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TWO-SIDED ESTIMATES OF THE ANALYTIC FUNCTION ASSOCIATED WITH THE EULER NUMBER

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We propose new two-sided estimates with sharp constants for the number e*. Minorants and majorants in the estimates are expressed as continued fractions. To justify the estimate, we essentially use the integral representation of a special analytic function obtained earlier by the authors. We discuss the validity of similar inequalities and relationship with known results. Bibliography*: 11 *titles.*

1 Statement of the Problem and the Main Result

In the works of the authors [1, 2] devoted to special aspects of the problem on the rational approximation of the number e the function

$$H(x) \equiv 1 - e^{-1}(1+x)^{1/x}, \quad x \in (-1, +\infty) \tag{1.1}$$

naturally arose and was studied. The study of properties of functions of similar structure turned out to be useful in the study of one difficult problem in finite difference theory (see [3]).

The analytic function (1.1) admits the power expansion

$$H(x) = \sum_{n=1}^{\infty} (-1)^{n-1} a_n x^n = \frac{1}{2}x - \frac{11}{24}x^2 + \frac{7}{16}x^3 - \frac{2447}{5760}x^4 + \dots, \quad x \in (-1, 1). \tag{1.2}$$

It is proved [1, Proposition 2.1] that all the coefficients a_n in the representation (1.2) are positive

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rational numbers and can be found by the recurrence rule

$$a_n = \frac{1}{n}\sum_{k=1}^{n}\frac{k}{k+1}\,a_{n-k}, \quad n \in \mathbb{N}, \tag{1.3}$$

with $a_0 = 1$. The number sequence (1.3) is strictly decreasing and tends to $1/e$. The proof of these properties is based on the integral representation

$$a_n = \frac{1}{e}\left(1 + \int\limits_0^1 \varphi(\tau)\,\tau^n\,d\tau\right), \quad n \in \mathbb{N}, \tag{1.4}$$

obtained in [1, Proposition 4.1]. The function φ in (1.4) has the form

$$\varphi(\tau) \equiv \frac{1}{\pi}\,\frac{\sin(\pi\tau)}{\tau^{1-\tau}\,(1-\tau)^{\tau}}, \quad \tau \in (0,1), \quad \varphi(0) = \varphi(1) = 1. \tag{1.5}$$

The elementary function (1.5) is symmetric, continuous on $[0,1]$, and infinitely differentiable on $(0,1)$. Analyzing formula (1.4), we derive the asymptotic law [2, § 2]

$$a_n = \frac{1}{e}\Big(1 = \frac{1}{n} - \frac{\ln n}{n^2} - \frac{\gamma}{n^2}\Big) + O\Big(\frac{\ln^2 n}{n^3}\Big), \quad n \to \infty,$$

where $\gamma = 0.57721\ldots$ is the Euler–Mascheroni constant.

Using (1.4), we can obtain the nice integral representation of the function (1.1)

$$H(x) = \frac{x}{e}\left(\frac{1}{1+x} + \int\limits_0^1 \frac{\tau\varphi(\tau)}{1+x\tau}\,d\tau\right), \quad x \in (-1,+\infty), \tag{1.6}$$

which follows from [2, Theorem 1].

Let us formulate the main result of the paper.

Theorem 1.1. *The analytic function* (1.1) *satisfies the following two-sided sharp estimates:*

$$\frac{e-2}{e}x \leqslant H(x) < \frac{x}{2}, \tag{1.7}$$

$$\frac{x}{2 + \dfrac{11x}{6}} \equiv \frac{1}{\dfrac{2}{x} + \dfrac{11}{6}} < H(x) \leqslant \frac{1}{\dfrac{2}{x} + \dfrac{4-e}{e-2}} \equiv \frac{x}{2 + \dfrac{(4-e)x}{e-2}} \quad \forall\, x \in (0,1]. \tag{1.8}$$

The left inequality in (1.7) becomes equality at the point $x = 1$, whereas the coefficient $1/2$ at x in the right inequality cannot be replaced by a smaller one in view of the asymptotics (1.2)

$$H(x) = \frac{x}{2} + O\left(x^2\right), \quad x \to 0.$$

The right inequality in (1.8) becomes equality at the point $x = 1$, the number $11/6$ in the denominator of the fraction in the left inequality cannot be replaced by a smaller one in view of the asymptotics

$$H(x) = \frac{x}{2 + \dfrac{11x}{6}} + O\left(x^3\right), \quad x \to 0,$$

which follows from (1.2).

2 Proof of Theorem 1.1

To derive (1.7) and (1.8), we need to use some properties of the auxiliary functions

$$\Phi_1(x) \equiv \frac{H(x)}{x} = \frac{1}{x}\,(1 - e^{-1}(1+x)^{1/x}), \quad \Phi_1(0) = a_1 = \frac{1}{2}, \tag{2.1}$$

$$\Phi_2(x) \equiv \frac{1}{x}\left(\frac{1}{\Phi_1(x)} - \frac{1}{\Phi_1(0)}\right) = \frac{1}{H(x)} - \frac{2}{x} = \frac{e}{e - (1+x)^{1/x}} - \frac{2}{x}, \quad \Phi_2(0) = \frac{a_2}{a_1^2} = \frac{11}{6}. \tag{2.2}$$

These functions are defined and continuous for all $x \in (-1, +\infty)$.

Lemma 2.1. *The function $\Phi_1(x)$ defined by* (2.1) *is decreasing on the ray $x > -1$.*

Proof. By (1.6),

$$\Phi_1(x) = \frac{1}{e}\left(\frac{1}{1+x} + \int_0^1 \frac{\tau\varphi(\tau)}{1+x\tau}\,d\tau\right), \quad x \in (-1, +\infty). \tag{2.3}$$

Since the function (1.5) is positive on $[0, 1]$, the derivative

$$\Phi_1'(x) = -\frac{1}{e}\left(\frac{1}{(1+x)^2} + \int_0^1 \frac{\tau^2\varphi(\tau)}{(1+x\tau)^2}\,d\tau\right) \tag{2.4}$$

is negative for all $x \in (-1, +\infty)$. Lemma 2.1 is proved. □

Lemma 2.2. *The function $F(x) \equiv 1/\Phi_1(x)$, where $\Phi_1(x)$ is defined by* (2.1)*, is increasing and strictly concave on the ray $x > -1$.*

Proof. By Lemma 2.1, the function $F(x)$ increases for $x > -1$. Let us show that F is convcave on this ray. Since Φ_1 is positive and

$$F'(x) = -\frac{\Phi_1'(x)}{\Phi_1^2(x)}, \quad F''(x) = -\frac{\Phi_1(x)\,\Phi_1''(x) - 2(\Phi_1'(x))^2}{\Phi_1^3(x)},$$

it suffices to prove the inequality

$$\Phi_1(x)\,\Phi_1''(x) > 2(\Phi_1'(x))^2, \quad x \in (-1, +\infty). \tag{2.5}$$

Differentiating (2.4), we find

$$\Phi_1''(x) = \frac{2}{e}\left(\frac{1}{(1+x)^3} + \int_0^1 \frac{\tau^3\varphi(\tau)}{(1+x\tau)^3}\,d\tau\right), \quad x \in (-1, +\infty). \tag{2.6}$$

Substituting (2.3), (2.4), (2.6) into (2.5), we get

$$\left(\frac{1}{1+x} + \int_0^1 \frac{\tau\varphi(\tau)}{1+x\tau}\,d\tau\right)\left(\frac{1}{(1+x)^3} + \int_0^1 \frac{\tau^3\varphi(\tau)}{(1+x\tau)^3}\,d\tau\right)$$

$$> \left(\frac{1}{(1+x)^2} + \int_0^1 \frac{\tau^2\varphi(\tau)}{(1+x\tau)^2}\,d\tau\right)^2. \tag{2.7}$$

Thus, it suffices to prove the relation (2.7) for all $x > -1$. For such x and $\tau \in [0,1]$ we have

$$\sqrt{\frac{\tau\varphi(\tau)}{1+x\tau}}\sqrt{\frac{\tau^3\varphi(\tau)}{(1+x\tau)^3}} = \frac{\tau^2\varphi(\tau)}{(1+x\tau)^2}.$$

However, in this case, from the Cauchy–Bunyakovsky inequality it follows that

$$\int_0^1 \frac{\tau\varphi(\tau)}{1+x\tau}\,d\tau \int_0^1 \frac{\tau^3\varphi(\tau)}{(1+x\tau)^3}\,d\tau \geqslant \left(\int_0^1 \frac{\tau^2\varphi(\tau)}{(1+x\tau)^2}\,d\tau\right)^2, \quad x > -1. \tag{2.8}$$

Moreover, the inequality of arithmetic and geometric means for the same x yields

$$\frac{1}{(1+x)^3}\int_0^1 \frac{\tau\varphi(\tau)}{1+x\tau}\,d\tau + \frac{1}{1+x}\int_0^1 \frac{\tau^3\varphi(\tau)}{(1+x\tau)^3}\,d\tau \geqslant \frac{2}{(1+x)^2}\sqrt{\int_0^1 \frac{\tau\varphi(\tau)}{1+x\tau}\,d\tau \int_0^1 \frac{\tau^3\varphi(\tau)}{(1+x\tau)^3}\,d\tau}.$$

Evaluating the resulting radical expression by using (2.8), for all $x > -1$ we have

$$\frac{1}{(1+x)^3}\int_0^1 \frac{\tau\varphi(\tau)}{1+x\tau}\,d\tau + \frac{1}{1+x}\int_0^1 \frac{\tau^3\varphi(\tau)}{(1+x\tau)^3}\,d\tau > \frac{2}{(1+x)^2}\int_0^1 \frac{\tau^2\varphi(\tau)}{(1+x\tau)^2}\,d\tau. \tag{2.9}$$

We have the sign $>$ in (2.9) because the inequality of arithmetic and geometric means is applied to distinct positive quantities.

Summarizing (2.8) and (2.9) and then adding $\dfrac{1}{(1+x)^4}$ to both sides, we obtain an inequality which implies (2.7). Lemma 2.2 is proved. □

The following assertion admits a simple geometric meaning and is a variant of the well-known three-chord lemma (see, for example, [4, Chapter 7, Section 1, Problem 1.2]). For the sake of completeness, we give a proof.

Lemma 2.3. *Let $G(x)$ be defined and strictly concave for $x \geqslant 0$. Then the difference ratio*

$$g(x) \equiv \frac{G(x) - G(0)}{x}$$

decreases on the ray $x > 0$.

Proof. Let $\alpha \in (0,1)$. By the definition of strict concavity, for any $0 \leqslant t_1 < t_2$

$$G(\alpha\, t_1 + (1-\alpha)\, t_2) > \alpha\, G(t_1) + (1-\alpha)\, G(t_2).$$

We show that for all $0 < x_1 < x_2$

$$g(x_1) \equiv \frac{G(x_1) - G(0)}{x_1} > \frac{G(x_2) - G(0)}{x_2} \equiv g(x_2).$$

Indeed, the latter is equivalent to the inequality

$$G(x_1) > \frac{x_2 - x_1}{x_2}\, G(0) + \frac{x_1}{x_2}\, G(x_2) \equiv \alpha\, G(0) + (1-\alpha)\, G(x_2),$$

which is valid by the strict concavity of $G(x)$. Lemma 2.3 is proved. □

Lemma 2.4. *The function $\Phi_2(x)$ defined by* (2.2) *is decreasing along the ray $x > 0$.*

Proof. By (2.2) and the definition of $F(x)$ (see Lemma 2.2), we have

$$\Phi_2(x) \equiv \frac{1}{x}\left(\frac{1}{\Phi_1(x)} - \frac{1}{\Phi_1(0)}\right) \equiv \frac{F(x)-F(0)}{x}, \quad x \in (-1, +\infty).$$

By Lemma 2.2, the function $F(x)$ is strictly concave on the ray $x > -1$ (even on the ray $x > 0$). However, in this case, from Lemma 2.3 it follows that the function $\Phi_2(x)$ is decreasing for $x > 0$. Lemma 2.4 is proved. □

Proof of Theorem 1.1. So, both functions $\Phi_1(x)$ and $\Phi_2(x)$ decrease on the ray $x > 0$ (see Lemmas 2.1 and 2.4). It follows that for all $x \in (0, 1]$ the following relations hold:

$$\frac{e-2}{e} = \Phi_1(1) \leqslant \Phi_1(x) \equiv \frac{1}{x}\left(1 - e^{-1}(1+x)^{1/x}\right) \equiv \frac{H(x)}{x} < \Phi_1(0) = \frac{1}{2},$$
$$\frac{4-e}{e-2} = \Phi_2(1) \leqslant \Phi_2(x) \equiv \frac{e}{e-(1+x)^{1/x}} - \frac{2}{x} \equiv \frac{1}{H(x)} - \frac{2}{x} < \Phi_2(0) = \frac{11}{6}.$$

Hence the inequalities (1.7) and (1.8) are valid. Theorem 1.1 is proved. □

The method of proving the main statement shows that the monotonicity considerations play an important role in deriving sharp estimates for numbers and functions required in analysis (see [5]).

3 Rational Approximations of e

There is a lot of publications on the best rate of rational approximations of the number e and other special numbers (see, for example, [6, 7]) We demonstrate how our results contribute to the problem of approximating the Euler number by the sequence $(1+1/m)^m$, $m \in \mathbb{N}$.

As shown in [2, Theorem 3], the series (1.2) envelops the function $H(x)$ for all $x > 0$. In other words, for each $x > 0$ we have a series of two-sided estimates

$$\sum_{n=1}^{2p} (-1)^{n-1} a_n x^n < 1 - e^{-1}(1+x)^{1/x} < \sum_{n=1}^{2q-1} (-1)^{n-1} a_n x^n, \quad p, q \in \mathbb{N}.$$

In particular, for $x = 1/m \in (0, 1]$ and $q = p + 1$

$$\sum_{n=1}^{2p} \frac{(-1)^{n-1} a_n}{m^n} < 1 - e^{-1}\left(1 + \frac{1}{m}\right)^m < \sum_{n=1}^{2p+1} \frac{(-1)^{n-1} a_n}{m^n} \tag{3.1}$$

for all $m, p \in \mathbb{N}$. For example, setting $p = 1$ in (3.1), we have

$$\frac{1}{2m} - \frac{11}{24m^2} < 1 - e^{-1}\left(1 + \frac{1}{m}\right)^m < \frac{1}{2m} - \frac{11}{24m^2} + \frac{7}{16m^3}, \quad m \in \mathbb{N}, \tag{3.2}$$

which improves the rational approximation of the number e

$$\frac{24m^2}{24m^2 - 12m + 11}\left(1 + \frac{1}{m}\right)^m < e < \frac{48m^3}{48m^3 - 24m^2 + 22m - 21}\left(1 + \frac{1}{m}\right)^m, \quad m \in \mathbb{N}.$$

Indeed, substituting the test value $m = 100$, we find the following bounds for e:

$$\frac{240000}{238811}\,1.01^{100} = \mathbf{2.71828}06\ldots < e < \mathbf{2.7182818}39\ \ldots = \frac{48000000}{47762179}\,1.01^{100}, \tag{3.3}$$

i.e., the minorant with five correct decimal places and the majorant with seven correct digits after the decimal point, despite the fact that

$$\left(1+\frac{1}{100}\right)^{100} = 1.01^{100} = \mathbf{2.7}048\ldots$$

with only one correct digit after the decimal point.

We note that the envelopment is also used to study the asymptotic behavior of the gamma function [8], central binomial coefficient [9], and remainders of number series [10].

Theorem 1.1 provides another useful format to estimate the amount of deviation

$$H\left(\frac{1}{m}\right) = 1 - e^{-1}\left(1+\frac{1}{m}\right)^{m}, \quad m \in \mathbb{N}. \tag{3.4}$$

Thus, choosing $x = 1/m$ in the inequality (1.7), we arrive at the simplest estimate

$$\frac{e-2}{em} \leqslant 1 - e^{-1}\left(1+\frac{1}{m}\right)^{m} < \frac{1}{2m}, \quad m \in \mathbb{N}, \tag{3.5}$$

in which the constants $(e-2)/e = 0.264\ldots$ and $1/2$ on the set of all $m \in \mathbb{N}$ cannot be improved because 1) due to the equality on the left-hand side of (3.5) for $m = 1$ and 2) due to enveloping (3.1). If we use the double inequality (1.8), then a strengthening (3.5) variant arises

$$\frac{1}{2m+\frac{11}{6}} < 1 - e^{-1}\left(1+\frac{1}{m}\right)^{m} \leqslant \frac{1}{2m+\frac{4-e}{e-2}}, \quad m \in \mathbb{N}, \tag{3.6}$$

where the choice of both constants $11/6 = 1.8(3)$ and $(4-e)/(e-2) = 1.784\ldots$ for a given pattern is again optimal by the same reasons as in (3.5). Comparing the lower bounds in (3.2) and (3.6), we see that the second one is always better. The situation is different for the upper bounds in (3.2) and (3.6): the first is better for all numbers $m \geqslant 7$, which is easily explained by the nonasymptotic nature of the constant $(4-e)/(e-2)$. The result (3.6) improves the well-known double inequality

$$\frac{1}{2m+2} < 1 - e^{-1}\left(1+\frac{1}{m}\right)^{m} < \frac{1}{2m+1}, \quad m \in \mathbb{N},$$

from the classic problem book [11, Part I, Chapter 4, Section 2].

For the number e itself the left-hand side of (3.6) gives for any $m \in \mathbb{N}$ the lower estimate

$$e > \frac{12m+11}{12m+5}\left(1+\frac{1}{m}\right)^{m}.$$

Testing it at the previous value $m = 100$, we write

$$e > \frac{1211}{1205}\,1.01^{100} = \mathbf{2.7182818}2\ldots,$$

where six correct digits after the decimal point demonstrate slight improvement compared to the lower bound of (3.3).

Let us mention another way to obtain nontrivial estimates for the Euler number based on the connection between power means and Rado means. In this case, one can set for e (respectively for (3.4)) the following inequalities valid for all $m \in \mathbb{N}$ and written out in order of increasing accuracy:

$$e < \sqrt{\frac{m+1}{m}}\Big(1+\frac{1}{m}\Big)^m \iff H\Big(\frac{1}{m}\Big) < \frac{1}{m+\sqrt{m(m+1)}}, \tag{3.7}$$

$$e < \frac{4(m+1)}{(\sqrt{m}+\sqrt{m+1}\,)^2}\Big(1+\frac{1}{m}\Big)^m \iff H\Big(\frac{1}{m}\Big) < \frac{2m+3-2\sqrt{m(m+1)}}{4(m+1)}, \tag{3.8}$$

$$e < \Big(\frac{2\sqrt[3]{(m+1)^2}}{\sqrt[3]{m^2}+\sqrt[3]{(m+1)^2}}\Big)^{3/2}\Big(1+\frac{1}{m}\Big)^m \iff H\Big(\frac{1}{m}\Big) < 1-\Big(\frac{\sqrt[3]{m^2}+\sqrt[3]{(m+1)^2}}{2\sqrt[3]{(m+1)^2}}\Big)^{3/2}. \tag{3.9}$$

Choosing and substituting $m = 100$ into (3.7)–(3.9), we consistently find that

$$e < \frac{\sqrt{101}}{10}\,1.01^{100} = \mathbf{2.718}304\ldots,$$

$$e < \frac{404}{(10+\sqrt{101}\,)^2}\,1.01^{100} = \mathbf{2.71828}743\ldots,$$

$$e < 202\sqrt{\frac{2}{(\sqrt[3]{10000}+\sqrt[3]{10201})^3}}\,1.01^{100} = \mathbf{2.7182818284}631\ldots,$$

where the accuracy of ten decimal places in the last estimate is achieved due to a significant complication of its format.

We show that such a high accuracy can be achieved by without changing the “rational” pattern of continued fractions chosen in the Theorem 1.1. However, it require a rigorous proof of one plausible hypothesis confirmed by numerical calculation.

4 Open Question

On the set $x \in (-1, +\infty)$, we define the functional sequence $\Phi_n(x)$ with numbering $n \in \mathbb{N}$ as follows. The first two elements Φ_1 and Φ_2 are given by formulas (2.1) and (2.2) respectively. Let

$$\Phi_3(x) \equiv \frac{1}{x}\Big(\frac{1}{\Phi_2(x)} - \frac{1}{\Phi_2(0)}\Big) = \frac{1}{\dfrac{x}{H(x)}-2} - \frac{6}{11x} = \frac{1}{\dfrac{x}{1-e^{-1}(1+x)^{\frac{1}{x}}}-2} - \frac{6}{11x} \tag{4.1}$$

by the natural agreement

$$\Phi_3(0) = \Big(\frac{a_1}{a_2}\Big)^2 a_3 - a_1 = \frac{5}{242}. \tag{4.2}$$

Then

$$\Phi_4(x) \equiv \frac{1}{x}\Big(\frac{1}{\Phi_3(x)} - \frac{1}{\Phi_3(0)}\Big) = \frac{1}{\dfrac{x}{\dfrac{x}{H(x)}-2} - \dfrac{6}{11}} - \frac{242}{5x}. \tag{4.3}$$

By continuity, we can assume that

$$\Phi_4(0) = \frac{5027}{250}. \tag{4.4}$$

The general recurrence rule is as follows:

$$\Phi_{n+1}(x) \equiv \frac{1}{x}\Big(\frac{1}{\Phi_n(x)} - \frac{1}{\Phi_n(0)}\Big), \quad n \in \mathbb{N}, \tag{4.5}$$

with definition by continuity at the point $x = 0$. Thus, $\Phi_n(x)$ is an elementary analytic function for $x > -1$ and any $n \in \mathbb{N}$.

Conjecture. For any $n \in \mathbb{N}$ the function $\Phi_n(x)$ from the sequence (4.5) is decreasing on the ray $x > 0$.

In Lemmas 2.1 and 2.4, the conjecture is confirmed for $n = 1$ and $n = 2$. Assuming that the conjecture is true $n = 3$ and taking into account (4.2), for the function (4.1) we obtain the following two-sided estimate:

$$\frac{17e - 46}{11(4-e)} = \Phi_3(1) \leqslant \Phi_3(x) < \Phi_3(0) = \frac{5}{242}, \quad x \in (0, 1], \tag{4.6}$$

with bounds

$$\frac{17e - 46}{11(4-e)} = 0.0149\ \ldots, \qquad \frac{5}{242} = 0.0206\ \ldots.$$

This estimate can be written as

$$\frac{x}{2 + \cfrac{x}{\cfrac{6}{11} + \cfrac{(17e-46)\,x}{11(4-e)}}} \leqslant H(x) < \frac{x}{2 + \cfrac{x}{\cfrac{6}{11} + \cfrac{5x}{242}}}, \quad x \in (0, 1]. \tag{4.7}$$

By (4.7), the deviation (3.4) satisfies the inequalities

$$\frac{1}{2m + \cfrac{11}{6 + \cfrac{17e - 46}{(4-e)\,m}}} \leqslant 1 - e^{-1}\Big(1 + \frac{1}{m}\Big)^m < \frac{1}{2m + \cfrac{11}{6 + \cfrac{5}{22\,m}}}, \quad m \in \mathbb{N}. \tag{4.8}$$

The right-hand side of (4.8) can be written as

$$e < \frac{264m^2 + 252m}{264m^2 + 120m - 5}\Big(1 + \frac{1}{m}\Big)^m, \quad m \in \mathbb{N}. \tag{4.9}$$

Setting $m = 100$ in (4.9), we get

$$e < \frac{2665200}{2651995}\, 1.01^{100} = \mathbf{2.718281828}651\ldots$$

with nine correct digits after the decimal point.

We assume that the conjecture is true for $n = 4$. Then, taking into account (4.4), for the function (4.3) a two-sided estimate we have

$$\frac{11(1032 - 379e)}{5(17e - 46)} = \Phi_4(1) \leqslant \Phi_4(x) < \Phi_4(0) = \frac{5027}{250}, \quad x \in (0, 1], \tag{4.10}$$

which can be written as

$$\frac{x}{2+\cfrac{x}{\cfrac{6}{11}+\cfrac{x}{\cfrac{242}{5}+\cfrac{5027\,x}{250}}}} < H(x) \leqslant \frac{x}{2+\cfrac{x}{\cfrac{6}{11}+\cfrac{x}{\cfrac{242}{5}+\cfrac{11(1032-379e)\,x}{5(17e-46)}}}}, \quad x \in (0,1], \tag{4.11}$$

with the same constants as in (4.10)

$$\frac{5027}{250} = 20.108\,\ldots, \qquad \frac{11(1032-379e)}{5(17e-46)} = 18.485\,\ldots,$$

found by a direct calculation on the basis of the expansion (1.2). In this case, from (4.11) we obtain for the deviation (3.4) the double inequality

$$\frac{1}{2m+\cfrac{1}{\cfrac{6}{11}+\cfrac{1}{\cfrac{242\,m}{5}+\cfrac{5027}{250}}}} < H\Big(\frac{1}{m}\Big) \leqslant \frac{1}{2m+\cfrac{1}{\cfrac{6}{11}+\cfrac{1}{\cfrac{242\,m}{5}+\cfrac{11(1032-379e)}{5(17e-46)}}}} \tag{4.12}$$

for all $m \in \mathbb{N}$. The left-hand side of (4.12) contains the estimate

$$e > \frac{145200m^2+198924m+55297}{145200m^2+126324m+22385}\Big(1+\frac{1}{m}\Big)^m, \quad m \in \mathbb{N}. \tag{4.13}$$

Taking $m = 100$ in (4.13), we get

$$e > \frac{12164857}{12104585}\,1.01^{100} = \mathbf{2.71828182845}882\,\ldots,$$

with eleven correct decimal places.

However, it is not yet known whether the “basic” relations (4.6) and (4.10) are valid. Apparently, confirming the formulated conjecture in full requires a more subtle approach compared to what was proposed in the proof of Theorem 1.1.

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