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ON THE SYMMETRY REDUCTION OF THE (1+3)-DIMENSIONAL INHOMOGENEOUS MONGE–AMPÈRE EQUATION TO ALGEBRAIC EQUATIONS

V. M. Fedorchuk[1,2] **and V. I. Fedorchuk**[1]

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We perform the procedure of symmetry reduction of (1+3)-dimensional inhomogeneous Monge–Ampère equation to algebraic equations. Some results obtained with the use of the classification of three-dimensional nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$ are presented.

Keywords: symmetry reduction, inhomogeneous Monge–Ampère equation, classification of the Lie algebras, nonconjugate subalgebras of the Lie algebras, Poincaré group $P(1,4)$.

Differential equations serve as one of the main tools in the construction of mathematical models of the processes running in the surrounding world. In numerous cases, the constructed differential equations have nontrivial symmetry. To study these equations, we can use, in particular, the classical Lie–Ovsyannikov method [1, 17]. With the help of this approach, it is possible, in particular, to perform symmetry reduction and construct the classes of invariant solutions of the investigated equations (see [1, 9, 10, 13, 17, 18, 19] and the references therein).

For the classification of symmetry reductions and invariant solutions of differential equations with nontrivial symmetry, the authors proposed [11] to use the structural properties of low-dimensional nonconjugate subalgebras of the same rank as the Lie algebras of the symmetry groups of the investigated equations.

In solving various problems of geometry, geometric analysis, string theory, cosmology, geometric optics, optimal transfer, one-dimensional gas dynamics, meteorology, and oceanography, we obtain the so-called Monge–Ampère equations in spaces of different dimensions and different types. At present, there is a significant number of available works devoted to the investigation of equations of this kind, in particular [2, 7, 8, 14–16, 20–23] (see also the references therein).

The present work is devoted to the study of relationships between the structural properties of the three-dimensional nonconjugate subalgebras [3] of the Lie algebra of the group $P(1,4)$, the types of symmetry reductions, and the invariant solutions of the (1+3)-dimensional inhomogeneous Monge–Ampère equation.

As a result of the symmetry reduction of the investigated equation, we obtain the following reduced equations:

- algebraic equations;
- first-order linear ordinary differential equations (ODE);
- nonlinear ODE of the first order;

[1] Pidstryhach Institute for Applied Problems in Mechanics and Mathematics, National Academy of Sciences of Ukraine, Lviv, Ukraine.
[2] Corresponding author; e-mail: vasfed@gmail.com.

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– nonlinear ODE of the second order;

– partial differential equations.

The results concerning the symmetry reduction of the (1+3)-dimensional inhomogeneous Monge–Ampère equation to the first-order ODE and its invariant solutions can be found in [4, 12].

In the present work, we give only our results concerning the symmetry reduction of the investigated equation to algebraic equations. For this purpose, we first consider some results obtained for the Lie algebra of the group $P(1,4)$ and its nonconjugate subalgebras.

1. Lie Algebra of the Group $P(1,4)$ and Its Nonconjugate Subalgebras

The Poincaré group $P(1,4)$ is a group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. Among the groups important for theoretical and mathematical physics, a special place is occupied by the group $P(1,4)$. This is the smallest group that contains, as subgroups, both the symmetry group of relativistic physics (Poincaré group $P(1,3)$) and the symmetry group of nonrelativistic physics (extended Galilean group $\tilde{G}(1,3)$ [5]).

The Lie algebra of the group $P(1,4)$ is specified by 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}$, $\mu,\nu = 0,1,2,3,4$, and P_μ, $\mu = 0,1,2,3,4$, satisfying the following commutation relations:

$$[P_\mu, P_\nu] = 0,$$

$$[M_{\mu\nu}, P_\sigma] = g_{\nu\sigma} P_\mu - g_{\mu\sigma} P_\nu,$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = g_{\mu\sigma} M_{\nu\rho} + g_{\nu\rho} M_{\mu\sigma} - g_{\mu\rho} M_{\nu\sigma} - g_{\nu\sigma} M_{\mu\rho},$$

where $g_{\mu\nu}$, $\mu,\nu = 0,1,2,3,4$, is a metric tensor with the following components:

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$$

and $g_{\mu\nu} = 0$ if $\mu \neq \nu$.

In the present work, we consider the following representation for the Lie algebra of the group $P(1,4)$ [6]:

$$P_0 = \frac{\partial}{\partial x_0}, \qquad P_1 = -\frac{\partial}{\partial x_1}, \qquad P_2 = -\frac{\partial}{\partial x_2},$$

$$P_3 = -\frac{\partial}{\partial x_3}, \qquad P_4 = -\frac{\partial}{\partial u}, \qquad M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu, \qquad x_4 \equiv u.$$

Further, we pass from $M_{\mu\nu}$ and P_μ to the following linear combinations:

$$G = M_{04}, \qquad L_1 = M_{23}, \qquad L_2 = -M_{13}, \qquad L_3 = M_{12},$$

$$P_a = M_{a4} - M_{0a}, \quad C_a = M_{a4} + M_{0a}, \quad a = 1,2,3,$$

$$X_0 = \frac{P_0 - P_4}{2}, \quad X_k = P_k, \quad k = 1,2,3, \quad X_4 = \frac{P_0 + P_4}{2}.$$

The classification of all nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ (whose dimensions do not exceed three) into the classes of isomorphic subalgebras was performed in [3]. As a result of the performed classification, it was established that there exist three-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ of the following types: $3A_1$, $A_2 \oplus A_1$, $A_{3,1}$, $A_{3,2}$, $A_{3,3}$, $A_{3,4}$, $A_{3,6}$, $A_{3,7}^a$, $A_{3,8}$, and $A_{3,9}$.

2. On the Symmetry Reduction of the (1+3)-Dimensional Inhomogeneous Monge–Ampère Equation to the Algebraic Equations

In the present work, we consider an inhomogeneous Monge–Ampère equation of the form

$$\det(u_{\mu\nu}) = \lambda(1 - u_\nu u^\nu)^3, \quad \lambda \neq 0, \tag{1}$$

where

$$u = u(x), \quad x = (x_0, x_1, x_2, x_3) \in M(1,3),$$

$$u_{\mu\nu} = \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}, \quad u^\nu = g^{\nu\alpha} u_\alpha, \quad u_\alpha \equiv \frac{\partial u}{\partial x_\alpha},$$

$$g_{\mu\nu} = (1,-1,-1,-1)\delta_{\mu\nu}, \quad \mu,\nu,\alpha = 0,1,2,3,$$

and $M(1,3)$ is a (1+3)-dimensional Minkowski space.

In 1983, Fushchych and Serov [6] studied the symmetry and constructed multiparameter families of exact solutions to the multidimensional Monge–Ampère equation. It follows from the results obtained in [6] that, in particular, the investigated inhomogeneous equation (1) is invariant under the group $P(1,4)$.

In what follows, we present our results concerning the symmetry reduction of the investigated equation to algebraic equations.

Subalgebras of the Type $3A_1$.

1. $\langle P_1 - \gamma X_3, \gamma > 0\rangle \oplus \langle P_2 - X_2 - \delta X_3, \delta \neq 0\rangle \oplus \langle X_4 \rangle$:

The ansatz

$$x_3(x_0 + u)^2 - (\gamma x_1 + \delta x_2 - x_3)(x_0 + u) - \gamma x_1 = \varphi(\omega), \quad \omega = x_0 + u.$$

The reduced equation

$$\omega^4 + 2\omega^3 + (\delta^2 + \gamma^2 + 1)\omega^2 + 2\gamma^2\omega + \gamma^2 = 0 .$$

The solution of the (1+3)-dimensional inhomogeneous Monge–Ampère equation is given by

$$(x_0 + u)^4 + 2(x_0 + u)^3 + (\delta^2 + \gamma^2 + 1)(x_0 + u)^2 + 2\gamma^2 (x_0 + u) + \gamma^2 = 0 .$$

2. $\langle P_1 - \gamma X_3, \gamma > 0 \rangle \oplus \langle P_2 - X_2 \rangle \oplus \langle X_4 \rangle$:

The ansatz

$$x_3 (x_0 + u)^2 - (\gamma x_1 - x_3)(x_0 + u) - \gamma x_1 = \varphi(\omega), \qquad \omega = x_0 + u .$$

The reduced equation

$$(\omega + 1)(\gamma^2 + \omega^2) = 0 .$$

The solutions of the reduced equation are

$$\omega + 1 = 0, \qquad \gamma^2 + \omega^2 = 0 .$$

The solutions of the (1+3)-dimensional inhomogeneous Monge–Ampère equation are

$$x_0 + u + 1 = 0, \qquad (x_0 + u)^2 + \gamma^2 = 0 .$$

3. $\langle P_1 \rangle \oplus \langle P_2 - X_2 - \delta X_3, \delta > 0 \rangle \oplus \langle X_4 \rangle$:

The ansatz

$$x_3 (x_0 + u) - \delta x_2 + x_3 = \varphi(\omega), \qquad \omega = x_0 + u .$$

The reduced equation

$$(\omega + 1)^2 + \delta^2 = 0 .$$

The solution of the reduced (1+3)-dimensional inhomogeneous Monge–Ampère equation

$$(x_0 + u + 1)^2 + \delta^2 = 0 .$$

Note that the left-hand sides of ansatzes 1, 2, and 3 are polynomials in the invariant $\omega = x_0 + u$.

4. $\langle P_1 - X_3 \rangle \oplus \langle P_2 \rangle \oplus \langle X_4 \rangle$:

The ansatz

$$x_3 - \frac{x_1}{x_0 + u} = \varphi(\omega), \qquad \omega = x_0 + u\,.$$

The reduced equation

$$\omega(\omega^2 + 1) = 0\,.$$

The solutions of the reduced equations are

$$\omega = 0, \qquad \omega^2 + 1 = 0\,.$$

The solutions of the (1+3)-dimensional inhomogeneous Monge–Ampère equation are

$$x_0 + u = 0, \qquad (x_0 + u)^2 + 1 = 0\,.$$

5. $\langle P_3 - X_2 \rangle \oplus \langle X_1 \rangle \oplus \langle X_4 \rangle$:

The ansatz

$$x_2 - \frac{x_3}{x_0 + u} = \varphi(\omega), \qquad \omega = x_0 + u\,.$$

The reduced equation

$$\omega(\omega^2 + 1) = 0\,.$$

The solutions of the reduced equations are

$$\omega = 0, \qquad \omega^2 + 1 = 0\,.$$

The solutions of the (1+3)-dimensional inhomogeneous Monge–Ampère equation is

$$x_0 + u = 0, \qquad (x_0 + u)^2 + 1 = 0\,.$$

Subalgebras of the Type $A_{3,1}$.

1. $\langle 4X_4, P_1 - X_2 - \gamma X_3, P_2 + X_1 - \mu X_2 - \delta X_3, \gamma > 0, \delta \neq 0, \mu > 0 \rangle$:

The ansatz

$$x_3(x_0+u)^2-(\gamma x_1+\delta x_2-\mu x_3)(x_0+u)+(\delta-\gamma\mu)x_1-\gamma x_2+x_3 = \varphi(\omega),$$

$$\omega = x_0+u.$$

The reduced equation

$$\omega^4+2\mu\omega^3+(\delta^2+\gamma^2+\mu^2+2)\omega^2+2\mu(\gamma^2+1)\omega+(\delta-\gamma\mu)^2+\gamma^2+1 = 0.$$

The solution of the (1+3)-dimensional inhomogeneous Monge–Ampère equation

$$(x_0+u)^4+2\mu(x_0+u)^3+(\delta^2+\gamma^2+\mu^2+2)(x_0+u)^2$$

$$+2\mu(\gamma^2+1)(x_0+u)+(\delta-\gamma\mu)^2+\gamma^2+1 = 0.$$

2. $\langle 4X_4, P_1-X_2-\gamma X_3, P_2+X_1-\mu X_2, \gamma>0, \mu>0\rangle$:

The ansatz

$$x_3(x_0+u)^2-(\gamma x_1-\mu x_3)(x_0+u)-\gamma\mu x_1-\gamma x_2+x_3 = \varphi(\omega),$$

$$\omega = x_0+u.$$

The reduced equation

$$\omega^4+2\mu\omega^3+(\gamma^2+\mu^2+2)\omega^2+2\mu(\gamma^2+1)\omega+\gamma^2(\mu^2+1)+1 = 0.$$

The solution of the (1+3)-dimensional inhomogeneous Monge–Ampère equation

$$(x_0+u)^4+2\mu(x_0+u)^3+(\gamma^2+\mu^2+2)(x_0+u)^2$$

$$+2\mu(\gamma^2+1)(x_0+u)+\gamma^2(\mu^2+1)+1 = 0.$$

3. $\langle 4X_4, P_1-X_2, P_2+X_1-\mu X_2-\delta X_3, \delta>0, \mu\neq 0\rangle$:

The ansatz

$$x_3(x_0+u)^2-(\delta x_2-\mu x_3)(x_0+u)+\delta x_1+x_3 = \varphi(\omega),$$

$$\omega = x_0+u.$$

The reduced equation

$$\omega^4 + 2\mu\omega^3 + (\delta^2 + \mu^2 + 2)\omega^2 + 2\mu\omega + \delta^2 + 1 = 0.$$

The solution of the (1+3)-dimensional inhomogeneous Monge–Ampère equation

$$(x_0 + u)^4 + 2\mu(x_0 + u)^3 + (\delta^2 + \mu^2 + 2)(x_0 + u)^2 + 2\mu(x_0 + u) + \delta^2 + 1 = 0.$$

4. $\langle 4X_4, P_1 - X_2, P_2 + X_1 - \delta X_3, \delta > 0\rangle$:

The ansatz

$$x_3(x_0 + u)^2 - \delta x_2(x_0 + u) + \delta x_1 + x_3 = \varphi(\omega),$$

$$\omega = x_0 + u.$$

The reduced equation

$$(\omega^2 + 1)(\omega^2 + \delta^2 + 1) = 0.$$

The solutions of the reduced equation are

$$\omega^2 + 1 = 0, \qquad \omega^2 + \delta^2 + 1 = 0.$$

The solutions of the (1+3)-dimensional inhomogeneous Monge–Ampère equation are

$$(x_0 + u)^2 + 1 = 0, \qquad (x_0 + u)^2 + \delta^2 + 1 = 0.$$

5. $\langle 4X_4, P_1 - X_2 - \beta X_3, P_2 + X_1, \beta > 0\rangle$:

The ansatz

$$x_3(x_0 + u)^2 - \beta x_1(x_0 + u) - \beta x_2 + x_3 = \varphi(\omega),$$

$$\omega = x_0 + u.$$

The reduced equation

$$(\omega^2 + 1)(\omega^2 + \beta^2 + 1) = 0.$$

The solutions of the reduced equation are

$$\omega^2 + 1 = 0, \qquad \omega^2 + \beta^2 + 1 = 0.$$

The solutions of the (1+3)-dimensional inhomogeneous Monge–Ampère equation are

$$(x_0 + u)^2 + 1 = 0, \qquad (x_0 + u)^2 + \beta^2 + 1 = 0.$$

6. $\langle 4X_4, P_1 - X_2, P_2 + X_1 - \mu X_2, \mu \neq 0 \rangle$:

The ansatz

$$x_3(x_0 + u)^2 + \mu x_3(x_0 + u) + x_3 = \varphi(\omega),$$

$$\omega = x_0 + u.$$

The reduced equation

$$\omega^2 + \mu\omega + 1 = 0.$$

The solution of the (1+3)-dimensional inhomogeneous Monge–Ampère equation

$$(x_0 + u)^2 + \mu(x_0 + u) + 1 = 0.$$

Note that the left-hand sides of ansatzes 1, 2, …, 6 are polynomials in the invariant

$$\omega = x_0 + u.$$

7. $\langle 2\mu X_4, P_3 - X_2, X_1 + \mu X_3, \mu > 0 \rangle$:

The ansatz

$$x_2 - \frac{x_3 - \mu x_1}{x_0 + u} = \varphi(\omega), \qquad \omega = x_0 + u.$$

The reduced equation

$$\omega(\omega^2 + \mu^2 + 1) = 0.$$

The solutions of the reduced equation are

$$\omega = 0, \qquad \omega^2 + \mu^2 + 1 = 0.$$

The solutions of the (1+3)-dimensional inhomogeneous Monge–Ampère equation are

$$x_0 + u = 0, \qquad (x_0 + u)^2 + \mu^2 + 1 = 0.$$

CONCLUSIONS

We establish the relationship between the types of three-dimensional nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$ and the symmetry reductions to the algebraic equations for the (1+3)-dimensional inhomogeneous Monge–Ampère equation. We also present some invariant solutions of the investigated equation.

As indicated above, there exist three-dimensional nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$ of the following types [3]: $3A_1$, $A_2 \oplus A_1$, $A_{3,1}$, $A_{3,2}$, $A_{3,3}$, $A_{3,4}$, $A_{3,6}$, $A_{3,7}^a$, $A_{3,8}$, and $A_{3,9}$.

For the (1+3)-dimensional inhomogeneous Monge–Ampère equation, we obtain reductions to the algebraic equations for some nonconjugate subalgebras of the following two types: $3A_1$ and $A_{3,1}$.

The constructed solutions of the (1+3)-dimensional inhomogeneous Monge–Ampère equation are polynomials of the first, second, and fourth degrees with respect to the invariant $x_0 + u$.

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