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CONDITIONS FOR EXISTENCE OF SOLUTIONS TO DISCRETE EQUATIONS WITH PRECOMPACT RANGE OF VALUES

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We establish conditions for the existence of solutions of discrete equations with precompact range of values by using c-continuous operators and admissible pairs of compact sets.

1. Main Notation and the Object of Research

Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ be, respectively, the sets of all positive, integer, and real numbers, let $\mathbb{R}^m$ be a real m-dimensional space, let $\mathbb{Z}^m$ be an Abelian group whose elements are vectors $\mathbf{n} = (n_1, n_2, \ldots, n_m) \in \mathbb{R}^m$ with $n_1, n_2, \ldots, n_m \in \mathbb{Z}$ with respect to the operation of addition:

$$\mathbf{n}_1 + \mathbf{n}_2 = (n_{1,1}, n_{1,2}, \ldots, n_{1,m}) + (n_{2,1}, n_{2,2}, \ldots, n_{2,m})$$

$$= (n_{1,1} + n_{2,1}, n_{1,2} + n_{2,2}, \ldots, n_{1,m} + n_{2,m}),$$

let M be an arbitrary complete metric space with a metric ρ_M and let a be an arbitrary element of this space.

By $\mathfrak{M}$ we denote a complete metric space of functions $\mathbf{x} = \mathbf{x}(\mathbf{n})$, $\mathbf{n} \in \mathbb{Z}^m$, with values in M for each of which

$$\sup_{\mathbf{n} \in \mathbb{Z}^m} \rho_M(\mathbf{x}(\mathbf{n}), a) < \infty \tag{1}$$

with the following metric:

$$\rho_{\mathfrak{M}}(\mathbf{x}_1, \mathbf{x}_2) = \sup_{\mathbf{n} \in \mathbb{Z}^m} \rho_M(\mathbf{x}_1(\mathbf{n}), \mathbf{x}_2(\mathbf{n})). \tag{2}$$

In equality (2), we have $\mathbf{x}_1 = \mathbf{x}_1(\mathbf{n})$ and $\mathbf{x}_2 = \mathbf{x}_2(\mathbf{n})$, $\mathbf{n} \in \mathbb{Z}^m$.

In view of (1), the elements of the space $\mathfrak{M}$ are functions bounded on $\mathbb{Z}^m$. The metric space $\mathfrak{M}$ is complete due to the completeness of the space M.

Consider an operator $\mathbf{F} : \mathfrak{M} \to \mathfrak{M}$. This operator is

1) bounded (maps every bounded set in the space $\mathfrak{M}$ into a bounded set of this space [1, p. 14]);

2) c-continuous (see Definition 3 in Sec. 2).

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Consider an equation

$$\mathbf{F}\mathbf{x} = \mathbf{h}, \tag{3}$$

where $\mathbf{h} \in \mathfrak{M}$. This equation is discrete and can be regarded as a generalization of difference equations.

The aim of the present paper is to establish the conditions under which, for every function $\mathbf{h} \in \mathfrak{M}$ with a precompact range of values in M, equation (3) has at least one solution $\mathbf{x} \in \mathfrak{M}$.

2. Locally Convergent Sequences and c-Continuous Mappings

In what follows, in the study of equation (3), an important role is played by locally convergent sequences of elements of the space $\mathfrak{M}$.

Definition 1. *By analogy with [2], we say that a sequence of elements* $\mathbf{y}_k = \mathbf{y}_k(\mathbf{n})$, $k \geq 1$, *of the space* $\mathfrak{M}$ *locally converges to an element* $\mathbf{y} = \mathbf{y}(\mathbf{n}) \in \mathfrak{M}$ *and write*

$$\mathbf{y}_k \xrightarrow{\text{loc},\, \mathfrak{M}} \mathbf{y} \quad \textit{as} \quad k \to \infty$$

if this sequence is bounded in $\mathfrak{M}$, *i.e.,*

$$\sup_{k \geq 2} \rho_{\mathfrak{M}}(\mathbf{y}_k, \mathbf{y}_1) < \infty,$$

and, for any $\mathbf{n} \in \mathbb{Z}^m$,

$$\lim_{k \to +\infty} \rho_M(\mathbf{y}_k(\mathbf{n}), \mathbf{y}(\mathbf{n})) = 0.$$

The concept of locally convergent sequences was introduced in [3, 4].

Definition 2. *A bounded sequence of elements* $\mathbf{y}_k \in \mathfrak{M}$, $k \geq 1$, *is called locally convergent if there exists an element* $\mathbf{z} \in \mathfrak{M}$ *such that*

$$\mathbf{y}_k \xrightarrow{\text{loc},\, \mathfrak{M}} \mathbf{z} \quad \textit{for} \quad k \to \infty.$$

Note that, in view of uniqueness of the limits of convergent sequences in the space $\mathcal{M}$, the element $\mathbf{z} \in \mathfrak{M}$ in Definition 2 is unique.

An important role is played by the following statement on the existence of locally coincident sequences of elements of the space $\mathfrak{M}$.

Lemma 1. *Let* $(\mathbf{y}_k)_{k \geq 1}$ *be an arbitrary bounded sequence of elements of the space* $\mathfrak{M}$ *for which the sets* $\{\mathbf{y}_k(\mathbf{n}) \colon k \geq 1\}$, $\mathbf{n} \in \mathbb{Z}^m$, *are precompact.*

There exists a locally convergent subsequence $(\mathbf{y}_{k_l})_{l \geq 1}$ *of the sequence* $(\mathbf{y}_k)_{k \geq 1}$, *which locally converges to* $\mathbf{z} = \mathbf{z}(g) \in \mathfrak{M}$ *as* $l \to \infty$, *where*

$$\mathbf{z}(\mathbf{n}) = \lim_{l \to \infty} \mathbf{y}_{k_l}(\mathbf{n}), \quad \mathbf{n} \in \mathbb{Z}^m,$$

for which

$$\sup_{\mathbf{n} \in \mathbb{Z}^m} \rho_M(\mathbf{z}(\mathbf{n}), a) \leq \sup_{l \geq 1,\, \mathbf{n} \in \mathbb{Z}^m} \rho_M(\mathbf{y}_{k_l}(\mathbf{n}), a). \tag{4}$$

Proof. Since the group $\mathbb{Z}^m$ is countable, its elements can be enumerated and, hence, this group can be represented in the form $\mathbb{Z}^m = \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3, \ldots\}$.

Consider the following subsequences of the sequence $(\mathbf{y}_k)_{k\geq 1}$:

$$\mathbf{y}_{k_{1,1}}, \mathbf{y}_{k_{1,2}}, \ldots, \mathbf{y}_{k_{1,p}}, \ldots,$$

$$\mathbf{y}_{k_{2,1}}, \mathbf{y}_{k_{2,2}}, \ldots, \mathbf{y}_{k_{2,p}}, \ldots,$$

$$\vdots$$

$$\mathbf{y}_{k_{l,1}}, \mathbf{y}_{k_{l,2}}, \ldots, \mathbf{y}_{k_{l,p}}, \ldots,$$

$$\vdots .$$

It is assumed that these sequences are such that

1) each subsequent sequence is a subsequence of the previous sequence;

2) the next sequences are convergent:

$$\mathbf{y}_{k_{1,1}}(\mathbf{n}_1), \mathbf{y}_{k_{1,2}}(\mathbf{n}_1), \ldots, \mathbf{y}_{k_{1,p}}(\mathbf{n}_1), \ldots,$$

$$\mathbf{y}_{k_{2,1}}(\mathbf{n}_2), \mathbf{y}_{k_{2,2}}(\mathbf{n}_2), \ldots, \mathbf{y}_{k_{2,p}}(\mathbf{n}_2), \ldots,$$

$$\vdots$$

$$\mathbf{y}_{k_{l,1}}(\mathbf{n}_l), \mathbf{y}_{k_{l,2}}(\mathbf{n}_l), \ldots, \mathbf{y}_{k_{l,p}}(\mathbf{n}_l), \ldots,$$

$$\vdots .$$

The set of sequences with these characteristics is nonempty because the sets $\{\mathbf{y}_k(\mathbf{n}) : k \geq 1\}$, $\mathbf{n} \in \mathbb{Z}^m$, are precompact.

In view of the properties of the analyzed sequences, the diagonal sequence

$$\mathbf{y}_{k_{1,1}}(\mathbf{n}), \mathbf{y}_{k_{2,2}}(\mathbf{n}), \ldots, \mathbf{y}_{k_{l,l}}(\mathbf{n}), \ldots$$

is convergent for every $\mathbf{n} \in \mathbb{Z}^m$ and, therefore, there are bounds

$$\mathbf{z}(\mathbf{n}) = \lim_{l\to\infty} \mathbf{y}_{k_{l,l}}(\mathbf{n}), \quad \mathbf{n} \in \mathbb{Z}^m. \tag{5}$$

In view of (5) and the boundedness of the sequence $(\mathbf{y}_k)_{k\geq 1}$ (by the conditions of Lemma 1), relation (4) is valid and the function given by the inequalities

$$\mathbf{z} = \mathbf{z}(\mathbf{n}), \quad \mathbf{n} \in \mathbb{Z}^m,$$

is an element of the space $\mathfrak{M}$.

Lemma 1 is proved.

Some specific cases of Lemma 1 were studied in [5, 6] and other works of the author by using the theory of c-continuous operators.

Definition 3. *An operator* $\mathbf{H}\colon \mathfrak{M} \to \mathfrak{M}$ *is called* c*-continuous if, for every sequence* $\mathbf{y}_k \in \mathfrak{M}$, $k \geq 1$, *and* $\mathbf{y} \in \mathfrak{M}$ *such that* $\mathbf{y}_{k_l} \xrightarrow{\text{loc},\, \mathfrak{M}} \mathbf{y}$ *as* $l \to \infty$, *the sequence* $\mathbf{H}\mathbf{y}_k \in \mathfrak{M}$, $k \geq 1$, *is locally convergent to* $\mathbf{H}\mathbf{y}$ *as* $k \to \infty$.

The notion of c-continuous operators was introduced "in the $\varepsilon,\ \delta$" language by Muhamadiev [7]. The definition of these operators based on the use of locally convergent sequences was proposed in [3, 8].

Note that, for the operators acting in the space $\mathfrak{M}$, the property of c-continuity does not follow from continuity and, vice versa, the property of continuity does not follow from c-continuity [9, p. 17–19].

3. The Set $\mathfrak{P}$ of Periodic Elements of the Space $\mathfrak{M}$

In solving the problem of existence of bounded solutions to equation (3), we use local approximations of elements of the space $\mathfrak{M}$ by periodic elements of a subset $\mathfrak{P}$ of this space.

The set $\mathfrak{P}$ contains only elements

$$\mathbf{x} = \mathbf{x}(\mathbf{n}) = \mathbf{x}((n_1, n_2, \ldots, n_m))$$

of the space $\mathfrak{M}$ that are $(T_1, T_2, \ldots, T_m)$-*periodic*, i.e., such that each of these elements satisfy the relation

$$\mathbf{x}((n_1 + t_1, n_2 + t_2, \ldots, n_m + t_m)) = \mathbf{x}((n_1, n_2, \ldots, n_m)$$

for all $t_1 \in \{0, T_1\}$, $t_2 \in \{0, T_2\}$, $\ldots$, $t_m \in \{0, T_m\}$, and $\mathbf{n} \in \mathbb{Z}^m$. Note that, in this case, the natural numbers $T_1, T_2, \ldots, T_m$ are not fixed.

4. Pairs of Compact Sets Admissible for F with Respect to the Elements of the Set $\mathfrak{P}$

Let $R(\mathbf{x})$ be the set of values of the function $\mathbf{x} \in \mathfrak{M}$, i.e., the set

$$R(\mathbf{x}) = \big\{\mathbf{x}(\mathbf{n}) \in M : \mathbf{n} \in \mathbb{Z}^m\big\}.$$

Definition 4. *A pair* (K_1, K_2) *of compact sets* $K_1, K_2 \subset M$ *is called admissible for a mapping* $\mathbf{F}\colon \mathfrak{M} \to \mathfrak{M}$ *[or for equation (3)] with respect to the elements of* $\mathfrak{P}$ *if, for each element* $\mathbf{h} \in \mathfrak{P}$ *such that* $R(\mathbf{h}) \subset K_2$, *equation (3) has a solution* $\mathbf{x} \in \mathfrak{M}$ *(which can be not unique) for which* $R(\mathbf{x}) \subset K_1$.

5. Example of Equation with a Nonempty Set of Admissible Pairs of Compact Sets

Consider a Banach space E with the norm $\|\cdot\|_E$ as a metric space M and an equation

$$\begin{aligned} \mathbf{x}((n_1, n_2, \ldots, n_m)) - \frac{q}{m}\Big(&\mathbf{x}((n_1 - 1, n_2, \ldots, n_m)) \\ &+ \mathbf{x}((n_1, n_2 - 1, \ldots, n_m)) + \ldots + \mathbf{x}((n_1, n_2, \ldots, n_m - 1))\Big) \\ &= \mathbf{h}((n_1, n_2, \ldots, n_m)), \quad (n_1, n_2, \ldots, n_m) \in \mathbb{Z}^m, \end{aligned} \tag{6}$$

where $q \in (0,1)$ and $\mathbf{h} = \mathbf{h}((n_1, n_2, \ldots, n_m))$ is a function bounded on $\mathbb{Z}^m$ with values in E.

Clearly, this equation can be represented in the form

$$\mathbf{x}((n_1, n_2, \ldots, n_m)) - q(\mathfrak{A}\mathbf{x})((n_1, n_2, \ldots, n_m)) = \mathbf{h}((n_1, n_2, \ldots, n_m)),$$
$$(n_1, n_2, \ldots, n_m) \in \mathbb{Z}^m, \tag{7}$$

where $\mathfrak{A}\colon \mathfrak{M} \to \mathfrak{M}$ is a continuous linear operator:

$$(\mathfrak{A}\mathbf{x})((n_1, n_2, \ldots, n_m)) = \frac{1}{m}\big(\mathbf{x}((n_1 - 1, n_2, \ldots, n_m))$$
$$+ \mathbf{x}((n_1, n_2 - 1, \ldots, n_m)) + \ldots + \mathbf{x}((n_1, n_2, \ldots, n_m - 1))\big)$$

with the norm $\|\mathfrak{A}\|_{L(\mathfrak{M},\mathfrak{M})}$ equal to 1. Here, the space $\mathfrak{M}$ is considered as in the case $M = E$.

In view of the inclusion $q \in (0,1)$, the unique solution $\mathbf{x}((n_1, n_2, \ldots, n_m))$ of equation (7) can be represented in the form

$$\mathbf{x}((n_1, n_2, \ldots, n_m)) = \sum_{k \geq 0} q^k \left(\mathfrak{A}^k \mathbf{h}\right)((n_1, n_2, \ldots, n_m)), \quad (n_1, n_2, \ldots, n_m) \in \mathbb{Z}^m. \tag{8}$$

Let K be an arbitrary nonempty absolutely convex [11, p. 15] set compact in E and let

$$\mathbf{h}(\mathbf{n}) \in K$$

for all $\mathbf{n} \in \mathbb{Z}^m$. In view of (8), the equality

$$\|\mathfrak{A}\|_{L(\mathfrak{M},\mathfrak{M})} = 1,$$

and the absolute convexity of the set K, we get

$$\mathbf{x}(\mathbf{n}) \in (1-q)^{-1} K$$

for all $\mathbf{n} \in \mathbb{Z}^m$, where $(1-q)^{-1}K$, just as K, is also an absolutely convex set compact in E.

This implies that the pair $((1-q)^{-1}K, K)$ of absolutely convex and compact sets in E is admissible for equation (6) with respect to the elements of the set $\mathfrak{P}$ for each absolutely convex compact set K.

6. Conditions for the Existence of Solutions to Equation (3) with Precompact Range of Values

The following theorem is true:

Theorem 1. *Let:*

(i) the operator $\mathbf{F}\colon \mathfrak{M} \to \mathfrak{M}$ *in equation (3) be bounded and c-continuous;*

(ii) a pair (K_1, K_2) *of compact sets* $K_1, K_2 \subset \mathcal{M}$ *be admissible for equation (3).*

Then, for any element $\mathbf{h} \in \mathfrak{M}$ *such that* $R(\mathbf{h}) \subset K_2$, *equation (3) has at least one solution* $\mathbf{x} \in \mathfrak{M}$ *for which* $R(\mathbf{x}) \subset K_1$.

Proof. We fix an arbitrary element $\mathbf{h} \in \mathfrak{M}$ with the range of values $R(\mathbf{h})$ in K_2 and consider a sequence of elements $\mathbf{h}_k \in \mathfrak{P}$, $k \geq 1$, such that

$$\mathbf{h}_k \xrightarrow{\text{loc}, \mathfrak{M}} \mathbf{h} \quad \text{as} \quad k \to \infty. \tag{9}$$

By virtue of the inclusion $R(\mathbf{h}) \subset K_2$, the elements of $\mathbf{h}_k$, $k \geq 1$, can be chosen to guarantee that

$$R(\mathbf{h}_k) \subset K_2, \quad k \geq 1. \tag{10}$$

In view of (10), the periodicity of $\mathbf{h}_k$, and the admissibility of the pair (K_1, K_2) for $\mathbf{F}$, the difference equation

$$\mathbf{F}\mathbf{x}_k = \mathbf{h}_k \tag{11}$$

possesses a solution $\mathbf{x}_k$ in the space $\mathfrak{M}$ and, moreover,

$$R(\mathbf{x}_k) \subset K_1 \quad \text{for each} \quad k \geq 1.$$

By Lemma 2, there exists a locally convergent subsequence $(\mathbf{x}_{k_l})_{l \geq 1}$ of the sequence $(\mathbf{x}_k)_{k \geq 1}$ and an element $\mathbf{x}_*$ such that

$$\mathbf{x}_{k_l} \xrightarrow{\text{loc}, \mathfrak{M}} \mathbf{x}_* \quad \text{as} \quad l \to \infty \tag{12}$$

and

$$R(\mathbf{x}_*) \subset K_1.$$

We now show that

$$(\mathbf{F}\mathbf{x}_*)(\mathbf{n}) = \mathbf{h}(\mathbf{n}) \quad \text{for all} \quad \mathbf{n} \in \mathbb{Z}^m. \tag{13}$$

Applying the triangle axiom to elements of space M [10, p. 41], we conclude that, for each $\mathbf{n} \in \mathbb{Z}^m$, the following inequality is true:

$$\rho_M((\mathbf{F}\mathbf{x}_*)(\mathbf{n}), \mathbf{h}(\mathbf{n})) \leq \rho_M((\mathbf{F}\mathbf{x}_*)(\mathbf{n}), (\mathbf{F}\mathbf{x}_{k_l})(\mathbf{n})) + \rho_M((\mathbf{F}\mathbf{x}_{k_l})(\mathbf{n}), \mathbf{h}_{k_l}(\mathbf{n})) + \rho_M(\mathbf{h}_{k_l}(\mathbf{n}), \mathbf{h}(\mathbf{n})).$$

Note that, in view of (9), (11), and (12) and the c-continuity of the operator $\mathbf{F}$, for any $\mathbf{n} \in \mathbb{Z}^m$, we get

$$\lim_{l \to \infty} \big(\rho_M((\mathbf{F}\mathbf{x}_*)(\mathbf{n}), (\mathbf{F}\mathbf{x}_{k_l})(\mathbf{n})) + \rho_M((\mathbf{F}\mathbf{x}_{k_l})(\mathbf{n}), \mathbf{h}_{k_l}(\mathbf{n})) + \rho_M(\mathbf{h}_{k_l}(\mathbf{n}), \mathbf{h}(\mathbf{n}))\big) = 0.$$

Hence,

$$\rho_M((\mathbf{F}\mathbf{x}_*)(\mathbf{n}), \mathbf{h}(\mathbf{n})) = 0 \quad \text{for all} \quad \mathbf{n} \in \mathbb{Z}^m,$$

i.e., relation (13) is valid and, hence, $\mathbf{x}_*$ is a solution of equation (3).

Theorem 1 is proved.

Corollary 1. *Suppose that, for an element* $\mathbf{h} \in \mathfrak{M}$ *with precompact range of values* $R(\mathbf{h})$, *there exists a set* K *compact in* M *for which the pair* $(K, \overline{R(\mathbf{h})})$ *is admissible for equation (3) with respect to the elements of the set* $\mathfrak{P}$.

Then equation (3) possesses at least one solution $\mathbf{x} \in \mathfrak{M}$ *such that* $R(\mathbf{x}) \subset K$.

Note that, in Theorem 1, the requirement of precompactness of the range of values of the function $\mathbf{h} \in \mathfrak{M}$ on the right-hand side of equation (3) is essential. This requirement is satisfied if, e.g., $\mathbf{h}$ is an almost periodic element of the space $\mathfrak{M}$ or an element of the set $\mathfrak{P}$. The cases of almost periodic discrete and, in particular, difference equations were investigated in [12–17].

7. Additional Remarks

1. Lemma 1 on the existence of a locally convergent sequence of elements of the metric space $\mathfrak{M}$ defined and bounded on the group $\mathbb{Z}^m$ is presented for the first time.

2. The concept of admissible pair of compact sets for the discrete equation (3) with respect to the set $\mathfrak{P}$ is introduced for the first time. This concept does not coincide with the notion of admissibility of a pair of Banach function spaces for differential equations, which was considered in [18].

3. The set of admissible pairs of compact sets for equation (3) is nonempty, as shown in Section 5.

The author states that there is no conflict of interest.

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