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ON RINGS WITH SEMIDISTRIBUTIVE MODULES

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Abstract. A module is said to be distributive if the lattice of its submodules is distributive. A direct sum of distributive modules is called a semidistributive module. In this paper, we consider rings A such that all right A-modules are semidistributive.

1. Introduction

We only consider associative unital nonzero rings and unitary modules. The words of type "a right Artinian ring A" ("an Artinian ring A") mean that the module A_A is Artinian (respectively, both modules A_A and ${}_AA$ are Artinian).

In the book *Dniester Notebook* [6] consisting of some problems of ring theory, L. A. Skornyakov posed the following two questions (Problem 1.116). Over which rings all right modules are semidistributive? Are there non-Artinian rings with this property?

Remark 1.1. In [16], it is proved that a ring over which all right modules are semidistributive is a right Artinian, right Köthe ring. (A ring A is called a *right Köthe ring* if every right A-module is a direct sum of cyclic modules.) It is well-known that right Köthe rings are Artinian. This gives negative answer to the second question of Skornyakov; also see [17, Theorem 11.6]. Therefore, all rings over which all right modules are semidistributive are right Köthe, Artinian rings. The problem of describing the right Köthe rings A, restricting ourselves only to the internal properties of the ring A, remains unsolved for arbitrary rings; this problem is called *Köthe's problem.* Köthe's problem is considered in many papers; see, e.g., [5, 7–9, 11, 12, 15].

Remark 1.2. See [16] and [17, Sec. 11.1] on the second question of Skornyakov. In [10], there was studied a partial case of rings over which all right modules are semidistributive.

For a module M, we denote by $J(M)$ the Jacobson radical of the module M. Let A be a semiprimary ring (a ring A is said to be *semiprimary* if the radical $J(A)$ is nilpotent and the factor ring $A/J(A)$ is a semisimple Artinian ring; every right or left Artinian ring is semiprimary) and let B be the *basic* ring of the semiprimary ring A, i.e., $B = eAe$, where e is a *basic* idempotent of the ring A; this means that $e = e_1 + \cdots + e_n$, where $\{e_1, \ldots, e_n\}$ is a set of local orthogonal idempotents of A such that $\{e_1A, \ldots, e_nA\}$ ($\{Ae_1, \ldots, Ae_n\}$) is the set of all pair-wise non-isomorphic indecomposable direct summands of the module A_A (respectively, ${}_AA$). If $A = B$, then the semiprimary ring A is said to be *self-basic.* It is well known that the basic ring B of the semiprimary ring A is self-basic and the category $\operatorname{Mod} A$ of all right A-modules is equivalent to the category of all right B-modules, i.e., the rings A and B are *Morita equivalent.* In addition, it is known that the property to be an Artinian self-basic ring is preserved under Morita equivalences.

2. Main Results

Remark 2.1. It follows from Remark 1.1 and the above that any ring A, over which all right modules are semidistributive, is an Artinian ring with self-basic basic ring B and the rings A and B are Morita equivalent. It is clear that the semidistributivity property of all right modules is preserved under Morita

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equivalences. Therefore, all right B-modules are semidistributive. Therefore, when studying rings over which all right modules are semidistributive, we can restrict ourself by Artinian self-basic Köthe rings B such that every ring that is Morita equivalent to the ring B is an Artinian self-basic ring.

A module is said to be *completely cyclic* if all of its submodules are cyclic.

Theorem 2.2 ([17, Theorem 11.6])**.** *For a ring A, the following conditions are equivalent.*

(1) *All right A-modules are semidistributive.*
(2) *A is an Artinian ring and every right A-module is a direct sum of completely cyclic distributive modules with composition series.*
(3) *A is an Artinian ring with basic idempotent e and for any right A-module M, the right eAe-module Me_{eAe} is a direct sum of completely cyclic modules with composition series.*

A ring A is said to be a ring *of finite representation type* if A is an Artinian ring that has up to isomorphism only a finite number of indecomposable right modules and a finite number of indecomposable left modules. A ring A is called a *right Kawada ring* if every ring that is Morita equivalent to the ring A is a right Köthe ring. We note that all right Köthe rings (in particular, all right Kawada rings) are rings of finite representation type.

Theorem 2.3 ([18])**.** *For a ring A, the following conditions are equivalent.*

(1) *All right A-modules are semidistributive.*
(2) *A is a right Kawada ring with basic ring B and every right A-module and every right B-module are direct sums of completely cyclic distributive modules.*
(3) *A is a right Kawada ring with basic ring B and every indecomposable right B-module is a completely cyclic module.*

Remark 2.4. In [11], Kawada solved Köthe's problem for finite-dimensional algebras over a field. Kawada's theorem completely describes self-basic finite-dimensional algebras A over a field such that every indecomposable A-module has the square-free socle and the square-free top; in the same work, all indecomposable A-modules are described. Köthe's problem remains unsolved in the general case. In [11], there are 19 rather complicated conditions for local idempotents of the basic ring B of the algebra A; all these conditions hold if and only if A is a right Köthe ring. In [15], Kawada's result is analyzed and commented. To the mentioned 19 conditions, we can add the following condition 20: for any local idempotent e of the basic algebra B, the module eB_B is completely cyclic. Therefore, we obtain a formal description of finite-dimensional algebras over a field over which all right modules are semidistributive. Of course, such a description is not very useful.

A ring is said to be *normal* or *Abelian* if all of its idempotents are central.

Corollary 2.5. *If the factor ring $A/J(A)$ of the ring A is normal, then all right A-modules are semidistributive if and only if A is an Artinian ring and every right A-module is a direct sum of completely cyclic modules with composition series.*

A ring in which all right ideals and all left ideals are principal is called a *principal ideal ring.*

Theorem 2.6 ([12])**.** *An Artinian principal ideal ring is a Köthe ring.*

Corollary 2.7 ([5, 12])**.** *A commutative ring A is a Köthe ring if and only if A is an Artinian principal ideal ring.*

A module is said to be *uniserial* if all of its submodules are linearly ordered with respect to inclusion. A direct sum of uniserial modules is called a *serial* module.

With the use of Corollary 2.5 and Theorem 2.6, it is easy to verify Theorem 2.8.

Theorem 2.8. *For a normal ring A, the following conditions are equivalent.*

(1) *All right A-modules are semidistributive.*

(2) *All left A-modules are semidistributive.*
(3) *A is an Artinian principal ideal ring.*
(4) *The ring A is isomorphic to a finite direct product of Artinian uniserial rings.*

For any module M, the *top* $\operatorname{top} M$ is the factor module $M/J(M)$. A module that does not contain direct sums of two nonzero isomorphic submodules is called a *square-free* module.

Theorem 2.9 ([4]). *A normal ring A is a Köthe ring if and only if A is an Artinian principal ideal ring.*

According to [2], a ring A is called a *strongly right* (*strongly left*) *Köthe ring* if every nonzero right (respectively, left) A-module is a direct sum of modules with nonzero cyclic square-free top. Right and left strongly Köthe rings are called *strongly Köthe rings*.

According to [2], a ring A is called a *right very strongly* (*left very strongly*) *Köthe ring* if every nonzero right (respectively, left) A-module is a direct sum of modules with simple top. Right and left very strongly Köthe rings are called *very strongly Köthe rings*.

The following proper inclusions are known (e.g., see [2]):

$$\text{very strongly right Köthe rings} \subsetneq \text{strongly right Köthe rings} \subsetneq \text{right Köthe rings.}$$

Theorem 2.10 ([2]). *For a ring A, the following conditions are equivalent.*
(1) *A is a right Köthe ring.*
(2) *Every nonzero right A-module is a direct sum of modules with nonzero cyclic top.*
(3) *The ring A is right Artinian and every right A-module is a direct sum of modules with cyclic top.*
(4) *A is a ring of finite representation type and every* (*finitely generated*) *indecomposable right A-module has the cyclic top.*
(5) *A is a ring of finite representation type and the top of every indecomposable right A-module U is isomorphically embedded in A/J.*

3. Addendum

In [1], the authors defined co-Köthe rings, which are close to Köthe rings: a ring A is called a *right* (*right strongly*, *right very strongly*) *co-Köthe ring* if every nonzero right A-module is a direct sum of modules with nonzero cyclic socle (respectively, with nonzero square-free socle, with simple socle). The left-side analogues of these notions are defined similarly.

Remark 3.1. By [1], there is a very strongly right co-Köthe ring over which there exists a nonsemidistributive right module.

A module is said to be *uniform* if the intersection of any two of its nonzero submodules is not equal to zero.

Example 3.2 (see also [1; 13; 14; 17, Example 1.22). Let A be the 5-dimensional algebra over the field $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ consisting of all (3×3)-matrices of the form

$$\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ 0 & f_{22} & 0 \\ 0 & 0 & f_{33} \end{bmatrix},$$

where $f_{ij} \in \mathbb{Z}_2$. Let e_{ij} be the matrix whose ijth entry is equal to 1 and all other entries are equal to 0. Then $\{e_{11}, e_{12}, e_{13}, e_{22}, e_{33}\}$ is a $\mathbb{Z}_2$-basis of the $\mathbb{Z}_2$-algebra A. Moreover, the following assertions are true.
(1) $1 = e_{11} + e_{22} + e_{33}$, where e_{11}, e_{22}, and e_{33} are primitive orthogonal idempotents, $e_{12}\mathbb{Z}_2 = e_{12}A$, $e_{13}\mathbb{Z}_2 = e_{13}A$, $J(A) = e_{12}\mathbb{Z}_2 + e_{13}\mathbb{Z}_2$, $J^2 = 0$, and the ring A/J is isomorphic to a direct product of three copies of the field $\mathbb{Z}_2$.
(2) $A_A = e_{11}A \oplus e_{22}A \oplus e_{33}A$, where $e_{22}A = e_{22}\mathbb{Z}_2$ and $e_{33}A = e_{33}\mathbb{Z}_2$ are simple projective right A-modules that are isomorphic to the modules $e_{12}A$ and $e_{13}A$, respectively.

(3) $e_{11}A = e_{11}\mathbb{Z}_2 + e_{12}\mathbb{Z}_2 + e_{13}\mathbb{Z}_2$ is an indecomposable distributive (and hence square-free) Noetherian Artinian completely cyclic A-module, but it is not uniform and every proper nonzero submodule of $e_{11}A$ coincides either with the projective module $e_{12}A \oplus e_{13}A$ or with one of the simple projective non-isomorphic modules $e_{12}A$ and $e_{13}A$.

It can be checked that A is a hereditary Artinian basic ring. So by [9], A is a Köthe ring. It can be shown that A has 32 elements, $|U(A)| = 4$, and non-unit elements of A form 3 isomorphism classes of cyclic indecomposable modules, which are as follows:

$$Q_1 = e_{11}A, \quad Q_2 = e_{11}A/e_{13}A, \quad Q_3 = e_{11}A/e_{12}A,$$
$$Q_4 = e_{11}A/J(A), \quad Q_5 = e_{22}A, \quad Q_6 = e_{33}A.$$

Also, every indecomposable cyclic right A-module has a square-free socle, since $\operatorname{Soc}(Q_1) = J(A)$ is square-free, $\operatorname{Soc} Q_2 = J(A)e_{13}A$, $\operatorname{Soc} Q_3 = J(A)/e_{12}A$, and Q_4, Q_5, and Q_6 are simple. Then every right A-module is a direct sum of square-free modules (A is a (strongly) right co-Köthe ring) while the right A-module $e_{11}A$ is not a direct sum of uniform modules (A is not a very strongly right co-Köthe ring), since A is not right serial.

We give several results from [1]. We recall that a semi-primary ring A is called a *right QF-2 ring* (*right co-QF-2 ring*) if every indecomposable projective right A-module has a simple essential socle (respectively, a simple top).

A semi-perfect ring A is a generalized left co-QF-2 ring if every indecomposable projective left A-module P has a square-free top. A semi-perfect ring A is a generalized co-QF-2 ring if A is a generalized left and a generalized right co-QF-2 ring.

Theorem 3.3 ([1]). *The following conditions are equivalent for a ring A*:

(1) *A is a right co-Köthe ring*;
(2) *every nonzero right A-module is a direct sum of modules with nonzero top and cyclic essential socle*;
(3) *A is of finite representation type and every* (*finitely generated*) *indecomposable right A-module has a cyclic* (*essential*) *socle*;
(4) *A is of finite representation type and the socle of every indecomposable right A-module U is isomorphically embedded in A/J.*

Let A be a ring of finite representation type and let $U = U_1 \oplus \cdots \oplus U_n$ and $\{U_1, \ldots, U_n\}$ be a complete set of representatives of the isomorphic classes of finitely generated indecomposable right A-modules. The *right Auslander ring* of A is $T = \operatorname{End} U_A$.

Theorem 3.4 ([1]). *The following conditions are equivalent for a ring A with $J = J(A)$*:

(1) *A is a strongly right co-Köthe ring*;
(2) *every right A-module is a direct sum of square-free modules*;
(3) *every nonzero right A-module is a direct sum of modules with nonzero top and square-free* (*cyclic*) *socle*;
(4) *A is of finite representation type and every* (*finitely generated*) *indecomposable right A-module has a square-free* (*cyclic*) *socle*;
(5) *A is of finite representation type and the right Auslander ring of A is a generalized right QF-2 ring*;
(6) *A is of finite representation type and the right Auslander ring of A is a generalized left co-QF-2 ring*;
(7) *A is of finite representation type with basic set of primitive idempotents $e_1, \ldots, e_n$ and the socle of each indecomposable right A-module U is isomorphically embedded in $(e_1A/e_1J) \oplus \cdots \oplus (e_nA/e_nJ)$.*

Theorem 3.5 ([1]). *Let all maximal right ideals of the ring A be ideals. The following conditions are equivalent*:

(1) *A is a right co-Köthe ring*;
(2) *A is a strongly right co-Köthe ring*;
(3) *A is of finite representation type and every indecomposable module has a square-free socle*;
(4) *A is of finite representation type and every indecomposable module has a cyclic socle*;
(5) *A is of finite representation type and the right Auslander ring of A is a generalized right QF-2 ring*;
(6) *A is of finite representation type and the right Auslander ring of A is a generalized left co-QF-2 ring.*

Theorem 3.6 ([1]). *Let A be a finite dimensional algebra over a field. If A is a strongly right co-Köthe ring, then A is a left Köthe ring.*

A module M is called an *extending* module if every submodule is essential in a direct summand of M.

Theorem 3.7 ([1]). *The following conditions are equivalent for a ring A*:

(1) *A is a very strongly right co-Köthe ring*;
(2) *every right A-module is a direct sum of co-cyclic modules*;
(3) *every nonzero right A-module is a direct sum of modules with nonzero top and simple socle*;
(4) *A is of finite representation type and every* (*finitely generated*) *indecomposable right A-module has a simple socle*;
(5) *every right A-module is a direct sum of extending modules*;
(6) *every right A-module is a direct sum of uniform modules*;
(7) *A is of finite representation type and the right Auslander ring of A is a right QF-2 ring*;
(8) *A is of finite representation type and the right Auslander ring of A is a left co-QF-2 ring.*

If these assertions (1)–(8) *hold, then A is an Artinian, right serial ring.*

A module M is called *lifting* if for every submodule N of M, there exists a direct sum decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq N$ and $N \cap M_2$ is superfluous in M_2.

Theorem 3.8 ([1]). *The following statements are equivalent for any ring A*:

(1) *A is a very strongly co-Köthe ring*;
(2) *A is a very strongly Köthe ring*;
(3) *A is an Artinian serial ring*;
(4) *every left and right A-module is a direct sum of uniform modules*;
(5) *every left and right A-module is a direct sum of extending modules*;
(6) *every left and right A-module is a direct sum of lifting modules*;
(7) *A is of finite representation type and the left* (*right*) *Auslander ring of A is a QF-2 ring*;
(8) *A is of finite representation type and the left (right) Auslander ring of A is a co-QF-2 ring*;
(9) *every left and right A-module is a direct sum of finitely generated modules with square-free top.*

Open question 3.9. Solve Köthe's problem in the general case.

Open question 3.10. Let over ring A all right modules be semidistributive. Is it true that all left A-modules are semidistributive?

Compliance with Ethical Standards

Conflict of interests. The author declares no conflict of interest.

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REFERENCES

1. Sh. Asgari, M. Behboodi, and S. Khedrizadeh, “Left co-Köthe rings and their characterizations,” *Commun. Algebra*, **51**, No. 12, 5107–5126 (2023); “Erratum to “Left co-Köthe rings and their characterizations”,” *Commun. Algebra*, **52**, No. 4, 1702 (2024).
2. Sh. Asgari, M. Behboodi, and S. Khedrizadeh, “Several characterizations of left Köthe rings,” *Rev. Real Acad. Cienc. Exact. Fis. Nat. Ser. A-Mat.*, **117**, No. 145 (2023).
3. S Baghdari and J. Öiner, “Pure semisimple and Köthe group rings,” *Commun. Algebra*, **51**, No. 7, 2779–2790 (2023).
4. M. Behboodi, A. Ghorbani, A. Moradzadeh-Dehkordi, and S. H. Shojaee, “On left Köthe rings and a generalization of a Köthe–Cohen–Kaplansky theorem,” *Proc. Amer. Math. Soc.*, **8**, 2625–2631 (2014).
5. I. S. Cohen and I. Kaplansky, “Rings for which every module is a direct sum of cyclic modules,” *Math. Z.*, **54**, 97–101 (1951).
6. *Dniester Notebook: Unsolved Problems in the Theory of Rings and Modules*, Math. Inst., Russ. Acad. Sci., Siber. Branch, Novosibirsk (1993).
7. Z. Fazelpour and A. Nasr-Isfahani, “Connections between representation-finite and Köthe rings,” *J. Algebra*, **514**, 25–39 (2018).
8. Z. Fazelpour and A. Nasr-Isfahani, *Auslander Correspondence for Kawada Rings*, `arXiv:math.RT/2105.10898`.
9. Z. Fazelpour and A. Nasr-Isfahani, “Coxeter diagrams and the Köthe’s problem,” *Canad. J. Math.*, **73**, No. 3, 656–686 (2021).
10. K. R. Fuller, “Rings of left invariant module type,” *Commun. Algebra*, **6**, 153–167 (1978).
11. Y. Kawada, “On Köthe’s problem concerning algebras for which every indecomposable module is cyclic. I–III,” *Sci. Rep. Tokyo Kyoiku Daigaku*, **7**, 154–230 (1962); **8**, 1–62 (1963); **9**, 165–250 (1964).
12. G. Köthe, “Verallgemeinerte Abelsche Gruppen mit hyperkomplexem Operatorenring,” *Math. Z.*, **39**, No. 1, 31–44 (1935).
13. T. Nakayama, “Note on uni-serial and generalized uni-serial rings,” *Proc. Imp. Acad. Tokyo*, **16**, 285–289 (1940).
14. T. Nakayama, “On Frobeniusean algebras. II,” *Ann. Math.*, **42**, 1–27 (1941).
15. C. M. Ringel, “Kawada’s theorem,” in: *Abelian Group Theorem. Proc. of the Oberwolfach Conf., January 12–17, 1981*, Lect. Notes Math., Vol. 874, Springer, Berlin (1981), pp. 431–447.
16. A. Tuganbaev, “Rings over which every module is a direct sum of distributive modules,” *Moscow Univ. Math. Bull.*, No. 1, 61–64 (1980).
17. A. Tuganbaev, *Semidistributive Modules and Rings*, Springer, Dordrecht (1998).
18. A. Tuganbaev, “$\aleph_0$-distributive modules and rings,” *Int. J. Algebra Comput.* (2023).

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