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On generalized convex sets and their applications

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Dedicated to the memory of Professor Yurii Borysovych Zelins'kyi

Abstract. Some properties and applications of generalized convex sets and generalized convex functions in multidimensional real, complex, and hypercomplex spaces are described.

Keywords. m-convex set, m-semiconvex set, m-hull of the set, m-semiconvex hull of the set, shadow problem, hypercomplex convex set, h-hull of the set, h-extreme point, h-extreme ray, $\mathbb{H}$-quasiconvex set, hypercomplex convex function, conjugate function.

1. Introduction

The concept of convexity plays an important role in mathematics.

Definition 1.1. ([1]) *A set $E \in \mathbb{R}^n$ is called convex if, together with every two points x_1 and x_2, E also contains the entire segment $[x_1, x_2]$ that connects these points.*

Definition 1.2. ([1]) *A subset X of an n-dimensional Euclidean space $\mathbb{R}^n$ is called the affine subspace of the space $\mathbb{R}^n$ if, together with any two points $x_1, x_2 \in \mathbb{R}^n$, it also contains a straight line passing through these points.*

All affine subspaces are convex sets because together with any two of their points, they also contain a line that passes through these points and therefore also contain a line segment connecting these points. Affine subspaces are a narrower class of sets than convex sets. This follows from the fact that a convex set together with any two of its points must contain not the entire straight line passing through these points but only a part of this straight line.

Consider analogs of convex sets in the Euclidean space

$$\mathbb{C}^n := \underbrace{\mathbb{C} \times \mathbb{C} \times ... \times \mathbb{C}}_{n},$$

where $\mathbb{C}$ is the algebra of complex numbers, and n is an arbitrary natural number. If $n = 1$, this space is the complex plane. In the space $\mathbb{C}^n$, similarly as in the space $\mathbb{R}^n$, a complex affine subspace is defined; see [2,3]. A complex Euclidean space of complex dimension m is a real Euclidean space of real dimension $2m$. Complex lines, m-planes, and hyperplanes are affine subspaces of complex dimension 1, m, and $n-1$, respectively.

No less important than convexity is the concept of linear convexity. The linear convexity of the set as $n = 2$ was first introduced in 1935 by H. Behnke and E. Peschl (see [4]) and has begun to be widely used since the 1960s thanks to the works by A. Martineau (see [5]) and L. Aizenberg (see [6, 7]).

Let us define the Martineau linearly convex set.

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Definition 1.3. ([5]) *A set in the space $\mathbb{C}^n$ is called linearly convex if its complement to the entire space $\mathbb{C}^n$ is a union of complex hyperplanes.*

Let us formulate the definition of the linearly convex domain and the linearly convex compact set (it is assumed that the compact set is connected) by Aizenberg.

Definition 1.4. ([6]) *A domain $D \subset \mathbb{C}^n$ is called linearly convex if, for an arbitrary point z of the boundary ∂D of the domain D, there exists a complex hyperplane that passes through z and does not intersect D.*

Definition 1.5. ([6]) *A set $E \subset \mathbb{C}^n$ is approximated from the outside (inside) by a sequence of domains D_k, $k = 1, 2, \ldots$, if $D_{k+1} \subset\subset D_k$ ($D_k \subset\subset D_{k+1}$) and $E = \bigcap_k D_k$ ($E = \bigcup_k D_k$). The notation $D_{k+1} \subset\subset D_k$ means that the closure $\overline{D_{k+1}}$ is bounded and, together with some of its neighborhood, belongs to D_k.*

Definition 1.6. ([6]) *A compact $K \subset \mathbb{C}^n$ is said to be linearly convex if there exists a sequence of linearly convex domains by which the compact K is approximated from the outside.*

Theorem 1.1. ([7]) *Each of the following two properties is equivalent to the linear convexity of the set $E \subset \mathbb{C}^n$ by Martineau:*

1) for an arbitrary point $z \notin E$, there exists a hyperplane that passes through z and does not intersect E;

2) if the point z is such that an arbitrary hyperplane passing through z intersects E, then $z \in E$.

From the Martineau linear convexity of a domain or a compact set, their Aizenberg linear convexity follows. The following theorem asserts that the inverse statement is false.

Theorem 1.2. ([7]) *There are Aizenberg linearly convex domains and compact sets that are not Martineau linearly convex.*

Definition 1.7. ([2,3]) *A set $E \subset \mathbb{C}^n$ is said to be strongly linearly convex if, for an arbitrary complex line γ, the sets $\gamma \bigcap E$ and $\gamma^o \setminus \gamma \bigcap E$ are connected ($\gamma^o = \gamma \bigcup (\infty)$).*

This definition generalizes the concept of convexity in the real case onto the complex case by using the internal properties of the set: a set $E \subset \mathbb{R}^n$ is convex if its intersection with an arbitrary real line is connected. Note that the definition of the linearly convex set is a generalization of the concept of the convexity of the set $E \subset \mathbb{R}^n$, which uses the external properties of convex sets, namely, the existence of a hyperplane that does not intersect the given set.

Note an important property of strongly linearly convex domains and compact sets.

Theorem 1.3. ([2, 3]) *Strongly linearly convex domains (compact sets) are linearly convex.*

2. Shadow problem

This section deals with issues related to the classic shadow problem and its generalizations.

Definition 2.1. ([8, 9]) *A set $E \subset \mathbb{R}^n$ is called m-convex with respect to the point $x \in \mathbb{R}^n \setminus E$, $m = 0, 1, ..., n-1$, if there is an m-dimensional plane L that passes through this point, $x \in L$, and does not intersect the given set, $L \bigcap E = \emptyset$.*

Definition 2.2. ([8, 9]) *A set $E \subset \mathbb{R}^n$ is called m-convex if it is m-convex with respect to every point $x \in \mathbb{R}^n \setminus E$ that belongs to the complement of this set.*

The concept of m-convex set in $\mathbb{R}^n$ allows us to look from a single point of view at some generalizations of convexity, including linear convexity. The definition of an m-convex set uses the generalization of the concept of convex set due to the existence of an m-plane that does not intersect the given set (the extrinsic property of convexity). On the other hand, for the generalization of convexity, as in the definition of convexity and strong linear convexity, it is possible to use restrictions on the intersections of this set by a straight line (the intrinsic property).

Proposition 2.1. ([8, 9]) *Let $\{E_i\}_{i \in I}$ be an arbitrary family of m-convex sets (here I is some finite or countable set of indices). Then the intersection of these sets $E = \bigcap_{i \in I} E_i$ is an m-convex set.*

It follows from Proposition 2.1 that for an arbitrary set $E \subset \mathbb{R}^n$, we can consider the minimum m-convex set that contains E and call it the m-hull of the set E.

Definition 2.3. ([8,9]) *An m-convex (by Proposition 2.1) intersection of all m-convex sets that contain a given set $E \subset \mathbb{R}^n$ is called the m-hull of the set E.*

Having defined the m-hull of the set E as the intersection of the m-convex sets containing the set E, we come to the following problem: find the criterion that the point $x \in \mathbb{R}^n \setminus E$ belongs to the m-hull of the set E. A partial case of a point belonging to the 1-hull of the union of a certain set of balls is the shadow problem posed by H. Khudaiberganov in 1982; see [10].

Shadow problem. *Find the minimum number of pairwise non-intersecting closed (open) balls in the space $\mathbb{R}^n$ with centers on the sphere S^{n-1} and radii smaller than the radius of the sphere, such that an arbitrary straight line passing through the center of the sphere would intersect at least one of these balls.*

In other words, this problem can be reformulated as follows:

What is the minimum number of pairwise non-intersecting closed (open) balls in the space $\mathbb{R}^n$ with centers on the sphere S^{n-1} and radii smaller than the radius of the sphere that ensures that the sphere center belongs to the 1-hull of the family of these balls?

Briefly, this problem is formulated as follows:

What is the minimum number of such balls that create a shadow for the center of the sphere?

For $n = 2$, this problem was solved by H. Khudaiberganov; see [10]. He showed that for a circle on a plane, two disks are necessary and sufficient to create a shadow. It was also proved that for $n > 2$ the minimum number of balls is equal to n. However, this proof turned out to be wrong.

In work [9], using the continuity of the change of straight lines, another solution to the shadow problem as $n = 2$ was given.

Theorem 2.1. ([10]) *There are two non-intersecting closed (open) disks with centers on the unit circle and radii less than one that ensure that the center of the circle belongs to the 1-hull of the family of these circles.*

Yu. Zelins'kyi together with his students I. Vygovs'ka and M. Stefanchuk completely solved the shadow problem as $n > 2$.

Theorem 2.2. ([8, 9]) *In order for the center of an $(n-1)$-sphere in an n-dimensional Euclidean space as $n > 2$ to belong to the 1-hull of the family of pairwise non-intersecting open (closed) balls with radii not larger (smaller) than the radius of the sphere, and centers located on the sphere, $n+1$ balls are necessary and sufficient.*

Yu. Zelins'kyi and M. Stefanchuk generalized the shadow problem to the case of an arbitrary point inside the sphere; see [11]:

What is the smallest number of pairwise non-intersecting open (closed) balls with centers on the sphere S^{n-1} and radii smaller (not larger) than the radius of the sphere that ensure that the sphere interior belongs to the 1-hull of the family of balls?

The following theorem holds in the case $n = 2$.

Theorem 2.3. ([11]) *In order for the interior of a circle to belong to the 1-hull of the family of pairwise non-intersecting open (closed) disks with centers on the circle and radii smaller than the radius of the circle, three disks are necessary and sufficient.*

The shadow problem can be generalized if, instead of the sphere, another surface is considered.

M. Tkachuk and T. Osipchuk formulated the shadow problem for the center of the ellipsoid of rotation in the space $\mathbb{R}^3$; see [12]:

Let a prolate ellipsoid of rotation be given. Find the minimum ratio between the lengths of its major and minor semi-axes such that three closed (open) pairwise non-intersecting balls with centers located on the ellipsoid and not intersecting the center of ellipsoid create a shadow for the center of ellipsoid.

In other words, this task can be formulated as follows; see [12]:

Find the minimum ratio between the lengths of the major and minor semi-axes of a prolate ellipsoid of rotation such that three closed (open) pairwise non-intersecting balls that do not intersect the center of ellipsoid and whose centers are located on the ellipsoid ensure that the center of ellipsoid belongs to the 1-hull of the family of balls.

The following theorem holds.

Theorem 2.4. ([12]) *Let a prolate ellipsoid of rotation be given with a ratio between its major and minor semi-axes that is strictly less than $2\sqrt{2}$. In order for the center of the given ellipsoid to belong to the 1-hull of the family of pairwise non-intersecting closed (open) balls that do not intersect the center of the ellipsoid and with centers located on it, three such balls are necessary and sufficient.*

M. Tkachuk and T. Osipchuk generalized the shadow problem by taking an arbitrary domain in the spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ instead of the sphere; see [13], [14].

The following theorems hold.

Theorem 2.5. ([13, 14]) *In order for an arbitrary fixed point x_0 in the domain $D \subset \mathbb{R}^2$ to belong to the 1-hull of closed (open) disks that pairwise do not intersect, do not contain the point x_0, and with centers on the boundary of the domain D, two such disks are necessary and sufficient.*

Theorem 2.6. ([14]) *In order for an arbitrary fixed point x_0 in the domain $D \subset \mathbb{R}^3$ to belong to the 1-hull of closed (open) balls that pairwise do not intersect, do not contain the point x_0, and with centers on the boundary of the domain D, four such balls are sufficient.*

T. Osipchuk formulated a problem that is close to the classical shadow problem; see [15]:

Find the minimum number of open (closed) and pairwise non-intersecting balls in the space $\mathbb{R}^n$ with centers on the sphere S^{n-1} and radii smaller than the radius of the sphere, which do not contain a fixed point inside the sphere and create a shadow at that point.

The following theorem gives a partial solution to the given problem.

Theorem 2.7. ([15]) *Let $S^2(r)$ be a sphere with center at the coordinate origin and radius r in the space $\mathbb{R}^3$. Let us denote by $n(x)$ the smallest number of non-intersecting open balls with centers on the sphere $S^2(r)$, and such that they do not contain a fixed point $x \in \mathbb{R}^3$ and create a shadow at this point. Then $n(x) = 3$ for every point $x \in \mathbb{R}^3$ such that $\frac{7}{9}r \leqslant |x| \leqslant r$.*

Yu. Zelins'kyi together with his students I. Vygovs'ka and H. Dakhil studied the shadow problem for balls of the equal radii; see [16]:

What is the minimum number of pairwise non-intersecting closed (open) balls with centers on the sphere S^{n-1} and equal radii smaller (not larger) than the radius of the sphere in an n-dimensional real Euclidean space $\mathbb{R}^n$ that is sufficient for an arbitrary straight line passing through the center of the sphere to intersect at least one of those balls?

The following theorems give a solution to this problem.

Theorem 2.8. ([16]) *$n+1$ closed balls of equal radii centered on the sphere S^{n-1} in the space $\mathbb{R}^n$ are sufficient to create a shadow in the center of the sphere if the balls can touch one another.*

Theorem 2.9. ([16]) *There is no set of open pairwise non-intersecting balls of equal radii in a three-dimensional real Euclidean space $\mathbb{R}^3$ with centers on the sphere S^2 and radii no larger than the radius of the sphere, such that an arbitrary straight line that passes through the center of the sphere intersects at least one of those balls.*

Theorem 2.10. ([16]) *There is no set of $m > 4$ pairwise non-intersecting (or tangent) closed balls of equal radii in a three-dimensional real Euclidean space $\mathbb{R}^3$ with centers on the sphere S^2 and radii smaller than the radius of the sphere, such that an arbitrary straight line that passes through the center of the sphere intersects at least one of those balls.*

Yu. Zelins'kyi considered the shadow problem for a family of balls whose centers are not connected to any predetermined set.

Theorem 2.11. ([17]) *In order for a selected point in an n-dimensional Euclidean space as $n \geqslant 2$ to belong to the 1-hull of the family of pairwise non-intersecting open (closed) balls that do not contain this point, n balls are necessary and sufficient.*

Also, Y. Zelins'kyi together with his students I. Vygovs'ka, H. Dakhil investigated an analog of this problem for balls of equal radii; see [18]:

What is the minimum number of pairwise non-intersecting closed (open) balls of equal radii in a three-dimensional real Euclidean space $\mathbb{R}^3$ that is necessary and sufficient for an arbitrary straight line that passes through a fixed point in space to intersect at least one of those balls?

An answer to this question is given by the following theorems:

Theorem 2.12. ([16]) *Four pairwise non-intersecting closed (open) balls of equal radii are sufficient in the space $\mathbb{R}^3$ to create a shadow at a fixed point.*

Theorem 2.13. ([18]) *Four pairwise non-intersecting closed (open) balls of equal radii are necessary and sufficient in the space $\mathbb{R}^3$ to create a shadow at a fixed point.*

T. Osipchuk solved this problem in an n-dimensional real Euclidean space $\mathbb{R}^n$, $n \geqslant 3$:

Theorem 2.14. ([19]) *$n+1$ pairwise non-intersecting closed (open) balls of equal radii are necessary and sufficient in the $\mathbb{R}^n$ space, $n \geqslant 3$, to create a shadow at a fixed point.*

By replacing balls in Theorem 2.11 by convex bodies with non-empty interior, we obtain another generalization of the shadow problem.

Let a convex set with non-empty interior be given in an n-dimensional Euclidean space $\mathbb{R}^n$, $n \geqslant 2$. A family of pairwise non-intersecting closed sets is obtained from this set by means of a group of geometric transformations. The following question arises: How many (the least number) of the elements of this family are sufficient for a selected point $x \in \mathbb{R}^n$ to belong to the 1-hull of this family (i.e., for an arbitrary straight line that passes through the point x to intersect at least one of those sets)? Yu. Zelins'kyi obtained a solution to this problem for a group of geometric transformations consisting of motions and homotheties of a convex set with non-empty interior.

Theorem 2.15. ([17]) *In order for a selected point in an n-dimensional Euclidean space as $n \geqslant 2$ to belong to the 1-hull of the family of pairwise non-intersecting closed sets obtained from a given convex set with non-empty interior and compact closure by means of the group of transformations consisting of motions and homotheties, n elements of this family are necessary and sufficient.*

Yu. Zelins'kyi and M. Stefanchuk solved a similar problem for a family of sets obtained from a convex set with non-empty interior using parallel transfers and homotheties.

Theorem 2.16. ([11]) *In order for a selected point in an n-dimensional Euclidean space as $n \geqslant 2$ to belong to the 1-hull of a family of pairwise non-intersecting closed sets obtained from a given convex set with non-empty interior by means of a group of transformations consisting of parallel translations and homotheties, n elements of this family are necessary and sufficient.*

Let us consider objects that are more general than those considered above.

Definition 2.4. ([8,9]) *A set $E \subset \mathbb{R}^n$ is called m-semiconvex with respect to a point $x \in \mathbb{R}^n \setminus E$, $m = 0, 1, ..., n-1$, if there is an m-dimensional half-plane L that passes through this point, $x \in L$, and does not intersect this set, $L \bigcap E = \emptyset$.*

Definition 2.5. ([8,9]) *A set $E \subset \mathbb{R}^n$ is called m-semiconvex if it is m-semiconvex with respect to every point $x \in \mathbb{R}^n \setminus E$ belonging to the complement of this set.*

Similarly as for m-convex sets, the convexity axiom holds for m-semiconvex ones.

Proposition 2.2. ([8,9]) *Let $\{E_i\}_{i \in I}$ be an arbitrary family of m-semiconvex sets (here I is some finite or countable set of indices). Then the intersection of these sets $E = \bigcap\limits_{i \in I} E_i$ is an m-semiconvex set.*

Therefore, for an arbitrary set $E \subset \mathbb{R}^n$, there always exists a minimum m-semiconvex set that is the intersection of all m-semiconvex sets containing E.

Definition 2.6. ([8,9]) *An m-semiconvex (by Proposition 2.2) intersection of all m-semiconvex sets containing a given set $E \subset \mathbb{R}^n$ is called the m-semiconvex hull of the set E.*

Let us consider an analog of the shadow problem for the semiconvexity formulated by Yu. Zelins'kyi and his students I. Vygovs'ka and M. Stefanchuk, which is a partial case of the point belonging to the 1-semiconvex hull of some family of balls; see [8,9].

What is the minimum number of pairwise non-intersecting closed (open) balls with centers on the sphere S^{n-1} and radii smaller (not larger) than the radius of the sphere that is sufficient for an arbitrary ray emanating from the center of the sphere to intersect at least one of those balls?

The following theorem gives a solution to this problem in the case $n = 2$.

Theorem 2.17. ([8,9]) *In order for the center of a circle $S^1 \subset \mathbb{R}^2$ to belong to the 1-semiconvex hull of the family of pairwise non-intersecting open (closed) disks with radii not greater (less) than the radius of the circle, and with centers located on this circle, three disks are necessary and sufficient.*

The following theorem gives sufficient conditions for the center of the sphere to belong to the 1-semiconvex hull of the family of balls with centers on this sphere.

Theorem 2.18. ([8, 9]) *In order for the center of a two-dimensional sphere in a three-dimensional Euclidean space to belong to the 1-semiconvex hull of a family of pairwise non-intersecting open (closed) balls with radii not greater (less) than the radius of the sphere and with centers located on the sphere, ten balls are sufficient.*

Yu. Zelins'kyi considered the shadow problem for a semiconvexity for a certain fixed point in the space $\mathbb{R}^n$.

Theorem 2.19. ([17]) *In order for a selected point in an n-dimensional Euclidean space as $n \geqslant 2$ to belong to the 1-semiconvex hull of the family of pairwise non-intersecting open (closed) balls that do not contain the given point, $n+1$ balls are necessary and sufficient.*

If the balls have equal radii in the space $\mathbb{R}^3$, then the following theorem holds.

Theorem 2.20. ([18]) *In order for a point in a three-dimensional real Euclidean space to belong to the 1-semiconvex hull of a family of open (closed) balls of equal radii, eight balls are sufficient.*

If the balls are replaced by convex bodies with non-empty interior, then the following question arises: What is the minimum number of pairwise non-intersecting closed (open) sets obtained from a given convex set with non-empty interior by means of some geometric transformations that is sufficient for a selected point $x \in \mathbb{R}^n$ to belong to the 1-semiconvex hull of this family (i.e., for an arbitrary ray emanating from this point to intersect at least one of those sets). In the work by Yu. Zelins'kyi [17], a solution to this problem was obtained for a group of geometric transformations consisting of motions and homotheties of a convex set with non-empty interior.

Theorem 2.21. ([17]) *In order for a selected point in an n-dimensional Euclidean space as $n \geqslant 2$ to belong to the 1-semiconvex hull of the family of pairwise non-intersecting closed sets obtained from a given convex set with non-empty interior and compact closure using a group of transformations consisting of motions and homotheties, $n+1$ elements of this family are necessary and sufficient.*

Yu. Zelins'kyi and M. Stefanchuk solved this problem for a family of sets obtained from a convex set with non-empty interior by means of parallel transfers and homotheties.

Theorem 2.22. ([11]) *In order for a selected point in an n-dimensional Euclidean space as $n \geqslant 2$ to belong to the 1-semiconvex hull of the family of pairwise non-intersecting closed sets obtained from a given convex set with non-empty interior by means of a group of transformations consisting of parallel transfers and homotheties, $2n$ elements of this family are necessary and sufficient.*

Consider the analogs of m-convex sets in the complex and hypercomplex spaces.

Let $\mathbb{H}$ be the algebra of quaternions $h = h_0 + e_1h_1 + e_2h_2 + e_3h_3$, where $h_0, h_1, h_2, h_3 \in \mathbb{R}$, and the imaginary units e_i, $i = 1, 2, 3$, satisfy the conditions

$$e_ie_i = -1, \quad e_ie_j = -e_je_i \quad (i \neq j, \quad j = 1, 2, 3),$$

$$e_1e_2 = e_3, \quad e_2e_3 = e_1, \quad e_3e_1 = e_2;$$

see [20].

Consider an n-dimensional hypercomplex space

$$\mathbb{H}^n := \underbrace{\mathbb{H} \times \mathbb{H} \times \ldots \times \mathbb{H}}_{n \geq 2},$$

whose elements are the points $x := (x^1, x^2, \ldots, x^n) \in \mathbb{H}^n$, where $x^j := h_0^j + e_1 h_1^j + e_2 h_2^j + e_3 h_3^j \in \mathbb{H}$, $j = \overline{1,n}$. Then each point $h = (h_0, h_1, h_2, h_3) \in \mathbb{R}^{4n}$, where $h_k = \{h_k^j\}_{j=1}^n$, $k = \overline{0,3}$, is identified with the point $x \in \mathbb{H}^n$.

m-dimensional complex (hypercomplex) planes in the space $\mathbb{C}^n$ ($\mathbb{H}^n$) are $2m$-dimensional ($4m$-dimensional) planes in the space $\mathbb{R}^n$.

Definition 2.7. ([17]) *A set $E \subset \mathbb{C}^n$ ($\mathbb{H}^n$) is called m-complex (m-hypercomplex) convex with respect to a point $x \in \mathbb{C}^n \setminus E$ ($x \in \mathbb{H}^n \setminus E$), $m = 0, 1, ..., n-1$, if there exists an m-dimensional complex (hypercomplex) plane L that passes through this point, $x \in L$, and does not intersect the given set, $L \bigcap E = \emptyset$.*

Definition 2.8. ([17]) *A set $E \subset \mathbb{C}^n$ ($\mathbb{H}^n$) is called m-complex (m-hypercomplex) convex if it is m-complex (m-hypercomplex) convex with respect to every point $x \in \mathbb{C}^n \setminus E$ ($x \in \mathbb{H}^n \setminus E$) belonging to the complement of this set.*

Similarly to the real case, for an arbitrary set $E \subset \mathbb{C}^n$ ($\mathbb{H}^n$) we can consider a minimum m-complex (m-hypercomplex) convex set that contains E and call it the m-complex (m-hypercomplex) hull of the set E.

Definition 2.9. ([17]) *The intersection of all m-complex (m-hypercomplex) convex sets containing a given set $E \subset \mathbb{C}^n$ ($\mathbb{H}^n$) is called the m-complex (m-hypercomplex) hull of the set E.*

Yu. Zelins'kyi formulated the shadow problem in complex and hypercomplex spaces; see [17].

What is the minimum number of pairwise non-intersecting closed balls with centers on the sphere $S^{2n-1} \subset \mathbb{C}^n$ ($S^{4n-1} \subset \mathbb{H}^n$) and radii smaller than the radius of the sphere that is sufficient for an arbitrary complex (hypercomplex) line passing through the center of the sphere to intersect at least one of those balls (that is, for the center of the sphere to belong to the 1-complex or 1-hypercomplex hull of those balls)?

Yu. Zelins'kyi found that two balls are necessary and sufficient to create a shadow in a complex (hypercomplex) space as $n = 2$.

Theorem 2.23. ([17]) *In order for a selected point in a 2-dimensional complex (hypercomplex) Euclidean space $\mathbb{C}^2$ ($\mathbb{H}^2$) to belong to the 1-complex (1-hypercomplex) hull of the family of pairwise non-intersecting open (closed) balls that do not contain this point, two balls are necessary and sufficient.*

In the work by Yu. Zelins'kyi and M. Stefanchuk [11], a sufficient number of such balls to create a shadow in complex and hypercomplex spaces as $n \geqslant 3$ was found.

Theorem 2.24. ([11]) *In order for the center of a sphere in an n-dimensional complex (hypercomplex) Euclidean space $\mathbb{C}^n$ ($\mathbb{H}^n$), $n \geqslant 3$, to belong to the 1-complex (1-hypercomplex) hull of the family of pairwise non-intersecting open (closed) balls with centers located on the sphere $S^{2n-1} \subset \mathbb{C}^n$ ($S^{4n-1} \subset \mathbb{H}^n$) and radii smaller than the radius of the sphere, $2n$ ($4n-2$) balls are sufficient.*

3. A null-measured set containing spheres of arbitrary radii

Many scientists dealt with the following problem: Find such families of sets which, after applying certain geometric transformations to them, belong to a set of rather small measure.

In this section, the generalization of the problem solved by A. Besicovitch and R. Rado [21] (they constructed a planar set of Lebesgue measure zero, which contains circles of arbitrary radii, for the 2-dimensional Euclidean space) to the case of n-dimensional Euclidean space as $n \geqslant 2$.

Let $\mathbf{M} = (M_i, i \in \mathbb{N})$ be a family of sets in an n-dimensional Euclidean space $\mathbb{R}^n$. We are interested in the families of sets that, after applying to them the family $\mathbf{T}$ of geometric transformations T_i, $(i \in \mathbb{N})$, firstly, belong to a set of sufficiently small measures and, secondly, their union has the zero measure.

In works [22–24], this issue was studied for sets of an n-dimensional Euclidean space $\mathbb{R}^n$. A. Besicovitch considered the cases when $\mathbf{M}$ is the family of all segments of finite length and arbitrary directions, as well as the family of straight lines of arbitrary directions.

Theorem 3.1. ([22,23]) *For $n \geqslant 2$, there exists a set $F \subset \mathbb{R}^n$ whose n-dimensional Lebesgue measure is zero and which contains a linear unit segment of arbitrary direction.*

Theorem 3.2. ([24]) *For $n > 2$, there exists a set $F \subset \mathbb{R}^n$ whose n-dimensional Lebesgue measure is zero and which contains a line of arbitrary direction.*

A. Besicovitch and R. Rado investigated this problem for the family of circles of arbitrary radii in the Euclidean plane. As a result of geometric transformations, the union of those families could be placed in a planar closed null-measured set.

Theorem 3.3. ([21]) *There is a planar closed set of measure zero that contains circles of arbitrary radii.*

M. Stefanchuk and M. Tkachuk solved this problem for the family of spheres of arbitrary radii in an n-dimensional Euclidean space $\mathbb{R}^n$. With the help of the family of geometric transformations, they obtained a set that is a union of spheres of arbitrary radii and whose Lebesgue measure is zero.

Theorem 3.4. ([25]) *In an n-dimensional Euclidean space $\mathbb{R}^n$, there exists a null-measured set that contains spheres of all radii.*

4. Extreme elements and quasiconvex sets in hypercomplex space

The natural analog of complex analysis is the hyper-complex analysis. Therefore, there arises a need to transfer some results of convex analysis known for the real and complex Euclidean spaces to an n-dimensional hypercomplex space $\mathbb{H}^n$, $n \in \mathbb{N}$, which is a direct product of n copies of the bodies of quaternions $\mathbb{H}$. G. Mkrtchyan worked on those problems; see [26, 27]. He introduced the concepts of hypercomplex convex and strongly hypercomplex convex sets and transferred some results of linear convex analysis to the hypercomplex space $\mathbb{H}^n$. Yu. Zelins'kyi (see [28]) and his students M. Tkachuk, T. Osipchuk, and B. Klishchuk continued to develop this direction.

In this section, some properties of extremal elements and $\mathbb{H}$-quasiconvex sets in the n-dimensional hypercomplex space $\mathbb{H}^n$ are presented; see [29, 30].

Let $E \subset \mathbb{H}^n$ be an arbitrary set that contains the coordinate origin $O = \{0, 0, ..., 0\}$. Put $x = (x_1, x_2, ..., x_n)$, $h = (h_1, h_2, ..., h_n)$, $\langle x, h\rangle = x_1h_1 + x_2h_2 + ... + x_nh_n$. The set $E^* = \{h | \langle x, h\rangle \neq 1, \ \forall x \in E\}$ is called *conjugate* with respect to the E set; see [26].

The hyperplane is a set $L \subset \mathbb{H}^n$ that satisfies one of the following conditions $\langle x, a\rangle = w$ or $\langle x - x_0, a\rangle = 0$, where x is an arbitrary point of the set L, x_0 is a fixed vector, w is a fixed scalar from $\mathbb{H}$, and a is a fixed covector. We will refer to the covector a as the *normal.* Accordingly, we will call as *affine* only the functions of the form: $l(x) = \langle x, a\rangle + b$, $b \in \mathbb{H}$.

If every point x is associated with a hyperplane $\{y|\langle x, y\rangle = 1\}$, then the conjugate set E^* can be interpreted as a set of hyperplanes that do not intersect the set E.

Definition 4.1. ([26]) *A set $E \subset \mathbb{H}^n$ is called hypercomplex convex if, for an arbitrary point $x_0 \in \mathbb{H}^n \setminus E$, there exists a hyperplane that passes through the point x_0 and does not intersect the set E.*

In the equation of hyperplane, the order of multiplication is essential because this operation is non-commutative in the algebra of quaternions. Therefore, for certainty, G. Mkrtchyan considers right hyperplanes, i.e., such hyperplanes that the point $x = (x_1, x_2, \ldots, x_n) \in \mathbb{H}^n$ with variable coordinates is multiplied by the fixed point $a = (a_1, a_2, \ldots, a_n) \in \mathbb{H}^n$ on the right. Note that the hypercomplex dimension of the hyperplane equals $n - 1$ in this case, whereas its real dimension equals $4n - 4$.

Definition 4.2. ([26]) *A set $E \subset \mathbb{H}^n$ is called strongly hypercomplex convex if its arbitrary intersection with the hypercomplex line γ is acyclic, i.e., $\widetilde{H}^i(\gamma \bigcap E) = 0$, $\forall i \geqslant 0$, where $\widetilde{H}^i(\gamma \bigcap E)$ is a reduced Alexandrov-Čech cohomology group of the set $\gamma \bigcap E$ with coefficients in the group of integer numbers.*

In work [27], it was proved that strongly hypercomplex convex compact sets are hypercomplex convex.

Let $E \subset \mathbb{H}$ be an arbitrary set. The complement to the union of the unbounded components of the set $\mathbb{H} \setminus E$ is called the *h-combination of the points of the set* E and is denoted as $[E]$. If E is an arbitrary set in the space $\mathbb{H}^n$, $n > 1$, then we say that the point x belongs to the h-combination of points from E if there is an intersection of the set E with a hypercomplex line γ such that $x \in [E \bigcap \gamma]$. The set of such points from $\mathbb{H}^n$ is called the *h-combination of points* of E and is denoted as $[E]$. By induction, the m-fold h-combination is determined as $[E]^m = [[E]^{m-1}]$; see [28].

Definition 4.3. ([26,28]) *The h-hull of a set $E \subset \mathbb{H}^n$ is the set $\widehat{E} = \bigcap\limits_{\pi} \pi^{-1}[\pi(E)]$, where $\pi : \mathbb{H}^n \to \lambda$ are all possible linear projections of the set onto hypercomplex lines, $[\pi(E)]$ is the h-combination of points of the set $\pi(E)$, and $\pi^{-1}[\pi(E)] = \{x \in \mathbb{H}^n|\ \pi(x) \in \pi(E)\}$ is its complete prototype.*

The following theorem asserts that for an arbitrary set of the space $\mathbb{H}^n$, the set of points of its h-hull coincides with the h-combination of the points of this set.

Theorem 4.1. ([29]) *If a set $E \subset \mathbb{H}^n$ is an h-hull, then $E = [E]$.*

The next theorem gives another way of constructing the h-hull of a set.

Theorem 4.2. ([29]) *For an arbitrary set $E \subset \mathbb{H}^n$, its h-hull can be represented in the form $\widehat{E} = (\bigcup\limits_{\lambda}]\lambda \bigcap E^*[)^*$.*

Definition 4.4. ([26]) *The h-interval of radius r and centered at the point x is the intersection of an open ball of radius r centered at the point x with a hypercomplex line passing through the point x.*

Definition 4.5. ([26]) *A point $x \in E \subset \mathbb{H}^n$ is called an h-extreme point of the set E if E has no h-interval that would contain x.*

Definition 4.6. ([29]) *The h-ray is a closed unbounded acyclic subset of a hypercomplex line with non-empty boundary.*

Definition 4.7. ([29]) *The extreme h-ray of a set $E \subset \mathbb{H}^n$ is the h-ray H that belongs to the set E if the set $E \setminus H$ is hypercomplex convex and every boundary point of the ray H is the h-extreme point of the set E. (This is equivalent to the statement that no point of the ray H is interior for an arbitrary h-interval that belongs to the set E and has at least one point outside H.)*

For a set $E \subset \mathbb{H}^n$, we use the following notation: hext E for the set of its h-extremal points, rhext E for the set of h-extremal rays, and hconv E for the h-hull of E.

The following theorem generalizes the Klee theorem of convex analysis (see [1]) to the hypercomplex case.

Theorem 4.3. ([29]) *Let $E \subset \mathbb{H}^n$ be a closed strongly hypercomplex convex body (i.e., $\mathrm{int}E \neq \emptyset$) with non-empty strongly hypercomplex convex boundary ∂E; then E has the form $E = E_1 \times \mathbb{H}^{n-1}$, where E_1 is an acyclic subset of the line $\mathbb{H}$ with non-empty interior with respect to this line.*

Theorem 4.4. ([29]) *Every closed strongly hypercomplex convex set $E \subset \mathbb{H}^n$ that does not contain a hypercomplex line is an h-hull of its h-extreme points and h-extreme rays, $E = \mathrm{hconv}(\mathrm{hext}E \bigcup \mathrm{rhext}E)$.*

The class of strongly hypercomplex convex sets is not closed with respect to intersections. Therefore, the basic axiom of convexity – the intersection of any number of convex sets must be convex – does not hold for it. M. Stefanchuk defined a class of sets that includes strongly hypercomplex convex sets and is closed with respect to intersections; see [29].

Definition 4.8. ([29]) *A hypercomplex convex set $E \subset \mathbb{H}^n$ is called the $\mathbb{H}$-quasiconvex set if its intersection with an arbitrary hypercomplex line γ does not contain a three-dimensional cocycle, i.e., $H^3(\gamma \bigcap E) = 0$.*

It is obvious that the class of $\mathbb{H}$-quasiconvex sets includes strongly hypercomplex convex domains and compact sets.

The following theorem demonstrates the closedness of the class of $\mathbb{H}$-quasiconvex sets in the sense that the intersection of an arbitrary family of compact $\mathbb{H}$-quasiconvex sets is an $\mathbb{H}$-quasiconvex set.

Theorem 4.5. ([29]) *The intersection of an arbitrary family of $\mathbb{H}$-quasiconvex compacts is an $\mathbb{H}$-quasiconvex compact set.*

Definition 4.9. ([28]) *The linear polyhedron is a set of the form $E = \{x |\ f_j(x) \in E_j,\ j \in J = \{1, 2, ..., N\}\}$, where $E_j \subset \mathbb{H}^1$, $f_j(x) = \sum_{k=1}^{n} a_{jk} x_k$, two arbitrary functions $f_k(x)$ and $f_j(x)$, $k \neq j$, are linearly independent, and each function f_j maps E to a subset of the hypercomplex line E_j.*

Let us present some examples of $\mathbb{H}$-quasiconvex sets.

Theorem 4.6. ([29]) *A compact linear polyhedron whose all faces do not contain three-dimensional cycles is the $\mathbb{H}$-quasiconvex set.*

Corollary 4.1. ([29]) *The intersection of strongly hypercomplex convex compact sets is an $\mathbb{H}$-quasiconvex set.*

Theorem 4.7. ([29]) *Every 3-dimensional acyclic hypercomplex convex compact set E is $\mathbb{H}$-quasiconvex.*

5. Hypercomplex convex and conjugate functions in hypercomplex space

Multi-valued linearly convex functions, whose graphs are given by linearly convex sets, were studied by Yu. Zelins'kyi in the n-dimensional complex space $\mathbb{C}^n$; see [2,3].

In this section, some results on multi-valued functions in complex space are generalized to the hypercomplex space $\mathbb{H}^n$, $n = 1, 2, \dots$, which is a direct product of n-copies of the bodies of quaternions $\mathbb{H}$ ($\mathbb{H}^1 := \mathbb{H}$). In particular, some properties of hypercomplex convex and conjugate functions in $\mathbb{H}^n$ are given; see [30–32].

Note that the multiplication operation of hypercomplex numbers may not satisfy the commutative law. For an arbitrary scalar $\lambda \in \mathbb{H}$ and a vector $x \in \mathbb{H}^n$, the multiplication of the vector by the scalar is given in the form: $\lambda x := (\lambda x_1, \dots, \lambda x_n)$, whereas the multiplication of the vector by a number on the right is not the product at all. Vectors x, y are called *collinear* if $x = \lambda y$ with a certain $\lambda \in \mathbb{H}$.

The functional $l\colon \mathbb{H}^n \longrightarrow \mathbb{H}$ with the characteristic property $l(ax + by) = al(x) + bl(y)$ for all x and y from $\mathbb{H}^n$ and arbitrary a and b from $\mathbb{H}$ is called the linear functional. We confine the consideration to linear functionals that can be written in the form $l(x) = x_1a_1 + \dots + x_na_n$, where $a = (a_1, \dots, a_n)$ is a fixed element from $\mathbb{H}^n$. Since an arbitrary $a \in \mathbb{H}^n$ generates a functional of this type, then $a = (a_1, \dots, a_n)$ is called the *covector* or *the element of the conjugate space* $\mathbb{H}^{n*}$.

Let the hyperplane $l \subset \mathbb{H}^n$ divide the space $\mathbb{H}^n$ into two half-spaces, H_1 and H_2.

Definition 5.1. ([1]) *A hyperplane $l \subset \mathbb{H}^n$ is called the support of the set $E \subset \mathbb{H}^n$ if the set E is contained in the closed half-space $\overline{H}_1 = H_1 \bigcup l$ but not in any other closed half-space belonging to the half-space $\overline{H}_1$.*

Definition 5.2. ([31]) *A hypercomplex convex set $E \subset \mathbb{H}^n$ is called strictly hypercomplex convex if, for an arbitrary support hyperplane l, the intersection $l \bigcap E$ does not contain points interior to l.*

In works [31, 32], the concept of multi-valued function was introduced. A function $f\colon \mathbb{H}^n \longrightarrow \mathbb{H}$ is called *multi-valued* if the set $f(x) \in \mathbb{H}$ is the image of the point $x \in \mathbb{H}^n$. The domain of such a function is denoted as $E_f := \{x \in \mathbb{H}^n : \exists y \in \mathbb{H},\ y = f(x)\}$. *The graph of the function* $f\colon \mathbb{H}^n \longrightarrow \mathbb{H}$ is a set of points of the form $\Gamma(f) = \{(x, y) \in \mathbb{H}^n \times \mathbb{H}\}$, which satisfies the condition $y \in f(x)$. A function $l\colon \mathbb{H}^n \longrightarrow \mathbb{H}$ is called *affine* if its graph is a hyperplane.

Definition 5.3. ([31, 32]) *A multi-valued function $f\colon E_f \longrightarrow \mathbb{H}$ is called hypercomplex convex if, for an arbitrary pair of points $(x_0, y_0) \in \mathbb{H}^{n+1} \setminus \Gamma(f)$, there exists an affine function l such that $y_0 = l(x_0)$ and $l(x) \bigcap f(x) = \emptyset$ for all $x \in \mathbb{H}^n$.*

Definition 5.4. ([31]) *A hypercomplex convex function $f\colon E_f \longrightarrow \mathbb{H}$ is called strongly hypercomplex convex (respectively, strictly hypercomplex convex) if its graph $\Gamma(f)$ is a strongly hypercomplex convex (respectively, strictly hypercomplex convex) set in $\mathbb{H}^{n+1}$ (in the strict case, we also require the openness of the function domain in order to avoid vertical tangents $x = x_0$ to the graph $\Gamma(f)$, which will be the support hyperplanes of the graph of the function f, and their intersection with the graph of the function f may have points internal to these hyperplanes).*

The definition of the hypercomplex convex function can be extended to multi-valued functions that take values in the extended hypercomplex plane $\overset{o}{\mathbb{H}} = \mathbb{H} \bigcup (\infty)$ compactified by a single point, while considering that at the points $x \in \mathbb{H}^n$ where $f(x)$ is not defined, $f(x) = \infty$ (in so doing, we consider that in an arbitrary neighborhood of the point x, there are points where the function is defined).

The effective set of a hypercomplex convex function f is the projection onto $\mathbb{H}^n$ of the graph of the function f. *The hypercomplex concave function* is a multi-valued function f for which the function

$\varphi = \mathbb{H} \backslash f$ is hypercomplex convex. *The multi-valued affine function* is a function that is simultaneously hypercomplex convex and hypercomplex concave, and for which there exists at least one point $x \in \mathbb{H}^n$ where each of the sets $(f(x) \bigcap \mathbb{H})$, $(\mathbb{H} \setminus f(x))$ is non-empty (i.e., at this point, the value of the function f differs from $\emptyset$ and $\mathbb{H}$). For a multi-valued affine function f, the equality $f(x) = f(\Theta) + l(x)$ holds, where l is a single-valued affine function, and $\Theta = (0, 0, ..., 0)$. A hypercomplex convex function is called the *eigenfunction* if the relationship $f(x) \bigcap \mathbb{H} \neq \emptyset$ holds for at least one x, and the inequality $\mathbb{H} \setminus f(x) \neq \emptyset$ holds for all x; see [31], [32].

Let us give some examples of hypercomplex convex functions.

For every normal vector y, consider all hyperplanes $l : \langle x, y \rangle = w$ that do not intersect some set E: $l \bigcap E = \emptyset$. For every y, we denote the set of all w such that $l \bigcap E = \emptyset$ as $W_E(y)$.

Definition 5.5. ([31, 32]) *The function*

$$W_E(y) = \overset{o}{\mathbb{H}} \setminus \bigcup_{x \in E} \langle x, y \rangle$$

is called the support function of the set $E \subset \mathbb{H}^n$.

Note that the graph of the support function $\Gamma(W_E) = \{(y, W_E(y)) : \ y \in \mathbb{H}^{n*}\}$ is a cone. If $z \in \Gamma(W_E)$, then $\forall \alpha \in \mathbb{H}$ has the inclusion $\alpha z \in \Gamma(W_E)$.

Definition 5.6. ([31, 32]) *If* $E \subset \mathbb{H}^n$ *is a hypercomplex convex set, then the function*

$$\delta(x|E) = \begin{cases} 0, & \text{if } x \in E, \\ \infty, & \text{if } x \notin E, \end{cases}$$

is called its indicator function.

Note that the support and indicator functions are hypercomplex convex.

Let us present some properties of hypercomplex convex functions.

Theorem 5.1. ([31, 32]) *If* f_α, $\alpha \in A$, *is a family of hypercomplex convex functions (here* A *is an arbitrary set of indices), then the function* $f = \bigcap\limits_{\alpha \in A} f_\alpha$ *given by the intersection of the family* f_α *is hypercomplex convex.*

Definition 5.7. ([31]) *We say that the function* $g = \operatorname{int} f$ *if its graph can be presented in the form* $\Gamma(g) = \operatorname{int}(\Gamma(f))$, *where* $\operatorname{int}(\cdot)$ *denotes the interiority of the corresponding set.*

Theorem 5.2. ([31]) *If* f *is a hypercomplex convex function and* $E_f = E_{\operatorname{int}(f)}$, *then* $\operatorname{int} f$ *is a hypercomplex convex function.*

Let $f \colon E_f \longrightarrow \overset{o}{\mathbb{H}} = \mathbb{H} \bigcup (\infty)$ be a multi-valued function. Consider the support function for the graph $\Gamma(f)$,

$$W_{\Gamma(f)}(z^*) = \overset{o}{\mathbb{H}} \setminus \bigcup_{z \in \Gamma(f)} \langle z, z^* \rangle = \overset{o}{\mathbb{H}} \setminus \bigcup_{x \in \mathbb{H}^n, y \in f(x)} (\langle x, x^* \rangle + y y^*),$$

where $z = (x, y)$, $z^* = (x^*, y^*)$, $x \in \mathbb{H}^n$, $x^* \in \mathbb{H}^{n*}$; y, $y^* \in \mathbb{H}$. Since the graph of the support function is a cone, it is completely determined by its cross-section, for example, the hyperplane $l : z^*_{n+1} = -1$,

$$W_{\Gamma(f)}(x^*) \bigcap l = \overset{o}{\mathbb{H}} \setminus \bigcup_{x \in \mathbb{H}^n} (\langle x, x^* \rangle - f(x)).$$

The function *conjugate* to f is a function defined by the equality

$$f^*(y) = \overset{o}{\mathbb{H}} \setminus \bigcup_x (\langle x, y\rangle - f(x)); \tag{5.1}$$

see [31, 32].

From this definition, a hypercomplex analog of the Young–Fenchel inequality follows (see [2]),

$$\langle x, y\rangle \notin f(x) + f^*(y). \tag{5.2}$$

Relationship (5.2) can be rewritten in the form

$$\langle x, y\rangle \in \mathbb{H} \setminus (f(x) + f^*(y)),$$

or

$$f(x) \bigcap (\langle x, y\rangle - f^*(y)) = \emptyset$$

for all $x \in \mathbb{H}^n$, $y \in \mathbb{H}^{n*}$.

The function conjugate to the function $f^*(y)$ has the following form:

$$f^{**}(x) = (f^*)^*(x) = \overset{o}{\mathbb{H}} \setminus \bigcup_y (\langle x, y\rangle - f^*(y)).$$

Consider examples of functions conjugate to the hypercomplex convex functions presented above.

Example 5.1. Conjugate to the multi-valued affine function $f(x) = \langle x, y_0\rangle + f(\Theta)$, where $f(\Theta)$ is a set, is the function

$$f^*(y) = \overset{o}{\mathbb{H}} \setminus \bigcup_x (\langle x, y\rangle - \langle x, y_0\rangle - f(\Theta)) = \overset{o}{\mathbb{H}} \setminus \bigcup_x (\langle x, y - y_0\rangle - f(\Theta)) =$$

$$= \begin{cases} \overset{o}{\mathbb{H}} \setminus (-f(\Theta)) & \text{if } y = y_0, \\ \infty & \text{if } y \neq y_0. \end{cases}$$

Example 5.2. Let $E \subset \mathbb{H}^n$, $\mathbb{H}^n \setminus E \neq \emptyset$, and $f(x) = \delta(x|E)$. Then,

$$f^*(y) = \overset{o}{\mathbb{H}} \setminus \bigcup_x (\langle x, y\rangle - \delta(x|E)) = \overset{o}{\mathbb{H}} \setminus \bigcup_{x \subset E} \langle x, y\rangle,$$

i.e., the conjugate function of the indicator function of the own subset of E is the support function of this set.

We write $f_1 \supseteq f_2$ if $f_1(x) \supseteq f_2(x)$ for all x, and do not exclude the case $f_2(x) = \emptyset$ for some points x. We also say that f_1 is a *continuation* of the function f_2, and f_2 is a *narrowing* of the function f_1. From the inclusions $f_1 \supseteq f_2$ and equalities (5.1) and (5.2), it follows that $f_1^* \subseteq f_2^*$.

Let us present some properties of conjugate functions.

Theorem 5.3. ([31, 32]) *For every function $f\colon \mathbb{H}^n \longrightarrow \mathbb{H}$, the inclusion $f \subset f^{**}$ is valid.*

Definition 5.8. ([31, 32]) *A multi-valued function $f\colon H^n \longrightarrow H$ is called open (respectively, closed or compact) if its graph is an open (respectively, closed or compact) set in $\mathbb{H}^{n+1}$.*

Theorem 5.4. ([31, 32]) *The function conjugate to an open function is closed and hypercomplex convex.*

Corollary 5.1. ([31,32]) *The function conjugate to the function $f\colon \mathbb{H}^n \longrightarrow \mathbb{H}$ is hypercomplex convex.*

Theorem 5.5. ([31,32]) *Let f be the hypercomplex convex eigenfunction. Then f^* is the eigenfunction.*

Theorem 5.6. ([31]) *Let we have a mapping $\Lambda\colon \mathbb{H}^n \longrightarrow \mathbb{H}^n$, which is a hypercomplex linear homeomorphism, and a function $g\colon \mathbb{H}^n \longrightarrow \mathbb{H}$. Let*

$$f(x) = \lambda g(\Lambda x + w_0) + \langle x, y_0\rangle + \gamma_0,$$

where $w_0 \in \mathbb{H}^n, y_0 \in \mathbb{H}^{n*}$, $\gamma_0 \in \mathbb{H}$, $\lambda \in \mathbb{H}\setminus\{0\}$. Then,

$$f^*(y) = \lambda g^*(\lambda^{-1}\Lambda^{-1*}(y - y_0)) - \langle \Lambda^{-1}w_0, y - y_0\rangle - \gamma_0.$$

From this theorem, we obtain formulas for calculating some conjugate functions:

$$f(x) = g(x + x_0) \Rightarrow f^*(y) = g^*(y) - \langle y_0, y\rangle$$

$$f(x) = g(x) + \langle x, y_0\rangle \Rightarrow f^*(y) = g^*(y - y_0);$$

$$f(x) = \lambda g(\mu x),\ \lambda \neq 0,\ \mu \neq 0 \Rightarrow f^*(y) = \lambda g^*(\lambda^{-1}\mu^{-1}y).$$

The following theorem is a hypercomplex analog of the Fenchel-Moreau theorem.

Theorem 5.7. ([31,32]) *Let a multi-valued function $f\colon \mathbb{H}^n \longrightarrow \mathbb{H}$ be such that $\mathbb{H}\setminus f(x) \neq \emptyset$ for all $x \in \mathbb{H}^n$. Then $f^{**} = f$ if and only if f is hypercomplex convex.*

Definition 5.9. ([31,32]) *A function f is called homogeneous if $f(\lambda x) = \lambda f(x)$ for all scalars $\lambda \in \mathbb{H}\backslash 0$.*

Theorem 5.8. ([31,32]) *Let $f\colon \mathbb{H}^n\setminus\Theta \longrightarrow \mathbb{H}$ be a hypercomplex convex homogeneous eigenfunction and $f(\Theta) = \mathbb{H}\setminus 0$. Then f is a support function of some set.*

Corollary 5.2. ([31, 32]) *If the homogeneous hypercomplex convex function $f\colon \mathbb{H}^n\setminus\Theta \longrightarrow \mathbb{H}$ is not affine, then $f^*(y) = \delta(y|E_{f^*})$.*

Theorem 5.9. ([31,32]) *If $f\colon \mathbb{H}^n\backslash\Theta \longrightarrow \mathbb{H}$ is the homogeneous hypercomplex convex function different from the affine function, then*

$$f(x) = \overset{o}{\mathbb{H}} \setminus \bigcup_{y\in E_{f^*}} \langle x, y\rangle.$$

Example 5.3. An example of homogeneous functions is the hypercomplex Minkowski function, which is defined as follows. Let E be a set in $\mathbb{H}^n$, and $\Theta \in E$. Put $R_E(x) = \{w \in \mathbb{H} \,|\, w^{-1}x \in E\}$ as $x \in \mathbb{H}^n\setminus\Theta$, $R_E(\Theta) = \mathbb{H}\setminus 0$. Let us demonstrate the homogeneity of the function R_E. We have $R_E(\lambda x) = \{w \in \mathbb{H} \,|\, w^{-1}(\lambda x) \in E\} = \{\lambda w \in \mathbb{H} \,|\, (\lambda w)^{-1}(\lambda x) = w^{-1}\lambda^{-1}(\lambda x) = w^{-1}x \in E\} = \lambda R_E(x)$.

Definition 5.10. ([31, 32]) *Let $f_\alpha\colon \mathbb{H}^n \longrightarrow \mathbb{H}$, $\alpha \in A$, be multi-valued functions. The function $(\bigcup\limits_\alpha f_\alpha)(x) := \bigcup\limits_\alpha f_\alpha(x)$ is called the union of functions f_α, and $(\bigcap\limits_\alpha f_\alpha)(x) := \bigcap\limits_\alpha f_\alpha(x)$ is called their intersection.*

The duality theorem holds for conjugate functions.

Theorem 5.10. ([31,32]) *Let $f_\alpha\colon \mathbb{H}^n \longrightarrow \mathbb{H}$, $\alpha \in A$, be multi-valued functions. Then, the following equality holds:*

$$\left(\bigcup_\alpha f_\alpha\right)^* = \bigcap_\alpha f_\alpha^*.$$

Declarations

Declaration of competing interest

The author declares no potential conflict of interest regarding the research, authorship, and publication of this article.

Data Availability

All data and materials used in the manuscript of the article are in the public domain. All previously obtained results used by the author in the manuscript are referenced. No additional special permissions are required to use these results.

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