\tilde{\Phi}(a^n)=\int x^n\mu(dx)$ for a positive compactly supported measure μ, we get:

$$\tilde{\Phi}(\log(P-a))=-\sum_{n=1}^{\infty}\frac{\tilde{\Phi}(a^n)}{n}=-\tilde{\Phi}(a)-\sum_{n=2}^{\infty}\frac{\tilde{\Phi}(a^n)}{n}.$$

Thus, substituting the moments of J_t, one has (Corollary 6.1) for all $t<T$:

$$
\begin{aligned}
&\tilde{\Phi}(\log(P - J_t))\\
&\quad = \tilde{\Phi}(\log(P - J_0)) - \theta t + \int_0^t \tilde{\Phi}\left(\sum_{n=1}^{\infty} J_s^n\right) ds - \theta \int_0^t \tilde{\Phi}\left(\sum_{n=1}^{\infty} J_s^n\right) ds\\
&\qquad - \lambda\theta \int_0^t \sum_{n=2}^{\infty}\sum_{k=0}^{n-2} \tilde{\Phi}(J_s^{n-1-k})\tilde{\Phi}(J_s^k(P - J_s))ds\\
&\quad = \tilde{\Phi}(\log(P - J_0)) - \theta t + (1-\theta)\int_0^t \tilde{\Phi}(J_s(P - J_s)^{-1})ds\\
&\qquad - \lambda\theta \int_0^t \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} \tilde{\Phi}(J_s^n)\tilde{\Phi}(J_s^k(P - J_s))ds\\
&\quad = \tilde{\Phi}(\log(P - J_0)) - \theta t + (1-\theta)\int_0^t \tilde{\Phi}(J_s(P - J_s)^{-1})ds\\
&\qquad - \lambda\theta \int_0^t \tilde{\Phi}(J_s(P - J_s)^{-1})\tilde{\Phi}(P)ds\\
&\quad = \tilde{\Phi}(\log(P - J_0)) - \theta t + (1-\theta-\lambda\theta)\int_0^t \tilde{\Phi}(J_s(P - J_s)^{-1})ds\\
&\quad = \tilde{\Phi}(\log(P - J_0)) - (1-\lambda\theta)t + (1-\theta-\lambda\theta)\int_0^t \tilde{\Phi}((P - J_s)^{-1})ds.
\end{aligned}
$$

When $\lambda \in]0, 1]$ and $1/\theta \geq 1+\lambda$ then $1-\theta-\lambda\theta \geq 0$ and $1-\lambda\theta \geq 0$. Hence if $P - J_0$ is invertible, then:

$$
\tilde{\Phi}(\log(P - J_t)) + (1-\lambda\theta)t \geq \tilde{\Phi}(\log(P - J_0)) > -\infty \quad \forall t < T
$$

which gives the injectivity of $P - J_t$, $\forall t \geq 0$. The second assertion follows since $P - J$ is a $FJP(\lambda\theta/(1-\theta), 1-\theta)$ and since J_0 is invertible. Indeed, $1/\theta \geq \lambda + 1 \Rightarrow \lambda\theta/(1-\theta) \leq 1$ and $\lambda \leq 1 \Rightarrow (\lambda-1)\theta \leq 0 \Rightarrow (\lambda\theta)/(1-\theta)+1 = (\theta(\lambda-1)+1)/(1-\theta) \leq 1/(1-\theta)$. Thus, similar computations applies when replacing J_t by $P - J_t$. □

Corollary 6.2 *Under the same conditions of Proposition 6.2, the $FJP(\lambda,\theta)$ satisfies for all $t \geq 0$ the following SDE*:

$$
dJ_t = \sqrt{\lambda\theta}\sqrt{P - J_t}dW_t\sqrt{J_t} + \sqrt{\lambda\theta}\sqrt{J_t}dW_t^{\star}\sqrt{P - J_t} + (\theta P - J_t)dt
$$

where W is a complex free Brownian motion.

Remark In the stationary case, one can compute explicitly $\tilde{\Phi}(\log(J_t))$. Let $\lambda \in]0, 1[$, $1/\theta > \lambda + 1$, $z \in [0, 1]$ so that both $\log(J_t)$, $\log(P - J_t)$ are well defined for all $t \geq 0$. Then

$$
\begin{aligned}
-\frac{d}{dz}\tilde{\Phi}(\log(P - zJ_t)) &= -\frac{1}{z}\left(\frac{1}{z}G_{J_t}\left(\frac{1}{z}\right) - 1\right)\\
&= \frac{(1+1/\lambda)z - r + \sqrt{Cz^2 - Bz + A}}{2z(1-z)}.
\end{aligned}
$$

Note that this derivative is well defined for $z=0$ and $z=1$. It follows that:

$$2\tilde{\Phi}(\log(P-J_t)) = -\int_0^1 \frac{(1+1/\lambda)z - r + \sqrt{Cz^2 - Bz + A}}{z(1-z)}dz. \tag{6}$$

Note first that $Cz^2 - Bz + A > 0 \forall\lambda \in]0,1[$, $1/\theta > \lambda + 1$ since $x_+, x_- \in]0,1[$ are the roots of $Az^2 - Bz + C$ (so that $z < 1/x_+$). In order to evaluate the integral in the right, we use the variable change $\sqrt{A}(1-uz) = \sqrt{Cz^2 - Bz + A}$, which gives:

$$z = \frac{2Au - B}{Au^2 - C}, \qquad 1 - z = \frac{Au^2 - 2Au + B - C}{Au^2 - C},$$
$$dz = -2A\frac{Au^2 - Bu + C}{(Au^2 - C)^2}du.$$

Moreover, since $A - B + C = A(1-\theta(\lambda+1))^2 \geq 0$ and $\theta(1+\lambda) \leq 1$, then the roots of $Au^2 - 2Au + B - C = 0$ are given by: $u_\pm = 1 \pm (1 - \theta(\lambda+1))$. On the other hand, $B/2A = (1/2)(x_+ + x_-) = \theta(\lambda + 1 - 2\lambda\theta)$. Hence our expression factorizes into:

$$\begin{aligned}
\tilde{\Phi}(\log(P-J_t)) &= -\frac{1}{\sqrt{A}}\int_{B/2A}^{\theta(\lambda+1)} \frac{(u - \theta(\lambda+1))(Au^2 - Bu + C)}{(u^2 - C/A)(Au^2 - 2Au + B - C)}du \\
&= -\frac{1}{\sqrt{A}}\int_{B/2A}^{\theta(\lambda+1)} \frac{(Au^2 - Bu + C)}{(u^2 - C/A)(u - u_+)}du \\
&= \int_{B/2A}^{\theta(\lambda+1)} \frac{C_1}{u - \theta(1-\lambda)} + \frac{C_2}{u + \theta(1-\lambda)} + \frac{C_3}{u - u_+}du
\end{aligned}$$

for some constants C_1, C_2, C_3 depending on both λ, θ, given by:

$$C_1 = 1, \qquad C_2 = 1/\lambda, \qquad C_3 = \frac{1 - \theta(\lambda+1)}{\lambda\theta}.$$

Thus, one gets:

$$\begin{aligned}
&\tilde{\Phi}(\log(P-J_t)) \\
&\quad = -[C_1\log(u - \theta(1-\lambda)) + C_2\log(u + \theta(1-\lambda)) + C_3\log(u_+ - u)]_{B/2A}^{\theta(\lambda+1)} \\
&\quad = \log(1-\theta) + \frac{1}{\lambda}\log(1-\lambda\theta) - C_3\log\left[\frac{(1-\theta(\lambda+1))}{1 - \theta(\lambda+1) + \lambda\theta^2}\right] \\
&\quad = (1 + C_3)\log(1-\theta) + \left(\frac{1}{\lambda} + C_3\right)\log(1-\lambda\theta) - C_3\log(1-\theta(\lambda+1)) \\
&\quad = \frac{(1-\theta)\log(1-\theta) + (1-\lambda\theta)\log(1-\lambda\theta) - (1-\theta(\lambda+1))\log(1-\theta(\lambda+1))}{\lambda\theta}.
\end{aligned}$$

Note that the result extends for all $\lambda \in]0,1]$, $1/\theta \geq \lambda + 1$. Since $P - J$ is still a $FJP(\lambda\theta/(1-\theta), 1-\theta)$, then:

$$\tilde{\Phi}(\log(J_t)) = \frac{\theta\log\theta + (1-\lambda\theta)\log(1-\lambda\theta) - \theta(1-\lambda)\log(\theta(1-\lambda))}{\lambda\theta}.$$

6.2 Free Martingales Polynomials

In this paragraph, we consider a stationary $FJP(1, 1/2)$ starting at J_0, the law of which is the Beta law $B(1/2, 1/2)$. Recall that a $\mathcal{A}_t$-adapted free process $(X_t)_{t\geq 0}$ is a $\mathcal{A}_t$-free martingale if and only if $\Phi(X_t|\mathcal{A}_s) = X_s$ (see [2, 6, 9]).

Proposition 6.3 *Let $\mathcal{J}_t$ denotes the von Neumann subalgebra generated by $(J_s, s \leq t)$ and let $0 < r < 1$. Then, the process $R_t := ((1+re^t)P - 2re^t J_t)((1+re^t)^2 P - 4re^t J_t)^{-1})_{t<-\ln r}$ is a $\mathcal{J}_t$-free martingale.*

Proof R_t can be written as:

$$\begin{aligned}
R_t &= \left[\frac{P}{1+re^t} - 2\frac{re^t}{(1+re^t)^2}J_t\right]\left[P - \frac{4re^t}{(1+re^t)^2}J_t\right]^{-1} \\
&= \left[\frac{1-re^t}{2(1+re^t)}P + \frac{1}{2}\left(P - \frac{4re^t}{(1+re^t)^2}J_t\right)\right]\left[P - \frac{4re^t}{(1+e^t)^2}J_t\right]^{-1} \\
&:= \frac{1-re^t}{2(1+re^t)}H_t + \frac{P}{2}
\end{aligned}$$

where

$$H_t = \left[P - \frac{4re^t}{(1+re^t)^2}J_t\right]^{-1} = \sum_{n\geq 0}\frac{(4re^t)^n}{(1+re^t)^{2n}}J_t^n$$

since $4re^t < (1+re^t)^2$ and $0 \leq J_t \leq P$ for all $t > 0$. It follows that:

$$2dR_t = \frac{1-re^t}{1+re^t}dH_t - \frac{2re^t}{(1+re^t)^2}H_t dt. \tag{7}$$

On the other hand, one has for $1 \leq l \leq n-1$ and $n \geq 2$:

$$\tilde{\Phi}(J_t^{n-l}) = \frac{\Gamma(n-l+1/2)}{\sqrt{\pi}(n-l)!}, \qquad \tilde{\Phi}(J_t^{n-l-1}(P-J_t)) = \frac{\Gamma(n-l-1/2)}{2\sqrt{\pi}(n-l)!}.$$

From the proof of Proposition 6.1, we deduce that, for all $t > 0$:

$$\begin{aligned}
dJ_t^n &= M_t + n\left(\frac{P}{2} - J_t\right)J_t^{n-1}dt + \frac{1}{2}(\Delta_U(J_t^n) + \Delta_V(J_t^n)) \\
&= M_t + \left[n\left(\frac{P}{2} - J_t\right)J_t^{n-1}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{l=1}^{n-1} l[(P-J_t)J_t^{l-1}\tilde{\Phi}(J_t^{n-l}) + J_t^l\tilde{\Phi}(J_t^{n-l-1}(P-J_t))]\Bigg]dt \\
&= M_t + \Bigg[n\left(\frac{P}{2}-J_t\right)J_t^{n-1} \\
&\quad + \sum_{l=1}^{n-1} l\left[(P-J_t)J_t^{l-1}\frac{\Gamma(n-l+1/2)}{2\sqrt{\pi}(n-l)!} + \frac{J_t^l}{2}\frac{\Gamma(n-l-1/2)}{2\sqrt{\pi}(n-l)!}\right]\Bigg]dt \\
&= M_t + \Bigg[n\left(\frac{P}{2}-J_t\right)J_t^{n-1} \\
&\quad + \sum_{l=1}^{n-1}\left[l\frac{\Gamma(n-l+1/2)}{2\sqrt{\pi}(n-l)!}J_t^{l-1} - l(n-l-1)\frac{\Gamma(n-l-1/2)}{2\sqrt{\pi}(n-l)!}J_t^l\right]\Bigg]dt \\
&= M_t + \Bigg[n\left(\frac{P}{2}-J_t\right)J_t^{n-1} \\
&\quad + \left[\sum_{l=1}^{n-1} l\frac{\Gamma(n-l+1/2)}{2\sqrt{\pi}(n-l)!}J_t^{l-1} - \sum_{l=2}^{n-1}\frac{(l-1)(n-l)\Gamma(n-l+1/2)}{2\sqrt{\pi}(n-l+1)!}J_t^{l-1}\right]\Bigg]dt \\
&= M_t + \Bigg[n\left(\frac{P}{2}-J_t\right)J_t^{n-1} \\
&\quad + \frac{1}{2\sqrt{\pi}}\left[\sum_{l=2}^{n-1} n\frac{\Gamma(n-l+1/2)}{(n-l+1)!}J_t^{l-1} + \frac{\Gamma(n-1/2)}{(n-1)!}P\right]\Bigg]dt \\
&= M_t + \left[n\left(\frac{P}{2}-J_t\right)J_t^{n-1} + \frac{n}{2\sqrt{\pi}}\sum_{l=1}^{n-1}\frac{\Gamma(n-l+1/2)}{(n-l+1)!}J_t^{l-1}\right]dt \\
&= M_t - nJ_t^n dt + \frac{n}{2\sqrt{\pi}}\sum_{l=1}^{n}\frac{\Gamma(n-l+1/2)}{(n-l+1)!}J_t^{l-1}\,dt
\end{aligned}
$$

where M_t stands for the martingale part. Note that this is true for $n=1$. Thus:

$$
\begin{aligned}
FV(dH_t) &= \sum_{n\ge 0}\frac{(4re^t)^n}{(1+re^t)^{2n}}FV(dJ_t^n) + \frac{1-re^t}{1+re^t}\sum_{n\ge 0}\frac{n(4re^t)^n}{(1+re^t)^{2n}}J_t^n dt \\
&= -\frac{2re^t}{1+re^t}\sum_{n\ge 0}\frac{n(4re^t)^n}{(1+re^t)^{2n}}J_t^n dt \\
&\quad + \frac{1}{2\sqrt{\pi}}\sum_{n\ge 0}\frac{n(4re^t)^n}{(1+re^t)^{2n}}\sum_{l=1}^{n}\frac{\Gamma(n-l+1/2)}{(n-l+1)!}J_t^{l-1}\,dt
\end{aligned}
$$

$$= -\frac{2re^t}{1+re^t}\sum_{n\geq 0}\frac{n(4re^t)^n}{(1+re^t)^{2n}}J_t^n\,dt$$

$$+\frac{1}{2}\sum_{l\geq 1}\frac{(4re^t)^l}{(1+re^t)^{2l}}\sum_{n\geq 0}\frac{(n+l)(1/2)_n}{(n+1)!}\frac{(4re^t)^n}{(1+re^t)^{2n}}J_t^{l-1}\,dt$$

$$= -\frac{2re^t}{1+re^t}\sum_{n\geq 0}\frac{n(4re^t)^n}{(1+re^t)^{2n}}J_t^n\,dt$$

$$+\frac{2re^t}{(1+re^t)^2}\sum_{l\geq 0}\frac{(4re^t)^l}{(1+re^t)^{2l}}\sum_{n\geq 0}\frac{(n+l+1)(1/2)_n}{(n+1)!}\frac{(4re^t)^n}{(1+re^t)^{2n}}J_t^l\,dt$$

$$= -\frac{2re^t}{1+re^t}\sum_{n\geq 0}\frac{n(4re^t)^n}{(1+re^t)^{2n}}J_t^n\,dt$$

$$+\frac{2re^t}{(1+re^t)^2}\sum_{l\geq 0}\frac{(4re^t)^l}{(1+re^t)^{2l}}J_t^l\sum_{n\geq 0}\frac{(1/2)_n}{n!}\frac{(4re^t)^n}{(1+re^t)^{2n}}\,dt$$

$$+\frac{1}{2}\sum_{l\geq 0}\frac{l(4re^t)^l}{(1+re^t)^{2l}}J_t^l\sum_{n\geq 0}\frac{(1/2)_n}{(n+1)!}\frac{(4re^t)^{n+1}}{(1+re^t)^{2n+2}}\,dt \tag{8}$$

where FV stands for the finite variation part. From (see [1]):

$${}_1\mathcal{F}_0(a,z) := \sum_{p=0}^{\infty}(a)_p\frac{z^p}{p!} = (1-z)^{-a}, \quad |z|<1,$$

one has for all t such that $re^t < 1$:

$$\sum_{n\geq 0}\frac{(1/2)_n}{n!}\frac{(4re^t)^n}{(1+re^t)^{2n}}\,dt = \left(1-\frac{4re^t}{(1+re^t)^2}\right)^{-1/2} = \frac{1+re^t}{1-re^t}.$$

Then, the second term in (8) cancels with the second one in (7). Similarly, from $\int_0^u dz/\sqrt{1-z} = 2-2\sqrt{1-u}$,

$$\frac{1}{2}\sum_{n\geq 0}\frac{(1/2)_n}{(n+1)!}\frac{(4re^t)^{n+1}}{(1+re^t)^{2n+2}} = 1-\left(1-\frac{4re^t}{(1+re^t)^2}\right)^{1/2} = \frac{2re^t}{1+re^t}$$

and the remaining terms cancel too. □

Corollary 6.3 *Let T_k denotes the kth-Tchebycheff polynomial of the first kind*:

$$T_k(x) := \cos(k\arccos(x)), \quad k\geq 0,\ x\in]-1,1[.$$

Thus the process $S(k)$ defined by $S_t(k) := e^{kt}T_k(2J_t - P)$ is a $\mathcal{J}_t$-free martingale.

Proof Let us first point out to the reader that these polynomials are orthogonal with respect to Beta distribution $B(1/2,1/2)$ which is the law of $FJP(1,1/2)$. The proof is standard (see [6] for the additive free BM) and uses the generating function of $(T_k)_{k\geq 0}$ which is given by [1]:

$$L(x,z) := \sum_{k=0}^{\infty} T_k(x) z^k = \frac{1-zx}{1-2zx+z^2}, \quad |z|<1.$$

Letting $z = re^t$ with $0 < r < e^{-t} < e^{-s}$ for $s < t$, then both $L(2J_t - P, re^t)$ and $L(2J_s - P, re^s)$ converge and an easy computation shows that:

$$L(2J_t - P, re^t) = R_t \quad \text{is a } \mathcal{J}_t\text{-free martingale,}$$

thus,

$$\tilde{\Phi}(R_t|\mathcal{J}_s) = R_s \quad \Leftrightarrow \quad \tilde{\Phi}[L(2J_t - P, re^t)|\mathcal{J}_s] = L(2J_s - P, re^s) \quad \Leftrightarrow$$
$$\sum_{k=0}^{\infty} \tilde{\Phi}(e^{kt} T_k(2J_t - P)|\mathcal{J}_s) r^k = \sum_{k=0}^{\infty} T_k(2J_s - P) e^{ks} r^k.$$

Taking the derivative of both sides at $r = 0$, we are done. □

7 On The Cauchy Transform of the Free Jacobi Process

Under the assumptions of the previous section, one has from (5):

$$\tilde{\Phi}(J_t) = (\tilde{\Phi}(J_0) - \theta)e^{-t} + \theta.$$

Then, a similar computation as in Proposition 6.2 using Corollary 6.1 gives for all u in the unit disk:

$$\begin{aligned}
&\tilde{\Phi}(\log(P - uJ_t)) \\
&\quad = -\sum_{n\geq 1} \frac{u^n}{n} \tilde{\Phi}(J_t^n) = -u\tilde{\Phi}(J_t) - \sum_{n\geq 2} \frac{u^n}{n} \tilde{\Phi}(J_t^n) \\
&\quad = u(\tilde{\Phi}(J_0) - \tilde{\Phi}(J_t)) + \tilde{\Phi}(\log(P - uJ_0)) \\
&\qquad + \int_0^t \tilde{\Phi}\left(\sum_{n\geq 2} (uJ_s)^n\right) ds - \theta u \int_0^t \tilde{\Phi}\left(\sum_{n\geq 2} (uJ_s)^{n-1}\right) ds \\
&\qquad - \lambda\theta \int_0^t \sum_{n\geq 2} \sum_{k=0}^{n-2} u^n \tilde{\Phi}(J_s^{n-k-1}) \tilde{\Phi}(J_s^k (P - J_s)) ds \\
&\quad = u(\tilde{\Phi}(J_0) - \theta)(1 - e^{-t}) + \tilde{\Phi}(\log(P - uJ_0)) + \int_0^t \tilde{\Phi}(u^2 J_s^2 (P - uJ_s)^{-1}) ds \\
&\qquad - \theta u \int_0^t \tilde{\Phi}(uJ_s(P - uJ_s)^{-1}) ds
\end{aligned}$$

$$-\lambda\theta u\int_0^t \tilde{\Phi}\Bigg(\sum_{n\geq 0}(uJ_s)^{n+1}\Bigg)\tilde{\Phi}\Bigg(\sum_{k\geq 0}(uJ_s)^k(P-J_s)\Bigg)ds.$$

Using once $u^2J_s^2=(uJ_s-P)(uJ_s+P)+P$ and twice $uJ_s=(uJ_s-P)+P$, we get:

$$\begin{aligned}
&\tilde{\Phi}(\log(P-uJ_t))\\
&\quad=u(\tilde{\Phi}(J_0)-\theta)(1-e^{-t})+\tilde{\Phi}(\log(P-uJ_0))+(1-\theta u)\int_0^t \tilde{\Phi}((P-uJ_s)^{-1})ds\\
&\qquad+\theta ut-\int_0^t \tilde{\Phi}((P+uJ_s))ds\\
&\qquad-\lambda\theta u\int_0^t \tilde{\Phi}(uJ_s(P-uJ_s)^{-1})\tilde{\Phi}((P-J_s)(P-uJ_s)^{-1})ds\\
&\quad=\tilde{\Phi}(\log(P-uJ_0))+(1-\theta u-\lambda\theta u)\int_0^t \tilde{\Phi}((P-uJ_s)^{-1})ds-(1-\lambda\theta u)t\\
&\qquad-\lambda\theta(u-1)\int_0^t \tilde{\Phi}((P-uJ_s)^{-1})\tilde{\Phi}(uJ_s(P-uJ_s)^{-1})ds\\
&\qquad+\lambda\theta(u-1)\int_0^t \tilde{\Phi}(uJ_s(P-uJ_s)^{-1})ds\\
&\quad=\tilde{\Phi}(\log(P-uJ_0))+(1-\theta u+\lambda\theta(u-2))\int_0^t \tilde{\Phi}((P-uJ_s)^{-1})ds-(1-\lambda\theta)t\\
&\qquad-\lambda\theta(u-1)\int_0^t \tilde{\Phi}^2((P-uJ_s)^{-1})ds.
\end{aligned}$$

Setting $h_t(u)=\tilde{\Phi}((P-uJ_t)^{-1})$, then $h_t(u)=(1/u)G_{J_t}(1/u):=(1/u)G_t(1/u)$ and

$$-\frac{d}{du}\tilde{\Phi}(\log(P-uJ_t))=\tilde{\Phi}(J_t(P-uJ_t)^{-1})=\frac{1}{u}(h_t(u)-1)=\frac{1}{u}\bigg(\frac{1}{u}G_t(u)-1\bigg).$$

Thus:

$$\begin{aligned}
G_t(1/u)&=G_0(1/u)+\theta(1-\lambda)u\int_0^t G_s(1/u)ds+\lambda\theta\int_0^t G_s^2(1/u)ds\\
&\quad+(1-\theta u+\lambda\theta(u-2))\int_0^t\bigg[G_s(1/u)+\frac{1}{u}G_s'(1/u)\bigg]ds\\
&\quad+2\lambda\theta\bigg(\frac{1-u}{u^2}\bigg)\int_0^t[uG_s^2(1/u)+G_s'(1/u)G_s(1/u)].
\end{aligned}$$

As a consequence, G satisfies:

Proposition 7.1

$$G_t(z) = G_0(z) + (1-2\lambda\theta)\int_0^t G_s(z)ds + \lambda\theta(2z-1)\int_0^t G_s^2(z)ds$$
$$+ ((1-2\lambda\theta)z - \theta(1-\lambda))\int_0^t G_s'(z)ds + 2\lambda\theta z(z-1)\int_0^t G_s(z)G_s'(z)ds.$$

Remark The expression above can be derived in a similar way by multiplying both sides of the recurrence formula in Corollary (6.1) by u^n and summing over n. Besides, it takes the p.d.e. form:

$$\partial_t G_t(z) = \partial_z\{[(1-2\lambda\theta)z - \theta(1-\lambda)]G_t(z) + \lambda\theta z(z-1)G_t^2(z)\}.$$

In the stationary case, one can see that $[(1-2\lambda\theta)z - \theta(1-\lambda)]G(z) + \lambda\theta z(z-1)G^2(z) = -\lambda\theta = -1/r$ with $G := G_{J_t}$ derived in Sect. 5. Thus the p.d.e. is satisfied.

8 Conclusion and Open Questions

The curious reader can check after browsing Chap. 3 in [15] that the study of the free Jacobi process is more handable than the one of its matrix analog, in the sense that, though both cases belong to a noncommutative context, more precise results on the law are derived in the infinite dimensional case, namely, the recurrence formula for the moments and the Cauchy transform though the nonlinear p.d.e. it satisfies. Nevertheless, we can not prove uniqueness of the solution of (5) as done for the matrix Jacobi process and for the free Wishart process as well. When the SDE (or free) is driven by Hölder-continuous coefficient operators, this uses mainly an invertibility argument as well as Gronwall Lemma. Hence, this can be done in the stationary case for $\lambda \in]0,1[$, $1/\theta > \lambda + 1$ (see Proposition 5.1). A general result is still an open problem.

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