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Exact Rate of Convergence of Some Approximation Schemes Associated to SDEs Driven by a Fractional Brownian Motion

Andreas Neuenkirch · Ivan Nourdin

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Abstract In this article, we derive the exact rate of convergence of some approximation schemes associated to scalar stochastic differential equations driven by a fractional Brownian motion with Hurst index H. We consider two cases. If $H > 1/2$, the exact rate of convergence of the Euler scheme is determined. We show that the error of the Euler scheme converges almost surely to a random variable, which in particular depends on the Malliavin derivative of the solution. This result extends those contained in J. Complex. **22**(4), 459–474, 2006 and C.R. Acad. Sci. Paris, Ser. I **340**(8), 611–614, 2005. When $1/6 < H < 1/2$, the exact rate of convergence of the Crank-Nicholson scheme is determined for a particular equation. Here we show convergence in law of the error to a random variable, which depends on the solution of the equation and an independent Gaussian random variable.

Keywords Fractional Brownian motion · Russo-Vallois integrals · Doss-Sussmann type transformation · Stochastic differential equations · Euler scheme · Crank-Nicholson scheme · Mixing law

1 Introduction

Let $B = (B_t, t \in [0, 1])$ be a fractional Brownian motion (in short: fBm) with Hurst parameter $H \in (0, 1)$, i.e., B is a continuous centered Gaussian process with covari-

A. Neuenkirch (✉)
Fachbereich Mathematik, Johann Wolfgang Goethe-Universität Frankfurt, Robert-Mayer-Straße 10, 60325 Frankfurt am Main, Germany
e-mail: neuenkirch@math.uni-frankfurt.de

I. Nourdin
LPMA, Université Pierre et Marie Curie Paris 6, Boîte courrier 188, 4 Place Jussieu, 75252 Paris Cedex 5, France
e-mail: nourdin@ccr.jussieu.fr

ance function

$$R_H(s,t)=\frac{1}{2}(s^{2H}+t^{2H}-|t-s|^{2H}),\quad s,t\in[0,1].$$

For $H=1/2$, B is a standard Brownian motion, while for $H\neq 1/2$, it is neither a semimartingale nor a Markov process. Moreover, it holds

$$(\mathrm{E}|B_t-B_s|^2)^{1/2}=|t-s|^H,\quad s,t\in[0,1],$$

and almost all sample paths of B are Hölder continuous of any order $\alpha\in(0,H)$.

In this paper, we are interested in the pathwise approximation of the equation

$$X_t=x_0+\int_0^t\sigma(X_s)dB_s+\int_0^t b(X_s)ds,\quad t\in[0,1], \tag{1}$$

with a deterministic initial value $x_0\in\mathbb{R}$. Here, σ and b satisfy some standard smoothness assumptions and the integral equation (1) is understood in the sense of Russo-Vallois. Let us recall briefly the significant points of this theory.

Definition 1 (Following [27]) Let $Z=(Z_t)_{t\in[0,1]}$ be a stochastic process with continuous paths.

- A family of processes $(H_t^{(\varepsilon)})_{t\in[0,1]}$ is said to converge to the process $(H_t)_{t\in[0,1]}$ in the *ucp sense*, if $\sup_{t\in[0,1]}|H_t^{(\varepsilon)}-H_t|$ goes to 0 in probability, as $\varepsilon\to 0$.
- The (*Russo-Vallois*) *forward integral* $\int_0^t Z_s d^- B_s$ is defined by

$$\lim_{\varepsilon\to 0}\text{-ucp }\varepsilon^{-1}\int_0^t Z_t(B_{t+\varepsilon}-B_t)dt, \tag{2}$$

  provided the limit exists.
- The (*Russo-Vallois*) *symmetric integral* $\int_0^t Z_s d^\circ B_s$ is defined by

$$\lim_{\varepsilon\to 0}\text{-ucp }(2\varepsilon)^{-1}\int_0^t (Z_{t+\varepsilon}+Z_t)(B_{t+\varepsilon}-B_t)dt, \tag{3}$$

  provided the limit exists.

Now we state the exact meaning of (1) and give conditions for the existence and uniqueness of its solution. We consider two cases, according to the value of H:

Case $H>1/2$. Here the integral with respect to B is defined by the forward integral (2).

Proposition 1 *If* $\sigma\in\mathcal{C}_b^2$ *and if* b *satisfies a global Lipschitz condition, then the equation*

$$X_t=x_0+\int_0^t\sigma(X_s)d^-B_s+\int_0^t b(X_s)ds,\quad t\in[0,1] \tag{4}$$

admits a unique solution X in the set of processes whose paths are Hölder continuous of order $\alpha > 1 - H$. Moreover, we have a Doss-Sussmann type [7, 29] *representation*:

$$X_t = \phi(A_t, B_t), \quad t \in [0,1], \tag{5}$$

where ϕ and A are given respectively by

$$\frac{\partial \phi}{\partial x_2}(x_1, x_2) = \sigma(\phi(x_1, x_2)), \qquad \phi(x_1, 0) = x_1, \quad x_1, x_2 \in \mathbb{R} \tag{6}$$

and

$$A'_t = \exp\left(-\int_0^{B_t} \sigma'(\phi(A_t, s))ds\right) b(\phi(A_t, B_t)), \qquad A_0 = x_0, \quad t \in [0,1]. \tag{7}$$

Proof If X and Y are two real-valued processes, whose paths are a.s. Hölder continuous of index $\alpha > 0$ and $\beta > 0$ with $\alpha + \beta > 1$, then $\int_0^t Y_s d^- X_s$ coincides with the Young integral $\int_0^t Y_s dX_s$ (see [28, Proposition 2.12]). Consequently, Proposition 1 is a consequence of, e.g., [11] or [24]. □

Case $1/6 < H < 1/2$. When $H < 1/2$, in particular the forward integral $\int_0^t B_s d^- B_s$ does not exist. Thus, in this case, the use of the symmetric integral (3) is more adequate. Here we consider only the case $b = 0$: for the general case see [19, 20] and Remark 1.

Proposition 2 *If $H > 1/6$ and if $\sigma \in \mathcal{C}^5$ satisfies a global Lipschitz condition, then the equation*

$$X_t = x_0 + \int_0^t \sigma(X_s) d^\circ B_s, \quad t \in [0,1] \tag{8}$$

admits a unique solution X in the set of processes of the form $X_t = f(B_t)$ with $f \in \mathcal{C}^5$. The solution is given by $X_t = \phi(x_0, B_t)$, $t \in [0,1]$, where ϕ is defined by (6).

Proof See [19, Theorem 2.10]. □

Remark 1 In [21], Nourdin and Simon developed recently a new concept, namely the Newton-Côtes integral corrected by Lévy areas, in order to study (1) for any $H \in (0,1)$. It allows to use a fixed point theorem to obtain existence and uniqueness in the set of processes whose paths are Hölder continuous of index $\alpha \in (0,1)$ and not only in the more restrictive—and a little arbitrary—set of processes of the form $X_t = f(B_t, A_t)$ with $f : \mathbb{R} \times [0,1] \to \mathbb{R}$ regular enough and A a process with $\mathcal{C}^1$-trajectories, as shown in [19].

Approximation schemes for stochastic differential equations of the type (1) are studied only in few articles, see, e.g., [17] and the references therein. In [18], the second-named author considers the approximation of autonomous differential equations driven by Hölder continuous functions (of any fractal index $0 < \alpha < 1$). He determines upper bounds for the order of convergence of the Euler scheme and a Milshtein-type scheme, see also [30], and applies then his results to the case of the fBm. In [17], the first-named author studies the following equation with additive fractional noise

$$X_t = x_0 + \int_0^t \sigma(s)dB_s + \int_0^t b(s, X_s)ds, \quad t \in [0, 1] \tag{9}$$

under the hypothesis $H > 1/2$. For a mean-square-L^2-error criterion, he derives by means of the Malliavin calculus the exact rate of convergence of the Euler scheme, also for non-equidistant discretizations. Moreover, the optimal approximation of (9) is also studied in [17].

In this paper, we are interested in the exact rate of convergence of the Euler scheme associated to (4) and of the Crank-Nicholson schemes associated to (8). Thus here, compared to [17], we study the non-additive case. We obtain two types of results (see Sect. 3 for precise statements):

(i) If $H > 1/2$ and under standard assumptions on σ and b, then the classical Euler scheme $\overline{X}^n$ with step-size $1/n$ for (4) defined by

$$\begin{cases} \overline{X}^n_0 = x_0, \\ \overline{X}^n_{(k+1)/n} = \overline{X}^n_{k/n} + \sigma(\overline{X}^n_{k/n})(B_{(k+1)/n} - B_{k/n}) + b(\overline{X}^n_{k/n})\frac{1}{n}, \quad k \in \{0, \dots, n-1\}, \end{cases} \tag{10}$$

and $\overline{X}^n_t = \overline{X}^n_{[nt]/n}$ for $t \in [0, 1]$ verifies

$$n^{2H-1}[\overline{X}^n_1 - X_1] \xrightarrow{\text{a.s.}} -\frac{1}{2}\int_0^1 \sigma'(X_s)D_sX_1ds. \tag{11}$$

Here D_sX_t, $s, t \in [0, 1]$ denotes the Malliavin derivative at time s of X_t with respect to the fBm B. This result is somewhat surprising because it does not have an analogue in the case of the standard Brownian motion. Indeed, in this framework, or more generally when SDEs driven by semimartingales are considered, it is generally shown that $\overline{X}^n_1$ converges a.s. to X_1 and then that the correctly renormalized difference converges in law, see, e.g, [13] and Remark 2, point 3. For the approximation of Itô-SDEs with respect to mean square error criterions, see, e.g., [12] or [16] and Remark 2, point 3.

Moreover, if we consider the global error on the interval [0, 1] of the Euler scheme, we obtain

$$n^{2H-1}\|\overline{X}^n - X\|_\infty \xrightarrow{\text{a.s.}} \frac{1}{2}\sup_{t\in[0,1]}\left|\int_0^t \sigma'(X_s)D_sX_tds\right|.$$

(ii) Assume that $1/6 < H < 1/2$ and let us consider the Crank-Nicholson scheme $\widehat{X}^n$ with step-size $1/n$ associated to (8):

$$\begin{cases} \widehat{X}^n_0 = x_0, \\ \widehat{X}^n_{(k+1)/n} = \widehat{X}^n_{k/n} + \frac{1}{2}(\sigma(\widehat{X}^n_{k/n}) + \sigma(\widehat{X}^n_{(k+1)/n}))(B_{(k+1)/n} - B_{k/n}), \\ \quad k \in \{0, \dots, n-1\}, \end{cases} \tag{12}$$

and $\widehat{X}^n_t = \widehat{X}^n_{[nt]/n}$ for $t \in [0, 1]$. Here we obtain the following rates of convergence:

- *(Exact rate)* If the diffusion coefficient $\sigma \in \mathcal{C}^1$ satisfies

$$\sigma(x)^2 = \alpha x^2 + \beta x + \gamma \quad \text{with some } \alpha, \beta, \gamma \in \mathbb{R},$$

we have

$$n^{3H-1/2}[\widehat{X}_1^n - X_1] \xrightarrow{\mathcal{L}} \sigma_H \frac{\alpha}{12} \sigma(X_1) G, \tag{13}$$

with $G \sim \mathrm{N}(0, 1)$ independent of X_1 and σ_H^2 given by (22). We prove also an equivalent of (13), at the global level:

$$n^{3H-1/2} \sup_{k \in \{0,\ldots,n\}} |\widehat{X}_{k/n}^n - X_{k/n}| \xrightarrow{\mathcal{L}} \sigma_H \frac{\alpha}{12} \sup_{t \in [0,1]} |\sigma(X_t) W_t|,$$

with W a standard Brownian motion independent of X. Compared to the above result for the Euler scheme, the convergence to a mixing law obtained here is classical in the semimartingale framework. In the fBm framework, such a phenomenon was already obtained in three recent papers for $1/2 < H < 3/4$: in [5], the authors study the asymptotic behavior of the power variation of processes of the form $\int_0^T u_s dB_s$, while in [14] and [15] the asymptotic behavior of $\int_0^t f(\overline{X}_s^n) G(\dot{\overline{X}}_s^n n^{H-1}) ds$ is studied, where $\overline{X}^n$ denotes the broken-line approximation with stepsize $1/n$ of the solution X of (8) and $\dot{\overline{X}}^n$ its derivative.
- *(Upper bound)* If $1/3 < H < 1/2$ and $\sigma \in \mathcal{C}_b^\infty$ is bounded, we have

$$\text{for any } \alpha < 3H - 1/2, \quad n^\alpha [\widehat{X}_1^n - X_1] \xrightarrow{\text{Prob}} 0. \tag{14}$$

Note that the first case covers in particular linear diffusion coefficients, while in the second case we consider smooth diffusion coefficients, which are bounded and therefore nonlinear. The exact rate of convergence in the general case remains an open problem although it seems that it is again $3H - 1/2$, as in (13).

The paper is organized as follows. In Sect. 2, we recall a few facts about the Malliavin calculus with respect to the fBm B. Section 3 contains the results concerning the exact rates of convergence for the Euler and the Crank-Nicholson schemes associated to (4) and (8) respectively. The proofs of the results for the Euler scheme are postponed to Sect. 4.

2 Recalls of Malliavin Calculus with Respect to a fBm

Let us give a few facts about the Gaussian structure of fBm and its Malliavin derivative process, following Sect. 3.1 in [23] and Chap. 1.2 in [22]. Let $\mathcal{E}$ be the set of step-functions on $[0, 1]$. Consider the Hilbert space $\mathcal{H}$ defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s), \quad s, t \in [0, 1].$$

More precisely, if we set

$$K_H(t, s) = \Gamma(H + 1/2)^{-1} (t - s)^{H-1/2} F(H - 1/2, 1/2 - H; H + 1/2, 1 - t/s),$$

where F denotes the standard hypergeometric function, and if we define the linear operator K_H^* from $\mathcal{E}$ to $L^2([0,1])$ by

$$(K_H^*\varphi)(s) = K_H(T,s)\varphi(s) + \int_s^T (\varphi(r) - \varphi(s))\frac{\partial K_H}{\partial r}(r,s)dr, \quad \varphi \in \mathcal{H},\ s \in [0,1],$$

then $\mathcal{H}$ is isometric to $L^2([0,1])$ due to the equality

$$\langle \varphi, \rho\rangle_{\mathcal{H}} = \int_0^T (K_H^*\varphi)(s)(K_H^*\rho)(s)ds, \quad \varphi, \rho \in \mathcal{H}. \tag{15}$$

The process B is a centered Gaussian process with covariance function R_H, hence its associated Gaussian space is isometric to $\mathcal{H}$ through the mapping $\mathbf{1}_{[0,t]} \mapsto B_t$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with compact support and consider the random variable $F = f(B_{t_1}, \ldots, B_{t_n})$ (we then say that F is a smooth random variable). The derivative process of F is the element of $L^2(\Omega, \mathcal{H})$ defined by

$$D_s F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B_{t_1}, \ldots, B_{t_n})\mathbf{1}_{[0,t_i]}(s), \quad s \in [0,1].$$

In particular $D_s B_t = \mathbf{1}_{[0,t]}(s)$. As usual, $\mathbb{D}^{1,2}$ is the closure of the set of smooth random variables with respect to the norm

$$\|F\|_{1,2}^2 = \mathrm{E}[|F|^2] + \mathrm{E}[\|D.F\|_{\mathcal{H}}^2].$$

The divergence operator δ is the adjoint of the derivative operator. If a random variable $u \in L^2(\Omega, \mathcal{H})$ belongs to the domain of the divergence operator, then $\delta(u)$ is defined by the duality relationship

$$\mathrm{E}(F\delta(u)) = \mathrm{E}\langle DF, u\rangle_{\mathcal{H}},$$

for every $F \in \mathbb{D}^{1,2}$.

Finally, let us recall the following result proved in [21]:

Proposition 3 *Let $H > 1/2$, $\sigma \in \mathcal{C}_b^2$ and $b \in \mathcal{C}_b^1$. Then we have for the unique solution $X = (X_t, t \in [0,1])$ of (4) that $X_t \in \mathbb{D}^{1,2}$ for any $t \in (0,1]$ and*

$$D_s X_t = \sigma(X_s)\exp\left(\int_s^t b'(X_u)du + \int_s^t \sigma'(X_u)d^- B_u\right), \quad 0 \le s \le t \le 1,\ t > 0. \tag{16}$$

3 Exact Rates of Convergence

In the sequel, we will assume that $[0,1]$ is partitioned by $\{0 = t_0 < t_1 < \cdots < t_n = 1\}$ with $t_k = k/n$, $0 \le k \le n$. Rates of convergence will thus be given relative to this partition scheme. For simplicity, we write $\Delta B_{k/n}$ instead of $B_{(k+1)/n} - B_{k/n}$. For the usage of non-equidistant discretization in the simulation of fBm, see Remark 3, point 2.

3.1 Euler Scheme

In this section, we assume that $H > 1/2$ and we consider (4), i.e., the integral with respect to B is defined by (2). The Euler scheme $\overline{X}^n$ for (4) is defined by (10). The following theorem shows that the exact rate of convergence of the Euler scheme is in general n^{-2H+1} for the error at the single point $t = 1$.

Theorem 1 *Let $b \in \mathcal{C}_b^2$ and $\sigma \in \mathcal{C}_b^3$. Then, as $n \to \infty$, we have*

$$n^{2H-1}[\overline{X}_1^n - X_1] \xrightarrow{\text{a.s.}} -\frac{1}{2}\int_0^1 \sigma'(X_s)D_sX_1 ds. \tag{17}$$

Proof We postpone it to Sect. 4. □

Remark 2 (1) The asymptotic constant of the error does not vanish, e.g., for the linear equation with constant coefficients, i.e.,

$$X_t = x_0 + \gamma\int_0^t X_s d^- B_s + \beta\int_0^t X_s ds, \quad t \in [0, 1] \tag{18}$$

with $\gamma, \beta \in \mathbb{R}$, $\gamma \neq 0$ and $x_0 \neq 0$.

(2) The appearance of the Malliavin derivative in the asymptotic constant seems to be due to the fact that $(D_t X_1)_{t\in[0,1]}$ measures the functional dependency of X_1 on the driving fBm, see [25].

(3) For the Itô-SDE, i.e., the case $H = 1/2$, it is shown in [13] that

$$n^{1/2}[\overline{X}_1^n - X_1^{\text{Itô}}] \xrightarrow{\mathcal{L}} -\frac{1}{\sqrt{2}}Y_1\int_0^1 \sigma\sigma'(X_s^{\text{Itô}})Y_s^{-1}dW_s$$

as $n \to \infty$, with a Brownian motion W, which is independent of the Brownian motion B, and

$$Y_s = \exp\left(\int_0^s b'(X_u^{\text{Itô}}) - \frac{1}{2}\sigma\sigma'(X_u^{\text{Itô}})du + \int_0^s \sigma'(X_u^{\text{Itô}})dB_u\right), \quad s \in [0, 1].$$

In [3] the analogous assertion for the mean-square error is established, i.e.,

$$n\mathrm{E}|\overline{X}_1^n - X_1^{\text{Itô}}|^2 \longrightarrow \frac{1}{2}\mathrm{E}\left|Y_1\int_0^1 \sigma\sigma'(X_s^{\text{Itô}})Y_s^{-1}dW_s\right|^2$$

as $n \to \infty$.

(4) Assume that $b \in \mathcal{C}_b^2$, $\sigma \in \mathcal{C}_b^3$ and that additionally b and σ are bounded with $\inf_{x\in\mathbb{R}} |\sigma(x)| > 0$. Under these stronger assumptions, which are a priori only of technical nature, we can show—by applying the same techniques—that Theorem 1 is also valid with respect to the mean square error, i.e.,

$$n^{2H-1}(\mathrm{E}|\overline{X}_1^n - X_1|^2)^{1/2} \longrightarrow \frac{1}{2}\left(\mathrm{E}\left|\int_0^1 \sigma'(X_s)D_sX_1 ds\right|^2\right)^{1/2}$$

as $n \to \infty$. Note that under the above assumptions on b and σ (7) simplifies to

$$A'_t = \sigma(A_t)\frac{b(\phi(A_t, B_t))}{\sigma(\phi(A_t, B_t))}, \qquad A_0 = x_0, \quad t \in [0, 1],$$

which allows us to control the integrability of the remainder terms in the error expansions made in the proof of Theorem 1.

(5) For (18) and when $H < 3/4$, it is possible to go further and to obtain a convergence in law for the third term in the asymptotic development of $\overline{X}_1^n$, see also Theorem 3: We have, as $n \to +\infty$,

$$\begin{aligned} \overline{X}_1^n &\xrightarrow{\text{a.s.}} X_1, \\ n^{2H-1}[\overline{X}_1^n - X_1] &\xrightarrow{\text{a.s.}} -\tfrac{\gamma^2}{2} X_1, \\ n^{2H-1/2}[\overline{X}_1^n - X_1 + \tfrac{\gamma^2}{2} X_1 n^{1-2H}] &\xrightarrow{\mathcal{L}} -\tfrac{\gamma^2}{2} X_1 G \end{aligned} \tag{19}$$

with G a centered Gaussian random variable. For $H = 3/4$ the last convergence is again valid if one replaces $n^{2H-1/2}$ by $n^{2H-1/2}(\log n)^{-1/2}$. Indeed, we have

$$\overline{X}_1^n = x_0 \exp\left(\gamma B_1 + \beta - \frac{1}{2}\gamma^2 \sum_{k=0}^{n-1} (\Delta B_{k/n})^2 + R_n\right)$$

with $n^{2H-1/2}|R_n| \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$. Thus it holds, as $n \to +\infty$,

$$\overline{X}_1^n - X_1 + \frac{\gamma^2}{2} X_1 n^{1-2H} \approx -\frac{\gamma^2}{2n^{2H}} X_1 \sum_{k=0}^{n-1} [(n^H \Delta B_{k/n})^2 - 1].$$

Hence Theorem 3 (or Theorem 6 for the case $H = 3/4$) in [5] allows us to obtain the convergence in law in (19). When $H > 3/4$, it seems to be hard to derive a result in law since, in this case, arguments used in the proof of Theorem 3 in [5] are not valid anymore. Indeed, in this case, we do not work in a Gaussian framework, see, e.g., Theorem 8 in [5].

To overcome this problem one can modify the Euler scheme for the linear equation such that the second order quadratic variation of B appears in the error expansion. See, e.g., [15] for a similar strategy in the case of weighted p-variations of fractional diffusions. The second order quadratic variation of fractional Brownian motion is given by

$$V_n^2(B) = \sum_{k=1}^{n-1} (B_{(k+1)/n} - 2B_{k/n} + B_{(k-1)/n})^2.$$

It is well known, compare for example [1], that

$$\begin{aligned} n^{2H-1} V_n^2(B) &\xrightarrow{\text{a.s.}} 4 - 2^{2H}, \\ n^{2H-1/2} V_n^2(B) - n^{1/2}(4 - 2^{2H}) &\xrightarrow{\mathcal{L}} G, \end{aligned}$$

as $n \to \infty$, where G is a centered Gaussian random variable with known variance $c_H^2 > 0$. Moreover, by an obvious modification of Proposition 4 we also have

$$(B_1, n^{2H-1/2} V_n^2(B) - n^{1/2}(4 - 2^{2H})) \xrightarrow{\mathcal{L}} (B_1, G)$$

as $n \to \infty$, with G independent of B_1.

For the following approximation scheme for the linear equation (18)

$$\begin{cases} \widetilde{X}_0^n = x_0, \\ \widetilde{X}_{(k+1)/n}^n = \widetilde{X}_{k/n}^n + \gamma \widetilde{X}_{k/n}^n \Delta_{k/n} B + \frac{\gamma^2}{2} \widetilde{X}_{k/n}^n \Delta_{k/n} B \Delta_{(k-1)/n} B + \beta \widetilde{X}_{k/n}^n \frac{1}{n}, \\ \quad k \in \{0, \dots, n-1\}, \end{cases}$$

we get by straightforward calculations

$$\overline{X}_1^n = X_1 \exp\left(-\frac{1}{4} \gamma^2 V_n^2(B) - \frac{\beta^2}{2} \frac{1}{n} - \beta \gamma B_1 \frac{1}{n} + R_n \right)$$

with $n^{\min\{2H-1/2,1\}} |R_n| \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$. Hence we obtain

$$n^{2H-1} [\widetilde{X}_1^n - X_1] \xrightarrow{\text{a.s.}} -\frac{\gamma^2}{4} (4 - 2^{2H}) X_1$$

as $n \to \infty$. Furthermore, we get the following error expansions according to the different values of H.

(i) Case $1/2 < H < 3/4$:

$$n^{2H-1/2} \left[\widetilde{X}_1^n - X_1 + \frac{\gamma^2}{4} (4 - 2^{2H}) X_1 n^{1-2H} \right] \xrightarrow{\mathcal{L}} -\frac{\gamma^2}{4} X_1 G.$$

(ii) Case $H = 3/4$:

$$n \left[\widetilde{X}_1^n - X_1 + \frac{\gamma^2}{4} (4 - 2^{3/2}) X_1 n^{-1/2} \right] \xrightarrow{\mathcal{L}} -\frac{\gamma^2}{4} X_1 G - \frac{\beta^2}{2} X_1 - \beta \gamma X_1 B_1.$$

(iii) Case $3/4 < H < 1$:

$$n \left[\widetilde{X}_1^n - X_1 + \frac{\gamma^2}{4} (4 - 2^{2H}) X_1 n^{1-2H} \right] \xrightarrow{\mathcal{L}} -\frac{\beta^2}{2} X_1 - \beta \gamma X_1 B_1.$$

Thus for this scheme we get—according to the values of H—different error expansions due to the drift part of the equation.

For the global error on the interval [0, 1], we obtain the following result.

Theorem 2

$$n^{2H-1} \|\overline{X}^n - X\|_\infty \xrightarrow{\text{a.s.}} \frac{1}{2} \sup_{t \in [0,1]} \left| \int_0^t \sigma'(X_s) D_s X_t ds \right|. \tag{20}$$

Proof We postpone it to Sect. 4. □

Hence the Euler scheme obtains the same exact rate of convergence for the global error on the interval [0,1] as for the error at the single point $t=1$. Moreover we have a.s.

$$\sup_{t\in[0,1]}\left|\int_0^t \sigma'(X_s)D_sX_1ds\right|=0$$

if and only if a.s.

$$X_t\in(\sigma\sigma')^{-1}(\{0\})\quad \text{for all } t\in[0,1].$$

Remark 3 (1) If $b=0$, Theorem 1 and 2 are again valid under the weaker assumption that $\sigma\in\mathcal{C}_b^1$. Since in this case

$$D_sX_t=\sigma(X_s)\exp\left(\int_s^t\sigma'(X_u)d^-B_u\right)=\sigma(X_t),\qquad s\in[0,t],$$

which is an obvious consequence of the change of variable formula for fBm, we have here

$$n^{2H-1}[\overline{X}_1^n-X_1]\xrightarrow{\text{a.s.}}-\frac{1}{2}\sigma(X_1)\int_0^1\sigma'(X_s)ds$$

and

$$n^{2H-1}\|\overline{X}^n-X\|_\infty\xrightarrow{\text{a.s.}}\frac{1}{2}\sup_{t\in[0,1]}\left|\sigma(X_t)\int_0^t\sigma'(X_s)ds\right|,$$

respectively.

(2) For $H\neq 1/2$ the increments of fractional Brownian motion are correlated. Therefore the exact simulation of $B_{t_1},\dots,B_{t_n}$ is in general computationally very expensive. The Cholesky decomposition method, which is to our best knowledge the only known exact method for the non-equidistant simulation of fractional Brownian, requires $O(n^3)$ operations. Moreover the covariance matrix, which has to be decomposed, is ill-conditioned. If the discretization is equidistant, i.e., $t_i=i/n$, $i=1,\dots,n$, the computational cost can be lowered considerably, making use of the stationarity of the increments of fractional Brownian motion. For example, the Davies-Harte algorithm for the equidistant simulation of fractional Brownian motion has computational cost $O(n\log(n))$, see, e.g., [6]. For a comprehensive survey of simulation methods for fractional Brownian motion, see, e.g., [4].

(3) In the Skorohod setting, it is in general difficult to write an Euler type scheme associated to (1), even if $H>1/2$ and $b=0$. Indeed, in this case, by using the integration by parts rule $\delta(Fu)=F\delta(u)-\langle DF,u\rangle_{\mathcal{H}}$ for the Skorohod integral and by approximating $X_{(k+1)/n}$ by $X_{k/n}+\int_{k/n}^{(k+1)/n}\sigma(X_{k/n})\delta B_s$ (as in the case $H=1/2$), one obtains

$$\overline{X}^n_{(k+1)/n}=\overline{X}^n_{k/n}+\sigma(\overline{X}^n_{k/n})(B_{(k+1)/n}-B_{k/n})-\sigma'(\overline{X}^n_{k/n})\langle D\overline{X}^n_{k/n},1_{[k/n,(k+1)/n]}\rangle_{\mathcal{H}}.$$

The problem is that the Malliavin derivative $D\overline{X}^n_{k/n}$ appears, which is difficult to compute directly. Moreover, the error analysis of such an approximation seems also to be very difficult, because the L^2-norm of the Skorohod integral involves the first Malliavin derivative of the integrand. Thus, for analyzing such an approximation scheme, we need also to control the difference between the Malliavin derivative of the solution and the Malliavin derivative of the approximation. But this involves the second Malliavin derivative etc. and we cannot have closable formulas. It is one of the reasons for which we preferred here to work within the Russo-Vallois framework, instead of the Skorohod one. Another reason is that the Russo-Vallois framework is, from our point of view, simpler in the one-dimensional case than the Skorohod one, as it is shown in [19].

3.2 Crank-Nicholson Scheme

In this section, we assume that $1/6 < H < 1/2$ and we consider (8). Let $\widehat{X}^n$ be the Crank-Nicholson scheme defined by (12), which is the canonical scheme associated to (8), since the integral with respect to the driving fBm B is defined by the symmetric integral. It is an implicit scheme, but it is nevertheless well-defined, since for n sufficiently large, $x \mapsto x - \frac{1}{2}\Delta B_{k/n}\sigma(x)$ is invertible. Although (12) seems to be rather close to (10) with $b = 0$, the situation is in fact here significantly more difficult. That is why we study the rate of convergence for the Crank-Nicholson scheme only in the following particular cases:

- Case 1: $1/6 < H < 1/2$ and $\sigma \in \mathcal{C}^1$ satisfies $\sigma(x)^2 = \alpha x^2 + \beta x + \gamma$ for some $\alpha, \beta, \gamma \in \mathbb{R}$,
- Case 2: $1/3 < H < 1/2$ and $\sigma \in \mathcal{C}_b^\infty$ bounded.

3.2.1 Case 1

Compared to Theorem 1, we have here a convergence in law. Moreover, the limit of the error is expressed as a mixed law between B_1 and an independent standard Gaussian random variable G, see also Remark 2, point 4.

Theorem 3 *Assume that* $1/6 < H < 1/2$ *and* $\sigma \in \mathcal{C}^1$ *satisfies* $\sigma(x)^2 = \alpha x^2 + \beta x + \gamma$ *for some* $\alpha, \beta, \gamma \in \mathbb{R}$. *Then, as* $n \to \infty$, *we have*

$$n^{3H-1/2}[\widehat{X}_1^n - X_1] \xrightarrow{\mathcal{L}} \sigma_H \frac{\alpha}{12}\sigma(X_1)G. \tag{21}$$

Here $G \sim \mathrm{N}(0,1)$ *is independent of* X_1 *and*

$$\sigma_H^2 = 4/3 + 1/3\sum_{\ell=1}^{\infty}\theta(\ell)^3, \quad \textit{where } 2\theta(\ell) = (\ell+1)^{2H} + (\ell-1)^{2H} - 2\ell^{2H}. \tag{22}$$

In fact, we have also a result at a functional level:

Theorem 4 *Assume that* $1/6 < H < 1/2$ *and* $\sigma \in \mathcal{C}^1$ *satisfies* $\sigma(x)^2 = \alpha x^2 + \beta x + \gamma$ *for some* $\alpha, \beta, \gamma \in \mathbb{R}$. *Then, as* $n \to \infty$, *we have*

$$n^{3H-1/2}\sup_{k\in\{0,\ldots,n\}}|\widehat{X}_{k/n}^n - X_{k/n}| \xrightarrow{\mathcal{L}} \sigma_H \frac{\alpha}{12}\sup_{t\in[0,1]}|\sigma(X_t)W_t|. \tag{23}$$

Here W is a standard Brownian motion independent of X and $\sigma_H > 0$ is once again given by (22).

Remark 4 (1) In [19], equation (8) is also studied and it is shown that $\widehat{X}_1^n$ converges in probability if and only if $H > 1/6$ and that, in this case, the limit is X_1, the solution of (8) at $t = 1$. Of course, this fact is also an obvious consequence of Theorem 3.

(2) Let us show how the constant in (22) appears. Set

$$Y_n = n^{-1/2} \sum_{k=0}^{n-1} (n^H \Delta B_{k/n})^3.$$

We claim that $\mathrm{Var}[Y_n] \to \sigma_H^2$ as $n \to +\infty$. Indeed, using $(n^H B_{t/n})_{t\in[0,\infty)} \overset{\mathcal{L}}{=} (B_t)_{t\in[0,\infty)}$, we have

$$\mathrm{Var}[Y_n] = n^{-1} \sum_{k,\ell=0}^{n-1} \mathrm{E}[(B_{k+1} - B_k)^3 (B_{\ell+1} - B_\ell)^3].$$

Since $x^3 = H_3(x) + 3H_1(x)$, where $H_1(x) = x$ and $H_3(x) = x^3 - 3x$ denote the first and third Hermite polynomial, we can write, by using the well-known identity $\mathrm{E}[H_i(X)H_j(Y)] = 0$ for $i \neq j$ and $\mathrm{E}[H_i(X)H_i(Y)] = \mathrm{E}[XY]^i/i!$ for a centered Gaussian vector (X, Y) with $\mathrm{Var}[X] = \mathrm{Var}[Y] = 1$ (see, e.g., [22, Lemma 1.1.1]):

$$\mathrm{Var}[Y_n] = (6n)^{-1} \sum_{k,\ell=0}^{n-1} \theta(\ell - k)^3 + 9n^{-1} \sum_{k,\ell=0}^{n-1} \theta(\ell - k),$$

with

$$2\theta(\ell-k) = 2\mathrm{E}[(B_{k+1}-B_k)(B_{\ell+1}-B_\ell)] = |\ell-k+1|^{2H} + |\ell-k-1|^{2H} - 2|\ell-k|^{2H}.$$

On the one hand, let us remark that $n^{-1}\sum_{k,\ell=0}^{n-1} \theta(\ell - k) = n^{-1}\mathrm{E}(\sum_{k=0}^{n-1} B_{k+1} - B_k)^2 = n^{2H-1} \longrightarrow 0$ as $n \to +\infty$, if $H < 1/2$. On the other hand, we can write

$$\sum_{k,\ell=0}^{n-1} \theta(\ell - k)^3 = \sum_{k=0}^{n-1} \theta(0)^3 + 2\sum_{k=0}^{n-1} \sum_{\ell=k+1}^{n-1} \theta(\ell - k)^3 = 8n + 2\sum_{k=0}^{n-1} \sum_{\ell=1}^{n-k-1} \theta(\ell)^3.$$

Consequently, since $\theta(\ell) < 0$ for $H < 1/2$, we deduce by using Cesaro's theorem that $\mathrm{Var}[Y_n] \longrightarrow \sigma_H^2$ given by (22), as $n \to +\infty$.

(3) In [10], in particular the approximation of fBm by its piecewise linear interpolation

$$\widetilde{B}_t^n = B_{k/n} + (nt - [nt])(B_{(k+1)/n} - B_{k/n}), \qquad t \in [k/n, (k+1)/n]$$

is studied. It is shown that the correct renormalization of $\|B - \widetilde{B}^n\|_\infty$ converges to the Gumbel distribution, i.e.,

$$P(\|\widetilde{B}^n - B\|_\infty \leq \sigma_n(\nu_n + x/\nu_n)) \longrightarrow \exp(-\exp(-x))$$

as $n \to$ for $x \in \mathbb{R}$, where $\sigma_n \approx c_H n^{-H}$ with $c_H > 0$ and ν_n is in terms of $\log(n)$. Since $3H - 1/2 > H$ for $H > 1/4$, the analogue of Theorem 2 in the setting of Theorem 4, i.e.,

$$n^{3H-1/2} \|\widehat{X}^n - X\|_\infty \xrightarrow{\mathcal{L}} \sigma_H \frac{\alpha}{12} \sup_{t\in[0,1]} |\sigma(X_t) W_t|,$$

as $n \to \infty$, can not hold without further restriction of the Hurst parameter.

For the proof of Theorem 3, we need the following lemma.

Lemma 1 (i) *We have, for* $H < 1/2$,

$$\left(B_1, n^{3H-1/2} \sum_{k=0}^{n-1} (\Delta B_{k/n})^3\right) \xrightarrow{\mathcal{L}} (B_1, G), \quad \text{as } n \to +\infty, \tag{24}$$

where G *is a centered Gaussian random variable with variance* σ_H^2 *given by* (22), *independent of* B_1.

(ii) *We have*

$$n^{5H-1/2} \sum_{k=0}^{n-1} (\Delta B_{k/n})^5 \xrightarrow{\mathcal{L}} G', \quad \text{as } n \to +\infty,$$

where G' *is a centered Gaussian random variable.*

(iii) *We have*

$$n^{6H-1} \sum_{k=0}^{n-1} (\Delta B_{k/n})^6 \xrightarrow{\text{Prob}} 15, \quad \text{as } n \to +\infty.$$

Proof of Lemma 1 The second and the third point are classical: we refer to [2]. Thus we have only to prove the first point. Let us denote by $H_1(x) = x$ and $H_3(x) = x^3 - 3x$ the first and third Hermite polynomial. Since $H < 1/2$, we have

$$n^{-1/2} \sum_{k=0}^{n-1} H_1(n^H \Delta B_{k/n}) = n^{H-1/2} B_1 \xrightarrow{\text{Prob}} 0$$

and we deduce that the convergence in law (24) will hold if and only if

$$\left(B_1, n^{-1/2} \sum_{k=0}^{n-1} H_3(n^H \Delta B_{k/n})\right) \xrightarrow{\mathcal{L}} (B_1, G), \tag{25}$$

as $n \to +\infty$. In [2], it is shown that $n^{-1/2} \sum_{k=0}^{n-1} H_3(n^H \Delta B_{k/n}) \xrightarrow{\mathcal{L}} G$. See also the second point of Remark 4. Then, the proof (25) is finished by the following proposition, which is an obvious consequence of the main result contained in [26]. □

Proposition 4 *Let* (F_1^n, F_3^n) *be a random vector such that, for every* n, F_i^n $(i = 1, 3)$ *is in the* i*-th Wiener chaos associated to the fBm* B. *If* $F_1^n \xrightarrow{\mathcal{L}} G_1$ *and* $F_3^n \xrightarrow{\mathcal{L}} G_3$ *with* G_i $(i = 1, 3)$ *some Gaussian variables, then* $\mathcal{L}(F_1^n, F_3^n) \longrightarrow \mathcal{L}(G_1) \otimes \mathcal{L}(G_3)$.

Proof of Theorem 3 In [19], the second-named author proved

$$\widehat{X}_1^n = \phi\left(x_0, B_1 + \frac{\alpha}{12}\sum_{k=0}^{n-1}(\Delta B_{k/n})^3 + \frac{\alpha^2}{80}\sum_{k=0}^{n-1}(\Delta B_{k/n})^5 + O\left(\sum_{k=0}^{n-1}(\Delta B_{k/n})^6\right)\right). \quad (26)$$

Note that $X_1 = \phi(x_0, B_1)$ and $\frac{\partial \phi}{\partial x_2}(x_1, x_s) = \sigma(\phi(x_1, x_2))$. Consequently by a Taylor expansion $\widehat{X}_1^n - X_1$ equals

$$\frac{\alpha}{12}\sigma(X_1)\sum_{k=0}^{n-1}(\Delta B_{k/n})^3 + \frac{\alpha^2}{80}\sigma(X_1)\sum_{k=0}^{n-1}(\Delta B_{k/n})^5$$
$$+ O\left(|X_1| \times \sum_{k=0}^{n-1}(\Delta B_{k/n})^6 \wedge \left(\sum_{k=0}^{n-1}(\Delta B_{k/n})^3\right)^2\right).$$

By the second point of Lemma 1, we deduce that $n^{3H-1/2}\sum_{k=0}^{n-1}(\Delta B_{k/n})^5 \longrightarrow 0$ in law, hence in probability. By the third point of Lemma 1, and since $H > 1/6$, we have that $n^{3H-1/2}\sum_{k=0}^{n-1}(\Delta B_{k/n})^6 = n^{1/2-3H} \times n^{6H-1}\sum_{k=0}^{n-1}(\Delta B_{k/n})^6 \longrightarrow 0$ in probability. By the first point of Lemma 1, and since $H > 1/6$, we have that $n^{3H-1/2}(\sum_{k=0}^{n-1}(\Delta B_{k/n})^3)^2 = n^{1/2-3H}(n^{3H-1/2}\sum_{k=0}^{n-1}(\Delta B_{k/n})^3)^2 \longrightarrow 0$ in probability. Then, using again the first point of Lemma 1 and Slutsky's Lemma, we obtain (21). □

Proof of Theorem 4 Exactly as in the proof of Theorem 3 in [5], we can prove

$$(B_t, n^{3H-1/2}V_3^n(B)_t) \xrightarrow{\mathcal{L}} (B_t, \sigma_H W_t), \quad \text{as } n \to +\infty,$$

in the space $\mathcal{D}([0,1])^2$ equipped with the Skorohod topology. Here, W is a standard Brownian motion independent of B and $V_3^n(B)$ is defined by $V_3^n(B)_t = \sum_{\ell=0}^{[nt]-1}(\Delta B_{\ell/n})^3$. To obtain Theorem 4, it suffices then to adapt the proof of Theorem 3 above. □

3.2.2 Case 2

Now assume that $\sigma \in \mathcal{C}_b^\infty$ is bounded and that $1/3 < H < 1/2$.

In the sequel, we will need some fine properties concerning the m-order variation of B on the interval $[0, 1]$. Let us state them in the following proposition:

Proposition 5 *Let $h \in \mathcal{C}_b^1$.*

1. *If $m \in \mathbb{N}$ is even then, for any $H \in (0,1)$:*

$$n^{mH-1}\sum_{k=0}^{n-1} h(B_{k/n})(B_{(k+1)/n} - B_{k/n})^m \xrightarrow{\text{Prob}} \frac{m!}{2^{m/2}(m/2)!}\int_0^1 h(B_s)ds,$$

as $n \to +\infty$.

2. *If $m \in \mathbb{N} \setminus \{1\}$ is odd then, for any $H \in (1/4, 1/2)$ and $\alpha < mH - 1/2$:*

$$n^{\alpha}\sum_{k=0}^{n-1}\left[h(B_{k/n}) + \frac{1}{2}h'(B_{k/n})(B_{(k+1)/n} - B_{k/n})\right](B_{(k+1)/n} - B_{k/n})^m \xrightarrow{\text{Prob}} 0,$$

as $n \to +\infty$. (27)

Proof (1) When $h \equiv 1$, it is a classical result: we refer to [2] for instance. To obtain the general case, it suffices to adapt the methodology developed in step 5 of [8, p. 8] or in the proof of Theorem 1 in [5].

(2) Using the same linear regression as in the proof of Theorem 4.1 in [9], we can prove that, when $H < 1/2$,

$$n^{\alpha}\sum_{k=0}^{n-1}[h(B_{(k+1)/n}) + h(B_{k/n})](B_{(k+1)/n} - B_{k/n})^m$$

converges in probability to 0, for any $\alpha < mH - 1/2$. Convergence (27) can then be obtained using a Taylor expansion and the fact that $H > 1/4$.

The details are left to the reader. □

Using the above proposition we can show the following result.

Theorem 5 *Assume that $H \in (1/3, 1/2)$ and $\sigma \in \mathcal{C}_b^\infty$ is bounded. Then we have:*

$$\textit{For any } \alpha < 3H - 1/2, \quad n^{\alpha}[\widehat{X}_1^{(n)} - X_1] \xrightarrow{\text{Prob}} 0, \tag{28}$$

as $n \to \infty$.

Proof of Theorem 5 In the following, denote

$$\Delta^j Z_{k/n} = (Z_{(k+1)/n} - Z_{k/n})^j$$

for $j, n \in \mathbb{N}$, $k \in \{0, \ldots, n-1\}$ and a process $Z = (Z_t)_{t\in[0,1]}$. When $j = 1$, we prefer the notation $\Delta Z_{k/n}$ instead of $\Delta^1 Z_{k/n}$ for simplicity. Denote also, for $p \in \mathbb{N}$,

$$\Delta^p(B) = \max_{k=0,\ldots,n-1} |\Delta^p B_{k/n}|.$$

Moreover, recall that ϕ given by (6) verifies the semigroup property:

$$\forall x, y, z \in \mathbb{R}, \quad \phi(\phi(x, y), z) = \phi(x, y + z) \tag{29}$$

and we have

$$X_t = \phi(x, B_t), \quad t \in [0, 1].$$

Simple but tedious computations (see, for instance, [19]) allow us to obtain:

$$\widehat{X}^{(n)}_{(k+1)/n} = \phi(\widehat{X}^{(n)}_{k/n}, \Delta B_{k/n} + f(\widehat{X}^{(n)}_{k/n})\Delta^3 B_{k/n} + g(\widehat{X}^{(n)}_{k/n})\Delta^4 B_{k/n} + O(\Delta^5(B))),$$

with $f = \frac{(\sigma^2)''}{24}$ and $g = \frac{\sigma(\sigma^2)'''}{12}$. We deduce from the semi-group property (29) that for every $\ell \in \{1, \dots, n\}$:

$$\widehat{X}^{(n)}_{\ell/n} = \phi\left(x, B_{\ell/n} + \sum_{k=0}^{\ell-1} f(\widehat{X}^{(n)}_{k/n})\Delta^3 B_{k/n} + \sum_{k=0}^{\ell-1} g(\widehat{X}^{(n)}_{k/n})\Delta^4 B_{k/n} + O(n\Delta^5(B))\right). \tag{30}$$

In particular, we have

$$\sup_{\ell\in\{0,\dots,n\}} |\widehat{X}^{(n)}_{\ell/n} - X_{\ell/n}| = \sup_{\ell\in\{0,\dots,n\}} |\widehat{X}^{(n)}_{\ell/n} - \phi(x, B_{\ell/n})| = O(n\Delta^3(B)) \tag{31}$$

and then (30) becomes

$$\widehat{X}^{(n)}_{\ell/n} = \phi\left(x, B_{\ell/n} + \sum_{k=0}^{\ell-1} f(\widehat{X}^{(n)}_{k/n})\Delta^3 B_{k/n} + \sum_{k=0}^{\ell-1} g(X_{k/n})\Delta^4 B_{k/n} + O(n^2\Delta^7(B))\right). \tag{32}$$

Due to the assumptions on σ we have that

$$\phi(x, y_2) = \phi(x, y_1) + \sum_{j=1}^{m} \frac{1}{j!}\frac{\partial^j \phi}{(\partial x_2)^j}(x, y_1)(y_2 - y_1)^j + O((y_2 - y_1)^{m+1}).$$

Thus we get

$$\begin{aligned}\widehat{X}^{(n)}_{k/n} &= X_{k/n} + \sum_{j=1}^{m} \frac{1}{j!}\frac{\partial^j \phi}{(\partial x_2)^j}(x, B_{k/n}) \\ &\quad\times \left(\sum_{k_1=0}^{k-1} f(\widehat{X}^{(n)}_{k_1/n})\Delta^3 B_{k_1/n} + \sum_{k_1=0}^{k-1} g(X_{k_1/n})\Delta^4 B_{k_1/n} + O(n^2\Delta^7(B))\right)^j \\ &\quad + O(n^{m+1}\Delta^{3(m+1)}(B)).\end{aligned} \tag{33}$$

(i) Now assume for a moment that $H > 5/12$. By using (33) with $m = 1$ and $\frac{\partial\phi}{\partial x_2}(x_1, x_2) = \sigma(\phi(x_1, x_2))$ we get

$$\widehat{X}^{(n)}_{k/n} = X_{k/n} + \sigma(X_{k/n}) \sum_{k_1=0}^{k-1} f(\widehat{X}^{n}_{k_1/n})\Delta^3 B_{k_1/n} + O(n^2\Delta^6(B))$$

and

$$\widehat{X}^{(n)}_{k/n} = X_{k/n} + \sigma(X_{k/n}) \sum_{k_1=0}^{k-1} f(X_{k_1/n})\Delta^3 B_{k_1/n} + O(n^2\Delta^6(B)). \tag{34}$$

By inserting the previous equality in (32) with $\ell = n$, we obtain

$$\begin{aligned}\widehat{X}^{(n)}_1 = \phi\Bigg(x, B_1 &+ \sum_{k=0}^{n-1} f(X_{k/n})\Delta^3 B_{k/n} \\ &+ \sum_{k=0}^{n-1} f'\sigma(X_{k/n})\Delta^3 B_{k/n} \sum_{k_1=0}^{k} f(X_{k_1/n})\Delta^3 B_{k_1/n} \\ &+ \sum_{k=0}^{n-1} g(X_{k/n})\Delta^4 B_{k/n} + O(n^3\Delta^9(B))\Bigg).\end{aligned} \tag{35}$$

But we have, due to the second point of Proposition 5 and the fact that $g = 2\sigma f'$ and $X_t = \phi(x, B_t)$:

$$n^\alpha \left(\sum_{k=0}^{n-1} f(X_{k/n})\Delta^3 B_{k/n} + \sum_{k=0}^{n-1} g(X_{k/n})\Delta^4 B_{k/n} \right) \xrightarrow{\text{Prob}} 0.$$

Moreover, since $H > 5/12$ and $\alpha < 3H - 1/2$, we have

$$n^{\alpha+3}\Delta^9(B) \xrightarrow{\text{a.s.}} 0.$$

On the other hand, since

$$\begin{aligned}&2\sum_{k=0}^{n-1} f'\sigma(X_{k/n})\Delta^3 B_{k/n} \sum_{k_1=0}^{k} f(X_{k_1/n})\Delta^3 B_{k_1/n} \\ &\quad = \left(\sum_{k=0}^{n-1} f'\sigma(X_{k/n})\Delta^3 B_{k/n}\right)\left(\sum_{k_1=0}^{n-1} f(X_{k_1/n})\Delta^3 B_{k_1/n}\right) - \sum_{k=0}^{n-1} ff'\sigma(X_{k/n})\Delta^6 B_{k/n}\end{aligned}$$

we deduce, this time due to the first and the second point of Proposition 5, that

$$n^\alpha \sum_{k=0}^{n-1} f'\sigma(X_{k/n})\Delta^3 B_{k/n} \sum_{k_1=0}^{k} f(X_{k_1/n})\Delta^3 B_{k_1/n} \xrightarrow{\text{Prob}} 0.$$

Finally, we obtain (28) when $H > 5/12$.

(ii) To prove the announced result, that is (28) for arbitrary $H \in (1/3, 1/2)$, it suffices to use (33) with the appropriate m for the considered H and then to proceed as in (i). The remaining details are left to the reader. □

4 Proof of Theorems 1 and 2

Throughout this section we assume that $b \in \mathcal{C}_b^2$, $\sigma \in \mathcal{C}_b^3$ and $H > 1/2$. For $g : [0,1] \to \mathbb{R}$ and $\lambda \in (0,1)$ we will use the usual notations

$$\|g\|_\infty = \sup_{t\in[0,1]} |g(t)|, \qquad \|g\|_\lambda = \sup_{s,t\in[0,1],s\neq t} \frac{|g(t)-g(s)|}{|t-s|^\lambda}.$$

Moreover positive constants, depending only on b, σ, their derivatives, x_0 and H, will be denoted by c, regardless of their value. We will write Δ instead of $1/n$.

The following properties of the function ϕ are taken from Lemma 2.1 in [30].

Lemma 2 *Let ϕ given by (6). Then we have*

(a) $$\phi(x_1,x_2) = \phi(\phi(x_1,y), x_2 - y), \quad x_1 = \phi(\phi(x_1,x_2), -x_2),$$

(b) $$\frac{\partial\phi}{\partial x_2}(x_1,x_2) = \sigma(x_1)\frac{\partial\phi}{\partial x_1}(x_1,x_2),$$

(c) $$\sigma^2(x_1)\frac{\partial^2\phi}{\partial x_1^2}(x_1,-x_2) - 2\sigma(x_1)\frac{\partial^2\phi}{\partial x_1\partial x_2}(x_1,-x_2) + (\sigma\sigma')(x_1)\frac{\partial\phi}{\partial x_1}(x_1,-x_2) + \frac{\partial^2\phi}{\partial x_2^2}(x_1,-x_2) = 0,$$

(d) $$1 = \frac{\partial\phi}{\partial x_1}(\phi(x_1,x_2),-x_2)\frac{\partial\phi}{\partial x_1}(x_1,x_2),$$

(e) $$\frac{\partial\phi}{\partial x_1}(x_1,x_2) = \exp\left(\int_0^{x_2} \sigma'(\phi(x_1,s))ds\right)$$

for all $x_1, x_2, y \in \mathbb{R}$.

The following is well known and easy to prove.

Lemma 3 *Let $n \in \mathbb{N}$ and $a_i, b_i \in \mathbb{R}$ for $i = 1,\dots,n$.*

(a) *For x_j, $j = 0,\dots,n$ given by the recursion*

$$x_{j+1} = x_j b_j + a_j, \quad j = 0,\dots,n-1,$$

with $x_0 = 0$, we have

$$x_j = \sum_{i=0}^{j-1} a_i \prod_{k=i+1}^{j-1} b_k, \quad j = 1,\dots n.$$

(b) *If*

$$|x_{j+1}| \le |x_j||b_j| + |a_j|, \quad j = 0,\dots,n-1,$$

with $x_0 = 0$ *and* $|b_j| \geq 1$ *for all* $j = 0, \ldots, n-1$, *then*

$$\max_{j=0,\ldots,n} |x_j| \leq \sum_{i=0}^{n-1} |a_i| \prod_{k=1}^{n-1} |b_k|.$$

We will also require that the Euler approximation of the solution and the process $(A_t)_{t\in[0,1]}$ given by (7) can be uniformly bounded in terms of the driving fBm.

Lemma 4 *We have*

$$\sup_{n\in\mathbb{N}} \sup_{k=0,\ldots,n} |\overline{X}^n_{k/n}| \leq \exp(c \exp(c(\|B\|_\infty + \|B\|^2_{1/2}))) \quad a.s., \tag{36}$$

$$\|A\|_\infty \leq \exp(c \exp(c\|B\|_\infty)) \quad a.s. \tag{37}$$

Proof We prove only the first assertion, following the proof of Lemma 2.4. in [30]. The second assertion can be obtained by a straightforward application of Gronwall's Lemma to (7). By Lemma 2(a) we have

$$\phi(\overline{X}^n_{k/n}, -B_{k/n}) = \phi(\phi(\overline{X}^n_{k/n}, \Delta B_{k/n}), -B_{(k+1)/n}).$$

Using this, we obtain by the mean value theorem

$$\begin{aligned}
&\phi(\overline{X}^n_{(k+1)/n}, -B_{(k+1)/n}) - \phi(\overline{X}^n_{k/n}, -B_{k/n}) \\
&\quad = [\overline{X}^n_{(k+1)/n} - \phi(\overline{X}^n_{k/n}, \Delta B_{k/n})] \frac{\partial\phi}{\partial x_1}(\xi_k, -B_{(k+1)/n})
\end{aligned}$$

with ξ_k between $\overline{X}^n_{(k+1)/n}$ and $\phi(\overline{X}^n_{k/n}, \Delta B_{k/n})$. Moreover

$$\phi(\overline{X}^n_{k/n}, \Delta B_{k/n}) = \phi(\overline{X}^n_{k/n}, 0) + \Delta B_{k/n} \frac{\partial\phi}{\partial x_2}(\overline{X}^n_{k/n}, 0) + \frac{1}{2}(\Delta B_{k/n})^2 \frac{\partial^2\phi}{\partial x_2^2}(\overline{X}^n_{k/n}, \zeta_k)$$

with $|\zeta_k| \leq |\Delta B_{k/n}|$. Thus we get

$$\phi(\overline{X}^n_{k/n}, \Delta B_{k/n}) = \overline{X}^n_{k/n} + \sigma(\overline{X}^n_{k/n})\Delta B_{k/n} + \frac{1}{2}(\sigma\sigma')(\phi(\overline{X}^n_{k/n}, \zeta_k))(\Delta B_{k/n})^2$$

by the definition of ϕ. So we finally obtain

$$\begin{aligned}
&\phi(\overline{X}^n_{(k+1)/n}, -B_{(k+1)/n}) - \phi(\overline{X}^n_{k/n}, -B_{k/n}) \\
&\quad = \left[b(\overline{X}^n_{k/n})\Delta - \frac{1}{2}(\sigma\sigma')(\phi(\overline{X}^n_{k/n}, \zeta_k))(\Delta B_{k/n})^2 \right] \frac{\partial\phi}{\partial x_1}(\xi_k, -B_{(k+1)/n}).
\end{aligned}$$

Since

$$\frac{\partial\phi}{\partial x_1}(x_1, x_2) = \exp\left(\int_0^{x_2} \sigma'(\phi(x_1, s)) ds \right)$$

by Lemma 2(e), we have

$$\left|\frac{\partial\phi}{\partial x_1}(\xi_k, -B_{(k+1)/n})\right| \le \exp(c\|B\|_\infty).$$

Due to the assumptions, the drift and diffusion coefficients satisfy a linear growth condition, i.e.,

$$|b(x)| \le c(1+x), \qquad |\sigma(x)| \le c(1+x)$$

for $x \in \mathbb{R}$. Hence we get

$$|\phi(\overline{X}^n_{(k+1)/n}, -B_{(k+1/n)})| \le |\phi(\overline{X}^n_{k/n}, -B_{k/n})| + c\exp(c\|B\|_\infty)(1+|\overline{X}^n_{k/n}|)\Delta$$
$$+ c\exp(c\|B\|_\infty)(1+|\phi(\overline{X}^n_{k/n}, \zeta_k)|)(\Delta B_{k/n})^2.$$

Since by Lemma 2(a)

$$\overline{X}^n_{k/n} = \phi(\phi(\overline{X}^n_{k/n}, -B_{k/n}), B_{k/n}),$$

and $\phi(0,0)=0$, we have by Lemma 2(b) and (e)

$$|\overline{X}^n_{k/n}| \le c\exp(c\|B\|_\infty)(|\phi(\overline{X}^n_{k/n}, -B_{k/n})| + \|B\|_\infty)$$

and furthermore, since $|\zeta_k| \le |\Delta B_{k/n}|$

$$|\phi(\overline{X}^n_{k/n}, \zeta_k)| \le c\exp(c\|B\|_\infty)(|\overline{X}^n_{k/n}| + \|B\|_\infty)$$
$$\le c\exp(c\|B\|_\infty)(|\phi(\overline{X}^n_{k/n}, -B_{k/n})| + \|B\|_\infty).$$

Together with

$$|\Delta B_{k/n}| \le \|B\|_{1/2}\Delta^{1/2},$$

this yields

$$|\phi(\overline{X}^n_{(k+1)/n}, -B_{(k+1/n)})| \le |\phi(\overline{X}^n_{k/n}, -B_{k/n})|[1 + c\exp(c\|B\|_\infty)(1+\|B\|^2_{1/2})\Delta]$$
$$+ c\exp(c\|B\|_\infty)(1+\|B\|^2_{1/2})\Delta.$$

Setting $M = \|B\|_\infty + \|B\|^2_{1/2}$ it follows by Lemma 3

$$|\phi(\overline{X}^n_{k/n}, -B_{k/n})| \le \sum_{k=1}^n c\exp(cM)\Delta\prod_{j=1}^n(1+c\exp(cM)\Delta)$$
$$\le \exp(c\exp(cM)).$$

Thus with

$$\overline{X}^n_{k/n} = \phi(\phi(\overline{X}^n_{k/n}, -B_{k/n}), B_{k/n}),$$

we get the estimate

$$|\overline{X}^n_{k/n}| \le \exp(c\|B\|_\infty)|\phi(\overline{X}^n_{k/n}, -B_{k/n})| + c\exp(c\|B\|_\infty)\|B\|_\infty \le \exp(c\exp(cM)).$$

□

Now we will state some lemmas, which will be needed to determine the asymptotic constant of the error of the Euler scheme. The following Lemma 5 can be shown by straightforward calculations.

Lemma 5 *Denote*

$$f(x,y) = \exp\left(-\int_0^x \sigma'(\phi(y,s))ds\right)b(\phi(y,x)), \quad x, y \in \mathbb{R}.$$

Then we have $f \in \mathcal{C}^{1,2}$ *and in particular*

$$f_y(x,y) = b'(\phi(y,x)) - f(x,y)\int_0^x \sigma''(\phi(y,s))\frac{\partial\phi}{\partial x_1}(y,s)ds, \quad x, y \in \mathbb{R}.$$

Lemma 6 *We have a.s.*

$$\begin{aligned}&\exp\left(\int_s^t b'(X_u)du + \int_s^t \sigma'(X_u)d^- B_u\right)\\ &\quad= \frac{\partial\phi}{\partial x_1}(A_t, B_t)\frac{\partial\phi}{\partial x_1}(X_s, -B_s)\exp\left(\int_s^t f_y(B_u, A_u)du\right), \quad 0 \le s \le t \le 1.\end{aligned}$$

Proof By Lemma 2(d) and (e) we have

$$\begin{aligned}\frac{\partial\phi}{\partial x_1}(A_t, B_t) &= \exp\left(\int_0^{B_t} \sigma'(\phi(A_t, u))du\right),\\ \frac{\partial\phi}{\partial x_1}(X_s, -B_s) &= \exp\left(-\int_0^{B_s} \sigma'(\phi(A_s, u))du\right).\end{aligned}$$

Using the notation

$$g(x,y) = \int_0^x \sigma'(\phi(y,u))du$$

we get by Lemma 5

$$\begin{aligned}&\frac{\partial\phi}{\partial x_1}(A_t, B_t)\frac{\partial\phi}{\partial x_1}(X_s, -B_s)\exp\left(\int_s^t f_y(B_u, A_u)du\right)\\ &\quad= \exp\left(g(B_t, A_t) - g(B_s, A_s) + \int_s^t f_y(B_u, A_u)du\right)\\ &\quad= \exp\left(\int_s^t b'(X_u)du\right)\exp(g(B_t, A_t) - g(B_s, A_s))\\ &\qquad \exp\left(-\int_s^t \int_0^{B_u} \sigma''(\phi(A_u, \tau))\frac{\partial\phi}{\partial x_1}(A_u, \tau)d\tau f(B_u, A_u)du\right).\end{aligned}$$

Since

$$g_x(x,y)=\sigma'(\phi(y,x))$$

and

$$g_y(x,y)=\int_0^x \sigma''(\phi(y,s))\frac{\partial\phi}{\partial x_1}(y,s)ds$$

we have by the change of variable formula for Riemann-Stieltjes integrals, see e.g., [11],

$$\begin{aligned}g(B_t,A_t)-g(B_s,A_s)&=\int_s^t \sigma'(\phi(A_u,B_u))d^-B_u\\&\quad+\int_s^t\int_0^{B_u}\sigma''(\phi(A_u,v))\frac{\partial\phi}{\partial x_1}(A_u,v)dv\,dA_u.\end{aligned}$$

Since $A_t'=f(B_t,A_t)$ we finally get

$$\begin{aligned}&g(B_t,A_t)-g(B_s,A_s)-\int_s^t\int_0^{B_u}\sigma''(\phi(A_u,v))\frac{\partial\phi}{\partial x_1}(A_u,v)dv f(B_u,A_u)du\\&=\int_s^t\sigma'(X_u)d^-B_u,\end{aligned}$$

which shows the assertion. □

The next lemma can be shown by a density argument.

Lemma 7 *Let $g,h\in\mathcal{C}([0,1])$ and denote $\Delta h_{k/n}=h((k+1)/n)-h(k/n)$ for $k=0\dots,n-1$, $n\in\mathbb{N}$. If*

$$\sup_{t\in[0,1]}\left|n^{2H-1}\sum_{k=0}^{n-1}\mathbf{1}_{[0,t]}(k/n)(\Delta h_{k/n})^2-t\right|\longrightarrow 0,$$

as $n\to\infty$, then it follows

$$\sup_{t\in[0,1]}\left|n^{2H-1}\sum_{k=0}^{n-1}g(k/n)\mathbf{1}_{[0,t]}(k/n)(\Delta h_{k/n})^2-\int_0^t g(s)ds\right|\longrightarrow 0,$$

as $n\to\infty$.

Now we finally prove Theorem 1 and 2. In the following we will denote by C random constants, which depend only on b, σ, their derivatives, x_0, H, $\|B\|_\infty$ and $\|B\|_\lambda$ with $\lambda<H$, regardless of their value. We start with the proof of Theorem 2.

Proof of Theorem 2 (1) We first establish a rough estimate for the pathwise error of the Euler scheme. For this, we follow the lines of the proof of Theorem 2.6 in [30]. Set

$$\widehat{A}_k^n=\phi(\overline{X}_{k/n}^n,-B_{k/n}),\quad k=0,\dots,n$$

for $n \in \mathbb{N}$. By a Taylor expansion, the properties of ϕ and Lemma 4 we have

$$\begin{aligned}\widehat{A}^n_{k+1} - \widehat{A}^n_k &= \frac{\partial \phi}{\partial x_1}(\overline{X}^n_{k/n}, -B_{k/n})(\overline{X}^n_{(k+1)/n} - \overline{X}^n_{k/n}) - \frac{\partial \phi}{\partial x_2}(\overline{X}^n_{k/n}, -B_{k/n})\Delta B_{k/n} \\ &\quad + \frac{1}{2}\frac{\partial^2 \phi}{\partial x_1^2}(\overline{X}^n_{k/n}, -B_{k/n})\sigma(\overline{X}^n_{k/n})^2(\Delta B_{k/n})^2 \\ &\quad + \frac{1}{2}\frac{\partial^2 \phi}{\partial x_2^2}(\overline{X}^n_{k/n}, -B_{k/n})(\Delta B_{k/n})^2 \\ &\quad - \frac{\partial^2 \phi}{\partial x_2 \partial x_1}(\overline{X}^n_{k/n}, -B_{k/n})\sigma(\overline{X}^n_{k/n})(\Delta B_{k/n})^2 + R^{(1)}_k\end{aligned}$$

with

$$|R^{(1)}_k| \le C((\Delta B_{k/n})^3 + \Delta \cdot \Delta B_{k/n} + \Delta^2). \tag{38}$$

Since

$$\begin{aligned}&-\frac{1}{2}\frac{\partial \phi}{\partial x_1}(\overline{X}^n_{k/n}, -B_{k/n})(\sigma\sigma')(\overline{X}^n_{k/n}) \\ &\quad = \frac{1}{2}\frac{\partial^2 \phi}{\partial x_1^2}(\overline{X}^n_{k/n}, -B_{k/n})\sigma(\overline{X}^n_{k/n})^2 + \frac{1}{2}\frac{\partial^2 \phi}{\partial x_2^2}(\overline{X}^n_{k/n}, -B_{k/n}) \\ &\qquad - \frac{\partial^2 \phi}{\partial x_2 \partial x_1}(\overline{X}^n_{k/n}, -B_{k/n})\sigma(\overline{X}^n_{k/n})\end{aligned}$$

by Lemma 2(c), we have

$$\widehat{A}^n_{k+1} = \widehat{A}^n_k + b(\overline{X}^n_{k/n})\frac{\partial \phi}{\partial x_1}(\overline{X}^n_k, -B_{k/n})\Delta + \widehat{Q}_k + R^{(1)}_k,$$

for $k = 0, \dots, n-1$, $n \in \mathbb{N}$, with

$$\widehat{Q}_k = -\frac{1}{2}(\sigma\sigma')(\overline{X}^n_{k/n})(\Delta B_{k/n})^2 \frac{\partial \phi}{\partial x_1}(\overline{X}^n_k, -B_{k/n}), \quad k = 0, \dots, n.$$

Since $\overline{X}^n_{k/n} = \phi(\widehat{A}^n_k, B_{k/n})$ and using Lemma 2(d) and (e) we get

$$\widehat{A}^n_{k+1} = \widehat{A}^n_k + f(B_{k/n}, \widehat{A}^n_k)\Delta + \widehat{Q}_k + R^{(1)}_k$$

for $k = 0, \dots, n-1$, $n \in \mathbb{N}$, with the function f given in Lemma 5. Note that

$$\sup_{n\in\mathbb{N}} \sup_{k=0,\dots,n} |\widehat{A}^n_k| \le \exp(c \exp(c(\|B\|_\infty + \|B\|^2_{1/2}))),$$

as a consequence of Lemma 4. Now set

$$e_k = A_{k/n} - \widehat{A}^n_k, \quad k = 0, \dots, n.$$

We have $e_0 = A_0 - \phi(x_0, 0) = 0$ and

$$|e_{k+1}| \le |e_k|(1 + C\Delta) + |Q_k| + |R_k^{(1)}| + \left|\int_{k/n}^{(k+1)/n} f(B_\tau, A_\tau) - f(B_{k/n}, A_{k/n})d\tau\right|.$$

Since

$$\left|\int_{k/n}^{(k+1)/n} f(B_\tau, A_\tau) - f(B_{k/n}, A_{k/n})d\tau\right| \le C\int_{k/n}^{(k+1)/n} |B_\tau - B_{k/n}|d\tau + C\Delta^2,$$

we can rewrite the above recursion as

$$|e_{k+1}| \le |e_k|(1 + C\Delta) + |Q_k| + |R_k^{(2)}|$$

with

$$|R_k^{(2)}| \le C(\|B\|_{H-\varepsilon}^3 \Delta^{3H-3\varepsilon} + \|B\|_{H-\varepsilon}\Delta^{H+1-\varepsilon} + \Delta^2) \le C\Delta^{H+1-\varepsilon}. \tag{39}$$

Since also

$$|\widehat{Q}_k| \le C(\Delta B_{k/n})^2 \le C\Delta^{2H-2\varepsilon},$$

we get by Lemma 3

$$\max_{k=0,\dots,n} |e_k| \le \prod_{i=0}^{n-1}(1 + C\Delta)\sum_{j=0}^{n-1}(|\widehat{Q}_j| + |R_j^{(2)}|) \le C\Delta^{2H-1-2\varepsilon}. \tag{40}$$

Moreover, we have

$$\max_{k=0,\dots,n} |X_{k/n} - \overline{X}_{k/n}^n| \le C\Delta^{2H-1-2\varepsilon}, \tag{41}$$

due to $X_t = \phi(A_t, B_t)$, $t \in [0, 1]$, and $\overline{X}_{k/n}^n = \phi(\widehat{A}_k^n, B_{k/n})$, $k = 0, \dots, n$.

(2) Now we derive the exact asymptotics of the error of the Euler scheme. We can write the recursion for the error $e_k = A_{k/n} - \widehat{A}_k^n$ as

$$e_{k+1} = e_k + f_y(B_{k/n}, A_{k/n})e_k\Delta + \widehat{Q}_k + R_k^{(2)} + \frac{1}{2}f_{yy}(B_{k/n}, \eta_k)e_k^2\Delta$$

with η_k between $A_{k/n}$ and $\widehat{A}_k^n$. Put

$$Q_k = -\frac{1}{2}(\sigma\sigma')(X_{k/n})(\Delta B_{k/n})^2\frac{\partial\phi}{\partial x_1}(X_{k/n}, -B_{k/n}).$$

By (40) we have

$$|Q_k - \widehat{Q}_k| \le C\Delta^{4H-1-4\varepsilon}.$$

Since moreover

$$\left|\int_{k/n}^{(k+1)/n} f_y(B_t, A_t)dt - f_y(B_{k/n}, A_{k/n})\Delta\right| \le C\Delta^{H+1-\varepsilon},$$

we get by (39) and (41)

$$e_{k+1} = e_k + e_k \int_{k/n}^{(k+1)/n} f_y(B_t, A_t)dt + Q_k + R_k^{(3)}$$

with

$$|R_k^{(3)}| \le C\Delta^{\min\{4H-1-4\varepsilon, H+1-\varepsilon\}}.$$

Applying Lemma 3 yields

$$A_{k/n} - \widehat{A}_k^n = \sum_{i=0}^{k-1} Q_i \prod_{j=i+1}^{k-1} \left(1 + \int_{j/n}^{(j+1)/n} f_y(B_t, A_t)dt\right) + R_k^{(4)},$$

with

$$\begin{aligned}\sup_{k=0,\dots,n} |R_k^{(4)}| &= \sup_{k=1,\dots,n} \left| \sum_{i=0}^{k-1} R_i^{(3)} \prod_{j=i+1}^{k-1} \left(1 + \int_{j/n}^{(j+1)/n} f_y(B_t, A_t)dt\right)\right| \\ &\le C\Delta^{\min\{4H-2-4\varepsilon, H-\varepsilon\}}.\end{aligned}$$

Thus it remains to consider the term

$$\sum_{i=0}^{k-1} Q_i \prod_{j=i+1}^{k-1} \left(1 + \int_{j/n}^{(j+1)/n} f_y(B_t, A_t)dt\right).$$

Now set

$$a_j = \int_{j/n}^{(j+1)/n} f_y(B_t, A_t)dt, \quad j = 0, \dots, n-1.$$

Since

$$\begin{aligned}&\left| \prod_{j=i+1}^{k-1} (1 + a_j) - \exp\left(\sum_{j=i}^{k-1} a_j\right)\right| \\ &\quad \le \exp\left(-\sum_{j=0}^{i-1} a_j\right) \left| \exp\left(\sum_{j=0}^{i-1} a_j\right) \prod_{j=i+1}^{k-1} (1 + a_j) - \exp\left(\sum_{j=0}^{k-1} a_j\right)\right|\end{aligned}$$

we get

$$\sup_{0\le j<k-1\le n} \left| \prod_{j=i+1}^{k-1} (1 + a_j) - \exp\left(\sum_{j=i}^{k-1} a_j\right)\right| \le C\Delta$$

by a straightforward application of Lemma 3. Hence we have

$$A_{k/n} - \widehat{A}_k^n = \sum_{i=0}^{k-1} Q_i \exp\left(\int_{i/n}^{k/n} f_y(B_s, A_s)ds\right) + R_k^{(5)} \tag{42}$$

with

$$\sup_{k=0,\dots,n} |R_k^{(5)}| \le C\Delta^{\min\{4H-2-4\varepsilon, H-\varepsilon\}}.$$

Moreover, since $X_t = \phi(A_t, B_t)$, $t \in [0,1]$ and $\overline{X}^n_{k/n} = \phi(\widehat{A}^n_k, B_{k/n})$, $k = 0, \dots, n$ we have

$$X_{k/n} - \overline{X}^n_{k/n} = \frac{\partial \phi}{\partial x_1}(A_{k/n}, B_{k/n})(A_{k/n} - \widehat{A}^n_k) + \frac{1}{2}\frac{\partial^2 \phi}{\partial x_1^2}(\theta_k, B_{k/n})(A_{k/n} - \widehat{A}^n_k)^2$$

with θ_k between $A_{k/n}$ and $\widehat{A}^n_k$. It follows by (40) and (42)

$$X_{k/n} - \overline{X}^n_{k/n} = \frac{\partial \phi}{\partial x_1}(A_{k/n}, B_{k/n}) \sum_{i=0}^{k-1} Q_i \exp\left(\int_{i/n}^{k/n} f_y(B_s, A_s) ds\right) + R_k^{(6)}$$

with

$$|R_k^{(6)}| \le C\Delta^{\min\{4H-2-4\varepsilon, H-\varepsilon\}}.$$

Since finally by Lemma 6

$$\begin{aligned}
&\frac{\partial \phi}{\partial x_1}(A_{k/n}, B_{k/n}) Q_i \exp\left(\int_{i/n}^{k/n} f_y(B_s, A_s) ds\right) \\
&\quad = -\frac{1}{2}\frac{\partial \phi}{\partial x_1}(A_{k/n}, B_{k/n})(\sigma\sigma')(X_{i/n}) \frac{\partial \phi}{\partial x_1}(X_{i/n}, -B_{i/n}) \\
&\qquad \times \exp\left(\int_{i/n}^{k/n} f_y(B_s, A_s) ds\right)(\Delta B_{i/n})^2 \\
&\quad = -\frac{1}{2}\sigma'(X_{i/n}) D_{i/n} X_{k/n} (\Delta B_{i/n})^2,
\end{aligned}$$

we have

$$X_{k/n} - \overline{X}^n_{k/n} = -\frac{1}{2}\sum_{i=0}^{k-1} \sigma'(X_{i/n}) D_{i/n} X_{k/n} (\Delta B_{i/n})^2 + R_k^{(6)}. \tag{43}$$

Define $\widetilde{X}^n_t = X_{k/n}$ for $t \in [k/n, (k+1)/n[$. Since clearly

$$\|X - \widetilde{X}^n\|_\infty \le C\Delta^{H-\varepsilon},$$

we have

$$\|X - \overline{X}^n\|_\infty = \max_{k=0,\dots,n} |X_{k/n} - \overline{X}^n_{k/n}| + R_k^{(7)}$$

with

$$|R_k^{(7)}| \le C\Delta^{H-\varepsilon}.$$

Thus it follows

$$\lim_{n\to\infty} n^{2H-1}\|X-\overline{X}^n\|_\infty = \lim_{n\to\infty}\max_{k=0,\dots,n} n^{2H-1}|X_{k/n}-\overline{X}^n_{k/n}|$$

and we get by (43)

$$\lim_{n\to\infty} n^{2H-1}\|X-\overline{X}^n\|_\infty = \lim_{n\to\infty}\max_{k=0,\dots,n} \frac{n^{2H-1}}{2}\left|\sum_{i=0}^{k-1}\sigma'(X_{i/n})D_{i/n}X_{k/n}(\Delta B_{i/n})^2\right|.$$

Furthermore it holds

$$\begin{aligned}
&\lim_{n\to\infty}\max_{k=0,\dots,n} \frac{n^{2H-1}}{2}\left|\sum_{i=0}^{k-1}\sigma'(X_{i/n})D_{i/n}X_{k/n}(\Delta B_{i/n})^2\right| \\
&\quad = \lim_{n\to\infty}\sup_{t\in[0,1]} \frac{n^{2H-1}}{2} Z_t\left|\sum_{i=0}^{n-1}\mathbf{1}_{[0,t]}(i/n)\sigma'(X_{i/n})D_{i/n}X_1(\Delta B_{i/n})^2\right|
\end{aligned}$$

with

$$Z_t = \exp\left(-\int_t^1 b'(X_u)du - \int_t^1 \sigma'(X_u)d^-B_u\right), \quad t\in[0,1].$$

This is due to the fact that the sample paths of Z are Hölder continuous of any order $\lambda < H$. It is well known that

$$n^{2H-1}\sum_{k=0}^{n-1}\mathbf{1}_{[0,t]}(k/n)(\Delta B_{k/n})^2 \xrightarrow{\text{a.s.}} t$$

as $n\to\infty$ for all $t\in[0,1]$. Since $n^{2H-1}\sum_{k=0}^{n-1}\mathbf{1}_{[0,t]}(k/n)(\Delta B_{k/n})^2$ is monotone in t, the exceptional set of the almost sure convergence can be chosen independent of $t\in[0,1]$. Thus we get by Dini's second theorem that a.s.

$$\lim_{n\to\infty}\sup_{t\in[0,1]} |n^{2H-1}\sum_{k=0}^{n-1}\mathbf{1}_{[0,t]}(k/n)(\Delta B_{k/n})^2 - t| = 0. \tag{44}$$

Hence it follows by Lemma 7

$$\begin{aligned}
&\lim_{n\to\infty}\sup_{t\in[0,1]} n^{2H-1}Z_t\left|\sum_{i=0}^{n-1}\mathbf{1}_{[0,t]}(i/n)\sigma'(X_{i/n})D_{i/n}X_1(\Delta B_{i/n})^2\right| \\
&\quad = \sup_{t\in[0,1]} Z_t\left|\int_0^t \sigma'(X_u)D_uX_1du\right| \quad \text{a.s.,}
\end{aligned}$$

which finally shows the assertion. □

Proof of Theorem 1 By (43) we have

$$X_1 - \overline{X}_1^n = -\frac{1}{2}\sum_{i=0}^{n-1}\sigma'(X_{i/n})D_{i/n}X_1(\Delta B_{i/n})^2 + R_n^{(6)}$$

with

$$|R_n^{(6)}| \le C\Delta^{\min\{4H-2-4\varepsilon, H-\varepsilon\}}.$$

The assertion follows then by (44) and Lemma 7, as in the previous proof. □

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