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On the Heyde Theorem for Finite Abelian Groups*

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It is well-known Heyde's characterization theorem for the Gaussian distribution on the real line: if ξ_j are independent random variables, α_j, β_j are nonzero constants such that $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \neq 0$ for all $i \neq j$ and the conditional distribution of $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ given $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ is symmetric, then all random variables ξ_j are Gaussian. We prove some analogs of this theorem, assuming that independent random variables take on values in a finite Abelian group X and the coefficients α_j, β_j are automorphisms of X.

KEY WORDS: Characterization of probability distributions; idempotent distributions; finite Abelian groups.

1. INTRODUCTION

The following characterization theorem for the Gaussian distribution on the real line was proved in Heyde[8] (see also section 13.4.1 of Kagan *et al.*[10]).

Theorem A. Let $\xi_1, \ldots, \xi_n, n \geqslant 2$ be independent random variables, α_j, β_j be nonzero constants such that $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \neq 0$ for all $i \neq j$. If the conditional distribution of $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ given $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ is symmetric, then all random variables ξ_j are Gaussian (can be degenerate ones).

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Discuss the following general statement of the problem. Let X be a locally compact Abelian separable metric group, $\mathrm{Aut}(X)$ be the set of topological automorphisms of X. Assume that $\alpha_j, \beta_j \in \mathrm{Aut}(X), 1 \leqslant j \leqslant n$, $n \geqslant 2$ and $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \in \mathrm{Aut}(X)$ for all $i \neq j$ (this condition one can regarded as a natural analog of the relevant condition for the real line). Let ξ_j be independent random variables taking on values in X and with distributions μ_j. Consider the linear statistics $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$. The problem consists in the description of groups X for which the symmetry of the conditional distribution of L_2 given L_1 implies that either all distributions μ_j are Gaussian or μ_j belong to a class of distributions which can be regarded as an analog of the Gaussian distributions (see e.g. Rukhin[11]; Heyer and Rall[9]; Feldman[1–6]; Feldman and Graczyk,[7] where group analogs of the well-known characterization theorems of Bernstein, Skitovich-Darmois and Polya for the Gaussian distribution are studied).

The present article is the first step in solving this problem. Namely, we shall consider the case of finite Abelian groups. At first we agree about notation. For a locally compact Abelian group X let $Y = X^*$ be its character group, (x, y) be the value of a character $y \in Y$ on an element $x \in X$. If G is a subgroup of X, then denote by $A(Y, G) = \{x \in Y : (x, y) = 1 \text{ for all } x \in G\}$ its annihilator. For $\alpha \in \mathrm{Aut}(X)$ define the conjugate automorphism $\tilde{\alpha} \in \mathrm{Aut}(X)$ by the formula $(x, \tilde{\alpha}y) = (\alpha x, y)$ for all $x \in X, y \in Y$. We recall that a subgroup G of a group X is said to be characteristic if G is invariant with respect to any $\alpha \in \mathrm{Aut}(X)$. For any natural n denote by $f_n : X \to X$ the homomorphism $f_n(x) = nx$ and put $X_{(n)} = \mathrm{Ker}\, f_n$, $X^{(n)} = \mathrm{Im}\, f_n$. Denote by $\mathbb{Z}(n) = \{0, 1, \ldots, n-1\}$ the finite cyclic group of order n with addition module n as a group operation.

Let $M^1(X)$ be the convolution semigroup of probability distributions on X, $\hat{\mu}(y) = \int_X (x, y) d\mu(x)$ be the characteristic function of a distribution $\mu \in M^1(X)$, $\sigma(\mu)$ be the support of μ. It is useful to remark that if H is a close subgroup of Y and $\hat{\mu}(y) \equiv 1$, $y \in H$, then $\hat{\mu}(y + h) = \hat{\mu}(y)$ for all $y \in Y$, $h \in H$ and $\sigma(\mu) \subset A(X, H)$. Denote by $I(X)$ the set of the idempotent distributions on X, i.e. the set of shifts of the Haar distributions m_K of compact subgroups K of X. It should be observed that for a finite Abelian group X the class $I(X)$ can be regarded as an analog of the class of the Gaussian distributions. Note that the characteristic function of the Haar distribution m_K is of the following form

$$\hat{m}_K(y) = \begin{cases} 1, & y \in A(Y, K), \\ 0, & y \notin A(Y, K). \end{cases}$$

For $\mu \in M^1(X)$ we define the distribution $\bar{\mu} \in M^1(X)$ by the formula $\bar{\mu}(E) = \mu(-E)$ for all Borel sets $E \subset X$. Note that $\hat{\bar{\mu}}(y) = \overline{\hat{\mu}(y)}$. If ξ is a random variable taking on values in X and with a distribution μ, then $\hat{\mu}(y)=\mathbf{E}[(\xi, y)]$.

2. THE CASE OF TWO INDEPENDENT RANDOM VARIABLES

Study at first the case when a number of independent random variables $n=2$. The following theorem is valid.

Theorem 1. Let X be a finite Abelian group satisfying the condition: (i) $X_{(2)}=\{0\}$, i.e. the group X contains no elements of order two. Let ξ_1, ξ_2 be independent random variables with values in X and with distributions μ_1, μ_2. Assume that α_1, α_2, β_1, $\beta_2 \in \mathrm{Aut}(X)$ and $\beta_1\alpha_1^{-1} \pm \beta_2\alpha_2^{-1} \in \mathrm{Aut}(X)$. If the conditional distribution of $L_2=\beta_1\xi_1+\beta_2\xi_2$ given $L_1=\alpha_1\xi_1+\alpha_2\xi_2$ is symmetric, then $\mu_1, \mu_2 \in I(X)$.

Proof. Passing to the random variables $\xi_j'=\alpha_j\xi_j$, $j=1,2$ we can suppose, without loss of generality, that $L_1=\xi_1+\xi_2$ and $L_2=\delta_1\xi_1+\delta_2\xi_2$, where $\delta_j \in \mathrm{Aut}(X)$ and $\delta_1 \pm \delta_2 \in \mathrm{Aut}(X)$. It is obvious that the conditional distribution of L_2 given L_1 is symmetric if and only if the conditional characteristic function $\mathbf{E}[(L_2, y)|L_1]$ is real-valued, i.e.

$$\mathbf{E}[(L_2, v)|L_1]=\mathbf{E}[(L_2, -v)|L_1], \quad v \in Y.$$

It is easily verified that this equality is equivalent to the fact that for all $u, v \in Y$ the following equality

$$\mathbf{E}[\{(L_2, v)-(L_2, -v)\}(L_1, u)]=0 \tag{1}$$

is fulfilled. Taking into account that ξ_1 and ξ_2 are independent (1) holds true if and only if the characteristic functions $\hat{\mu}_j(y)=\mathbf{E}[(\xi_j, y)]$ satisfy the equation

$$\hat{\mu}_1(u+\tilde{\delta}_1 v)\hat{\mu}_2(u+\tilde{\delta}_2 v)=\hat{\mu}_1(u-\tilde{\delta}_1 v)\hat{\mu}_2(u-\tilde{\delta}_2 v), \quad u, v \in Y. \tag{2}$$

Note that the characteristic functions of the distributions $\nu_j=\mu_j * \bar{\mu}_j$ satisfy equation (2) too, besides $\hat{\nu}_j(y)=|\hat{\mu}_j(y)|^2 \geqslant 0$, and $\hat{\nu}_j(-y)=\hat{\nu}_j(y)$. It is obvious that we may assume, without loss of generality, that $\tilde{\delta}_1=I$, where I is the identity automorphism. Put $f(y)=\hat{\nu}_1(y)$, $g(y)=\hat{\nu}_2(y)$, $\epsilon=\tilde{\delta}_2$ and rewrite Eq. (2) using these notation. We obtain

$$f(u+v)g(u+\epsilon v)=f(u-v)g(u-\epsilon v), \quad u, v \in Y. \tag{3}$$

We shall prove that if $X_{(2)}=\{0\}$, then $f(y)=g(y)=\hat{m}_K(y)$, where K is a subgroup of X. Hence, Theorem 1 will be proved.

Set $a = I - \epsilon$, $b = I + \epsilon$, $c = ab^{-1}$. It follows from the conditions of Theorem 1 that $a, b \in \mathrm{Aut}(Y)$. Putting $v = -u$ in (3) we obtain

$$g(au) = f(2u)g(bu), \quad u \in Y,$$

and hence

$$g(cu) = f(2b^{-1}u)g(u), \quad u \in Y. \tag{4}$$

Since $0 \leqslant f(y) \leqslant 1$, we infer

$$g(cu) \leqslant g(u), \quad u \in Y.$$

Inasmuch as Y is a finite group, $\mathrm{Aut}(Y)$ is a finite group too and hence $c^n = I$ for some natural n. We shall assume that n is the smallest one here. We have

$$g(y) = g(c^n y) \leqslant \cdots \leqslant g(cy) \leqslant g(y), \quad y \in Y.$$

It follows from this that

$$g(y) = g(cy) = \cdots = g(c^{n-1}y), \quad y \in Y. \tag{5}$$

Putting $u = -\epsilon v$ in (3) and taking into account that $f(-y) = f(y)$, $g(-y) = g(y)$, we obtain $f(av) = f(bv)g(2\epsilon v), v \in Y$ and hence

$$f(cv) = f(v)g(2\epsilon b^{-1}v), \quad v \in Y. \tag{6}$$

Reasoning as above, we infer

$$f(y) = f(cy) = \cdots = f(c^{n-1}y), \quad y \in Y. \tag{7}$$

Thus for each orbit $O_y = \{y, cy, \ldots, c^{n-1}y\}$ the functions $f(y)$ and $g(y)$ take on a constant value, generally, depending on y.

Put $E_f = \{y \in Y : f(y) \neq 0\}$, $B_f = \{y \in Y : f(y) = 1\}$. Analogously we shall introduce notation E_g and B_g. Since $X \approx Y$ and $X_{(2)} = \{0\}$, then $Y_{(2)} = \{0\}$, so that $f_2 \in \mathrm{Aut}(Y)$. For arbitrary finite set F denote by $|F|$ the number of elements of F. It follows from (4) and (5) that if $y \in E_g$, then

$$f(2b^{-1}y) = 1. \tag{8}$$

It follows from (8) that $|E_g| \leqslant |B_f|$. Similarly, it follows from (6) and (7) that if $y \in E_f$, then $g(2\epsilon b^{-1}y) = 1$. This implies that $|E_f| \leqslant |B_g|$. We finally obtain $|E_g| \leqslant |B_f| \leqslant |E_f|$, $|E_f| \leqslant |B_g| \leqslant |E_g|$. From this it follows that $|E_f| = |B_f|$, $|E_g| = |B_g|$. Thus, $E_f = B_f$, $E_g = B_g$. It follows from $f_2 \in \mathrm{Aut}(Y)$ and $c(B_g) = B_g$ that $\epsilon(B_g) = B_g$. Hence $b(B_g) = B_g$ and (8) implies that $E_g = B_g \subset B_f$, so that $B_f = B_g$. Thus, $f(y) = g(y) = \hat{m}_K(y)$, where $K = A(X, B_f)$. Theorem 1 is proved. □

Corollary 1. Suppose that $X_{(2)} = \{0\}$ and $f(y)$ and $g(y)$ are arbitrary characteristic functions on the group Y satisfying Eq. (3). Then $f(y)$ and $g(y)$ have the form $f(y) = (x_1, y)\hat{m}_K(y)$, $g(y) = (x_2, y)\hat{m}_K(y)$, where $x_1, x_2 \in X$ and K is a subgroup of X.

Corollary 2. Let X be a finite Abelian group, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathrm{Aut}(X)$ and $\beta_1\alpha_1^{-1} + \beta_2\alpha_1^{-1} \in \mathrm{Aut}(X)$. Let ξ_1, ξ_2 independent random variables with values in X and with distributions μ_1, μ_2 such that their characteristic functions do not vanish. If the conditional distribution of $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ given $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ is symmetric, then $\sigma(\mu'_j) \subset (X_{(2)})$, $j = 1, 2$ for some shifts μ'_j of distribution μ_j.

Proof. We use the fact that if $\mu \in M^1(X)$ and $\hat{\mu}(y) \equiv 1$ for $y \in H$, where H is a subgroup of X, then $\sigma(\mu) \subset A(X, H)$. Note that $A(X, Y^{(2)}) = X_{(2)}$. With the notation of Theorem 1 μ_j is a divisor of ν_j. It means that the required assertion will be proved if we check that $\sigma(\nu_j) \subset X_{(2)}$. Thus, it suffices to show that $f(y) \equiv g(y) \equiv 1$ for $y \in Y^{(2)}$. The equality $f(y) \equiv 1$ for $y \in Y^{(2)}$ follows directly from (4), (5) and the fact that $b \in \mathrm{Aut}(Y)$. Similarly, it follows from (6), (7) and $\epsilon, b \in \mathrm{Aut}(Y)$ that $g(y) \equiv 1$ for $y \in Y^{(2)}$. □

Remark 1. If condition (i) Theorem 1 is not fulfilled, then Theorem 1 is false. To prove this note that if μ is an arbitrary distribution on X and $\sigma(\mu) \subset G$, where G is a subgroup of X, then $\hat{\mu}(y + l) = \hat{\mu}(y)$ for all $y \in Y, l \in A(Y, G)$. Taking into account that $A(Y, X_{(2)}) = Y^{(2)}$ we see that if $\sigma(\mu) \subset X_{(2)}$, then $\hat{\mu}(y + 2h) = \hat{\mu}(y)$ and hence, $\hat{\mu}(y + h) = \hat{\mu}(y - h)$ for all $y, h \in Y$. Therefore, if ξ_1 and ξ_2 are arbitrary independent random variables with values in the subgroup $X_{(2)} \subset X$ and with distributions μ_1, μ_2 then equation (2) holds true. Thus, the conditional distribution of $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ given $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ is symmetric.

Remark 2. The condition $\beta_1\alpha_1^{-1} - \beta_2\alpha_2^{-2} \in \mathrm{Aut}(X)$ in Theorem 1 can be omitted. Realy, note that this condition is equivalent to the condition $a \in \mathrm{Aut}(Y)$. Assume that $a \notin \mathrm{Aut}(Y)$, i.e. $B = \mathrm{Ker}\ a \neq \{0\}$ and take $u \in B$. We have $\epsilon u = u$ and $bu = 2u$. Putting $u = v \in B$ in (3) we obtain

$$f(2u)g(2u) = 1, \quad u \in B,$$

and hence, $f(y) = g(y) = 1$ for all $y \in B$. For this reason the functions $f(y)$ and $g(y)$ are B-invariant, and they induce functions $\tilde{f}$ and $\tilde{g}$ on the factor-group Y/B, namely $\tilde{f}([y]) = f(y), \tilde{g}([y]) = g(y), y \in [y]$. The automorphism ϵ also induces an automorphism $\hat{\epsilon}$ on the factor-group Y/B by the rule $\hat{\epsilon}[y] = [\epsilon y], y \in [y]$ and we can consider now equation (3) on the factor-group Y/B. If the induced automorphism $\hat{\epsilon}$ does not satisfy the condition $\hat{a} \notin \mathrm{Aut}(Y)$, we repeat this procedure. In a finite number of steps we arrive at an induced automorphism satisfying already this condition.

We shall supplement Theorem 1 with the following statement.

Proposition 1. Let X be a finite Abelian group. Assume that $\delta, I \pm \delta \in \mathrm{Aut}(X)$ and ξ_2, ξ_2 are independent identically distributed with a distribution m_K random variables taking on values in X. The conditional distribution of $L_2 = \xi_1 + \delta\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric if and only if

$$\gamma(K) = K, \tag{9}$$

where $\gamma = (I + \delta)^{-1}(I - \delta)$.

Proof. We shall retain the designations used in the proof of Theorem 1. Put $L = A(Y, K)$ It is obvious that (9) is fulfilled if and only if

$$c(L) = L. \tag{10}$$

If the conditional distribution of L_2 given L_1 is symmetric, then the characteristic functions $f(y) = g(y) = \hat{m}_K(y)$ satisfy Eq. (3). Hence (4) implies that if $cy \in L$, then $y \in L$. Since $c^n = I$, (10) holds true.

Suppose that (10) is fulfilled. We shall verify that the characteristic functions $f(y)$ and $g(y)$ satisfy equation (3). Assume that for some $u, v \in Y$ the left-hand side of (3) is equal to 1. This implies that

$$u + v \in L, \qquad u + \epsilon v \in L, \tag{11}$$

and therefore $av = (I - \epsilon)v \in L$. Since $av = cbv$, (10) implies that

$$bv = (I + \epsilon)v \in L. \tag{12}$$

It follows from (11) and (12) that $u - v \in L, u - \epsilon v \in L$. Thus, the right-hand side of (3) is equal to 1 too. Similarly one can check that if the right-hand side of (3) is equal to 1 for some $u, v \in Y$, then the left-hand side is equal to 1 too. We proved that the characteristic functions $f(y) = g(y) = \hat{m}_K(y)$ satisfy equation (3). It means that the conditional distribution of L_2 given L_1 is symmetric. □

Remark 3. It is obvious that if $\delta(K) = K$, then $\gamma(K) = K$. Note that $I + \gamma = f_2(I + \delta)^{-1}$ and suppose that $X_{(2)} = \{0\}$. This implies that $f_2 \in \mathrm{Aut}(X)$, so that $I + \gamma \in \mathrm{Aut}(X)$. Hence, $\delta = (I + \gamma)^{-1}(I - \gamma)$ and it follows from (9) that $\delta(K) = K$. Thus, if $X_{(2)} = \{0\}$, then condition (9) is equivalent to the condition $\delta(K) = K$.

Note that in the proof of Theorem 1 we assumed that there exist automorphisms $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathrm{Aut}(X)$ such that $\beta_1\alpha_1^{-1} \pm \beta_2\alpha_2^{-1} \in \mathrm{Aut}(X)$. Describe now the finite Abelian groups X which possess this property. Afterwards we shall use the obtained result to prove a group analog of the Heyde theorem for an arbitrary number of independent random variables. Clearly, it suffices to find out when there exists $\delta \in \mathrm{Aut}(X)$ such that $I \pm \delta \in \mathrm{Aut}(X)$.

Proposition 2. Let X be a finite Abelian group, $X = \sum X_p$ be the decomposition of X in a direct sum of its p-components. Then the following statements are equivalent:

(α) for both $p=2$ and $p=3$ either $X_p = \{0\}$ or the decomposition of X_p in a direct sum of its cyclic subgroups contains each cyclic summand with multiplicity not less than two;

(β) there exists $\delta \in \mathrm{Aut}(X)$ such that $I \pm \delta \in \mathrm{Aut}(X)$.

Proof. Note that each p-component is a characteristic subgroup of X. It follows from this that there exists the required automorphism $\delta \in \mathrm{Aut}(X)$ if and only if there exists $\delta \in \mathrm{Aut}(X_p)$ for each prime p such that $X_p \neq \{0\}$. If $p > 3$ we can set $\delta = f_2$. Then $\delta, I \pm \delta \in \mathrm{Aut}(X_p)$. In what follows we restrict ourself considering the group X_3. The reasoning for the group X_2 is similarly. Denote by $K = X_3$ and represent K as a direct sum of its cyclic subgroups

$$K = \sum_i (\mathbb{Z}(3^{k_i}))^{n_i}, \quad k_i < k_{i+1}. \tag{13}$$

It is known that the numbers k_i and n_i are uniquely determined by the group K. We shall prove that there exists $\delta \in \mathrm{Aut}(X)$ such that $I \pm \delta \in \mathrm{Aut}(X)$ if and only if all $n_i \geqslant 2$ in (13). Really, assume that $n_{i_0} = 1$ for some i_0. We observe that for any natural n the subgroups $X_{(n)}$ and $X^{(n)}$ are characteristic. For this reason the subgroups $H_i = K^{(3^{k_i}-1)} \cap K_{(3)}$ are characteristic too. Hence, $\delta(H_{i_0} \setminus H_{i_0+1}) = H_{i_0} \setminus H_{i_0+1}$ for any $\delta \in \mathrm{Aut}(K)$. If $I \pm \delta \in \mathrm{Aut}(K)$, then

$$(I \pm \delta)(H_{i_0} \setminus H_{i_0+1}) = H_{i_0} \setminus H_{i_0+1}. \tag{14}$$

Let π be the natural projection of K on $\mathbb{Z}(3^{k_{i_0}})$. If $x \in H_{i_0} \setminus H_{i_0+1}$, then $\pi(x) \in \{\lambda, 2\lambda\}$, where λ is an element of order 3 in $\mathbb{Z}(3^{k_{i_o}})$. Assume for definiteness that $\pi(x) = \lambda$. If $\pi(\delta x) = \lambda$, then $\pi((I-\delta)x) = 0$, and if $\pi(\delta x) = 2\lambda$, then $\pi((I+\delta)x) = 0$, contrary to (14). The case $\pi(x) = 2\lambda$ can be considered similarly. Thus we proved that the condition $n_i \geqslant 2$ in (13) is necessary for the existence of δ.

It remains to prove the sufficiency. Consider $G_1 = (\mathbb{Z}(3^r))^2$ and set $\delta_1(k,l) = (k+l, k)$, $(k,l) \in G_1$. It is evident that $\delta_1, I \pm \delta_1 \in \mathrm{Aut}(G_1)$. For $G_2 = (\mathbb{Z}(3^r))^3$ we put $\delta_1(k,l,m) = (k+l+m, k+l, k)$, $(k,l,m) \in G_2$. Then $\delta_2, I \pm \delta_2 \in \mathrm{Aut}(G_2)$. If all $n_i \geqslant 2$ in (13) then K is a direct sum of groups of the form either G_1 or G_2 and the required automorphism $\delta \in \mathrm{Aut}(X)$ can be constructed as a direct sum of the automorphisms δ_1 and δ_2.

We use now Theorem 1 for proving the following statement.

Theorem 2. Let $X = \mathbb{R} + G$, where G is a finite Abelian group such that $G_{(2)} = \{0\}$. Suppose that $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathrm{Aut}(X)$ and $\beta_1\alpha^{-1} \pm \beta_2\alpha_1^{-1} \in \mathrm{Aut}(X)$, and ξ_1, ξ_2 are independent random variables with values in X and with distributions μ_1, μ_2. If the conditional distribution of $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ given $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ is symmetric, then $\mu_j = \gamma_j * \pi_j$, where γ_j are Gaussian distributions on $\mathbb{R}$, and $\pi_j \in I(X)$, $j = 1, 2$.

Proof. We have $Y = X^* \approx \mathbb{R} + H$, where $H = G^*$. Denote by (s,h), $s \in \mathbb{R}, h \in H$ elements of Y. If $d \in \mathrm{Aut}(Y)$, then $d(\mathbb{R}) = \mathbb{R}$ because $\mathbb{R}$ is the connected component of zero of Y. It is obvious that $d(H) = H$. We shall retain the notation d for the restrictions of d on $\mathbb{R}$ and on H, and we shall write $d(s,h) = (ds, dh)$, $(s,h) \in Y$. Reasoning as in the proof of Theorem 1 we reduce the proof of Theorem 2 to the case when $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$, where $\delta, I \pm \delta \in \mathrm{Aut}(X)$, and hence, to solving of Eq. (3), which becomes

$$\begin{aligned} &f(s+s', h+h')g(s+\epsilon s', h+\epsilon h') \\ &\quad = f(s-s', h-h')g(s-\epsilon s', h-\epsilon h'), \quad (s,h), (s',h') \in Y, \end{aligned} \tag{15}$$

where $f(y) = \hat{\mu}_1(y)$, $g(y) = \hat{\mu}_2(y)$. Putting $h = h' = 0$ in (15) we obtain by Theorem A

$$f(s,0) = \exp\{-\sigma_1 s^2 + it_1 s\}, \quad g(s,0) = \exp\{-\sigma_2 s^2 + it_2 s\}, \tag{16}$$

where $\sigma_j \geqslant 0, -\infty < t_j < \infty$, $j = 1, 2$. Putting $s = s' = 0$ in (15) we infer the functional equation

$$f(0, h+h')g(0, h+\epsilon h') = f(0, h-h')g(0, h-\epsilon h'), \quad h, h' \in H. \tag{17}$$

Applying Corollary 1 we see that each solution of Eq. (17) is of the form

$$f(0,h) = \hat{m}_K(h)(g_1, h), \quad g(0,h) = \hat{m}_K(h)(g_2, h), \quad h \in H, \tag{18}$$

where K is a subgroup of G and $g_1, g_2 \in G$. Substituting, if necessary, the distributions μ_j on their shifts we can suppose that $g_1 = g_2 = 0$ in (18). Set $B = A(H, K)$, then

$$f(0,h) = g(0,h) = \begin{cases} 1, & h \in B, \\ 0, & h \notin B. \end{cases} \tag{19}$$

It follows from (19) that the characteristic functions $f(s,h)$ and $g(s,h)$ are B-invariant. On the other hand as appears from Proposition 1 and Remark 3 $\delta(K) = K$ and hence, B is invariant with respect to ϵ. So, we can consider Eq. (19) on the factor-group Y/B letting $\tilde{f}([(s,h)]) = f(s,h), \tilde{g}([(s,h)\}) = g(s,h), \epsilon[(s,h)] = [\epsilon(s,h)]$, where $\hat{\epsilon}$ is a homomorphism induced by ϵ. Since $\epsilon(B) = B$, we have $\hat{\epsilon} \in \mathrm{Aut}(Y/B)$. Similarly, $\hat{a}, \hat{b} \in \mathrm{Aut}(Y/B)$, where $a = I - \epsilon, b = I + \epsilon$. The passage from Eq. (15) on Y to Eq. (15) on the factor-group Y/B means that we pass from consideration of random variables taking on values in X to consideration of random variables with values in $\mathbb{R} + K$. As appears from the above we can assume that

$$f(0,h) = g(0,h) = \begin{cases} 1, & h = 0, \\ 0, & h \neq 0 \end{cases} \tag{20}$$

holds true from the beginning. Put $s' = s, h' = -h$ in (15). We obtain

$$f(2s,0)g(bs,ah) = f(0,2h)g(as,bh). \tag{21}$$

Since $2h \neq 0$ for any $h \in H, h \neq 0$, it follows from (20) that the right-hand side of (21) vanishes for $h \in H, h \neq 0$. Taking into account (16) it follows from this that $g(bs,ah) = 0$ for all $s \in \mathbb{R}$ and $h \in H, h \neq 0$. Hence, $g(s,h) = 0$ for all $s \in \mathbb{R}$ and $h \neq 0$. Thus we obtain the representation

$$g(s,h) = \begin{cases} \exp\{-\sigma_2 s^2 + it_2 s\}, & h = 0, \\ 0, & h \neq 0. \end{cases}$$

Reasoning similarly we infer the analogous representation for $f(s,h)$ too. The statement of Theorem 2 follows directly from the obtained representations. □

3. THE CASE OF *n* INDEPENDENT RANDOM VARIABLES

Consider now the case of arbitrary number n of independent random variables.

Theorem 3. Let X be a finite Abelian group satisfying condition (α) of Proposition 2, let $X=\sum X_p$ be the decomposition of X in a direct sum of its p-component.

(I) Let X satisfy the conditions: (i) $X_{(2)}=\{0\}$; (ii) the decomposition of the group X_5 in a direct sum of its cyclic subgroups contains at least one cyclic summand with multiplicity one. Let $\alpha_j, \beta_j \in \mathrm{Aut}(X), 1 \leqslant j \leqslant n, n \geqslant 2$ and $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \in \mathrm{Aut}(X)$ for all $i \neq j$. Let ξ_j be independent random variables taking on values in X and with distributions μ_j. If the conditional distribution of $L_2=\beta_1\xi_1+\cdots+\beta_n\xi_n$ given $L_1=\alpha_1\xi_1+\cdots+\alpha_n\xi_n$ is symmetric, then all $\mu_j \in I(X)$.

(II) If $X_{(2)} \neq \{0\}$, then there exist $\delta \in \mathrm{Aut}(X)$ such that $I \pm \delta \in \mathrm{Aut}(X)$ and independent random variables ξ_1, ξ_2 taking on values in X and with distributions μ_1, μ_2 such that the conditional distribution of $L_2=\xi_1+\delta\xi_2$ given $L_1=\xi_1+\xi_2$ is symmetric, whereas $\mu_1, \mu_2 \notin I(X)$. If $X_{(2)}=\{0\}$ and condition (ii) is not fulfilled, then there exist $\alpha, \beta \in \mathrm{Aut}(X)$ such that

$$I \pm \alpha, I \pm \beta, \alpha \pm \beta \in \mathrm{Aut}(X), \tag{22}$$

and independent random variables $\xi_j, j=1,2,3$ taking on values in X and with distributions μ_j such that the conditional distribution of $L_2=\xi_1+\alpha\xi_2+\beta\xi_3$ given $L_1=\xi_1+\xi_2+\xi_3$ is symmetric, whereas all $\mu_j \notin I(X)$.

Proof. Let

$$X_5=\sum_i (\mathbb{Z}(5^{k_i}))^{n_i}, \quad k_i<k_{i+1}$$

be the decomposition of X_5 in a direct sum of its cyclic subgroups. At first we shall prove (I). Assume that (ii) is fulfilled, i.e. $n_{i_0}=1$ for some n_{i_0}. We shall verify that there exist no automorphisms $\alpha, \beta \in \mathrm{Aut}(X)$ such that (22) is fylfilled. Since the subgroup X_5 is characteristic, without loss of generality, we may assume that $X=X_5$. Set $H_i=X^{(5^{k_i}-1)} \cap X_{(5)}$. Since H_i is a characteristic subgroup, we have

$$\delta(H_{i_0} \backslash H_{i_0+1})=H_{i_0} \backslash H_{i_0+1} \tag{23}$$

for any $\delta \in \mathrm{Aut}(X)$. Let π be the natural projection of X on $\mathbb{Z}(5^{k_{i_0}})$. If $x \in H_{i_0} \backslash H_{i_{0+1}}$, then $\pi(x) \in \{\lambda, 2\lambda, 3\lambda, 4\lambda\}$, where λ is an element of order 5 in $\mathbb{Z}(5^{k_{i_0}})$. Assume for definiteness that $\pi(x)=\lambda$. The rest cases can be considered similarly. Set $\pi(\alpha x)=k\lambda, \pi(\beta x)=l\lambda$, where $k, l \in \{1,2,3,4\}$.

It follows from (22) and (23) that $\pi((I-\alpha)x)\neq 0$, so that $k\neq 1$. Since $\pi((I+\alpha)x)\neq 0$, we have $k\neq 4$. Similarly, $l\neq 1, l\neq 4$. Furthermore $\pi((\alpha-\beta)x)\neq 0$ implies that $k\neq l$. Hence, either $k=2, l=3$ or $k=3, l=2$. But then $\pi((\alpha+\beta)x)=0$. The obtained contradiction shows that if (ii) is fulfilled, then the number of independent random variables in Theorem 3 $n=2$. Statement (I) follows from (i) and Theorem 1. To prove (II) we need

Lemma 1. Let G be a group of the form $(\mathbb{Z}(3^r))^2$, $(\mathbb{Z}(3^r))^3$, $(\mathbb{Z}(5^r))^2$, $(\mathbb{Z}(5^r))^3, \mathbb{Z}(p^r)$, p is prime, $p\geqslant 7$. Then there exist $\alpha, \beta\in \mathrm{Aut}(G)$ such that (22) is fulfilled, and independent random variables ξ_j, $j=1,2,3$ taking on values in G and with distributions μ_j such that the conditional distribution of $L_2=\xi_1+\alpha\xi_2+\beta\xi_3$ given $L_1=\xi_1+\xi_2+\xi_3$ is symmetric, whereas all $\mu_j\notin I(G)$.

Proof. Let $G=\mathbb{Z}(p^r))^2$, $p=3,5$. Then $H=G^*\approx G$. Put $\alpha(k,l)=(k+2l, 2k+2l), \beta(k,l)=(2k+2l, 2k+l)$. Obviously that $\alpha,\beta\in\mathrm{Aut}(G)$ and (22) is fulfilled. Let ξ_j, $j=1,2,3$ be independent identically distributed random variables taking on values in G and with the distribution $\mu\in M^1(G)$ of the form $\mu(\{x\})=\frac{1}{p^{2r}}\big[1+\mathrm{Re}(x,(1,0))\big]$. Then the characteristic function $\hat{\mu}(y)$ is of the form

$$\hat{\mu}(y)=\begin{cases}1, & y=(0,0)\\ 1/2, & y\in\big\{(1,0),(p^r-1,0)\big\},\\ 0, & y\notin\big\{(0,0),(1,0),(p^r-1,0)\big\}.\end{cases}$$

We shall check that the conditional distribution of L_2 given L_1 is symmetric. To this end it suffices to verify that the characteristic function $\hat{\mu}(y)$ satisfies the equation

$$\hat{\mu}(u+v)\hat{\mu}(u+\tilde{\alpha}v)\hat{\mu}(u+\tilde{\beta}v)=\hat{\mu}(u-v)\hat{\mu}(u-\tilde{\alpha}v)\hat{\mu}(u-\tilde{\beta}v),\quad u,v\in H. \tag{24}$$

Observe that $\alpha=\tilde{\alpha}, \beta=\tilde{\beta}$. Obviously that (24) is fulfilled if $v=0$. Assume that $u=(k,l), v=(k',l')\neq 0$. We shall check that the left-hand side in (24) vanishes. Really, in the opposite case we have

$$\begin{aligned}&l+l'=0 \pmod{p^r},\\ &l+2k'+2l'=0 \pmod{p^r},\\ &l+2k'+l'=0 \pmod{p^r},\end{aligned}$$

This implies that $k'=l'=0$, i.e. $v=0$ contrary to the assumption. We verify similarly that for $v\neq 0$ the right-hand side in (24) vanishes too. So (24) is fulfilled for all $u,v\in H$. Thus for the groups $G=(\mathbb{Z}(p^r))^2$, $p=3,5$ Lemma 1 is proved. For the rest groups G we restrict ourself to indicate

$\alpha, \beta \in \mathrm{Aut}(G)$, and $\mu \in M^1(G)$. For $G = (\mathbb{Z}(p^r))^3$, $p = 3, 5$ put $\alpha(k, l, m) = (k + l + m, k + l, k)$, $\beta(k, l, m) = (2k + l + m, k + 2l, k + m)$, $\mu(\{x\}) = \frac{1}{p^{3r}}[1 + \mathrm{Re}(x, (1, 0, 0))]$. For $G = \mathbb{Z}(p^r)$, p is prime, $p \geqslant 7$ put $\alpha x = 2x$, $\beta x = 4x$, $\mu(\{x\}) = \frac{1}{p^r}[1 + \mathrm{Re}(x, 1)]$.

We may complete now the proof of Theorem 3. If $X_{(2)} \neq \{0\}$, then we have $X = X_2 + G$. By Proposition 2 there exists $\delta \in \mathrm{Aut}(X_2)$ such that $I \pm \delta \in \mathrm{Aut}(X_2)$. Extend δ to an automorphism of X (we keep the notation δ for the extended automorphism) in such a manner that $I \pm \delta \in \mathrm{Aut}(X)$ for the extended automorphism too. Consider arbitrary independent random variables ξ_1, ξ_2 taking on values in $X_{(2)}$ and with distributions $\mu_1, \mu_2 \notin I(X_{(2)})$. Taking into account Remark 1 we see that the conditional distribution of $L_2 = \xi_1 + \delta\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric whereas $\mu_1, \mu_2 \notin I(X)$.

If $X_{(2)} = \{0\}$ and (ii) is not fulfilled, then by Proposition 2 X is decomposed in a direct sum of groups G enumerated in Lemma 1. Let G_0 be one of these groups. Apply Lemma 1 and consider independent identically distributed random variables ξ_J, $j = 1, 2, 3$ taking on values in $G_0 \subset X$ with the distribution $\mu \notin I(G_0)$ and consider corresponding automorphisms $\alpha, \beta \in \mathrm{Aut}(G_0)$. Extend α and β to automorphisms of X in such a manner that for the extended automorphisms (22) remains true (we keep the notation α and β for the extended automorphisms). The conditional distribution of $L_2 = \xi_1 + \alpha\xi_2 + \beta\xi_3$ given $L_1 = \xi_1 + \xi_2 + \xi_3$ is symmetric by the construction, whereas all ξ_j have the distribution $\mu \notin I(X)$. Theorem 3 is proved completely. $\square$

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