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FRACTIONAL WAVE EQUATION WITH CHANGING DIRECTION OF EVOLUTION

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We study the fractional wave equation with changing direction of evolution. The existence and uniqueness of a generalized solution are proved. Bibliography: 5 *titles.*

1 Introduction

Assume that $T > 0$ and $\Omega \subset R^m$ is a bounded domain with smooth boundary $\Gamma = \partial\Omega$. In the cylinder $Q = (0, T) \times \Omega$, $S = (0, T) \times \Gamma$, $0 < \nu < 1$, we consider the mixed problem for the model equation with fractional Gerasimov–Caputo derivative

$$\partial_t^\nu (k(t, x) u_t(t, x)) - \Delta u(t, x) + \gamma u_t(t, x) = f(t, x) \tag{1.1}$$

where the coefficient $k(t, x) \in C^1(\overline{Q})$ is of arbitrary sign.

The fractional diffusion equation was considered in [2], where the solvability of some boundary value problems was established by the method of a priori estimates of the form

$$\int_0^T \psi(v) D^\nu (kv)\, dt \geqslant C(\|v\|_B).$$

More general inequalities and there applications can be found in [1] (see the references in [2]).

In the present paper, we use a similar technique to study the fractional wave equation. The statement of the problem is similar to that for mixed type equations with usual derivative.

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Namely, under some conditions, the problem

$$\begin{aligned}
&(k(t,x)u_t(t,x))_t - \Delta u(t,x) + \gamma u_t(t,x) = f(t,x),\\
&u(t,x)|_S = 0,\\
&u(0,x) = 0,\\
&u_t(0,x) = u_0(x), \quad x \in \Omega_0^+ = \{x | k(0,x) > 0\},\\
&u_t(T,x) = u_1(x), \quad x \in \Omega_T^- = \{x | k(T,x) < 0\}
\end{aligned}$$

is well-posed. A similar mixed problem is well-posed for Equation (1.1). We prove the existence of a generalized solution to this mixed problem. The uniqueness of a solution is established under additional conditions on the sign of $k(0,x)$ and $k(T,x)$.

2 Auxiliaries

We use the following definitions and properties of fractional derivatives (see [3]). Assume that $0 < \nu < 1$ and $t > 0$. The fractional integral of order ν with the origin at a point a is defined by

$$J_a^\nu y(t) = \frac{\operatorname{sgn}(t-a)}{\Gamma(\nu)} \int\limits_a^t \frac{y(s)}{|t-s|^{1-\nu}}\, ds.$$

The fractional Riemann–Liouville derivative of order ν with the origin at a is defined by

$$D_a^\nu y(t) = \frac{1}{\Gamma(1-\nu)} \frac{d}{dt} \int\limits_a^t \frac{y(s)}{|t-s|^{\nu}}\, ds.$$

The fractional Gerasimov–Caputo derivative is defined by

$$\partial_a^\nu y(t) = \frac{1}{\Gamma(1-\nu)} \int\limits_a^t \frac{y'(s)}{|t-s|^{\nu}}\, ds.$$

In the case $a = 0$, we write $J^\nu y(t)$, $D^\nu y(t)$, $\partial^\nu y(t)$. As known, for $\nu \neq 1/2$

$$C_0(\nu)\|y(t)\|^2_{W_2^\nu(0,T)} \leqslant \int\limits_0^T (y^2(t) + (D^\nu y(t))^2)\, dt \leqslant C_1(\nu)\|y(t)\|^2_{W_2^\nu(0,T)}.$$

We use some assertions proved in [2]. Let $y(t) \in C^1(0,T)$. Then for some constant $C > 0$ we have (see [1, Lemma 4.1] and [2, formula (2.1)])

$$\int\limits_0^T y(t) D^\nu y(t)\, dt \geqslant C \int\limits_0^T y^2(t) \Big(\frac{1}{t^\nu} + \frac{1}{(T-t)^\nu}\Big)\, dt + C\|y(t)\|^2_{W_2^{\nu/2}(0,T)}; \qquad (2.1)$$

moreover, if $y(0) = 0$, then (see [1, Lemma 5.4])

$$\int\limits_0^T y'(t) D^\nu y(t)\, dt = \int\limits_0^T D^{1-\nu}(D^\nu y(t)) D^\nu y(t)\, dt \geqslant C\|D^\nu y(t)\|^2_{W_2^{(1-\nu)/2}(0,T)}$$

and

$$\int_0^T y'(t)D^\nu y(t)\,dt \geqslant C\|y(t)\|^2_{W_2^{(1+\nu)/2}(0,T)}. \tag{2.2}$$

Further, for any smooth functions $f(t)$ and $g(t)$ we set

$$J(f,g) = \int_0^T f(t)g(t)\,dt.$$

Then for any $0 < \mu < 1/2$ (see [2, formula (2.2)])

$$|J(f,g)| \leqslant C_3(\mu,T)\|J^\mu f(t)\|_{L_2(0,T)}\|D^\mu g(t)\|_{L_2(0,T)}. \tag{2.3}$$

Assume that $0 < \mu < 1$ and $0 < \theta < 1$. We put

$$J_{\mu,\theta}y(t) = \frac{1}{\Gamma(\mu)}\int_0^t \frac{y(s)}{(t-s+\theta)^{1-\mu}}\,ds,$$

$$K_{\mu,\theta}y(t) = \frac{1}{\Gamma(1-\mu)}\frac{d}{dt}\int_0^t \frac{y(s)}{(t-s+\theta)^{\mu}}\,ds.$$

It is easy to see that

$$K_{\mu,\theta}v(t,x) = \frac{d}{dt}J_{1-\mu,\theta}.$$

For some constant $C_K = C_K(\mu,T) > 0$ independent of θ we have (see [2, formula (2.3)])

$$\int_0^T y(t)K_{\mu,\theta}y(t)\,dt \geqslant C_K\|y(t)\|_{L_2(0,T)}. \tag{2.4}$$

We set

$$D_{\mu,\theta}(y(t),z(t)) = \int_0^T (D^\mu y(t) - K_{\mu,\theta}y(t))z(t)\,dt.$$

Lemma 2.1 ([2, Lemma 2.1]). *There exists a constant $C(\mu,T) > 0$ independent of θ such that for any smooth functions $y(t)$ amd $z(t)$ the following inequality holds*:

$$|D_{\mu,\theta}(y(t),z(t))| \leqslant C(\mu,T)\theta^{(1-\mu)/2}\|y(t)\|_{W_2^1(0,T)}\|z(t)\|_{W_2^1(0,T)}. \tag{2.5}$$

Corollary 2.1 ([2, Corollary 2.1]). *For any smooth function $y(t)$ the following estimate holds*:

$$\int_0^T y(t)K_{\mu,\theta}y(t)\,dt \geqslant C_{K1}\|y(t)\|^2_{W_2^{\mu/2}(0,T)} - C_{K2}\theta^{(1-\mu)/2}\|y(t)\|^2_{W_2^1(0,T)} \tag{2.6}$$

with some constants $C_{K1}(\mu,T) > 0$ and $C_{K2}(\mu,T) > 0$.

Corollary 2.2. *For any smooth function $y(t)$ such that $y(0) = 0$*

$$\int_0^T y'(t) K_{\mu,\theta} y(t)\, dt \geqslant C_{K3} \|y(t)\|^2_{W_2^{(1+\mu)/2}(0,T)} - C_{K3} \theta^{(1-\mu)/2} \|y(t)\|^2_{W_2^2(0,T)} \tag{2.7}$$

with some constants $C_{K3}(\mu, T) > 0$ and $C_{K4}(\mu, T) > 0$.

Proof. First of all, we note that

$$\int_0^T y'(t) K_{\mu,\theta} y(t)\, dt = \int_0^T y'(t) D^\mu y(t)\, dt - D_{\mu,\theta}(y(t), y'(t)).$$

Then we apply the inequality (2.2) and Lemma 2.1. □

3 Statement of the Problem and Existence Theorem

By technical reasons, we consider a bit more general problem

$$\partial^\nu (k(t,x) u_t(t,x)) - \Delta u(t,x) + \gamma_1 u_t(t,x) + \gamma_2 \partial^\nu u(t,x) = f(t,x), \tag{3.1}$$
$$u(t,x)|_S = 0, \tag{3.2}$$
$$u(0,x) = 0, \tag{3.3}$$
$$u_t(0,x) = 0, \quad x \in \Omega_0^+ = \{x | k(0,x) > 0\}, \tag{3.4}$$
$$u_t(T,x) = 0, \quad x \in \Omega_T^- = \{x | k(T,x) < 0\}. \tag{3.5}$$

A generalized solution to this problem is determined similarly to [2]. We denote

$$\chi_0(x) = k(0,x) u_t(0,x), \quad \chi_T(x) = k(T,x) u_t(T,x).$$

Note that (3.4) and (3.5) imply

$$\operatorname{supp} \chi_0(x) \subseteq \Omega_0^-, \quad \operatorname{supp} \chi_T(x) \subseteq \Omega_T^+, , \tag{3.6}$$

where Ω_0^- and Ω_T^+ are defined as in (3.4) and (3.5). Formally applying the operator J^ν to Equation (3.1), we obtain the equality for $t \in [0,T]$

$$k(t,x) u_t(t,x) - J^\nu \Delta u(t,x) + \gamma_1 J^\nu u_t(t,x) + \gamma_2 u(t,x) = J^\nu f(t,x) + \chi_0(x) \tag{3.7}$$

and

$$\chi_T(x) - \chi_0(x) - J^\nu \Delta u(T,x) + \gamma_1 J^\nu u_t(T,x) + \gamma_2 u(T,x) = J^\nu f(T,x). \tag{3.8}$$

A function $u(t,x) \in L_2(Q)$ is called a *generalized solution* to the problem (3.1)–(3.5) if

$$u_t(t,x) \in L_2(Q), \quad u(0,x) = 0,$$
$$\partial^{1-\nu} u(t,x) \in C([0,T]; W_2^{-1}(\Omega)), \quad k(t,x) u_t(t,x) \in C([0,T], W_2^{-1}(\Omega)),$$
$$u(t,x) \in L_2(0,T; \mathring{W}_2^1(\Omega)), \quad J^\nu u(t,x) \in C([0,T]; \mathring{W}_2^1(\Omega));$$

moreover, (3.7), (3.8), and (3.6) hold for some functions $\chi_0(x)$, $\chi_T(x) \in L_2(\Omega)$.

Everywhere below, we set $\mu = 1 - \nu$.

Theorem 3.1. *Assume that* $\gamma_1 \geqslant 0$, $f(t,x) \in W_2^{\mu/2}(0,T;L_2(\Omega))$, *and for some* $\gamma_0 > 0$

$$2C_K(\mu,T)\gamma_1 + 2\gamma_2 + k_t(t,x) \geqslant \gamma_0, \quad (t,x) \in Q, \tag{3.9}$$

where the constant $C_K(\mu,T)$ *is taken from the inequality* (2.4). *Then the problem* (3.1)–(3.5) *has a generalized solution such that*

$$u(t,x) \in W_2^{\frac{1+\mu}{2}}(0,T;\mathring{W}_2^1(\Omega)), \quad u_t(t,x) \in W_2^{\mu/2}(0,T;L_2(\Omega)) \tag{3.10}$$

and

$$\|\chi_0\|^2_{L_2(\Omega)} + \|\chi_T\|^2_{L_2(\Omega)} + \|u_t\|^2_{W_2^{\mu/2}(0,T;L_2(\Omega))} + \|u\|^2_{W_2^{\frac{1+\mu}{2}}(0,T;\mathring{W}_2^1(\Omega))} \leqslant C\|f\|^2_{W_2^{\mu/2}(0,T;L_2(\Omega))}.$$

Moreover, if $D^\mu f(t,x) \in L_2(Q)$, *then*

$$\int_Q (C_K(\mu,T)\gamma_1 + \gamma_2 + k_t(t,x)/2)u_t^2(t,x)\,dQ \leqslant \int_Q D^\mu f(t,x)u_t(t,x)\,dQ. \tag{3.11}$$

Proof. As in [2], we use the regularization method proposed in [4]. Let $0 < \varepsilon < 1$. We introduce a family of smooth functions $f_\varepsilon(t,x)$ such that

$$f_\varepsilon(0,x) = 0, \tag{3.12}$$

$$\lim_{\varepsilon\to 0} \|f_\varepsilon(t,x) - f(t,x)\|_{W_2^{\mu/2}(0,T;L_2(\Omega))} = 0, \tag{3.13}$$

$$\lim_{\varepsilon\to 0} \varepsilon\|f_\varepsilon(t,x)\|^2_{W_2^1(0,T;L_2(\Omega))} = 0. \tag{3.14}$$

Note that the condition (3.12) is compatible with the condition (3.13) since $\mu < 1$. Then we consider the problem (see [4])

$$\begin{aligned}
&-\varepsilon u_{ttt} + (k(t,x)u_t(t,x))_t - D^\mu \Delta u(t,x) + \gamma_1 D^\mu u_t(t,x) + \gamma_2 u_t(t,x) = D^\mu f_\varepsilon(t,x),\\
&u(t,x)|_S = 0, \quad u(0,x) = 0,\\
&-\varepsilon u_{tt}(0,x) + k^+(0,x)u_t(0,x) = 0,\\
&-\varepsilon u_{tt}(T,x) + k^-(T,x)u_t(T,x) = 0.
\end{aligned}$$

As usual,

$$\eta^+ = \begin{cases} \eta, & \eta > 0,\\ 0, & \eta \leqslant 0,\end{cases} \qquad \eta^- = \begin{cases} \eta, & \eta < 0,\\ 0, & \eta \geqslant 0.\end{cases}$$

To establish the solvability of the regularized problem, we use the special Galerkin method. Let $\{w_k(x)\}_{k\in\mathbb{N}}$ be the system of eigenfunctions of the problem

$$-\Delta w_k = \lambda_k w_k, \quad w_k(x)|_\Gamma = 0,$$

that are orthonormal in $L_2(\Omega)$. For any $n > 0$ we denote by $E_n \subset L_2(\Omega)$ the subspace of functions spanned by the vectors w_k, $k = \overline{1,n}$. We note that the space E_n is finite-dimensional and, consequently, for some constant $C(n)$ we have the inequality

$$\|h\|^2_{W_2^1(\Omega)} \leqslant C(n)\|h\|^2_{L_2(\Omega)}, \quad \forall h \in E_n. \tag{3.15}$$

We denote by P_n the orthogonal projection in $L_2(\Omega)$ onto the space E_n. Assuming that $\theta = \theta(\varepsilon, n)$ (the exact value will be indicated later), we consider the problem

$$\begin{aligned} &-\varepsilon v_{nttt}(t,x) + P_n(k(t,x)v_{nt}(t,x))_t - K_{\mu,\theta}\Delta v_n(t,x) \\ &\quad + \gamma_1 K_{\mu,\theta} v_{nt}(t,x) + \gamma_2 v_{nt} = K_{\mu,\theta} P_n f_\varepsilon(t,x), \end{aligned} \tag{3.16}$$

$$v_n(t,x)|_S = 0, \quad v_n(0,x) = 0, \tag{3.17}$$

$$-\varepsilon v_{ntt}(x,0) + P_n(k^+(0,x)v_{nt}(0,x)) = 0, \tag{3.18}$$

$$-\varepsilon v_{ntt}(T,x) + P_n(k^-(T,x)v_{nt}(T,x)) = 0, \tag{3.19}$$

where $v_n(t,x) = \sum\limits_{k=1}^{n} V_k(t) w_k(x)$. If no comfusion arises, we omit the superscript n. It is easy to see that the solvability of this system follows from the uniqueness of a solution. Therefore, it suffices to derive a suitable a priori estimate for the solution.

We multiply Equation (3.16) by $2v_t(t,x)$ and integrate over the cylinder Q

$$\begin{aligned} &\int\limits_\Omega (|k(0,x)|v_t^2(0,x) + |k(T,x)|v_t^2(T,x)) + 2\varepsilon \int\limits_Q v_{tt}^2\,dQ + \int\limits_Q (2\gamma_2 + k_t)v_t^2\,dQ \\ &+ 2\int\limits_Q ((K_{\mu,\theta}\nabla v, \nabla v_t) + \gamma_1 v_t K_{\mu,\theta} v_t)\,dQ = 2\int\limits_Q v_t K_{\mu,\theta} f_\varepsilon\,dQ. \end{aligned} \tag{3.20}$$

Let $\delta > 0$. Using (2.4), (2.6), (2.7), (3.15), and (2.3), we get

$$\begin{aligned} &\int\limits_\Omega (|k(0,x)|v_t^2(0,x) + |k(T,x)|v_t^2(T,x)) + 2\varepsilon \int\limits_Q v_{tt}^2\,dQ + 2\delta C_{K1}\|v_t\|^2_{W_2^{\mu/2}(0,T;L_2(\Omega))} \\ &+ \int\limits_Q (2C_K(\gamma_1 - \delta) + 2\gamma_2 + k_t)v_t^2\,dQ + C_{K3}\|\nabla v\|_{W_2^{(1+\mu)/2}(0,T;L_2(\Omega))} \\ &\leqslant 2\theta^{(1-\mu)/2}\|v_{tt}\|^2_{L_2(Q)}(\delta C_{K2} + C(n)C_{K4}) + C\|f\|_{W_2^{\mu/2}(0,T;L_2(\Omega))}\|u_t\|_{W_2^{\mu/2}(0,T;L_2(\Omega))}. \end{aligned}$$

Choosing δ and θ sufficiently small, we get the required estimate

$$\begin{aligned} &\int\limits_\Omega (|k(0,x)|v_t^2(0,x) + |k(T,x)|v_t^2(T,x)) + \varepsilon \int\limits_Q v_{tt}^2\,dQ \\ &+ \|v_t\|^2_{W_2^{\mu/2}(0,T;L_2(\Omega))} + \|\nabla v\|_{W_2^{(1+\mu)/2}(0,T;L_2(\Omega))} \leqslant C\|f\|^2_{W_2^{\mu/2}(0,T;L_2(\Omega))}. \end{aligned} \tag{3.21}$$

This estimate guarantees that the system (3.16)–(3.19) is uniquely solvable. Note that for all $(t,x) \in Q$ we have the equality

$$(-\varepsilon v_{ntt} + P_n(kv_{nt}))\big|_0^t = J_{\nu,\theta}(P_n f_\varepsilon + \Delta v_n - \gamma_1 v_{nt}) - \gamma_2 v_n$$

and, due to the conditions (3.17), (3.18), and (3.19),

$$P_n(k^+(T,x)v_{nt}(T,x)) - P_n(k^-(0,x)v_{nt}(0,x)) = J_{\nu,\theta}(P_n f_\varepsilon + \Delta v_n - \gamma_1 v_{nt})(T,x) - \gamma_2 v_n(T,x).$$

Now, we can pass to the limit as $n \to \infty$. Passing, if necessary, to a subsequence, we assume that for some functions $z_{\varepsilon 0}(x)$, $z_{\varepsilon T}(x)$, $u_\varepsilon(t,x)$ the following convergences take place:

$$\begin{aligned}
&\sqrt{|k(0,x)|}v_{nt}(0,x) \rightharpoonup z_{\varepsilon 0}(x), && \text{weakly in } L_2(\Omega),\\
&\sqrt{|k(T,x)|}v_{nt}(T,x) \rightharpoonup z_{\varepsilon T}(x), && \text{weakly in } L_2(\Omega),\\
&v_n(t,x) \rightharpoonup u_\varepsilon(t,x), && \text{weakly in } W_2^{(1+\mu)/2}(0,T;W_2^1(\Omega)),\\
&v_{nt}(t,x) \rightharpoonup u_{\varepsilon t}(t,x), && \text{weakly in } W_2^{\mu/2}(0,T;L_2(\Omega))
\end{aligned}$$

as $n \to \infty$. In this case, due to the estimate (3.21),

$$\varepsilon\|u_{\varepsilon tt}\|^2_{L_2(Q)} + \|J^\nu u_{\varepsilon t}\|_{W_2^{\frac{1+\nu}{2}}(0,T;W_2^1(\Omega))} \leqslant C. \tag{3.22}$$

In particular, $J^\nu u_{\varepsilon t}(t,x) \in C([0,T];W_2^1(\Omega))$ and $J^\nu u_{\varepsilon t}(0,x) = 0$. Next, we denote

$$\chi_{0\varepsilon}(x) = -\sqrt{|k^-(0,x)|}z_{\varepsilon 0}(x), \quad \chi_{T\varepsilon}(x) = \sqrt{|k^+(T,x)|}z_{\varepsilon T}(x).$$

It is clear that these functions satisfy the conditions (3.6) and, by virtue of (3.18) and (3.19), the following equalities hold:

$$-\varepsilon u_{\varepsilon tt} + k u_{\varepsilon t} - \chi_{0\varepsilon} = J_\nu(f_\varepsilon + \Delta u_\varepsilon - \gamma_1 u_{\varepsilon t}) - \gamma_2 u_{\varepsilon t},$$

$$\chi_{T\varepsilon}(x) - \chi_{0\varepsilon}(x) = J_\nu(f_\varepsilon + \Delta u_\varepsilon - \gamma_1 u_{\varepsilon t})(T,x) - \gamma_2 u_{\varepsilon t}(T,x).$$

Now, we can pass to the limit as $\varepsilon \to 0$. All actions are standard, and we omit them.

Now, assume that $D^\mu f(t,x) \in L_2(Q)$. Passing to the limit as $n \to \infty$ and then as $\varepsilon \to 0$ in (3.20), we get (3.11). □

4 Uniqueness Theorems

In a more general case, the study of the uniqueness of such a generalized solution is difficult even in the case of ordinary derivatives (see [4]). We consider a simpler case where the functions $k(0,x)$ and $k(T,x)$ do not change the sign.

Theorem 4.1. *Assume that $k(0,x) \geqslant 0$, $k(T,x) \geqslant 0$, $\gamma_1 > 0$, and $\gamma_2 > 0$ is large enough. Then a generalized solution to the problem* (3.1)–(3.5) *is unique.*

Proof. Consider a solution $u(t,x)$ to the problem (3.1)–(3.5) with $f(t,x) \equiv 0$. Then

$$k(t,x)u_t(t,x) - J^\nu \Delta u(t,x) + \gamma_1 \partial^{1-\nu}u(t,x) + \gamma_2 u(t,x) = \chi_0(x). \tag{4.1}$$

By (3.6) and the inequality $k(0,x) \geqslant 0$, we have $\chi_0(x) \equiv 0$. Multiplying (4.1) by $2u(t,x)$ and integrating over Q, we have

$$\int_\Omega k(T,x)u^2(T,x)\,dx + 2\int_Q (J^\nu \nabla u(t,x), \nabla u(t,x))\,dQ$$
$$+ 2\gamma_1 \int_Q u(t,x)\partial^{1-\nu}u(t,x)\,dQ + \int_Q (2\gamma_2 - k_t(t,x))u^2(t,x)\,dQ = 0.$$

Thus, $\int_Q (2\gamma_2 - k_t(t,x))u^2(t,x)\,dQ \leqslant 0$ and $u(t,x) \equiv 0$ provided that $2\gamma_2 - k_t(t,x) \geqslant \overline{\gamma} > 0$. □

We note that the above proof is similar to that of the uniqueness theorem for the wave equation [5, Theorem 3.1].

Corollary 4.1. *Let the assumptions of Theorem* 3.1 *hold. Assume that* $k(0,x) \geqslant 0$ *and* $k(T,x) \geqslant 0$. *Then a generalized solution to the problem* (3.1)–(3.5) *is unique.*

Proof. Consider a solution $u(t,x)$ to the problem (3.1)–(3.5) with $f(t,x) \equiv 0$. It is clear that for any λ

$$\partial^\nu (k(t,x)u_t(t,x)) - \Delta u(t,x) + \gamma_1 u_t(t,x) + (\gamma_2 + \lambda)\partial^\nu u(t,x) = \lambda \partial^\nu u(t,x). \tag{4.2}$$

We note that $D^\mu \partial^\nu u(t,x) = u_t(t,x) \in L_2(Q)$. By Theorem 4.1, the solution to problem (4.2), (3.2)–(3.5) is unique, and, by Theorem 3.1 (see (3.11))

$$\int\limits_Q (C_K(\mu,T)\gamma_1 + \gamma_2 + \lambda + k_t(t,x)/2)u_t^2(t,x)\,dQ \leqslant \int\limits_Q \lambda u_t^2(t,x)\,dQ.$$

Thus, $\gamma_0 \int\limits_Q u_t^2(t,x)\,dQ \leqslant 0$ ant $u(t,x) \equiv 0$. □

A simple idea proposed in Theorem 4.1 does not work if $k(0,x) \leqslant 0$ or $k(T,x) \leqslant 0$. In such cases, we use convolution mollifiers. The arguments are not difficult, but a rigorous proof requires a lot of tedious technical work.

For any a and b we set

$$Q_{a,b} = (a,b) \times \Omega.$$

First of all, we need to extend the functions $k(t,x)$ and $u(t,x)$ inside $Q_{-T,2T}$. We set

$$u(t,x) = \begin{cases} 0, & t < 0, \\ u(2T - t, x), & t > T. \end{cases}$$

The function $k(t,x)$ is extended in such a way $k(t,x) \in C^1(\overline{Q}_{-T,2T})$, and the condition (3.9) is satisfied in $\overline{Q}_{-T,2T}$. Let $\rho(t) \geqslant 0$ be a smooth even function such that $\operatorname{supp}\rho(t) \subset (-1,1)$, and let

$$\operatorname{sgn}(t)\rho'(t) \leqslant 0, \qquad \int\limits_{-1}^{1} \rho(t)\,dt = 1. \tag{4.3}$$

We note that

$$\int\limits_{-1}^{1} t\rho'(t)\,dt = -1. \tag{4.4}$$

For any $\delta > 0$ we denote

$$\rho_\delta(t) = \frac{1}{\delta}\rho(t/\delta), \quad u_\delta(t,x) = \rho_\delta(t) * u(t,x).$$

Lemma 4.1. *Assume that* $a < b$, $w(t,x) \in L_2(Q_{a,b})$, $0 < 2\delta < b - a$. *Denote* $\widetilde{Q}_\delta = Q_{a+\delta,b-\delta}$ *and* $F(\delta,t,x) = k(t,x)\rho_\delta(t) * w(t,x) - \rho_\delta(t) * (k(t,x)w(t,x))$. *Then* $F_t(\delta,t,x) \in L_2(\widetilde{Q}_\delta)$ *and* $\|F_t(\delta,t,x)\|_{L_2(\widetilde{Q}_\delta)} \to 0$ *as* $\delta \to 0$.

Proof. Let $(t,x)\in\widetilde{Q}_\delta$. It is easy to see that

$$F_t(\delta,t,x)=\int\limits_a^b(\rho_\delta(t-s)k'(t,x)-\rho'_\delta(t-s)(k(s,x)-k(t,x)))w(s,x)\,ds.$$

By the Taylor formula $k(s,x)=k(t,x)+k'(t,x)(s-t)+o(|s-t|)$, it follows that

$$F_t(\delta,t,x)(t,x)=k'(t,x)(\rho_\delta(t)+t\rho'_\delta(t))*w(t,x)+o(\delta)\int\limits_a^b|\rho'_\delta(t-s)||w(s,x)|\,ds.$$

Taking into account (4.3) and (4.4), we obtain the required statement. □

Lemma 4.2. *Assume that $a<b$, $0<\alpha<1/2$, $w(t,x)\in W_2^\alpha(a,b;L_2(\Omega))$, $0<2\delta<b-a$. Then for some $\beta=\beta(\alpha,a,b)>0$ and any $t_0\in[a+\delta,b-\delta]$*

$$\|w_\delta(t_0,x)\|^2_{L_2(\Omega)}\leqslant C\delta^{\beta-1}\|w(t,x)\|^2_{W_2^\alpha(a,b;L_2(\Omega))}.$$

Proof. Let $1/q=1/2-\alpha$ and $1/p=1-1/q$. By the embedding theorem,

$$|w_\delta(t_0,x)|\leqslant C\|\rho_\delta(t)\|_{L_p(-\delta,\delta)}\|w(t,x)\|_{W_2^\alpha(a,b)}\leqslant C\delta^{(1-p)/p}\|w(t,x)\|_{W_2^\alpha(a,b)}.$$

Note that $1<p<2$. So, we put $\beta=(2-p)/p$ and get

$$\|w_\delta(t_0,x)\|^2_{L_2(\Omega)}\leqslant C\delta^{\beta-1}\|w(t,x)\|^2_{W_2^\alpha(a,b;L_2(\Omega))}.$$

The lemma is proved. □

Lemma 4.3. *Assume that $a<b$, $w(t,x)\in L_2(Q_{a,b})$, $\widetilde{k}(t,x)\in C^1(Q_{a,b})$, $\widetilde{k}(a,x)=0$. $0<2\delta<b-a$. Extend by zero $\widetilde{k}(t,x)$, $w(t,x)$ for $t<a$ and consider $(\widetilde{k}w)_\delta(t,x)$. Then*

$$\|(\widetilde{k}w)_{\delta t}(t,x)\|_{L_2(Q_{a-\delta,a})}\leqslant C\|w(t,x)\|_{L_2(Q_{a,a+\delta})}.$$

Proof. It is easy to see that for $t<a$

$$(\widetilde{k}w)_{\delta t}(t,x)=\widetilde{k}(t,x)\rho'_\delta*w(t,x)+\int\limits_a^{a+\delta}\rho'_\delta(t-s)(\widetilde{k}(s,x)-\widetilde{k}(t,x)))w(s,x)\,ds.$$

It remains to note that $\widetilde{k}(s,x)-\widetilde{k}(t,x)=O(|s-t|)$ and $\int\limits_{-\delta}^{\delta}|\rho'_\delta(t)|\,dt\leqslant C/\delta$. □

Lemma 4.4. *Assume that $a<b$, $0<\alpha<1$, $w(t)\in W_2^\alpha(a,b)$, and (an even continuation) $w(t)=w(2b-t)$ for $t>b$. Then $w(t)\in W_2^\alpha(a,2b-a)$.*

Proof. Without loss of generality we can assume that $a=0$ and $b=1$. According to the definition, we consider the integral

$$J=\int\limits_0^2dt\int\limits_0^2\frac{(w(t)-w(\tau))^2}{|t-\tau|^{1+2\alpha}}\,d\tau.$$

After trivial transformations we get

$$J = 2\int_0^1 dt \int_0^1 \frac{(w(t)-w(\tau))^2}{|t-\tau|^{1+2\alpha}}\,d\tau + 2\int_0^1 dt \int_0^1 \frac{(w(\tau)-w(t))^2}{|2-\tau-t|^{1+2\alpha}}\,d\tau.$$

It remains to note that $2-\tau-t \geqslant |t-\tau|$ for any $0<t,\tau<1$. □

Lemma 4.5. *Assume that $a<b$, $1/2<\alpha<1$, $w(t)\in W_2^\alpha(a,b)$, $w(a)=0$, and*

$$z(t) = \begin{cases} 0, & t<a, \\ w(t)-w(a), & t\in(a,b). \end{cases}$$

Then $z(t)\in W_2^\alpha(2a-b,b)$.

Proof. Without loss of generality we can assume, $a=0$, $b=1$. Then

$$\int_{-1}^1 dt \int_{-1}^1 \frac{(z(t)-z(\tau))^2}{|t-\tau|^{1+2\alpha}}\,d\tau \leqslant \int_{-1}^1 dt \int_0^1 \frac{(w(t)-w(\tau))^2}{|t-\tau|^{1+2\alpha}}\,d\tau + C\int_0^1 \frac{(w(t)-w(a))^2}{t^{2\alpha}}\,dt.$$

Note, that $w(t)-w(a) = J_a^\alpha \partial_a^\alpha w(t)$. Thus the required statement follows from the inequality (3.17) in [3])

$$\int_0^1 \frac{(w(t)-w(a))^2}{t^{2\alpha}}\,dt \leqslant C\|w(t)\|^2_{W_2^\alpha(a,b)}.$$

The lemma is proved. □

Theorem 4.2. *Let the condition* (3.9) *holds. The problem* (3.1)-(3.5) *has at most one solution satisfying* (3.10) *provided that the functions $k(0,x)$ and $k(T,x)$ do not change sign.*

Proof. Let $f(t,x)\equiv 0$, and $u(t,x)$ a solution to problem (3.1)-(3.5), satisfying (3.10). We have four slightly different cases to consider.

Case 1. Assume that $k(0,x)\geqslant 0$ and $k(T,x)\geqslant 0$. The required statement follows from Corollary 4.1.

Case 2. Assume that $k(0,x)\leqslant 0$ and $k(T,x)\geqslant 0$. Assume also that $0<\delta<T/2$, $v(t,x) = u_\delta(t,x)$, and $F(\delta,t,x) = k(t,x)v_t(t,x) - \rho_\delta(t) * (k(t,x)u_t(t,x))$. Then for $(t,x)\in Q_{\delta,T-\delta}$

$$k(t,x)v_t(t,x) - J^\nu \Delta v(t,x) + \gamma_1 J^\nu v_t(t,x) + \gamma_2 v(t,x) = \chi_0 + F(\delta,t,x),$$

$$(k(t,x)v_t(t,x))_t - D^\mu \Delta v(t,x) + \gamma_1 D^\mu v_t(t,x) + \gamma_2 v_t(t,x) = F_t(\delta,t,x).$$

As above, multiplying by $2v_t(t,x)$ and integrating over $Q_{\delta,T-\delta}$, we get

$$\int_\Omega (k(T,x)v_t^2(T-\delta,x) - k(0,x)v_t^2(\delta,x))\,d\Omega + \gamma_0 \int_{Q_{0,T-\delta}} v_t^2\,dQ$$

$$\leqslant \int_\Omega (k(T,x)-k(T-\delta,x))v_t^2(T-\delta,x)\,d\Omega - \int_\Omega (k(0,x)-k(\delta,x))v_t^2(\delta,x)\,d\Omega$$

$$+\,2\int_{Q_{0,\delta}} v_t(-D^\mu\Delta v(t,x) + \gamma_1 D^\mu v_t(t,x) + \gamma_2 v_t(t,x))\,dQ$$

$$+\int\limits_{Q_{\delta,T-\delta}} |F_t(\delta,t,x)v_t(t,x)|\,dQ = J_1+J_2+J_3.$$

By Lemmas 4.1 and 4.2, we have $|J_1| = O(\delta^\beta)$ and $|J_3| = o(1)$ as $\delta\to 0$. By the inequality (2.3),

$$|J_2| \leqslant C(\|v_t\|^2_{W_2^{\mu/2}(0,\delta;L_2(\Omega))} + \|v\|^2_{W_2^{(1+\mu)/2}(0,\delta;\mathring{W}_2^1(\Omega))}).$$

Passing to the limit as $\delta\to 0$, we get $u(t,x)\equiv 0$.

Case 3. Assume that $k(0,x)\leqslant 0$ and $k(T,x)\leqslant 0$. First of all, we extend the equality (3.7) inside $Q_{0,2T}$. By Lemma 4.4, $u(t,x)\in W_2^{\frac{1+\mu}{2}}(0,2T;\mathring{W}_2^1(\Omega))$ and (see [3, Theorem 11.6]) $u_t(t,x)\in W_2^{\mu/2}(0,2T;L_2(\Omega))$ since $\mu<1$. We set

$$H(t,x) = -J^\nu\Delta u(t,x)+\gamma_1 J^\nu u_t(t,x)+\gamma_2 u(t,x),$$

$$\widetilde{H}(t,x) = H(t,x)+H(2T-t,x).$$

By (3.6) and (3.8), $\chi_1(x)\equiv 0$ and $H(T,x)=\chi_0$. Next, let $T<t<2T$. By definition,

$$k(t,x)u_t(t,x) = -k(2T-t,x)u_t(2T-t,x)+\widetilde{k}(t,x)u_t(2T-t,x),$$

where $\widetilde{k}(t,x)=k(2T-t,x)-k(t,x)$. So, for $T<t<2T$

$$k(t,x)u_t(t,x)+H(t,x) = \widetilde{k}(t,x)u_t(2T-t,x)+\widetilde{H}(t,x)-\chi_0(x).$$

Finally, we have

$$k(t,x)u_t(t,x)-J^\nu\Delta u(t,x)+\gamma_1 J^\nu u_t(t,x)+\gamma_2 u(t,x)-\chi_0(x) = G(t,x),$$

where

$$G(t,x) = \begin{cases} 0, & t<T,\\ \widetilde{k}(t,x)u_t(2T-t,x)+\widetilde{H}(t,x)-2H(T,x), & t>T.\end{cases}$$

Let $0<\delta<T/2$, $v(t,x)=u_\delta(t,x)$. Then for $(t,x)\in Q_{\delta,T}$

$$(k(t,x)v_t(t,x))_t - D^\mu\Delta v(t,x)+\gamma_1 D^\mu v_t(t,x)+\gamma_2 v_t(t,x) = F_t(\delta,t,x)+G_{\delta t}(t,x).$$

As above, we multiply this equation by $2v_t(t,x)$ and integrate over $Q_{\delta,T}$. Since $v_t(T,x)=0$, we get

$$\int\limits_\Omega -k(0,x)v_t^2(\delta,x)\,d\Omega + \gamma_0\int\limits_{Q_{0,T}} v_t^2\,dQ \leqslant \int\limits_\Omega (k(\delta,x)-k(0,x))v_t^2(\delta,x)\,d\Omega$$

$$+2\int\limits_{Q_{0,\delta}} v_t(-D^\mu\Delta v(t,x)+\gamma_1 D^\mu v_t(t,x)+\gamma_2 v_t(t,x))\,dQ$$

$$+\int\limits_{Q_{\delta,T}} |F_t(\delta,t,x)v_t(t,x)|\,dQ + \int\limits_{Q_{T-\delta,T}} |G_{\delta t}(t,x)v_t(t,x)|\,dQ.$$

Compared to Case 2, we need to estimate the term

$$J = \int\limits_{Q_{T-\delta,T}} |G_{\delta t}(t,x) v_t(t,x)| \, dQ.$$

Let $H_1(t,x) = J^\nu \Delta u(t,x) - J^\nu \Delta u(T,x)$, $H_2(t,x) = J^\nu u_t(t,x) - J^\nu u_t(T,x)$, $H_3(t,x) = u_t(t,x) - u_t(T,x)$,

$$G_0(t,x) = \begin{cases} 0, & t < T, \\ \widetilde{k}(t,x) u_t(2T-t,x), & t > T, \end{cases}$$

and for $k = 1,2,3$

$$G_k(t,x) = \begin{cases} 0, & t < T, \\ H_k(t,x) + H_k(2T-t,x), & t > T. \end{cases}$$

Then we can write

$$|J| \leqslant \sum_{k=0}^{3} \int\limits_{Q_{T-\delta,T}} |G_{k\delta t}(t,x) v_t(t,x)| \, dQ = J_0 + J_1 + J_2 + J_3.$$

It is obvious that

$$|J_3| \leqslant C \|u_t\|^2_{L_2(Q_{T-2\delta,T})}.$$

By Lemma 4.3,

$$|J_0| \leqslant C \|u_t\|_{L_2(Q_{T-\delta,T})} \|u_t\|_{L_2(Q_{T-2\delta,T})}.$$

We consider the term $G_2(t,x)$. By Lemmas 4.4 and 4.5, $G_2(t,x) \in W_2^{\nu+\mu/2}(T-\delta,T;L_2(\Omega))$, and, by (2.3),

$$|J_2| \leqslant C \|v_t\|_{W_2^{\mu/2}(T-\delta,T;L_2(\Omega))} \|J^{\mu/2}_{T-\delta} G_{2\delta t}(t,x)\|_{L_2(Q_{T-2\delta,T})} \leqslant C \|u_t\|_{W_2^{\mu/2}(T-\delta,T;L_2(\Omega))}.$$

Finally, we consider the term $G_1(t,x)$. By Lemmas 4.4 and 4.5, $G_{1\delta t}(t,x) \in W_2^{\nu/2}(0,T;W_2^{-1}(\Omega))$ and $G_{1\delta t}(t,x) = 0$ if $t < T-\delta$. Using (2.3), we get

$$|J_1| \leqslant C \|J^{\nu/2}_T v\|_{L_2(T-\delta,T;\mathring{W}_2^1(\Omega))} \|G_{1\delta t}(t,x)\|_{W_2^{\nu/2}(0,T;W_2^{-1}(\Omega))} \leqslant C \|v\|^2_{W_2^{(1+\mu/2)}(T-\delta,T;\mathring{W}_2^1(\Omega))}.$$

Thus, we have established all the required estimates and can pass to the limit as $\delta \to 0$.

Case 4. Assume that $k(0,x) \geqslant 0$ and $k(T,x) \leqslant 0$. According to (3.6), $\chi_0 \equiv 0$ and

$$k(t,x) u_t(t,x) - J^\nu_{-T/2} \Delta u(t,x) + \gamma_1 J^\nu_{-T/2} u_t(t,x) + \gamma_2 u(t,x) = G(t,x).$$

Now, we multiply this equation by $2v_t(t,x)$ and integrate over the cylinder $Q_{-T/2,T}$. The rest of the proof is the same as in Case 3. □

Declarations

Data availability This manuscript has no associated data.

Ethical Conduct Not applicable.

Conflicts of interest The author declare that there is no conflict of interest.

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