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ON PERIODIC SOLUTIONS OF A SECOND-ORDER ORDINARY DIFFERENTIAL EQUATION

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Abstract. We consider a differential equation containing first- and second-order forms with respect to the phase variable and its derivative with constant coefficients and a periodic inhomogeneity. Using the method of constructing a positively invariant rectangular domain, we examine the existence of a asymptotically stable (in the Lyapunov sense) periodic solution. Criteria for the existence of a periodic solution are formulated in terms of properties of isoclines. We consider cases where the zero isocline is a nondegenerate second-order curve.

***Keywords and phrases*:** second-order differential equation, qualitative theory, periodic solution, stability, nonlinear oscillator.

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1. Introduction.

Consider the differential equation

$$\ddot{x} + ax + b\dot{x} + c(t) + ax^2 + \beta x\dot{x} + \gamma(\dot{x})^2 = 0, \tag{1}$$

where the coefficients a, b, α, β, and γ are constants and $c(t)$ is a bounded T-periodic function. We state conditions for the existence of a stable T-periodic solution for Eq. (1).

Equations of the form (1) are often used as a model of nonlinear oscillator; it belongs to the class of generalized Rayleigh-type equations. Classical results related to the existence of periodic solutions of equations of the Lienard and Rayleigh types can be found in [2]. In this paper, we apply the method of canonical domains based on the following assertion (see [1]).

Theorem 1. *Let a system $\dot{x} = f(t,x)$ have a T-periodic in t right-hand side and possess the property of existence and uniqueness of a solution of the Cauchy problem in a set $\mathbb{R}\times D$. If there exists a convex compact set $\Omega \subset D$ bounded by smooth curves $\Phi_i(x) = 0$ on which the estimates* $(\operatorname{grad}\Phi_i(x), f(t,x)) \leq 0$ *hold, $i = \overline{1,m}$, then the system $\dot{x} = f(t,x)$ has a T-periodic solution with an initial value from* $\operatorname{Int}\Omega$.

Any domain Ω satisfying Theorem 1 is said to be canonical.

2. Auxiliary results.

We reduce Eq. (1) to an equivalent second-order normal system. Let $\dot{x} = kx - y$, where k is a parameter. Then $\ddot{x} = k\dot{x} - \dot{y}$. Substituting $\dot{x}$ and $\ddot{x}$ into Eq. (1), we obtain the relation

$$k^2x - ky - \dot{y} + ax + bkx - by + c(t) + \alpha x^2 + \beta xy + \gamma k^2x^2 + \gamma y^2 - 2\gamma kxy = 0;$$

then

$$\dot{y} = x^2(\alpha + \beta k + \gamma k^2) + y^2\gamma + x(a + bk + k^2) - xy(\beta + 2\gamma k) - y(k + b) + c(t).$$

Below we assume that the parameter k satisfies the condition $\beta + 2\gamma k = 0$. Then Eq. (1) corresponds to the system

$$\begin{cases} \dot{x} = f_1(x,y) = kx - y, \\ \dot{x} = f_2(t,x,y) = x^2(\alpha - \gamma k^2) + \gamma y^2 + x(a + bk + k^2) - y(k + b) + c(t). \end{cases} \tag{2}$$

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Thus, the problem on periodic solutions of Eq. (1) is reduced to the problem on periodic solutions of the system (2).

We state necessary conditions of the existence of a periodic solution. For this purpose, consider the location of the infinity isocline I^∞: $\dot{x} = 0$ and the movable zero isocline I_t^∞: $\dot{x} = 0$ of the system (2).

For each t, the isocline I^∞ (I_t^0) divides the plane xOy into two parts: interior, where $\dot{x} < 0$ ($\dot{y} < 0$), and exterior, where $\dot{x} > 0$ ($\dot{y} > 0$). We denote these parts by $X^{\rm int}$ and $X^{\rm ext}$ (or $Y_t^{X^{\rm int}}$ and $Y_t^{\rm ext}$), respectively. For the isocline I_t^0, construct the corresponding fixed (stationary) interior and exterior parts of the phase plane:

$$Y^{\rm ext} = \{(x,y) : f_2(t,x,y) > 0,\ t \in \mathbb{R}\}, \quad Y^{\rm int} = \{(x,y) : f_2(t,x,y) < 0,\ t \in \mathbb{R}\}.$$

Obviously, for all $t \in \mathbb{R}$, the isoclines satisfy the conditions

$$I^\infty = X = \partial X = \partial X^{\rm int} \cup \partial X^{\rm ext}, \quad I_t^0 \subset Y, \quad \partial Y = \partial Y^{\rm int} \cup \partial Y^{\rm ext},$$

where Y is the domain containing all isoclines of the origin I_t^0. Since in each of the domains $X^{\rm int}$, $X^{\rm ext}$, $Y^{\rm int}$, and $Y^{\rm ext}$ the components of the vector solution of the system (2) are strictly monotonic, the trajectory of the periodic solution of the system (2) cannot entirely lie in these domains. Therefore, the following assertion on the localization of a canonical domain holds.

Lemma 1. *If Ω is a canonical domain for the system* (2), *then $\Omega \supset X \cap Y$.*

We rewrite the system (2) in the vector form $\dot{z} = f(z) + \mathrm{colon}(0, c(t))$, where

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad f(z) = \begin{pmatrix} kx - y \\ x^2(\alpha - \gamma k^2) + \gamma y^2 + x(a + bk + k^2) - y(k+b) \end{pmatrix}.$$

Choose some solutions $z_1 = z_1(t)$, $z_2 = z_2(t)$ of the system (2). For the function $V(z_1, z_2) = (z_1 - z_2)^2$, calculate the derivative with respect to the system (2). By the Lagrange formula we obtain

$$\dot{V}(z_1, z_2) = 2(z_1 - z_2)^T F(\tilde{z})(z_1 - z_2),$$

where

$$F(\tilde{z}) = \frac{1}{2}\left(\frac{\partial f(\tilde{z})}{\partial z} + \left(\frac{\partial f(\tilde{z})}{\partial z}\right)^T\right)$$

is the Hermitian component of the Jacobi matrix and $\tilde{z}$ is a vector depending on z_1 and z_2.

Assume that Λ is a positive invariant domain for the system (2). Choose $z_1(0)$ and $z_2(0)$ arbitrarily. Then $z_1(t), z_2(t) \in \Lambda$ for any $t \geq 0$. Assume that z_1 and z_2 are different T-periodic solutions whose trajectories lie in the positive invariant canonical domain Ω. If $\dot{V}(z_1, z_2) < 0$ for $t \geq 0$, then $V(z_1, z_2)$ strictly decreases for $t \geq 0$. Moreover,

$$z_1(0) \neq z_2(0), \quad V(z_1(0), z_2(0)) > V(z_1(T), z_2(T)),$$

that is,

$$\|z_1(0) - z_2(0)\|_2 > \|z_1(T) - z_2(T)\|_2.$$

Therefore, the function $z(t) = z_1(t) - z_2(t)$ is not T-periodic. Thus, if $\dot{V}(z_1, z_2) < 0$ for $t \geq 0$ in the domain Ω, then this domain contains a unique trajectory of the periodic solution of the system (2).

Moreover, if $\dot{V}(z_1, z_2) < 0$ for $t \geq 0$, where $z_1(t) \in \Omega$ is a T-periodic solution of the system (2) and $z_2(t) \in \Omega$ is an arbitrary solution of the system (2), then

$$V(z_1(0), z_2(0)) > V(z_1(T), z_2(T)).$$

Therefore, the operator of shift by trajectories maps a neighborhood of the initial value $z_1(0)$ into itself. By induction, this implies the stability of the periodic solution z_1 (see [1]).

Thus, if $F(z) < 0$ in a positive invariant domain Ω, then $\dot{V}(z_1, z_2) < 0$ for $t \geq 0$ for any pair of solutions with initial values from Ω. Moreover, the domain Ω contains a unique trajectory of the stable periodic solution of the system (2).

For the system (2) we have

$$F(z) = \begin{bmatrix} k & \dfrac{2x(\alpha-\gamma k^2)+(a+bk+k^2)-1}{2} \\ \dfrac{2x(\alpha-\gamma k^2)+(a+bk+k^2)-1}{2} & 2\gamma y-(k+b) \end{bmatrix}.$$

By the Sylvester criterion, the condition $F(z) < 0$ in Ω is equivalent to the system of estimates for principal minors:

$$F(z) = \begin{cases} \Delta_1 = k < 0, \\ \Delta_2(x,y) = (2\gamma y - (k+b)) - \dfrac{(2x(\alpha-\gamma k^2)+(a+bk+k^2)-1)^2}{4} > 0. \end{cases}$$

Therefore, everywhere below, we will assume that the condition

$$k < 0 \tag{3}$$

holds. Moreover, the condition $\beta + 2\gamma k = 0$ implies that the coefficients β and γ have the same sign.

In this paper, we search for the boundary $\partial\Omega$ of the canonical domain as a rectangle $ABCD$:

$$AB : \Phi_1(x,y) = y_1 - y = 0; \quad \operatorname{grad}\Phi_1 = (0;-1); \quad BC : \Phi_2(x,y) = x - x_2 = 0; \quad \operatorname{grad}\Phi_2 = (1;0);$$
$$AD : \Phi_3(x,y) = x_1 - x = 0; \quad \operatorname{grad}\Phi_3 = (-1;0); \quad BC : \Phi_4(x,y) = y - y_2 = 0; \quad \operatorname{grad}\Phi_4 = (0;1).$$

In this case, Theorem 1 can be applied if the inequalities with nonlinear terms of the same order as in the right-hand side of the system considered hold. Moreover, to verify the stability conditions, it suffices to consider the vertices of the rectangle. Then Theorem 1 can be formulated as follows.

Theorem 2. *Assume that the right-hand side of the system* (2) *on the boundary $ABCD$ satisfies the conditions*

$$\begin{cases} f_2(t,x,y) \ge 0 & \text{if } \Phi_1 = 0, \\ f_1(t,x,y) \ge 0 & \text{if } \Phi_2 = 0, \\ f_1(t,x,y) \ge 0 & \text{if } \Phi_3 = 0, \\ f_2(t,x,y) \ge 0 & \text{if } \Phi_4 = 0. \end{cases}$$

Moreover, at the vertices of the rectangle $ABCD$, the estimate $\Delta_2(x,y) > 0$ holds. Then the system (2) *has a stable T-periodic solution with an initial data inside $ABCD$.*

3. Main results. Consider nondegenerate particular cases of the correspondence between the isocline I_t^0 and one of the standard forms for second-order curves if the system (2) is not linear.

We use the notation

$$c_1 = \inf_{0,T} c(t), \quad c_2 = \sup_{0,T} c(t).$$

Assume that $\gamma \neq 0$ and $\alpha - \gamma k^2 \neq 0$. Extract the complete squares in the equation for I_t^0:

$$(\alpha - \gamma k^2)(x - x_0)^2 + \gamma(y - y_0)^2 = m - c(t), \tag{4}$$

where

$$(x_0, y_0) = \left(\frac{-a - bk - k^2}{2(\alpha - \gamma k^2)}; \frac{k+b}{2\gamma} \right)$$

is the center of the curve,

$$m = \frac{(a+bk+k^2)^2}{2(\alpha-\gamma k^2)} + \frac{(k+b)^2}{4\gamma}.$$

Assume that the following estimates hold:

$$\gamma < 0, \quad \alpha - \gamma k^2 > 0, \quad c_1 > m. \tag{5}$$

Then the curve (4) is a hyperbola with the semiaxes

$$\sqrt{\frac{c(t)-m}{\alpha-\gamma k^2}}, \quad \sqrt{\frac{m-c(t)}{\gamma}}.$$

To apply Theorem 2, we assume that the lines I^∞: $y = kx$ and

$$I_1^0 : \frac{(y-y_0)^2}{s_1^2} - \frac{(x-x_0)^2}{q_1^2} = 1, \quad y > y_0,$$

do not intersect; here

$$q_1 = \sqrt{\frac{|m-c_2|}{\alpha-\gamma k^2}}, \quad s_1 = \sqrt{\frac{|m-c_2|}{|\gamma|}}.$$

Thus, the discriminant of the equation

$$\frac{(kx-y_0)^2}{s_1^2} - \frac{(x-x_0)^2}{q_1^2} = 1 \tag{6}$$

must be positive. This condition holds if the following inequality is fulfilled:

$$(kx_0-y_0)^2 + (k^2q_1^2 - s_1^2) > 0. \tag{7}$$

Thus, under the condition (7), the lines I^∞ and I_1^0, $y > y_0$, intersect at some points (h_1, kh_1) and (h_2, kh_2), where h_1 and h_2 are the roots of Eq. (6), $h_1 < h_2$.

For the hyperbola I_2^0, consider the vertex (x_0, y_0+s_2), $s_2 = \sqrt{|m-c_1|/|\gamma|}$. Under the condition (7), the lines I^∞ and I_2^0 intersect twice. Consider the intersection point with the largest abscissa l. Consider the straight lines

$$AB : y = y_l = \begin{cases} kl, & l \le x_0, \\ y_0 + s_2, & l > x_0, \end{cases} \quad BC : x = x_2 = \frac{y_1}{k}; \tag{8}$$

then $AB \subset Y^{\text{ext}}$, $BC \subset X^{\text{int}}$.

Calculate the ordinate y_2 of the intersection point of the straight line BC with the line I_1^0. If the condition

$$kh_2 \le y_2 \le kh_1 \tag{9}$$

holds, then we choose the straight lines

$$CD : y = y_2, \quad DA : x = x_1 = \frac{y_2}{k}. \tag{10}$$

The stability condition holds in the rectangle Ω bounded by the lines (8) and (10) if the following inequalities are fulfilled:

$$\Delta_2(x_1, y_1) > 0, \quad \Delta_2(x_2, y_1) > 0. \tag{11}$$

Thus, by Theorem 2, the following assertion holds.

Theorem 3. *If the conditions* (3), (5), (7), (9), *and* (11) *are fulfilled, then the system* (2) *has a stable* T*-periodic solution whose trajectory lies inside the domain bounded by the straight lines* (8) *and* (10).

Similarly we can consider the other cases in which the zero isocline is a hyperbola and the case where the line (4) is an ellipse or a parabola.

Assume that the following estimates hold:

$$\gamma > 0, \quad \alpha - \gamma k^2 > 0, \quad c_2 < m. \tag{12}$$

Then for any $t \in \mathbb{R}$ the isocline I_t^0 defined by Eq. (4) is an ellipse with the semiaxes

$$\sqrt{\frac{m-c(t)}{\alpha-\gamma k^2}}, \quad \sqrt{\frac{m-c(t)}{\gamma}}.$$

Then the numbers

$$q_1 = \inf_{[0,T]} \left(\sqrt{\frac{m - c(t)}{\alpha - \gamma k^2}} \right) = \sqrt{\frac{m - c_2}{\alpha - \gamma k^2}}, \quad s_1 = \inf_{[0,T]} \left(\sqrt{\frac{m - c(t)}{\gamma}} \right) = \sqrt{\frac{m - c_2}{\gamma}}$$

are the semiaxes of the boundary I_t^0 of the domain Y^{int}, which contains the center (x_0, y_0) of the ellipse and the numbers

$$q_2 = \sup_{[0,T]} \left(\sqrt{\frac{m - c(t)}{\alpha - \gamma k^2}} \right) = \sqrt{\frac{m - c_1}{\alpha - \gamma k^2}}, \quad s_2 = \sup_{[0,T]} \left(\sqrt{\frac{m - c(t)}{\gamma}} \right) = \sqrt{\frac{m - c_1}{\gamma}}$$

are the semiaxes of the boundary I_2^0 of the domain Y^{ext}. To apply Theorem 2 in this case, we assume that the lines I^∞ and I_t^0 intersect twice at some points (h_1, kh_1) and (h_2, kh_2), where $h_1 < h_2$. This is possible if

$$(k^2 q_1^2 + s_1^2) - (kx_0 - y_0)^2 > 0. \tag{13}$$

Under the condition (13), the lines I^∞ and I_2^0 also intersect twice. Assume that l is the largest of the abscissas of their intersection points. We choose the straight lines

$$AB : y = y_l = \begin{cases} kl, & l \le x_0, \\ y_0 + s_2, & l > x_0, \end{cases} \quad BC : x = x_2 = \frac{y_1}{k}. \tag{14}$$

Let y_2 be the ordinate of the intersection point of the straight line BC with the isocline I_1^0. If

$$kh_2 \le y_2 \le kh_1, \tag{15}$$

then we choose the straight lines

$$CD : y = y_2, \quad DA : x = x_1 = \frac{y_2}{k}. \tag{16}$$

In the rectangle Ω bounded by the lines (14) and (18), the stability condition holds if the following inequalities are fulfilled:

$$\Delta_2(x_i, y_j) > 0; \quad i, j = 1, 2. \tag{17}$$

Thus, due to Theorem 2, the following assertion holds.

Theorem 4. *If the conditions* (3), (12), (13), (15), *and* (17) *are fulfilled, then the system* (2) *has a stable T-periodic solution whose trajectory lies inside the domain bounded by the straight lines* (14) *and* (17).

Assume that in the system (3)

$$\gamma = 0, \quad \alpha > 0, \quad k + b > 0. \tag{18}$$

Then for any t the zero isocline is the parabola

$$y = \frac{\alpha}{k + b} x^2 + \frac{a + bk + k^2}{k + b} x + \frac{c(t)}{k + b} \tag{19}$$

directed downward and the abscissa of the vertex is $x_0 = (-a - bk - k^2)/(2\alpha)$. Under the condition (18), the domains Y^{ext} and Y^{int} are located below and above the parabola (19), respectively. The ordinates of the vertices of the fixed zero isoclines I_1^0 and I_2^0 are

$$s_1 = \frac{c_2}{k + b}, \quad s_2 = \frac{c_1}{k + b},$$

respectively. Assume that

$$a^2 - 4\alpha c_2 > 0. \tag{20}$$

Then the lines I^∞ and I_1^0 have two intersection points (h_1, kh_1) and (h_2, kh_2), where $h_1 < h_2$.

Choose the straight lines

$$AB: y = y_l = \begin{bmatrix} kl, & l \le x_0, \\ y_0 + s_2, & l > x_0, \end{bmatrix}; \quad BC: x = x_2 = \frac{y_1}{k}. \tag{21}$$

where l is the largest of the abscissas of the intersection points of the lines I^{∞} and I_2^0. Calculate the ordinate y_2 of the intersection point of BC and I_1^0. Under the condition

$$kh_2 \le y_2 \le kh_1, \tag{22}$$

we choose the straight lines

$$CD: y = y_2, \quad DA: x = x_1 = \frac{y_2}{k}. \tag{23}$$

In the rectangle $ABCD$, the stability condition holds if the following estimates are fulfilled:

$$\begin{cases} \Delta_2(x_1, y_1) > 0, \\ \Delta_2(x_2, y_1) > 0. \end{cases} \tag{24}$$

Thus, the following assertion holds.

Theorem 5. *If the conditions* (3), (18), (20), (22), *and* (24) *are fulfilled, then the system* (2) *has a stable T-periodic solution whose trajectory lies inside the domain bounded by the straight lines* (21) *and* (23).

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