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ON THE LIMIT DISTRIBUTION FUNCTION OF THE VALUE OF A DIFFUSION SEMI-MARKOV PROCESS ON INTERVAL WITH UNATTAINABLE BOUNDARIES

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A diffusion semi-Markov process on a finite interval with unattainable boundaries is considered. It is assumed that unattainable property is not connected with process stop in the interval. A limit theorem for alternating renewal processes is applied to derive the limit distribution function of the diffusion process. Bibliography: 8 titles

1. A diffusion semi-Markov process

Let $\mathcal{C}$ be the standard metric space of continuous functions ξ defined on the interval $[0,\infty)$ with values in $\mathbb{R}$ (sample trajectories of the process), $\mathcal{F}$ the standard sigma-algebra of subsets of $\mathcal{C}$, and P a probability measure on $\mathcal{F}$. A continuous random process of the general form $\mathcal{X}$ is completely characterized by the triple $(\mathcal{C},\mathcal{F},P)$, where in some cases it is convenient to consider a consistent set of probability measures rather than a single probability measure.

Let X_t $(t\ge 0)$ be a function with parameter t on $\mathcal{C}$, defined by the equality $X_t(\xi)=\xi(t)$ (one-coordinate projection of the trajectory), and let θ_t be a shift operator on $\mathcal{C}$, where $\theta_t(\xi)\in\mathcal{C}$ and $X_s(\theta_t(\xi))=\xi(t+s)$ for all $s\ge 0$. We also consider the class $\mathcal{T}$ of measurable mappings τ, where $0\le\tau(\xi)\le\infty$ (random moments of time admitting infinite values) for all ξ as well as the class $\mathcal{T}_0$ of Markov moments (times) defined in the standard way.

Let $\tau\in\mathcal{T}$. On the set $\{\tau<\infty\}$, we define a functional X_τ and an operator θ_τ with random parameter, where

$$X_\tau(\xi)\equiv X_{\tau(\xi)}(\xi),\quad \theta_\tau(\xi)\equiv\theta_{\tau(\xi)}(\xi).$$

Let $f_2\circ f_1\equiv f_2(f_1)$ (superposition of two functions). On the set

$$\{\tau_1<\infty,\,\tau_2\circ\theta_{\tau_1}<\infty\},$$

we can write

$$\theta_{\tau_2}\circ\theta_{\tau_1}=\theta_{\tau_2}(\theta_{\tau_1})=\theta_{\tau_2(\theta_{\tau_1})}(\theta_{\tau_1})=\theta_{\tau_3},$$

where

$$\tau_3\equiv\tau_1+\tau_2(\theta_{\tau_1})\equiv\tau_1\dot{+}\tau_2.$$

This is the so-called shift addition defined on the set $\tau_1<\infty$. Note that the operation $\dot{+}$ is associative but not commutative.

Thus,

$$\theta_{\tau_1\dot{+}\tau_2}=\theta_{\tau_2}\circ\theta_{\tau_1}.$$

Then

$$\theta_{\tau_1\dot{+}\ldots\dot{+}\tau_n}=\theta_{\tau_n}\circ\ldots\circ\theta_{\tau_1}. \tag{1}$$

In what follows, we put $X_\tau\equiv X(\tau)$, assuming that both of these functions are random (depend on ξ).

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Let (P_x), $x \in \mathbb{R}$, be a family of probability measures on $\mathcal{F}$ such that $P_x(S) \equiv P_x(X(0)=x,\ S)$ for all x and any set $S \in \mathcal{F}$. We assume that this family is measurable with respect to the parameter x.

A Markov moment τ_1 is called a regeneration moment of the family of measures (P_x) $(x \in \mathbb{R})$ if for all x, any $\mathcal{F}_{\tau_1}$-measurable function f_1, and any $\mathcal{F}$-measurable function f_2, we have

$$E_x(f_1 \cdot (f_2 \circ \theta_{\tau_1});\ \tau_1 < \infty,\ \tau_2 \circ \theta_{\tau_1} < \infty) = E_x(f_1 \cdot E_{X(\tau_1)}(f_2;\ \tau_2 < \infty);\ \tau_1 < \infty),$$

where $\mathcal{F}_{\tau_1}$ is the standard sigma-algebra of events (sets) "preceding" the moment τ_1, and $E_x(f;\ S)$ is the integral of the function f on the set S over the measure P_x.

For continuous semi-Markov processes, the moments of the first exit from open sets play a special role. Let $\Delta = (a, b)$, where $a < b$. Denote by σ_Δ the moment of the first exit of the process $\mathcal{X}$ from the set Δ. It is known (see [1, p. 194]) that σ_Δ is a Markov moment with respect to natural filtering.

A continuous semi-Markov process is a process given by a family of measures (P_x) such that the Markov moment σ_Δ is a regeneration moment for any Δ.

Put

$$g_\Delta(\lambda, x) = E_x\left(\exp(-\lambda\sigma_\Delta), \quad \sigma_\Delta < \infty,\ \ X(\sigma_\Delta) = a\right),$$
$$h_\Delta(\lambda, x) = E_x\left(\exp(-\lambda\sigma_\Delta), \quad \sigma_\Delta < \infty,\ \ X(\sigma_\Delta) = b\right).$$

These are the so-called semi-Markov transition generating functions. It is known (see [2]) that the system of these functions defines a semi-Markov process.

Let $\Delta_1 = (c, d)$ and $\Delta = (a, b)$, where $\Delta_1 \subset \Delta$ and $x \in \Delta_1$. From the definition of a continuous semi-Markov process, it follows that the system of two equations below is satisfied:

$$g_\Delta(\lambda, x) = g_{\Delta_1}(\lambda, x) g_\Delta(\lambda, c) + h_{\Delta_1}(\lambda, x) g_\Delta(\lambda, d), \tag{2}$$
$$h_\Delta(\lambda, x) = g_{\Delta_1}(\lambda, x) h_\Delta(\lambda, c) + h_{\Delta_1}(\lambda, x) h_\Delta(\lambda, d). \tag{3}$$

A continuous semi-Markov process is called a diffusion semi-Markov process on the interval Δ if each of these functions satisfies the differential equation

$$\frac{1}{2} y'' + A(x) y' - B(\lambda,\ x) y = 0 \tag{4}$$

with boundary values

$$g_\Delta(\lambda,\ a) = h_\Delta(\lambda,\ b) = 1, \qquad g_\Delta(\lambda,\ b) = h_\Delta(\lambda,\ a) = 0.$$

In this equation, $A(x)$ is a continuously differentiable function, $B(\lambda,\ x)$ is a positive function continuous on the second argument, nondecreasing, continuously differentiable on the first argument, and having a completely monotone partial derivative on the first argument. The reason for this definition and, in particular, the property that the coefficient $A(x)$ is independent of λ, follows from the properties of the Laplace transform (see [2, pp. 159–163]).

When $\lambda = 0$, equation (4) becomes the equation

$$\frac{1}{2} u'' + A(x) u' - B(0, x) u = 0,$$

where the solution does not depend on λ. It is known (see, e.g., [3, p. 15]) that the case $B(0, x) \equiv 0$ relates to a Markov diffusion process without break. The same property is also true for a semi-Markov diffusion process without infinite constancy interval (i.e., without stopping "forever"). This property follows from [4, formula (18)].

The boundary a of the interval (a, b) is said to be regular for a continuous semi-Markov process if

$$P_x(\sigma_{(a,b)} < \infty, X(\sigma_{(a,b)}) = a) > 0$$

for all $x \in (a, b)$. The boundary a of the interval (a, b) is said to be unattainable for a continuous semi-Markov process if

$$P_x(\sigma_{(a,b)} < \infty, X(\sigma_{(a,b)}) = a) = 0$$

for all $x \in (a, b)$. The regularity and unattainability of the right boundary of the interval (a, b) are defined similarly.

We are interested in a diffusion semi-Markov process with values on the interval (a_0, b_0) for which both bounds are unattainable.

The conditions of unattainability of the boundaries of the value interval of a diffusion semi-Markov process whose transient derivative functions are given by the differential equation (4) without the summand $B(\lambda, x)u$ were considered in [5]. Necessary and sufficient conditions for unattainability in terms of the coefficient $A(x)$ were obtained there. However, the limit distribution was not considered in that paper.

2. Alternating recovery process

To find the limit (as $t \to \infty$) distribution, we use the results of recovery theory.

The recovery process is a random piecewise constant process $\mathcal{Z}$ with nondecreasing trajectories for which the interval lengths between jumps are mutually independent positive random variables (independence is understood with respect to some probability measure P).

An alternating recovery process $X(t)$ is a recovery process for which all odd intervals between jumps have the same distribution F_1 and all even intervals between jumps have the same distribution F_2, where, in general, $F_1 \neq F_2$ (the names odd and even are defined according to the sequence of jump points of a piecewise constant process $\mathcal{Z}$) (see, e.g., [6]). We use the following theorem of the theory of alternating recovery processes.

Theorem 1. *The limit as $t \to \infty$ of the probability that $X(t)$ belongs to the even interval of constancy of this process is equal to p_1, where*

$$p_1 = \frac{m_1}{m_1 + m_2},$$

and m_1 and m_2 are the expectations of the lengths of the odd and even intervals, respectively.

The proof is given in [6, p. 98].

Let $a_0 < a < b < b_0$, and let $(\Delta_n)_{n=1}^{\infty}$ be a sequence of intervals, where $\Delta_n \subset (a_0, b_0)$ and the boundaries of the last interval are unreachable and every interior point of the interval is regular. This sequence gives rise to a set of nondecreasing sequences $(T_k^n)_{n=1}^{\infty}$, $1 \le k \le n$ points on the time axis, where

$$T_1^1 = \sigma_{\Delta_1}, \quad T_k^{n+1} = T_k^n \dot{+} \sigma_{\Delta_{n+1}},$$

whence

$$T_k^n = \sigma_{\Delta_k} \dot{+} \sigma_{\Delta_{k+1}} \dot{+} \ldots \dot{+} \sigma_{\Delta_n}.$$

Also put $T_k^{k-1} = 0$.

Let (t_n) be an arbitrary sequence of numbers ($t_n \in \mathbb{R}$). Consider the set of sequences of random variables $(S_k^n)_{k=1}^n$, where

$$S_k^n = t_k \sigma_{\Delta_k} + S_{k+1}^n \circ \theta_{\sigma_{\Delta_k}}.$$

Also put $S_{n+1}^n = 0$.

It follows that

$$S_k^n = \sum_{m=k}^{n} t_m \, \sigma_{\Delta_m} \circ \theta_{T_k^{m-1}},$$

and hence

$$S_1^n = t_1\sigma_{\Delta_1} + \sum_{k=2}^{n} t_k\,\sigma_{\Delta_k}\circ\theta_{T_1^{k-1}}$$

Let $\Delta_n = (a_0, b)$ if n is odd, and $\Delta_n = (a, b_0)$ if n is even.

Condition A. For any interval Δ, where $\Delta \subset (a_0, b_0)$ and $\Delta \neq (a_0, b_0)$, and for any $x \in \Delta$, we have $P_x(\sigma_\Delta < \infty) = 1$ and $P_x(\sigma_{(a_0,b_0)} < \infty) = 0$.

From this condition and the definition of unattainability of the boundaries of the interval (a_0, b_0), it follows that $X_{T_1^n} = b$ with P_a-probability one if n is odd, and $X_{T_1^n} = a$ if n is even.

Furthermore, from Condition A, it follows that $P_x(T_k^n < \infty) = 1$ for all $n \geq 1$ and $1 \leq k \leq n$, and therefore $E_x(f;\, T_k^n < \infty) = E_x(f)$.

Lemma 1. *Let a be initial point of the process $\mathcal{X}$ on the interval (a_0, b_0) with unreachable boundaries. If Condition A is satisfied, then for any $n \geq 2$, the random variables $T_1^1, T_1^2 - T_1^1, \ldots T_1^n - T_1^{n-1}$ are mutually independent random variables with respect to the measure P_a.*

Proof. To simplify the notation, throughout the proof of the lemma, we put

$$\tau_k \equiv \sigma_{\Delta_k},\quad \beta_k \equiv \theta_{\tau_k},\quad \phi_k \equiv t_k\tau_k,$$

and omit the symbol "$\circ$" between operators where it is undoubted.

In these notation, we have

$$S_k^n = \phi_k + S_{k+1}^n\beta_k,$$

where $T_1^1 = \tau_1$, $T_1^0 = 0$, and θ_0 is the identity transformation operator (i.e., $\theta_0(\xi) = \xi$).

By the method of inverse mathematical induction, we obtain

$$S_k^n = \sum_{m=k}^{n} \phi_m\theta_{T_k^{m-1}}. \tag{5}$$

From the condition A, it follows that

$$E_a(\exp(iS_1^n)\,;\, T_1^{n-1} < \infty) = E_a(\exp(iS_1^n)),$$

where $i \equiv \sqrt{-1}$ (imaginary unit).

By the semi-Markov property of the process, we obtain

$$E_a(\exp(iS_1^n)) = E_a(\exp(i\phi_1))E_b(\exp(iS_2^n)).$$

Since a constant function (e.g., $f \equiv 1$) is measurable with respect to any sigma-algebra, we have

$$E_b(\exp(iS_2^n)) = E_a(\exp(iS_2^n\beta_1)).$$

Consequently,

$$\begin{aligned}
E_a(\exp(iS_1^n)) &= E_a(\exp(i\phi_1))E_a(\exp(iS_2^n\beta_1))\\
&= E_a(\exp(i\phi_1))E_a(\exp(i\phi_2\beta_1))E_a(\exp(i\phi_3\beta_2\beta_1)\ldots E_a(\exp(i\phi_n\beta_{n-1}\ldots\beta_1))\\
&= E_a(\exp(i\phi_1))E_a(\exp(i\phi_2\theta_{T_1^1})E_a(\exp(i\phi_3^n\theta_{T_1^2})\ldots E_a(\exp(i\phi_n\theta_{T_1^{n-1}}))\\
&= E_a(\exp(it_1\tau_1))E_a(\exp(it_2\tau_2\theta_{T_1^1})E_a(\exp(it_3\tau_3\theta_{T_1^2})\ldots E_a(\exp(it_n\tau_n\theta_{T_1^{n-1}})).
\end{aligned}$$

On the other hand,

$$S_1^n = t_1\tau_1 + t_2\tau_2\theta_{T_1^1} + t_3\tau_3\theta_{T_1^2} + \cdots + t_n\tau_n\theta_{T_1^{n-1}}.$$

From here and from the well-known theorem on multivariate characteristic functions (see, for example, [7, p. 304]), it follows that the random variables

$$\tau_1,\ \tau_2\theta_{T_1^1},\ \tau_3\theta_{T_1^2}, \ldots \tau_n\theta_{T_1^{n-1}}$$

are mutually independent random variables with respect to the measure P_a.

From the definition of T_1^n and the associative property of the operation $\dot{+}$, it follows that $T_1^n - T_1^{n-1} = \tau_n\theta_{T_1^{n-1}}$, where $n \geq 1$.

The lemma is proved. □

From Lemma 1, it follows that the point process $\mathcal{Z} \equiv \mathcal{Z}(a,b)$ defined by the sequence of points T_1^n on the time axis is an alternating recovery process with distribution functions $F_1(t) \equiv P_a(\sigma_{(a_0,b)} < t)$ on odd intervals and $F_2(t) \equiv P_b(\sigma_{(a,b_0)} < t)$ on even intervals.

3. Limit distribution function

Put

$$l_{(\delta,\beta)}(x) \equiv E_x(\sigma_{(\delta,\beta)}, \quad \sigma_{(\delta,\beta)}) < \infty,\ X(\sigma_{(\delta,\beta)}) = \delta),$$
$$m_{(\delta,\beta)}(x) \equiv E_x(\sigma_{(\delta,\beta)}, \quad \sigma_{(\delta,\beta)}) < \infty,\ X(\sigma_{(\delta,\beta)}) = \beta),$$

where $\delta < x < \beta$.

Let $a_0 < a < b < b_0$, and let $N_t(a,b)$ be the event {at time t, the random variable X_t does not belong to the interval (b, b_0) of the alternating recovery process corresponding to the pair of points $\{a,b\}$}. From this definition, we see that the event $N_t(a,b)$ is equal to the event $\{X_t \in (a_0,b)\}$. Hence, according to Theorem 1, we have

$$\lim_{t\to\infty} P_a(X_t \in (a_0,b)) = \frac{m_{(a_0,b)}(a)}{l_{(a,b_0)}(b) + m_{(a_0,b)}(a)}.$$

It follows that on the interval (a, b_0) this fraction, as a function of b, is nondecreasing and tends to one as $b \to b_0$.

Let $a_0 < c < a < b_0$, and let $M_t(c,a)$ be the event {at time t, the random variable X_t does not belong to the interval (a_0, c) of the alternating recovery process corresponding to the pair of points $\{c,a\}$}. From this definition, we see that the event $M_t(c,a)$ is equal to the event $\{X_t \in (c,b_0)\}$. Hence, according to Theorem 1, we have

$$\lim_{t\to\infty} P_a(X_t \in (c,b_0)) = \frac{l_{(c,b_0)}(a)}{l_{(c,b_0)}(a) + m_{(a_0,a)}(c)}.$$

It follows that on the interval (a_0, a) this fraction, as a function of c, is nondecreasing and tends to one as $c \to a_0$.

As a result, we have proved the following theorem.

Theorem 2. *If Condition* A *is satisfied, then the diffusion semi-Markov process $X(t)$ with probability measure P_a on a finite interval (a_0, b_0) has the following distribution functional as $t \to \infty$:*

$$K_a(x) \equiv \lim_{t\to\infty} P_a(X(t) \in (a_0,x)) = \begin{cases} \dfrac{m(a_0,a,x)}{l(a,x,b_0)+m(a_0,a,x)} & \text{if } x \in (a,b_0), \\[2ex] \dfrac{m(a_0,x,a)}{l(x,a,b_0)+m(a_0,x,a)} & \text{if } x \in (a_0,a). \end{cases}$$

3.1. On calculating mathematical expectations. From the definition of transition derivative functions, it follows that there exist distribution functions $G_\Delta(t\,|\,x)$ and $H_\Delta(t\,|\,x)$ such that

$$g_\Delta(\lambda,x)=\int_0^\infty e^{-\lambda t}dG_\Delta(t\,|\,x),$$

$$h_\Delta(\lambda,x)=\int_0^\infty e^{-\lambda t}dH_\Delta(t\,|\,x).$$

By the uniform continuity of the functions $g_\Delta(\lambda,x)$ and $h_\Delta(\lambda,x)$ over λ and in view of these integral representations, the derivatives of them over λ can be obtained as the result of differentiation under the integrals

$$[g_\Delta(\lambda,x)]'_\lambda=\int_0^\infty(-t)e^{-\lambda t}dG_\Delta(t\,|\,x),$$

$$[h_\Delta(\lambda,x)]'_\lambda=\int_0^\infty(-t)e^{-\lambda t}dH_\Delta(t\,|\,x).$$

Setting $\lambda=0$ and $\Delta=(a,b)$, we obtain the equations

$$l_\Delta(x)=-[g_\Delta(\lambda,x)]'_{\lambda=0},$$
$$m_\Delta(x)=-[h_\Delta(\lambda,x)]'_{\lambda=0}.$$

Our next task is to use the equation

$$\frac{1}{2}u''+A(x)u'-B(\lambda,x)u=0$$

to find the expectations included in the definition of the distribution function $K_a(x)$.

Differentiating the terms of this differential equation by λ, changing the order of differentiation by x and λ, and setting $\lambda=0$, we obtain a differential equation with respect to the variable x for the function $l_\Delta(x)$,

$$l_\Delta(x)''+2A(x)\,l_\Delta(x)'+[2B(\lambda,x)]'_{\lambda=0}\,g_\Delta(0,x)-2B(0,x)\,l_\Delta(x)=0.$$

The last summand on the left-hand side of this equation is zero, because the process under consideration has no infinite constant interval (see above). It is also known (see, [2, p. 172]) that $B'_\lambda(0,x)>0$ for a nondegenerate continuous semi-Markov process. Hence

$$l_\Delta(x)''+2A(x)l_\Delta(x)'+2B'_\lambda(0,x)]g_\Delta(0,x)=0. \tag{6}$$

Similarly, we obtain the second equation

$$m_\Delta(x)''+2A(x)m_\Delta(x)'+2B'_\lambda(0,x)h_\Delta(0,x)=0. \tag{7}$$

Thus, we have two second-order differential equations of the form

$$y''_i+2A(x)y'_i+\gamma_i(x)=0\quad(i=1,\,2),$$

where

$$\gamma_1(x)\equiv 2B'_\lambda(0,x)g_\Delta(0,x)>0,$$
$$\gamma_2(x)\equiv 2B'_\lambda(0,x)h_\Delta(0,x)>0.$$

Put $z_i\equiv y'_i$. As a result, we obtain two first order differential equations

$$z'_i+2A(x)z_i+\gamma_i(x)=0\qquad(i=1,\,2).$$

The solutions of these equations can be written in explicit form (see, e.g., [8, p. 35]). On the interval (a, b), the solution of such an equation can be represented as

$$z_i(x) = e^{-F(x)}\left(z_i(c) - \int_c^x \gamma_i(t)e^{F(t)}\,dt\right), \tag{8}$$

where $a_0 < c < b_0$ and $F(x) \equiv \int_c^x 2A(t)\,dt$, and also we used the identity

$$\int_c^x f(t)\,dt \equiv -\int_x^c f(t)\,dt.$$

For each of these equations, we can write down two boundary representations of their solutions with respect to the interval $\Delta \equiv (a, b)$.

Condition B. According to the intuitive notion of an interval with regular boundaries for a and b, we define

$$m_{(a,b)}(a) = m_{(a,b)}(b) = 0, \qquad l_{(a,b)}(a) = l_{(a,b)}(b) = 0.$$

Keeping in mind the convergence of a to a_0, we represent the function $m_\Delta(x)$ in terms of the left end of the interval. If $x > a$, then

$$m'_{(a,b)}(x) \equiv z_2(x) = e^{-F_a(x)}\left(z_2(a) - \int_a^x \gamma_2(t)e^{F_a(t)}\,dt\right),$$

where

$$F_a(x) \equiv \int_a^x 2A(t)\,dt.$$

Next, we have

$$m_{(a,b)}(y) = C + \int_a^y m'_{(a,b)}(x)\,dx,$$

where from Condition B it follows that the arbitrary constant C is zero. Thus,

$$m_{(a,b)}(y) = \int_a^y e^{-F_a(x)}\left(m'_{(a,b)}(a) - \int_a^x \gamma_2(t)e^{F_a(t)}\,dt\right)dx.$$

For a final representation of $m_{(a,b)}(y)$ in terms of the coefficients of the original differential equation, only the value of $m'_{(a,b)}(a)$ is missing.

Let $y = b$. Then from Condition B, it follows

$$m'_{(a,b)}(a) = \left(\int_a^b e^{-F_a(x)}\,dx\right)^{-1} \int_a^b e^{-F_a(x)} \int_a^x \gamma_2(t)e^{F_a(t)}\,dt\,dx.$$

The desired representation of the expectation of $m_{(a,b)}(x)$ is obtained.

On the other hand,

$$l'_{(a,b)}(x) \equiv z_1(x) = e^{-F_b(x)}\left(l'_{(a,b)}(b) + \int_x^b \gamma_1(t)e^{F_b(t)}\,dt\right),$$

where $F_b(x) = \int\limits_b^x 2A(t)\,dt$. Hence

$$l_{(a,b)}(x) = C - \int\limits_x^b l'_{(a,b)}(t)\,dt.$$

Applying Condition B twice, we obtain $C = 0$ and

$$l'_{(a,b)}(b) = -\left(\int\limits_a^b e^{F_b(x)}\,dx\right)^{-1} \int\limits_a^b e^{-F_b(x)} \int\limits_x^b \gamma_1(t) e^{F_b(t)}\,dt\,dx.$$

The desired representation of the expectation of $l_{(a,b)}(x)$ is obtained.

We have found a representation of the function $K_a(x)$ in terms of expectations of the times of the first exit from finite intervals, one of whose boundaries is unattainable. These expectations are found as limits from the expectations on intervals with regular boundaries found above.

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Declarations

Data availability This manuscript has no associated data.
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REFERENCES

1. I. I. Gikman and A. V. Skorokhod, *The Theory of Stochastic Processes. Vol.* 2, Moscow, Nauka (1973).
2. B. P. Harlamov, *Continuous Semi-Markov Processes* [in Russian], St. Petersburg, Nauka (2001).
3. E. B. Dynkin, *Markov Processes* [in Russian], Moscow (1963).
4. B. P. Harlamov, “Final distribution of a diffusion process with a final stop,” *Zap. Nauchn. Semin. POMI*, **431**, 209–241 (2014).
5. B. P. Harlamov, “On unattainable boundaries of a diffusion process range of values: semi-Markov approach,” *Zap. Nauchn. Sem. POMI*, **466**, 313–330 (2017).
6. D. Cox and V. Smith, *Renewal Theory* [Russian translation], Moscow (1967).
7. A. N. Shiryaev, *Probability*, Moscow, Nauka (1980).
8. E. Kamke, *Handbook of Ordinary Differential Equations*, Moscow, Nauka (1971).

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