Journal of Mathematical Sciences
https://doi.org/10.1007/s10958-024-07178-x

INTEGRAL OPERATORS WITH HOMOGENEOUS KERNELS OF DEGREE α IN LEBESGUE SPACES

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Accepted: 12 June 2024

Abstract
We consider multidimensional integral operators with homogeneous kernels of degree α. We obtain the conditions for α, which are necessary for the boundedness of these operators in Lebesgue spaces. Using the conditions for α, we establish sufficient conditions for the boundedness of integral operators with homogeneous kernels in Lebesgue spaces. In addition, concrete examples of operators are considered in this paper.

Keywords Integral operator · Homogeneous kernel · Rotation group · Boundedness

Introduction

At the present time, there are many papers dealing with multidimensional integral operators with homogeneous kernels of degree $(-n)$. The investigation of such operators was started by L. G. Mikhailov in connection with studying of elliptic differential equations with singular coefficients (see [1]–[2]). Further, the study of integral operators with homogeneous kernels was continued by N. K. Karapetyants, S. G. Samko, O. G. Avsyankin, V. M. Deundyak and other authors (e.g., see [3]–[11] and the bibliography therein). For operators, whose kernels are homogeneous of degree $(-n)$ and invariant with respect to the rotation group $SO(n)$, criteria for invertibility and the Fredholm property were obtained, the Banach algebras generated by these operators were studied, and the conditions for the projection method to apply were found. We especially note the articles [3, 5], and [10], in which the boundedness of operators with homogeneous kernels in various spaces was investigated.
The present paper is devoted to multidimensional integral operators whose kernels are homogeneous of degree α, where $\alpha \neq n$. We consider these operators either in the space $L_p(\mathbb{B}_n)$ or in the space $L_p(\mathbb{R}^n \setminus \mathbb{B}_n)$, where $\mathbb{B}_n$ is the unit ball in $\mathbb{R}^n$. We obtain the conditions for the homogeneity degree α, which are necessary for the boundedness of the integral operator in these spaces. Taking this into account, we establish sufficient conditions for the boundedness of integral operators with homogeneous kernels in these spaces. The case of kernels, which are invariant with respect to the rotation group $SO(n)$,

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Published online: 12 July 2024

is considered separately. We also give application our results to various concrete examples of operators with homogeneous kernels.

We use the following notation:

• $\mathbb{R}^n$ is the n-dimensional Euclidean space; $x = (x_1, \dots, x_n) \in \mathbb{R}^n$;
• $|x| = \sqrt{x_1^2 + \dots + x_n^2}$; $x' = x/|x|$; $x \cdot y = x_1 y_1 + \dots + x_n y_n$;
• $\mathbb{B}_n = \{x \in \mathbb{R}^n : |x| \leqslant 1\}$; $C\mathbb{B}_n = \mathbb{R}^n \setminus \mathbb{B}_n$;
• $\mathbb{S}_{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$;
• $\mathbb{R}_+ = (0, \infty)$.
• Let $1 \leqslant p \leqslant \infty$, and $D \subseteq \mathbb{R}^n$ be a measurable set. Then, $L_p(D)$ is the space of (classes of) measurable complex-valued functions with norm

$$\|f\|_{L_p(D)} = \left(\int\limits_D |f(x)|^p \, dx \right)^{1/p}, \ 1 \leqslant p < \infty; \quad \|f\|_{L_\infty(D)} = \operatorname*{ess\,sup}_{x \in D} |f(x)|.$$

The necessity of the condition $\alpha \geqslant -n$ (or $\alpha \leqslant -n$)

Let $1 \leqslant p \leqslant \infty$. In the space $L_p(\mathbb{B}_n)$, consider the integral operator

$$(K\varphi)(x) = \int\limits_{\mathbb{B}_n} k(x, y)\varphi(y) \, dy, \tag{1}$$

where the function $k(x, y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ (here and below, it is assumed that $n \geqslant 2$) is measurable and homogeneous of degree α ($\alpha \in \mathbb{R}$), i.e.,

$$k(\lambda x, \lambda y) = \lambda^\alpha k(x, y) \quad \forall \lambda > 0. \tag{2}$$

The purpose of this section is to find a condition for α, which is necessary for the boundedness of the operator K in the space $L_p(\mathbb{B}_n)$. To achieve it, define the operator U_δ, where $\delta > 0$, as follows:

$$(U_\delta \varphi)(x) = \begin{cases} \delta^{-n/p} \varphi(x/\delta), & |x| \leqslant \delta, \\ 0, & |x| > \delta, \end{cases}$$

if $0 < \delta < 1$, and

$$(U_\delta \varphi)(x) = \delta^{-n/p} \varphi(x/\delta),$$

if $\delta > 1$. Note that for $0 < \delta < 1$ the operator U_δ is left invertible, and the left inverse operator U_δ^{-1} is defined by the equation $U_\delta^{-1} = U_{\delta^{-1}}$. It is known (see [6]) that $\|U_\delta\| = 1$.

Lemma 1 *If the function $k(x, y)$ is homogeneous of degree α and the operator K is bounded in the space $L_p(\mathbb{B}_n)$, then $\alpha \geqslant -n$.*

Proof Let $0 < \delta < 1$. Taking into account the property (2), we have

$$(U_\delta^{-1} K U_\delta \varphi)(x) = \int\limits_{|y| \leqslant \delta} k(\delta x, y)\varphi(y/\delta) \, dy$$

$$= \delta^{n+\alpha} \int\limits_{|y|\leqslant 1} k(x,t)\varphi(t)\,dt = \delta^{n+\alpha}(K\varphi)(x).$$

Since $K\varphi = \delta^{-(n+\alpha)}U_\delta^{-1}KU_\delta\varphi$ and $\|U_\delta\| = 1$, then the inequality

$$\|K\varphi\|_{L_p(\mathbb{B}_n)} \leqslant \delta^{-(n+\alpha)}\|K\|\|\varphi\|_{L_p(\mathbb{B}_n)}$$

is valid. Assume that $n+\alpha < 0$. Then, letting δ tend to zero, we obtain that $\|K\varphi\|_{L_p(\mathbb{B}_n)} = 0$ for any function $\varphi \in L_p(\mathbb{B}_n)$. Hence, the operator K is the null operator. This contradiction leads to the inequality $n+\alpha \geqslant 0$.

In the space $L_p(C\mathbb{B}_n)$, we consider the operator

$$(\mathcal{K}\psi)(x) = \int\limits_{C\mathbb{B}_n} k(x,y)\psi(y)\,dy, \tag{3}$$

where the function $k(x,y)$ is defined on $\mathbb{R}^n \times \mathbb{R}^n$ and satisfies the condition (2). Define the operator $\mathcal{U}_\delta$, $\delta > 0$, in $L_p(C\mathbb{B}_n)$ by the formulas

$$(\mathcal{U}_\delta\psi)(x) = \begin{cases} \delta^{-n/p}\psi(x/\delta), & |x| \geqslant \delta, \\ 0, & |x| < \delta, \end{cases}$$

if $\delta > 1$, and

$$(\mathcal{U}_\delta\psi)(x) = \delta^{-n/p}\psi(x/\delta),$$

if $0 < \delta < 1$. Note that for $\delta > 1$ the operator $\mathcal{U}_\delta$ is left invertible, and the left inverse operator $\mathcal{U}_\delta^{-1}$ is given by the equation $\mathcal{U}_\delta^{-1} = \mathcal{U}_{\delta^{-1}}$.

Lemma 2 *If the function $k(x,y)$ is homogeneous of degree α and the operator $\mathcal{K}$ is bounded in the space $L_p(C\mathbb{B}_n)$, then $\alpha \leqslant -n$.*

Proof Let $\delta > 1$. Using the change of variables $y = \delta t$ and taking into account (2), we obtain

$$(\mathcal{U}_\delta^{-1}\mathcal{K}\mathcal{U}_\delta\psi)(x) = \int\limits_{|y|\geqslant\delta} k(\delta x,y)\psi(y/\delta)\,dy = \delta^{n+\alpha}(\mathcal{K}\psi)(x).$$

From here it follows that

$$\|\mathcal{K}\psi\|_{L_p(C\mathbb{B}_n)} \leqslant \delta^{-(n+\alpha)}\|\mathcal{K}\|\|\psi\|_{L_p(C\mathbb{B}_n)}.$$

If $n+\alpha > 0$, then letting δ tend to infinity, we obtain that $\|\mathcal{K}\psi\|_{L_p(C\mathbb{B}_n)} = 0$ for any function $\psi \in L_p(C\mathbb{B}_n)$. Therefore, $\mathcal{K}$ is a null operator, which is impossible. Hence, $n+\alpha \leqslant 0$.

Remark 1 In the space $L_p(\mathbb{R}^n)$, consider the operator

$$(\mathbf{K}\varphi)(x) = \int\limits_{\mathbb{R}^n} k(x,y)\varphi(y)\,dy,$$

where $k(x,y)$ satisfies the condition (2). It follows from Lemma 1 and Lemma 2 that if $\mathbf{K}$ is bounded, then it is necessary that $\alpha = -n$.

Sufficient conditions for boundedness

In the space $L_p(\mathbb{B}_n)$, $1 \leqslant p \leqslant \infty$, consider the operator K defined by the formula (1). The aim of this section is to find sufficient conditions for the boundedness of the operator K in $L_p(\mathbb{B}_n)$.

Theorem 1 *Let the function $k(x, y)$ be homogeneous of degree α, where $\alpha \geqslant -n$, and satisfy the following conditions*

$$\varkappa_1 = \operatorname*{ess\,sup}_{\sigma\in\mathbb{S}_{n-1}} \int\limits_{\mathbb{R}^n} |k(\sigma, t)||t|^{-n/p}\, dt < \infty, \tag{4}$$

$$\varkappa_2 = \operatorname*{ess\,sup}_{\sigma\in\mathbb{S}_{n-1}} \int\limits_{\mathbb{R}^n} |k(t, \sigma)||t|^{-n/p'-\alpha-n}\, dt < \infty, \tag{5}$$

where $p' = p/(p-1)$. Then, the operator K of the form (1) is bounded in the space $L_p(\mathbb{B}_n)$, $1 \leqslant p \leqslant \infty$, and

$$\|K\varphi\|_{L_p(\mathbb{B}_n)} \leqslant \varkappa_1^{1/p'} \varkappa_2^{1/p} \|\varphi\|_{L_p(\mathbb{B}_n)}. \tag{6}$$

Proof Let us consider two cases.

1) Let $1 \leqslant p < \infty$. Applying the Holder's inequality, we obtain

$$\begin{aligned}
|(K\varphi)(x)| &\leqslant \int\limits_{|y|\leqslant 1} |k(x, y)||\varphi(y)|\, dy \\
&= \int\limits_{|y|\leqslant 1} \big(|k(x, y)|^{1/p'}|y|^{-n/pp'}\big)\big(|k(x, y)|^{1/p}|y|^{n/pp'}|\varphi(y)|\big)\, dy \\
&\leqslant \Bigg(\int\limits_{|y|\leqslant 1} |k(x, y)||y|^{-n/p}\, dy\Bigg)^{1/p'} \Bigg(\int\limits_{|y|\leqslant 1} |k(x, y)||y|^{n/p'}|\varphi(y)|^p\, dy\Bigg)^{1/p}.
\end{aligned}$$

Let us transform the first integral. Using the change of variables $y = |x|t$ and taking into account (2), we have

$$\begin{aligned}
\int\limits_{|y|\leqslant 1} |k(x, y)||y|^{-n/p}\, dy &= |x|^{-n/p+\alpha+n} \int\limits_{|t|\leqslant \frac{1}{|x|}} |k(x', t)||t|^{-n/p}\, dt \\
&\leqslant |x|^{-n/p+\alpha+n} \int\limits_{\mathbb{R}^n} |k(x', t)||t|^{-n/p}\, dt \leqslant \varkappa_1 |x|^{-n/p+\alpha+n}.
\end{aligned}$$

Thus, we have the inequality

$$|(K\varphi)(x)| \leqslant \varkappa_1^{1/p'} |x|^{-n/(pp')+(\alpha+n)/p'} \Bigg(\int\limits_{|y|\leqslant 1} |k(x, y)||y|^{n/p'}|\varphi(y)|^p\, dy\Bigg)^{1/p}.$$

Hence,

$$\|K\varphi\|^p_{L_p(\mathbb{B}_n)} \leqslant \varkappa_1^{p/p'} \int\limits_{|x|\leqslant 1} |x|^{-n/p'+(\alpha+n)p/p'}\,dx \int\limits_{|y|\leqslant 1} |k(x,y)||y|^{n/p'}|\varphi(y)|^p\,dy$$
$$= \varkappa_1^{p/p'} \int\limits_{|y|\leqslant 1} |\varphi(y)|^p|y|^{n/p'}\,dy \int\limits_{|x|\leqslant 1} |k(x,y)||x|^{-n/p'+(\alpha+n)(p-1)}\,dx.$$

In the inner integral, we make the change of variables $x = |y|t$ and use the condition (2). As a result, we obtain

$$\|K\varphi\|^p_{L_p(\mathbb{B}_n)} \leqslant \varkappa_1^{p/p'} \int\limits_{|y|\leqslant 1} |\varphi(y)|^p|y|^{(\alpha+n)p}\,dy \int\limits_{|t|\leqslant \frac{1}{|y|}} |k(t,y')||t|^{-n/p'+(\alpha+n)(p-1)}\,dt.$$

Since $\alpha + n \geqslant 0$, then $|t|^{(\alpha+n)p} \leqslant |y|^{-(\alpha+n)p}$. Thus,

$$\|K\varphi\|^p_{L_p(\mathbb{B}_n)} \leqslant \varkappa_1^{p/p'} \int\limits_{|y|\leqslant 1} |\varphi(y)|^p\,dy \int\limits_{|t|\leqslant \frac{1}{|y|}} |k(t,y')||t|^{-n/p'-\alpha-n}\,dt$$
$$\leqslant \varkappa_1^{p/p'} \int\limits_{|y|\leqslant 1} |\varphi(y)|^p\,dy \int\limits_{\mathbb{R}^n} |k(t,y')||t|^{-n/p'-\alpha-n}\,dt$$
$$\leqslant \varkappa_1^{p/p'} \int\limits_{|y|\leqslant 1} |\varphi(y)|^p \left(\operatorname*{ess\,sup}_{y'\in\mathbb{S}_{n-1}} \int\limits_{\mathbb{R}^n} |k(t,y')||t|^{-n/p'-\alpha-n}\,dt \right) dy$$
$$= \varkappa_1^{p/p'}\varkappa_2 \int\limits_{|y|\leqslant 1} |\varphi(y)|^p\,dy \leqslant \varkappa_1^{p/p'}\varkappa_2 \|\varphi\|^p_{L_p(\mathbb{B}_n)}.$$

From here it follows inequality (6).

2) Let $p = \infty$. Then, we have

$$|(K\varphi)(x)| \leqslant \int\limits_{|y|\leqslant 1} |k(x,y)||\varphi(y)|\,dy \leqslant \|\varphi\|_{L_\infty(\mathbb{B}_n)} \int\limits_{|y|\leqslant 1} |k(x,y)|\,dy.$$

Using the change of variables $y = |x|t$ and the condition (2), we obtain

$$|(K\varphi)(x)| \leqslant \|\varphi\|_{L_\infty(\mathbb{B}_n)}|x|^{n+\alpha} \int\limits_{\mathbb{R}^n} |k(x',t)|\,dt.$$

Since $n + \alpha \geqslant 0$, then $|x|^{n+\alpha} \leqslant 1$ for all $x \in \mathbb{B}_n$. Thus,

$$|(K\varphi)(x)| \leqslant \|\varphi\|_{L_\infty(\mathbb{B}_n)} \operatorname*{ess\,sup}_{x'\in\mathbb{S}_{n-1}} \int\limits_{\mathbb{R}^n} |k(x',t)|\,dt \leqslant \varkappa_1 \|\varphi\|_{L_\infty(\mathbb{B}_n)}$$

for almost all $x \in \mathbb{B}_n$. Therefore,

$$\|K\varphi\|_{L_\infty(\mathbb{B}_n)} \leqslant \varkappa_1 \|\varphi\|_{L_\infty(\mathbb{B}_n)}. \tag{7}$$

The proof is complete.

Remark 2 For $p = 1$, it suffices to have only one condition (5). More precisely, if only condition (5) is satisfied, where $p = 1$, then the operator K is bounded in the space $L_1(\mathbb{B}_n)$ and

$$\|K\varphi\|_{L_1(\mathbb{B}_n)} \leqslant \varkappa_2 \|\varphi\|_{L_1(\mathbb{B}_n)}.$$

Similarly, if only condition (4) is satisfied, where $p = \infty$, then the operator K is bounded in $L_\infty(\mathbb{B}_n)$ and the inequality (7) holds.

Remark 3 For $\alpha = -n$ conditions (4) and (5) coincide with the well-known conditions of N.K. Karapetyants (see [3]).

For integral operators whose kernels are invariant under all rotations of $\mathbb{R}^n$, conditions (4) and (5) are significantly simplified. Recall that a function $k(x, y)$ is called *invariant under the rotation group SO(n)* if

$$k(\omega(x), \omega(y)) = k(x, y) \quad \forall \omega \in SO(n). \tag{8}$$

Corollary 1 *Let the function $k(x, y)$ be homogeneous of degree α, where $\alpha \geqslant -n$, invariant under the rotation group $SO(n)$, and satisfy the condition*

$$\varkappa = \int\limits_{\mathbb{R}^n} |k(e_1, t)||t|^{-n/p}\, dt = \int\limits_{\mathbb{R}^n} |k(t, e_1)||t|^{-n/p'-\alpha-n}\, dt < \infty, \tag{9}$$

where $e_1 = (1, 0, \dots, 0)$. Then, the operator K of the form (1) is bounded in the space $L_p(\mathbb{B}_n)$, $1 \leqslant p \leqslant \infty$, and

$$\|K\varphi\|_{L_p(\mathbb{B}_n)} \leqslant \varkappa \|\varphi\|_{L_p(\mathbb{B}_n)}. \tag{10}$$

Proof In (4) and (5), we make the change of variables $t = \omega_\sigma(\tau)$, where ω_σ is any element of the group $SO(n)$ such that $\omega_\sigma(e_1) = \sigma$. Then, taking into account (8), we reduce the formulas (4) and (5) to the form

$$\varkappa_1 = \operatorname*{ess\,sup}_{\sigma \in \mathbb{S}_{n-1}} \int\limits_{\mathbb{R}^n} |k(\sigma, t)||t|^{-n/p}\, dt = \int\limits_{\mathbb{R}^n} |k(e_1, \tau)||\tau|^{-n/p}\, d\tau < \infty,$$

$$\varkappa_2 = \operatorname*{ess\,sup}_{\sigma \in \mathbb{S}_{n-1}} \int\limits_{\mathbb{R}^n} |k(t, \sigma)||t|^{-n/p'-\alpha-n}\, dt = \int\limits_{\mathbb{R}^n} |k(\tau, e_1)||\tau|^{-n/p'-\alpha-n}\, d\tau < \infty.$$

To finish the proof, we show that $\varkappa_1 = \varkappa_2 = \varkappa$. Passing to the spherical coordinates $t = \rho\theta$, we have

$$\varkappa_1 = \int\limits_{\mathbb{R}^n} |k(e_1, \tau)||\tau|^{-n/p}\, d\tau = \int\limits_0^\infty \int\limits_{\mathbb{S}_{n-1}} |k(e_1, \rho\theta)|\rho^{n/p'-1}\, d\rho d\theta.$$

Using the change of variables $\rho = 1/r$ and using the condition (2), we obtain

$$\varkappa_1 = \int\limits_0^\infty \int\limits_{\mathbb{S}_{n-1}} |k(re_1, \theta)|r^{-n/p'-\alpha-1}\, dr d\theta.$$

Since $k(x, y)$ satisfies the condition (8), there exists a function $k_0(r, \rho, t)$ such that $k(x, y) = k_0(|x|, |y|, x' \cdot y')$ (see [4, p. 68]). Then,

$$k(re_1, \theta) = k_0(r, 1, e_1 \cdot \theta) = k_0(r, 1, \theta \cdot e_1) = k(r\theta, e_1).$$

Therefore,

$$\varkappa_1 = \int\limits_0^\infty \int\limits_{\mathbb{S}_{n-1}} |k(r\theta, e_1)|r^{-n/p'-\alpha-1}\, dr d\theta = \int\limits_{\mathbb{R}^n} |k(\tau, e_1)||\tau|^{-n/p'-\alpha-n}\, d\tau = \varkappa_2.$$

Since $\varkappa_1 = \varkappa_2 = \varkappa$, this consequence follows directly from Theorem 1.

If $k(x, y) \geqslant 0$ and $\alpha = -n$, then the condition (9) is necessary for the boundedness of the operator K (see [3, 4, p. 70]). But if $\alpha \neq n$, then (9) is not necessary for the boundedness of the operator K with a non-negative kernel. Indeed, in the space $L_2(\mathbb{B}_n)$ consider the operator

$$(K\varphi)(x) = \int\limits_{\mathbb{B}_n} \frac{\varphi(y)}{|x-y|^{n/2}}\, dy.$$

It is known (e.g., see [12, p. 212]) that this operator is bounded in $L_2(\mathbb{B}_n)$. However, (9) is not fulfilled because

$$\int\limits_{\mathbb{R}^n} \frac{dt}{|e_1 - t|^{n/2}|t|^{n/2}} = \infty.$$

Example 1 Let us provide an example of a function satisfying the conditions of Corollary 1. Consider the function

$$k(x, y) = \frac{\exp(i(x' \cdot y'))}{(|x|^\gamma + |y|^\gamma)|x-y|^\beta}, \tag{11}$$

where $i^2 = -1$, $\beta, \gamma > 0$ and $\beta + \gamma < n$. It is obvious that this function is homogeneous of degree $\alpha = -(\beta + \gamma)$ and invariant under the rotation group $SO(n)$. Moreover,

$$\varkappa = \int\limits_{\mathbb{R}^n} |k(e_1, t)||t|^{-n/p}\, dt = \int\limits_{\mathbb{R}^n} \frac{1}{(1 + |t|^\gamma)|e_1 - t|^\beta |t|^{n/p}}\, dt < \infty$$

if $1 < p < n/(n - \beta - \gamma)$. Thus, the operator K with the kernel $k(x, y)$ of the form (11) is bounded in the space $L_p(\mathbb{B}_n)$ for $1 < p < n/(n - \beta - \gamma)$.

In the case $\beta + \gamma = n$, where $\gamma > 0$, the operator K is bounded in $L_p(\mathbb{B}_n)$ for $1 < p < \infty$.

Further, in the space $L_p(\mathbb{B}_n)$, consider the operator of Volterra type

$$(\widetilde{K}\varphi)(x) = \int\limits_{|y|<|x|} k(x, y)\varphi(y)\, dy.$$

Corollary 2 *Let the function $k(x, y)$ be homogeneous of degree α, where $\alpha \geqslant -n$, invariant under the rotation group $SO(n)$, and satisfy the condition*

$$\widetilde{\varkappa} = \int\limits_{|t|<1} |k(e_1, t)||t|^{-n/p}\, dt = \int\limits_{|t|>1} |k(t, e_1)||t|^{-n/p' - \alpha - n}\, dt < \infty. \tag{12}$$

Then, the operator $\widetilde{K}$ is bounded in the space $L_p(\mathbb{B}_n)$, $1 \leqslant p \leqslant \infty$, and

$$\|\widetilde{K}\varphi\|_{L_p(\mathbb{B}_n)} \leqslant \widetilde{\varkappa}\|\varphi\|_{L_p(\mathbb{B}_n)}. \tag{13}$$

Proof Represent the operator $\widetilde{K}$ in the form (1) with the kernel $\widetilde{k}(x, y)$ given by the formula

$$\widetilde{k}(x, y) = \begin{cases} k(x, y), & |y| < |x|, \\ 0, & |y| > |x|. \end{cases}$$

Then, (12) is equivalent to the condition (9) for the function $\widetilde{k}(x, y)$. Therefore, by virtue of Corollary 1 the operator $\widetilde{K}$ is bounded in $L_p(\mathbb{B}_n)$ and the inequality (10) takes the form (13).

Example 2 Consider the operator $\widetilde{K}$ with the kernel $k(x, y)$ of the form (11), assuming that $\beta, \gamma > 0$ and $\beta + \gamma < n$. Since

$$\widetilde{\varkappa} = \int\limits_{|t|<1} \frac{1}{(1+|t|^{\gamma})|e_1 - t|^{\beta}|t|^{n/p}}\, dt < \infty$$

for $1 < p \leqslant \infty$, the operator $\widetilde{K}$ is bounded in $L_p(\mathbb{B}_n)$ for these values of p.

Let us proceed to the investigation of the operator $\mathcal{K}$ of the form (3). We consider this operator in the space $L_p(C\mathbb{B}_n)$.

Theorem 2 *Let the function $k(x, y)$ be homogeneous of degree α, where $\alpha \leqslant -n$, and satisfy conditions (4)–(5). Then, the operator $\mathcal{K}$ of the form (3) is bounded in the space $L_p(C\mathbb{B}_n)$, $1 \leqslant p \leqslant \infty$, and*

$$\|\mathcal{K}\psi\|_{L_p(C\mathbb{B}_n)} \leqslant \varkappa_1^{1/p'} \varkappa_2^{1/p} \|\psi\|_{L_p(C\mathbb{B}_n)}.$$

Proof The proof is similar to the proof of Theorem 1.

Corollary 3 *Let the function $k(x, y)$ be homogeneous of degree α, where $\alpha \leqslant -n$, invariant under the rotation group $SO(n)$, and satisfy condition (9). Then, the operator $\mathcal{K}$ of the form (3) is bounded in the space $L_p(C\mathbb{B}_n)$, $1 \leqslant p \leqslant \infty$, and*

$$\|\mathcal{K}\psi\|_{L_p(C\mathbb{B}_n)} \leqslant \varkappa \|\psi\|_{L_p(C\mathbb{B}_n)}.$$

Proof It follows from the equality $\varkappa_1 = \varkappa_2 = \varkappa$, which was obtained in the proof of Corollary 1.

Example 3 Consider the operator $\mathcal{K}$ with the kernel $k(x, y)$ of the form (11), where $\gamma > 0$, $0 < \beta < n$ and $\beta + \gamma > n$. Then,

$$\varkappa = \int\limits_{\mathbb{R}^n} \frac{1}{(1+|t|^{\gamma})|e_1 - t|^{\beta}|t|^{n/p}}\, dt < \infty,$$

if $1 < p \leqslant \infty$. It follows that the operator $\mathcal{K}$ is bounded in $L_p(C\mathbb{B}_n)$ for these values of p.

The one-dimensional case

In this section, we will make some clarifications related to one-dimensional integral operators with homogeneous kernels of degree α.
In the space $L_p(0, 1)$, where $1 \leqslant p \leqslant \infty$, consider the operator

$$(Qf)(x) = \int\limits_0^1 q(x, y) f(y)\, dy,$$

where the function $q(x, y)$ defined on $\mathbb{R}_+ \times \mathbb{R}_+$ is measurable and homogeneous of degree α ($\alpha \in \mathbb{R}$), i.e., $q(\lambda x, \lambda y) = \lambda^{\alpha} q(x, y)$ for any $\lambda > 0$.
As in Lemma 1, the boundedness of the operator Q in the space $L_p(0, 1)$ implies that $\alpha \geqslant -1$. The following theorem provides sufficient conditions for the boundedness of the operator Q.

Theorem 3 *Let the function $q(x, y)$ be homogeneous of degree α, where $\alpha \geqslant -1$, and satisfy the condition*

$$\vartheta = \int\limits_0^{\infty} |q(1, t)| t^{-1/p}\, dt = \int\limits_0^{\infty} |q(t, 1)| t^{-1/p' - \alpha - 1}\, dt < \infty. \tag{14}$$

Then, the operator Q is bounded in the space $L_p(0, 1)$, $1 \leqslant p \leqslant \infty$, and

$$\|Qf\|_{L_p(0,1)} \leqslant \vartheta \|f\|_{L_p(0,1)}.$$

Proof The equality of integrals in formula (14) is verified directly. The proof for the boundedness of the operator Q is analogous to the proof of Theorem 1.

Example 4 In the space $L_p(0,1)$, consider the Hardy-type operator

$$(Q\varphi)(x) = \frac{1}{x^\beta} \int_0^x \varphi(y)\, dy,$$

where $0 < \beta < 1$. The kernel of this operator is the function

$$q(x,y) = \begin{cases} 1/x^\beta, & y < x, \\ 0, & y > x. \end{cases}$$

It is easy to see that this function is homogeneous of degree $(-\beta)$ and satisfies the conditions (14) for $1 < p \leqslant \infty$.

In conclusion, it should be noted that the inequality $\alpha \leqslant -1$ is the necessary condition for boundedness of the operator

$$(Qf)(x) = \int_1^\infty q(x,y) f(y)\, dy$$

in the space $L_p(1,\infty)$, and the condition (14) is sufficient for the boundedness of this operator.

Funding This work was supported by the Regional Mathematical Center of the Southern Federal University with the Agreement No. 075-02-2024-1427 of the Ministry of Science and Higher Education of Russia.

Data availability The manuscript has no associated data.

Declarations

Conflict of interest The authors declare no competing interests.

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