DOI 10.1007/s10958-024-07101-4
Journal of Mathematical Sciences, Vol. 281, No. 2, May, 2024

QUASILINEAR INTERPOLATION BY MINIMAL SPLINES

L. P. Livshits,* A. A. Makarov,† and S. V. Makarova‡ UDC 519.6

The paper studies quasilinear interpolation by minimal splines constructed on nonuniform grids with multiple nodes. Asymptotic representations for normalized splines are obtained. The sharpness of biorthogonal approximation and the order of accuracy of quasilinear interpolation with respect to the grid stepsize are established. Results of numerical experiments on approximating some test functions, which demonstrate the effect of choosing a generating vector function in constructing the corresponding minimal spline, are presented. Bibliography: 35 *titles.*

To the memory of Yuri Kazimirovich Dem'yanovich

1. Introduction

The concept of *spline* as a piecewise polynomial function was introduced by Schoenberg [27]. The classical approach to construction of interpolation splines implies solving a system of linear algebraic equations, the order of which is determined by the number of interpolation conditions necessary to solve a particular interpolation problem (Lagrange, Hermite, or Hermite–Birkhoff problems) in a class of functions with "piecewise" properties with a certain smoothness at the nodes of the grid under consideration. The principal achievement of this theory was the construction of the B-splines (*basis* splines), having minimal support for a prescribed smoothness. This ensures that addition of new interpolation nodes implies only a local modification of the interpolating spline. On the other hand, in every specific case, the approximation properties and computational complexity of the resulting splines are studied individually [2, 28].

The first and one of the simplest examples of approximation by splines was the continuous piecewise linear interpolation (the Euler method) [35]. Generally speaking, such an approximation method is a *local* one, which does not require solving a large-order system of linear algebraic equations. Instead, a few systems of smaller order must be solved. In local methods, the coefficients at the basis functions are determined as the values of some approximation functionals, which are, for example, linear combinations of the values of the approximated function and its derivatives at some points. However, this does not necessarily lead to a loss of approximation accuracy. Local schemes in which the maximum order of accuracy is achieved are called quasi-interpolation schemes [1, 32]. In solving most applied problems, methods of approximation theory and numerical analysis somehow related to local approximation are used (for more detail, e. g., see [3, 17–20, 29, 33, 34]).

The local basis functions themselves can be determined, for instance, by solving small-order systems of linear algebraic equations. This approach appeared in connection with the theory of finite-element method and was applied by Goel [12], Strang and Fix [30], and Mikhlin and Dem'yanovich [5, 11, 25]. The Mikhlin–Dem'yanovich approach to constructing polynomial splines satisfying *approximation relations* and having a prescribed smoothness focuses on deriving the simplest approximation formulas. In this case, first one minimizes the *multiplicity of overlapping* (the so-called minimum multiplicity of overlapping supports of the basis functions)

*St.Petersburg State University, St.Petersburg, Russia, e-mail: st088046@student.spbu.ru.

†St.Petersburg State University; St.Petersburg State Electrotechnical University "LETI", St.Petersburg, Russia , e-mail: a.a.makarov@spbu.ru.

‡St.Petersburg State University, St.Petersburg, Russia, e-mail: st122359@student.spbu.ru.

Translated from *Zapiski Nauchnykh Seminarov POMI*, Vol. 524, 2023, pp. 94–111. Original article submitted October 16, 2023.

of the basis splines, and then the degree of the splines is minimized. The functions constructed in this way are called the *minimal splines* for chosen approximation relations and a prescribed smoothness. In view of the importance of these relations, Mikhlin [25] called them *fundamental* relations. If approximation relations of interpolation nature are used, then the resulting approximations will be accurate on polynomials of a certain degree (this degree is called *order of accuracy*). This allows one to find minimal splines with a local interpolation basis. An important characteristic of approximation is the number of derivatives of the approximated function it involves. This number is called *approximation height*. A zero-height spline only uses the values of the function being approximated but not its derivatives; such splines are said to be *Lagrangian*. A spline that uses consecutive ith derivatives of the approximated function ($i = 0, 1, \ldots, H$, where $H \in \mathbb{N}$) is called a *Hermitian* spline or a *spline of height* H. The works by Dem'yanovich [6–8] and by his students and colleagues (for more detail, see [9,13,15,21] and the references therein) are devoted to generalizing approximation relations and to developing the general theory of minimal splines on their basis (both with a local interpolation and an approximation bases, polynomial as well as nonpolynomial). The approximations constructed there are accurate on powers of an arbitrary sufficiently smooth function.

It can be seen that the classes of splines obtained from approximation relations have nonempty intersections with splines obtained by applying other approaches. For example, the well-known polynomial B-splines are minimal splines, which exhibits an intimate connection between the Schoenberg and Mikhlin–Dem'yanovich approaches. The Ryaben'ky splines [26] are minimal Lagrangian splines whose orders of smoothness and accuracy coincide. The classical Hermitian splines are a special case of the minimal Hermitian splines. The well-known quadratic and cubic continuous finite-element approximations [31] prove to be minimal splines. The piecewise polynomial Jenkins functions, known in the theory of osculatory interpolation, also are a special case of minimal splines [4]. For other examples, e.g., see [10, 14, 16].

This paper considers the minimal splines obtained from approximation relations using a complete chain of vectors and a generating vector function $\boldsymbol{\varphi}$. A certain method for choosing a complete chain of vectors allows one to consider the minimal splines that have maximum smoothness ($B_{\boldsymbol{\varphi}}$-splines) and to establish the uniqueness of the space of such splines among all spaces of minimal splines (that are determined by an arbitrary choice of the above-mentioned chain of vectors for a given grid and a given generating vector function).

The purpose of this work is to study quasilinear interpolation by minimal splines (with maximum smoothness), which are constructed on nonuniform grids with multiple nodes. In this paper, asymptotic representations for normalized splines are obtained. Theorems on the accuracy of biorthogonal approximation and on the order of accuracy of quasilinear interpolation with respect to the grid stepsize are proved. Results of numerical experiments on approximating some test functions using different generating vector functions for constructing the corresponding minimal spline are presented.

2. Space of coordinate splines

Let $\mathbb{Z}$ and $\mathbb{R}^1$ be the sets of integers and reals, respectively. By $C^r[a, b]$ we denote the set of r times continuously differentiable functions on an interval $[a, b]$, assuming that $C^0[a, b] := C[a, b]$. The space of piecewise continuous functions with finitely many discontinuities of the first kind on $[a, b]$ is denoted by $C^{-1}[a, b]$; in this paper, it is assumed that every function of this space is left-continuous.

On $[a, b] \subset \mathbb{R}^1$, consider a grid X with two extra nodes outside the interval $[a, b]$:

$$X \,:\, x_{-1} < a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b < x_{n+1}. \tag{1}$$

Let $J_{i,k} := \{i, i+1, \dots, k\}$, where $i, k \in \mathbb{Z}$, $i < k$. An ordered set of vectors $\mathbf{A} := \{\mathbf{a}_j \in \mathbb{R}^2 \mid j \in J_{-1,n-1}\}$ is called *a vector chain.* A chain $\mathbf{A}$ is said to be *complete* if the square matrices $(\mathbf{a}_{j-1}, \mathbf{a}_j)$, composed of the vectors $\mathbf{a}_{j-1}$ and $\mathbf{a}_j$, are invertible, i.e.,

$$\det(\mathbf{a}_{j-1}, \mathbf{a}_j) \neq 0, \quad j \in J_{-1,n-1}. \tag{2}$$

The union of all elementary grid intervals is denoted by $M := \cup_{j \in J_{-1,n}} (x_j, x_{j+1})$; $\mathbb{X}(M)$ is the linear space of real-valued functions defined on the set M.

Assume that $\mathbf{A}$ is a complete vector chain. Given a vector function $\boldsymbol{\varphi}\colon [a,b] \to \mathbb{R}^2$, define the functions $\omega_j \in \mathbb{X}(M)$, $j \in J_{-1,n-1}$, by the following relations:

$$\begin{gathered} \sum_{j'=k-1}^{k} \mathbf{a}_{j'}\, \omega_{j'}(t) \equiv \boldsymbol{\varphi}(t), \quad t \in (x_k, x_{k+1}), \quad k \in J_{-1,n-1}, \\ \omega_j(t) \equiv 0, \qquad t \notin [x_j, x_{j+2}] \cap M. \end{gathered} \tag{3}$$

For any fixed $t \in (x_k, x_{k+1})$, relations (3) can be regarded as a system of linear algebraic equations in the unknowns $\omega_j(t)$. By assumption (2), the system (3) has a unique solution, and $\operatorname{supp} \omega_j(t) \subset [x_j, x_{j+2}]$.

By Kramer's rule, from the linear algebraic equations (3) we obtain

$$\omega_j(t) = \begin{cases} \dfrac{\det(\mathbf{a}_{j-1}, \boldsymbol{\varphi}(t))}{\det(\mathbf{a}_{j-1}, \mathbf{a}_j)}, & t \in [x_j, x_{j+1}), \\ \dfrac{\det(\boldsymbol{\varphi}(t), \mathbf{a}_{j+1})}{\det(\mathbf{a}_j, \mathbf{a}_{j+1})}, & t \in [x_{j+1}, x_{j+2}), \\ 0 & \text{otherwise}. \end{cases} \tag{4}$$

The linear span of the functions $\omega_j(t)$ is called *the space of minimal coordinate* $(\mathbf{A}, \boldsymbol{\varphi})$*-splines* and is denoted by $\mathbb{S}(X, \mathbf{A}, \boldsymbol{\varphi})$. Identities (3) are referred to as *the approximation relations.* The vector function $\boldsymbol{\varphi}$ is called *the generating* vector function for the $(\mathbf{A}, \boldsymbol{\varphi})$-splines. The term *coordinate splines* is used for functions that form a basis of a spline space (in order to avoid using the term "basis splines," which is interpreted differently by different authors). The functions ω_j that solve approximation relations of the form (3) are called minimal coordinate splines of *Lagrangian type.*

Consider the chain of vectors $\mathbf{A}$ defined by the formula

$$\mathbf{a}_j := \boldsymbol{\varphi}_{j+1} = (1, \rho_{j+1})^T, \tag{5}$$

where the generating vector function is defined by the relation $\boldsymbol{\varphi}(t) := (1, \rho(t))^T$. Here, T means transposition, and $\boldsymbol{\varphi}_j := \boldsymbol{\varphi}(x_j)$, $\rho_j := \rho(x_j)$, $j \in J_{-1,n-1}$. The function $\rho(t)$ is also said to be *generating.*

In the sequel, we will assume that the function $\rho(t)$ satisfies the following conditions:

$$\rho \in C^1[a,b], \quad \rho'(t) \neq 0, \quad t \in [a,b]. \tag{6}$$

The completeness of the chain (5) is obvious. In view of the Lagrange mean-value theorem (formula of finite increments), the left-hand side of relation (2) can be written as

$$\det(\mathbf{a}_{j-1}, \mathbf{a}_j) = \rho_{j+1} - \rho_j = \rho'(\theta)(x_{j+1} - x_j), \quad \theta \in (x_j, x_{j+1}).$$

It follows that if the function $\rho(t)$ is strictly monotone on $[a,b]$, then the chain (5) is complete. For the grid (1) this follows from conditions (6).

For the complete chain (5), formulas (4) can be written in the form

$$\omega_j(t) = \begin{cases} \dfrac{\det(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}(t))}{\det(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_{j+1})} = \dfrac{\rho(t) - \rho_j}{\rho_{j+1} - \rho_j}, & t \in [x_j, x_{j+1}), \\ \dfrac{\det(\boldsymbol{\varphi}(t), \boldsymbol{\varphi}_{j+2})}{\det(\boldsymbol{\varphi}_{j+1}, \boldsymbol{\varphi}_{j+2})} = \dfrac{\rho_{j+2} - \rho(t)}{\rho_{j+2} - \rho_{j+1}}, & t \in [x_{j+1}, x_{j+2}), \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

The continuity of the functions (7) on the set M of elementary grid intervals is ensured by the continuity of the function $\rho(t)$. At the grid nodes, continuity immediately follows from the relation

$$\omega_j(x_i) = \delta_{j,i-1}, \tag{8}$$

where $\delta_{j,i}$ is the Kronecker symbol.

As has been shown above, the function $\rho(t)$ is strictly monotone by virtue of condition (6). Thus, the functions (7) are positive on their supports, i.e.,

$$\omega_j(t) > 0, \quad t \in (x_j, x_{j+2}). \tag{9}$$

The functions (7) yield a *partition of unity*, i.e.,

$$\sum_{j=-1}^{n-1} \omega_j(t) \equiv 1, \quad t \in [a, b], \tag{10}$$

which is established by considering the approximation relations component-by-component.

Below, for convenience, the vector components will be denoted by square brackets with nonnegative integer subscripts. For instance, a vector $\mathbf{a}_j \in \mathbb{R}^2$ can be represented as $\mathbf{a}_j := ([\mathbf{a}_j]_0, [\mathbf{a}_j]_1)^T$. Then, in view of (5), $[\mathbf{a}_j]_0 = 1$ and $[\boldsymbol{\varphi}(t)]_0 = 1$. Thus, the coefficient at the function $\omega_j(t)$ and the right-hand side of (3) are equal to 1, which implies (10).

The space $\mathbb{S}(X, \mathbf{A}, \boldsymbol{\varphi})$, where the chain $\mathbf{A}$ is determined from the vector function $\boldsymbol{\varphi}(t)$ via (5) and ρ satisfies conditions (6), is denoted by

$$\mathbb{S}(X) := \Big\{ u \mid u = \sum_{j=-1}^{n-1} c_j\, \omega_j, \quad c_j \in \mathbb{R}^1 \Big\}$$

and is called *the space of normalized linear minimal coordinate $B_{\boldsymbol{\varphi}}$-splines (of the second order)* on the grid X. The splines themselves will be called *the normalized minimal coordinate splines of maximum smoothness.*

If the generating vector function is defined by the equality $\boldsymbol{\varphi}(t) = (1, t)^T$, i.e., in (5) we set $\rho(t) = t$, then the functions (7) coincide with the known polynomial B-splines of the first degree (second order), i.e., with the one-dimensional Courant functions

$$\omega_j^B(t) = \begin{cases} \dfrac{t - x_j}{x_{j+1} - x_j}, & t \in [x_j, x_{j+1}), \\ \dfrac{x_{j+2} - t}{x_{j+2} - x_{j+1}}, & t \in [x_{j+1}, x_{j+2}), \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

In the case where the generating vector function $\boldsymbol{\varphi}$ has polynomial components, one can speak of the degree of the spline. Obviously, the polynomial splines of maximum smoothness (11) are splines of the first degree, i.e., linear splines. The difference between the degree of a polynomial spline and the order of its highest continuous derivative is called the *spline defect.* Thus, the splines (11) are splines with smallest defect (equal to 1).

3. Splines with multiple nodes

Let

$$h_j := x_{j+1} - x_j, \quad h := \max_j\{h_j\}, \quad j \in J_{-1,n}, \tag{12}$$

be the stepsizes and fineness characteristic of the grid (1).

Assume that $h \to +0$. The symbol $o(1)$ will be used for infinitesimals as $h \to 0$, i.e., $o(1) \underset{h\to 0}{\longrightarrow} 0$.

In what follows, we will use the Taylor expansion of the function $\rho(t)$,

$$\rho(t) = \rho_{j+k} + (t - x_{j+k})\rho'_{j+k} + (t - x_{j+k})o(1). \tag{13}$$

It is clear that for $t = x_{j+p}$ the above formula can be written as

$$\rho_{j+p} = \rho_{j+k} + (x_{j+p} - x_{j+k})\rho'_{j+k} + (x_{j+p} - x_{j+k})o(1). \tag{14}$$

Theorem 1. *The functions $\omega_j(t)$ of the form* (7) *possess the following asymptotic representation*:

$$\omega_j(t) = \begin{cases} \dfrac{t - x_j}{h_j}\left(1 + o(1)\right), & t \in [x_j, x_{j+1}), \\ \dfrac{x_{j+2} - t}{h_{j+1}}\left(1 + o(1)\right), & t \in [x_{j+1}, x_{j+2}). \end{cases} \tag{15}$$

Proof. By using expansions (13) and (14), for the function $\omega_j(t)$ and for $t \in [x_j, x_{j+1})$ from representation (7) we obtain

$$\omega_j(t) = \frac{(t - x_j)(\rho'_j + o(1))}{(x_{j+1} - x_j)(\rho'_j + o(1))} = \frac{t - x_j}{x_{j+1} - x_j}\left(1 + o(1)\right).$$

Similarly, for $t \in [x_{j+1}, x_{j+2})$ we have

$$\omega_j(t) = \frac{(t - x_{j+2})(\rho'_{j+2} + o(1))}{(x_{j+1} - x_{j+2})(\rho'_{j+2} + o(1))} = \frac{x_{j+2} - t}{x_{j+2} - x_{j+1}}\left(1 + o(1)\right).$$

With account for (12), the desired relation (15) is established. □

Remark 1. The main part of the asymptotics in (15) coincides with the representation of the B-spline (11).

On an interval $[a, b] \subset \mathbb{R}^1$, consider a grid X_n, where

$$x_{-1} \le a = x_0 \le x_1 \le \cdots \le x_{n-1} \le x_n = b \le x_{n+1}. \tag{16}$$

The grid nodes (16) whose values coincide are called *multiple nodes*. If a node x_j of the grid X_n occurs k times, i.e., $x_j = x_{j+1} = \cdots = x_{j+k-1}$, then it has multiplicity k.

Theorem 2. *At nodes of multiplicity* 2 *the function ω_j belongs to the space $C^{-1}[a, b]$; furthermore,*

(1) *if $x_j = x_{j+1} < x_{j+2}$, then $\omega_j(x_j + 0) = 1$;*
(2) *if $x_j < x_{j+1} = x_{j+2}$, then $\omega_j(x_{j+1} - 0) = 1$.*

Proof. Obviously, $\omega_j(x_j - 0) = 0$. By virtue of representation (15), we have

$$\omega_j(x_j + 0) = \lim_{t \to x_{j+1}+0} \frac{x_{j+2} - t}{h_{j+1}}(1 + o(1)) = 1,$$

which implies the first assertion of the theorem. The second assertion is established in a similar way. □

Remark 2. For grids of the form (1) the claim of Theorem 2 can be interpreted as follows. It is unnecessary to introduce extra nodes in the grid (1) outside of $[a, b]$. Then the boundary functions ω_{-1} and ω_{n-1} must be defined by the following formulas:

$$\omega_{-1}(t) = \begin{cases} \dfrac{\rho_1 - \rho(t)}{\rho_1 - \rho_0}, & t \in [x_0, x_1), \\ 0 & \text{otherwise;} \end{cases}$$

$$\omega_{n-1}(t) = \begin{cases} \dfrac{\rho(t) - \rho_{n-1}}{\rho_n - \rho_{n-1}}, & t \in [x_{n-1}, x_n], \\ 0 & \text{otherwise.} \end{cases}$$

4. Spline approximation

Consider a linear space $\mathfrak{U}$ over the field of reals and its conjugate space $\mathfrak{U}^*$ of linear functionals λ over the space $\mathfrak{U}$. The value of a functional λ on an element $u \in \mathfrak{U}$ is denoted by $\langle \lambda, u \rangle$.

A system of functionals $\{\mu_i\}_{i \in \mathbb{Z}}$ is said to be *biorthogonal* to the system of functions $\{f_j\}_{j \in \mathbb{Z}}$ if $\langle \mu_i, f_j \rangle = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker symbol. The functionals μ_i are the *biorthogonal* or *dual* functionals for the functions f_j.

Consider the splines (7), their derivatives

$$\omega_j'(t) = \begin{cases} \dfrac{\rho'(t)}{\rho_{j+1} - \rho_j}, & t \in [x_j, x_{j+1}), \\ -\dfrac{\rho'(t)}{\rho_{j+2} - \rho_{j+1}}, & t \in [x_{j+1}, x_{j+2}), \\ 0 & \text{otherwise,} \end{cases} \tag{17}$$

and the linear functionals $\lambda_j^{\langle r \rangle}$, $r = 0, 1$, $j \in J_{-1,n-1}$, given by the following formulas:

$$\langle \lambda_j^{\langle 0 \rangle}, u \rangle := u(x_j) + \frac{\rho_{j+1} - \rho_j}{\rho_j'} u'(x_j), \quad u \in C^1[a, b]; \tag{18}$$

$$\langle \lambda_j^{\langle 1 \rangle}, u \rangle := u(x_{j+1}), \quad u \in C[a, b]. \tag{19}$$

Theorem 3. *For any fixed $r \in \{0, 1\}$, the system of linear functionals $\{\lambda_j^{\langle r \rangle}\}$ defined by* (18)–(19) *is biorthogonal to the system of functions $\{\omega_{j'}\}$, i.e.,*

$$\langle \lambda_j^{\langle r \rangle}, \omega_{j'} \rangle = \delta_{j,j'}, \quad j, j' \in J_{-1,n-1}. \tag{20}$$

Proof. For $r = 1$, the theorem assertion is obvious in view of (8). For $r = 0$, the left-hand side of (20) can be represented in the form

$$\langle \lambda_j^{\langle 0 \rangle}, \omega_{j'} \rangle = \omega_{j'}(x_j) + \frac{\rho_{j+1} - \rho_j}{\rho_j'} \omega_{j'}'(x_j). \tag{21}$$

In view of the distribution of the supports of the functions $\omega_{j'}$ and $\omega_{j'}'$, occurring in (21), the equality $\langle \lambda_j^{\langle 0 \rangle}, \omega_{j'} \rangle = 0$ holds for all j such that $j \neq j'$ and $j \neq j' + 1$. If $j = j'$, then, by using (8) and (17), for $t = x_j$ we obtain $\langle \lambda_j^{\langle 0 \rangle}, \omega_j \rangle = 1$. Using the same properties, for $j = j' + 1$ we derive $\langle \lambda_j^{\langle 0 \rangle}, \omega_{j-1} \rangle = 0$. This completes the proof of the theorem. □

Remark 3. For a general method for constructing functionals biorthogonal to minimal splines, see [22, 23].

Consider the interpolation problem

$$\langle \lambda_i^{\langle r\rangle}, \widetilde{u}\rangle = v_i, \quad i \in J_{-1,n-1}, \quad \widetilde{u} \in \mathbb{S}(X), \tag{22}$$

where $\{v_i\}$ is a given sequence of numbers. For any fixed $r \in \{0,1\}$, in the space $\mathbb{S}(X)$ there is a unique solution of the direct interpolation problem (22), which is given by

$$\widetilde{u}(t) = \sum_{j=-1}^{n-1} v_j\, \omega_j(t), \quad t \in [a,b].$$

Given a function u on $[a,b]$, consider the spline

$$u_h(t) = \sum_{j=-1}^{n-1} \langle \mu_j, u\rangle\, \omega_j(t), \quad t \in [a,b], \tag{23}$$

where $\langle \mu_j, u\rangle$ are some linear functionals, which will be referred to as the *approximation* functionals. If the system of functionals $\{\mu_i\}$ is biorthogonal to the system of functions $\{\omega_j\}$, then the approximation (23) will be called the *biorthogonal* spline approximation.

Consider the biorthogonal spline approximation (23) for $\langle \mu_j, u\rangle = \langle \lambda_j^{\langle r\rangle}, u\rangle$. Taking into account the location of the supports of the functions ω_j for $t \in [x_k, x_{k+1}]$, we conclude that the sum (23) involves only two nonzero terms, whence

$$u_h^{\langle r\rangle}(t) = \sum_{j=k-1}^{k} \langle \lambda_j^{\langle r\rangle}, u\rangle\, \omega_j(t), \quad t \in [x_k, x_{k+1}]. \tag{24}$$

Theorem 4. *For any fixed $r \in \{0,1\}$, the approximation* (24) *is sharp on the components of the vector function $\boldsymbol{\varphi}$, i.e., if $u \in \{1, \rho(t)\}$, then*

$$u_h^{\langle r\rangle}(t) \equiv u(t).$$

Proof. The approximation relations (3) can be written in the following componentwise form:

$$\sum_{j=k-1}^{k} \langle \lambda_j^{\langle r\rangle}, [\boldsymbol{\varphi}]_i\rangle\, \omega_j(t) = [\boldsymbol{\varphi}]_i(t), \quad i = 0,1. \tag{25}$$

For $i = 0$ we have $[\boldsymbol{\varphi}]_i = 1$, and $\langle \lambda_j^{\langle r\rangle}, 1\rangle = 1$ for $r = 0,1$. Therefore, sharpness is equivalent to the partition of unity (10). For $[\boldsymbol{\varphi}(t)]_1 = \rho(t)$ we deduce that $\langle \lambda_j^{\langle r\rangle}, \rho\rangle = \rho_{j+1}$ for $r = 0,1$, whence sharpness on the function $u = \rho$ immediately follows from relations (24) and (25). □

Let $S(t)$ denote the approximation (24) with $r = 1$ and the approximation functional (19), i.e., set $S(t) := u_h^{\langle 1\rangle}(t)$. Then, for $t \in [x_k, x_{k+1}]$, we have

$$\begin{aligned} S(t) &= \sum_{j=k-1}^{k} u(x_{j+1})\, \omega_j(t) = u(x_k)\, \omega_{k-1}(t) + u(x_{k+1})\, \omega_k(t) \\ &= u(x_k)\,(1 - \omega_k(t)) + u(x_{k+1})\, \omega_k(t) = u(x_k) + \big(u(x_{k+1}) - u(x_k)\big)\, \omega_k(t). \end{aligned} \tag{26}$$

From the computational point of view, in order to reduce the number of operations, from all formulas (26), with account for (7), one should choose the formula

$$S(t) = u_k + \big(u_{k+1} - u_k\big)\, \frac{\rho(t) - \rho_k}{\rho_{k+1} - \rho_k}, \quad t \in [x_k, x_{k+1}], \tag{27}$$

where $u_k := u(x_k)$.

Now it is clear that the spline (27) is an interpolation spline, i.e.,

$$S(x_k) = u_k, \quad k \in J_{0,n}.$$

For the generating vector function $\boldsymbol{\varphi}(t) = (1,t)^T$, i.e., for $\rho(t) = t$, the approximation (27) is the well-known continuous piecewise linear interpolation. For other generators, the approximation (27) will be referred to as the *quasilinear* interpolation (by minimal splines).

5. Error of quasilinear interpolation

In this section, in addition to conditions (6) imposed on the function $\rho(t)$, we will also assume that $\rho \in C^2[a,b]$. This will allow us to continue using the Taylor expansion.

The norm in the space $C[a,b]$ will be defined by the relation

$$\|f(t)\|_{C[a,b]} = \max_{t\in[a,b]} |f(t)|, \quad f \in C[a,b].$$

Theorem 5. *For a function $u \in C^1[a,b]$, the approximation error satisfies the upper bound*

$$|u(t) - S(t)| \le 2h \, \|u'(t)\|_{C[a,b]},$$

and for a function $u \in C^2[a,b]$, it satisfies the upper bound

$$|u(t) - S(t)| \le Ch^2 \left(\|u'(t)\|_{C[a,b]} + \|u''(t)\|_{C[a,b]} \right),$$

where the constant $C > 0$ is independent of u and h and is given by a closed-form expression.

Proof. By using representation (26), for $t \in [x_k, x_{k+1}]$ we obtain

$$|S(t) - u(t)| = |u_k + \big(u_{k+1} - u_k\big)\, \omega_k(t) - u(t)|. \tag{28}$$

For a function $u \in C^1[a,b]$, the formulas of finite increments, valid for some intermediate values $\theta_{k,t} \in (x_k, t)$ and $\theta_{k,k+1} \in (x_k, x_{k+1})$, yield

$$u(t) = u_k + u'(\theta_{k,t})(t - x_k), \quad u_{k+1} = u_k + u'(\theta_{k,k+1})(x_{k+1} - x_k).$$

Substituting the latter expressions into (28), we find

$$|S(t) - u(t)| = |u'(\theta_{k,k+1})(x_{k+1} - x_k)\omega_k(t) - u'(\theta_{k,t})(t - x_k)|.$$

From (9) and (10) it follows that $|\omega_k(t)| \le 1$. Therefore, in the notation (12), we have

$$|S(t) - u(t)| \le 2 \max_{t\in[x_k,x_{k+1}]} |u'(t)|\, h,$$

and the first assertion of the theorem follows.

Now let expression (28) involve a function $u \in C^2[a,b]$, for which Taylor's formula with the Lagrange remainder yields

$$u(t) = u_k + u'_k(t - x_k) + \frac{1}{2}u''(\overline{\theta}_{k,t})(t - x_k)^2,$$
$$u_{k+1} = u_k + u'_k(x_{k+1} - x_k) + \frac{1}{2}u''(\overline{\theta}_{k,k+1})(x_{k+1} - x_k)^2,$$

where $\overline{\theta}_{k,t} \in (x_k, t)$ and $\overline{\theta}_{k,k+1} \in (x_k, x_{k+1})$.

Substituting the above expressions for $u(t)$ and $u_{k+1} - u_k$ into (28), we find

$$|S(t) - u(t)| = \Big| u'_k(x_{k+1} - x_k)\, \omega_k(t) - u'_k(t - x_k) + \frac{1}{2}u''(\overline{\theta}_{k,k+1})(x_{k+1} - x_k)^2\omega_k(t) - \frac{1}{2}u''(\overline{\theta}_{k,t})(t - x_k)^2 \Big|.$$

Denote

$$P := u'_k(x_{k+1} - x_k)\, \omega_k(t) - u'_k(t - x_k). \tag{29}$$

Then from the previous relation we obtain the upper bound

$$|S(t)-u(t)| \le |P| + \frac{1}{2}\left|u''(\overline{\theta}_{k,k+1})\right|(x_{k+1}-x_k)^2\,|\omega_k(t)| + \frac{1}{2}\left|u''(\overline{\theta}_{k,t})\right|(t-x_k)^2.$$

Using the notation (12) and the fact that $|\omega_k(t)| \le 1$, we derive

$$|S(t)-u(t)| \le |P| + \max_{t\in[x_k,x_{k+1}]}\left|u''(t)\right| h^2. \tag{30}$$

Consider the first term of the right-hand side of inequality (30). With account for (7) and (29), we have

$$P = u_k'(x_{k+1}-x_k)\left(\frac{\rho(t)-\rho_k}{\rho_{k+1}-\rho_k} - \frac{t-x_k}{x_{k+1}-x_k}\right). \tag{31}$$

The Taylor expansion of the function $\rho(t)$ with Lagrange remainder, including the formula of finite increments, has the form

$$\rho(t) = \rho_k + \rho_k'(t-x_k) + \frac{1}{2}\rho''(\overline{\tau}_{k,t})(t-x_k)^2, \quad \rho_{k+1} = \rho_k + \rho'(\tau_{k,k+1})(x_{k+1}-x_k), \tag{32}$$

where $\overline{\tau}_{k,t} \in (x_k, t)$ and $\tau_{k,k+1} \in (x_k, x_{k+1})$.

Upon substituting (32) into (31), we obtain

$$\begin{aligned} P &= u_k'(x_{k+1}-x_k)\left(\frac{\rho_k'(t-x_k)+\frac{1}{2}\rho''(\overline{\tau}_{k,t})(t-x_k)^2}{\rho'(\tau_{k,k+1})(x_{k+1}-x_k)} - \frac{t-x_k}{x_{k+1}-x_k}\right) \\ &= u_k'(t-x_k)\left(\frac{\rho_k' - \rho'(\tau_{k,k+1}) + \frac{1}{2}\rho''(\overline{\tau}_{k,t})(t-x_k)}{\rho'(\tau_{k,k+1})}\right). \end{aligned} \tag{33}$$

Note that for the function ρ' there is a point $\sigma_{k,\tau} \in (x_k, \tau_{k,k+1})$ such that

$$\rho_k' - \rho'(\tau_{k,k+1}) = \rho''(\sigma_{k,\tau})(x_k - \tau_{k,k+1}).$$

Therefore, we can write (33) in the form

$$P = u_k'(t-x_k)\left(\frac{\rho''(\sigma_{k,\tau})(x_k-\tau_{k,k+1}) + \frac{1}{2}\rho''(\overline{\tau}_{k,t})(t-x_k)}{\rho'(\tau_{k,k+1})}\right)$$

and bound $|P|$ from above as follows:

$$\begin{aligned} |P| &\le \left|u_k'\right|\,|t-x_k|\,\frac{|\rho''(\sigma_{k,\tau})|\,|x_k-\tau_{k,k+1}| + \frac{1}{2}|\rho''(\overline{\tau}_{k,t})|\,|t-x_k|}{|\rho'(\tau_{k,k+1})|} \\ &\le \frac{\frac{3}{2}\max\limits_{t\in[x_k,x_{k+1}]}|\rho''(t)|}{\min\limits_{t\in[x_k,x_{k+1}]}|\rho'(t)|}\max_{t\in[x_k,x_{k+1}]}|u'(t)|\,h^2. \end{aligned} \tag{34}$$

Finally, by combining inequalities (30) and (34), we obtain

$$|S(t)-u(t)| \le Ch^2\left(\max_{t\in[x_k,x_{k+1}]}\left|u'(t)\right| + \max_{t\in[x_k,x_{k+1}]}\left|u''(t)\right|\right), \tag{35}$$

where $C = \max\left\{1,\ \frac{3}{2}\max\limits_{t\in[a,b]}|\rho''(t)| \,/\, \min\limits_{t\in[a,b]}|\rho'(t)|\right\}$, and $\max\{\cdot,\cdot\}$ is the maximum of two numbers.

Now the second assertion of the theorem immediately follows from inequality (35). □

Remark 4. In order to derive an error bound for a certain approximation, one can use the method of integral representation of the remainder associated with an ordinary differential operator, for which the components of the generating vector function provide a fundamental system of solutions of the corresponding homogeneous equation (for more detail, see [9]).

6. Numerical experiments on quasilinear interpolation

Consider the quasilinear interpolation of the functions $u(t) = \arctan(t)$ and $u(t) = \sqrt{1-t^2}$ on the interval $[a, b] = [0.1, 0.6]$. To this end, we will use grids of the form (1) with account for Remark 2. Such grids will be denoted by $X_{n,[a,b]}$, and their nodes x_j, $j \in J_{0,n}$, will be determined by the formula $x_j = x_0 + j\,h$, where $x_0 = a$, $h = \frac{b-a}{n}$ is the stepsize, and $n = 10, 20, 30$.

In performing numerical experiments for functions $u(t)$ to be approximated, the quasilinear interpolation $S(t)$ will be constructed using formula (27) and different generating functions $\rho(t)$.

The error E_n of an approximation constructed is computed as the maximum of the absolute value of the deviation of the interpolant $S(t)$ from the original function $u(t)$ at the nodes of the auxiliary grid that is ten times finer than the original one, i.e.,

$$E_n = \max_{t_j \in X_{10n,[a,b]}} |u(t_j) - S(t_j)|. \tag{36}$$

Results of numerical experiments on evaluating the approximation error (36) as a function of the grid stepsize and the generating function $\rho(t)$ are presented in Tables 1 and 2. The first row of each of the tables corresponds to the case where $\rho(t) = t$, i.e., to the piecewise linear interpolation (using the B-splines of the first degree).

Table 1. Approximation errors for the function $u(t) = \arctan(t)$.

$\rho(t)$	$n = 10$	$n = 20$	$n = 30$
t	0.000203	0.000051	0.000023
$\sin(t)$	0.000072	0.000018	0.000008
$\tanh(t)$	**0.000041**	**0.000011**	**0.000005**

Table 2. Approximation errors for the function $u(t) = \sqrt{1-t^2}$.

$\rho(t)$	$n = 10$	$n = 20$	$n = 30$
t	0.000571	0.000147	0.000066
$\sqrt{1-t}$	0.000312	0.000079	0.000035
$\cosh(t)$	**0.000148**	**0.000040**	**0.000018**

As is seen from the above tables, the accuracy of the approximations obtained depends on the choice of the generating function $\rho(t)$. The last rows of the tables contain the best results obtained for the generating functions chosen. The results of numerical experiments agree with the order of accuracy with respect to the grid stepsize provided by Theorem 5. In view of Theorem 4, it is obvious that the best approximation ($E_n = 0$ up to round-off errors) is obtained for the generating function $\rho(t) = u(t)$. This result, being trivial, is not included in the tables.

Note that approximation of an arc of the circle (see Table 2) is of independent interest as it is widely used in computer-aided design systems (for more detail, see [24]).

Translated by the authors.

Grant Support

This work was supported by the Ministry of Science and Higher Education of the Russian Federation (grant No. FSEE-2021-0015).

Declarations

Data availability This manuscript has no associated data.
Ethical Conduct Not applicable.
Conflicts of interest The authors declare that there is no conflict of interest.

REFERENCES

1. M. Buhmann and J. Jäger, *Quasi-Interpolation*, Cambridge University Press (2022).
2. C. de Boor, *A Practical Guide to Splines*, Springer-Verlag, New York (2001).
3. G. Y. Deniskina, Y. I. Deniskin, and Y. I. Bityukov, "About some computational algorithms for local approximation splines, based on the wavelet transformation and convolution," *Lect. Notes Elect. Engin.*, **729**, 182–191 (2021).
4. A. Yu. Dem'yanovich and V. N. Malozemov, "Jenkins' functions and minimal splines," *Vestn. S.Peterb. Univ.*, Ser. 1., **2**, No. 8, 38–43 (1998).
5. Yu. K. Dem'yanovich, "On the construction of spaces of local functions on a nonuniform grid," *Zap. Nauchn. Semin. LOMI*, **124**, 140–163 (1983).
6. Yu. K. Dem'yanovich, *Local Approximation on a Manifold and Minimal Splines* [in Russian], St.Petersburg Univ. Press, St.Petersburg (1994).
7. Yu. K. Dem'yanovich, *Wavelets and Minimal Splines* [in Russian], St.Petersburg Univ. Press, St.Petersburg (2003).
8. Yu. K. Dem'yanovich, *Theory of Spline Wavelets* [in Russian], St.Petersburg Univ. Press, St.Petersburg (2013).
9. Yu. K. Dem'yanovich, V. O. Dron' , and O. N. Ivantsova, "Approximation by B_{φ}-splines," *Vestn. St.Peterburg. Univ.*, Ser. 10. Prikl. Matem. Inform. Prots. Upr., **3**, 67–72 (2013).
10. Yu. K. Dem'yanovich and A. A. Makarov, "Necessary and sufficient nonnegativity conditions for second-order coordinate trigonometric splines," *Vestn. St.Peterb. Univ.*, Ser. 1., **4(62)**, No. 1, 9–16 (2017).
11. Yu. K. Dem'yanovich and S. G. Mikhlin, "Grid approximation of functions of Sobolev spaces," *Zap. Nauchn. Semin. LOMI*, **35**, 6–11 (1973).
12. J. J. Goel, "Construction of basis functions for numerical utilization of Ritz's method," *Numer. Math.*, **12**, 435–447 (1968).
13. O. Kosogorov and A. Makarov, "On some piecewise quadratic spline functions," *Lect. Notes Comput. Sci.*, **10187**, 448–455 (2017).
14. E. K. Kulikov and A. A. Makarov, 'Approximation by hyperbolic splines," *Zap. Nauchn. Semin. POMI*, **472**, 179–194 (2018); English transl., *J. Math. Sci.*, **240**, No. 6, 822–832 (2019).
15. E. K. Kulikov and A. A. Makarov, "Quadratic minimal splines with multiple nodes," *Zap. Nauchn. Semin. POMI*, **482**, 220–230 (2019); English transl., *J. Math. Sci.*, **249**, No. 2, 256–262 (2020).
16. E. Kulikov and A. A. Makarov, "On de Boor–Fix type functionals for minimal splines," in: *Topics in Classical and Modern Analysis (Applied and Numerical Harmonic Analysis)*, 211–225 (2019).
17. E. Kulikov and A. A. Makarov, "On biorthogonal approximation of solutions of some boundary value problems on Shishkin mesh," *AIP Conf. Proc.*, **2302**, 110005 (2020).

18. E. K. Kulikov and A. A. Makarov, "On a modified spline collocation method for solving the Fredholm integral equation," *Diff. Uravn. Prots. Upr.*, **4**, 211–223 (2021).
19. E. K. Kulikov and A. A. Makarov, "A method for solving the Fredholm integral equation of the first kind," *Zap. Nauchn. Semin. POMI*, **514**, 113–125 (2022); English transl., *J. Math. Sci.*, **272**, No. 4, 558–565 (2023).
20. V. L. Leontiev, "Orthogonal splines in approximation of functions," *Math. Statistic*, **8**, No. 2, 167–172 (2020).
21. A. A. Makarov, "Construction of splines of maximal smoothness," *Probl. Mat. Anal.*, **60**, 25–38 (2011).
22. A. A. Makarov, "Biorthogonal systems of functionals and decomposition matrices for minimal splines," *Ukr. Mat. Visn.*, **9**, No. 2, 219–236 (2012).
23. A. A. Makarov, "On functionals dual to minimal splines," *Zap. Nauchn. Semin. POMI*, **453**, 198–218 (2016); English transl., *J. Math. Sci.*, **224**, No. 6, 942–955 (2017).
24. A. A. Makarov, "On an example of circular arc approximation by quadratic minimal splines," *Poincaré J. Anal. Appl.*, **2(II)**, 103–107 (2018).
25. S. G. Mikhlin, "Variational-grid approximation," *Zap. Nauchn. Semin. LOMI*, **48**, 32–188 (1974).
26. V. S. Ryaben'ky, *Stability of Finite-Difference Equations* [in Russian], Dis. kand. fiz.-mat. nauk, Moscow (1952).
27. I. J. Schoenberg, "Contributions to the problem of approximation of equidistant data by analytic functions," *Quart. Appl. Math.*, **4**, 45–99, 112–141 (1946).
28. L. L. Schumaker, *Spline Functions: Computational Methods*, Society for Industrial and Applied Mathematics (2015).
29. B. M. Shumilov, E. A. Esharov, and N. K. Arkabaev, "Construction and optimization of forecasts based on recurrent splines of the first degree," *Sib. Zh. Vychisl. Matem.*, **13**, No. 2, 227–241 (2010).
30. G. Strang and G. Fix, "Fourier analysis of the finite element method in Ritz–Galerkin theory," *Stud. Appl. Math.*, **48**, No. 3, 265–273 (1969).
31. G. Strand and J. Fix, *Theory of the Finite-Element Method* [in Russian], Moscow (1977).
32. Yu. S. Volkov and V. V. Bogdanov, "Error estimates for the simplest local approximation by splines," *Sib. Matem. Zh.*, **61**, No. 5, 1000–1008 (2020).
33. I. V. Yuyukin, "Interpolation of a navigation function by Lagrangian splines," *Vestn. Gos. Univ. Morsk. Rechn. Flota*, **12**, No. 1, 57–70 (2020).
34. A. I. Zadorin, "Interpolation method for a function with a singular component," *Lect. Notes Comput. Sci.*, **5434**, 612–619 (2009).
35. Yu. S. Zavyalov, B. I. Kvasov, and V. L. Miroshnichenko, *Methods of Spline Functions* [in Russian], Moscow (1980).

Submitted on January 18, 2024.