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DISTRIBUTION OF FUNCTIONALS OF BROWNIAN MOTION WITH LINEAR DRIFT AND ELASTICALLY KILLED AT ZERO

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Brownian motion with linear drift on positive half-line and killed elastically at zero is considered. A goal is to get a result that allows us to calculate the distributions of integral functionals with respect to spatial variable of local time of such a process. The explicit form of the distribution of the supremum with respect to spatial variable of local time is calculated for Brownian motion with linear drift reflecting at zero. Bibliography: 9 *titles*

We consider a Brownian motion with linear drift on positive half-line and elastically killed at zero. A brief description of this process is given in [1, Sec. 19, Appendix 1].

We are interested in a result that allows us to calculate distributions of integral functionals with respect to spatial variable of local time of such a process. This work continues the research started in papers [2] and [3] for skew Brownian motion and Brownian motion with discontinuous drift. For Brownian local time, this result is described in detail in [4, Chap. V, Sec. 5]. A starting point for our research is the Ray-Knight description of Brownian local time in space variable as a Markov process (see [5, 6]).

1. Brownian motion with linear drift on positive half-line and elastically killed at zero

Denote this process by $W_\mu^\circ(t)$, $t \geq 0$. Let $W_\mu^\circ(0) = x$, and let $W_+(t) = |W(t)|$ be the process of reflecting Brownian motion, where $W(t)$ is a Brownian motion process.

In what follows, the subscript of the probability and mathematical expectation means the initial state of the process.

By definition, W_μ° is a homogeneous Markov process on $[0, \infty)$, for which the Laplace transform of the transition density with respect to time, i.e., the function

$$G_z(x) := \lambda \int_0^\infty e^{-\lambda t} \frac{d}{dz} \mathbf{E}_x(W_\mu^\circ(t) < z)\, dt, \qquad x \in [0, \infty), \quad \lambda > 0,$$

is for every $z > 0$ and $\gamma > 0$ the unique continuous bounded solution to the problem

$$\frac{1}{2} G''(x) + \mu G'(x) - \lambda G(x) = 0, \ x \in (0, \infty) \setminus \{z\}, \tag{1.1}$$

$$G'(z+0) - G'(z-0) = -2\lambda, \tag{1.2}$$

$$G'(0+) = \gamma G(0). \tag{1.3}$$

Let τ be random time exponentially distributed with parameter $\lambda > 0$ and independent of the process $W_\mu^\circ(t)$, $t \geq 0$, and of the reflecting Brownian motion W_+. This moment is convenient to use the Laplace transforms with respect to the time. For example,

$$G_z(x) = \frac{d}{dz} \mathbf{P}_x(W_\mu^\circ(\tau) < z).$$

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Let us find an explicit solution to problem (1.1)–(1.3). We are looking for a solution in the form

$$G_z(x) = e^{\mu(z-x)} \frac{\lambda}{\sqrt{2\lambda+\mu^2}} \Big\{ e^{-|z-x|\sqrt{2\lambda+\eta^2}} + A e^{-(z+x)\sqrt{2\lambda+\mu^2}} \Big\}, \qquad x \ge 0. \tag{1.4}$$

In this representation, we took into account that the functions $e^{-\mu x \pm x\sqrt{2\lambda+\mu^2}}$ are solutions of homogeneous equation (1.1) on the whole real line, the resulting solution (1.4) is bounded and satisfies the condition on the jump of the derivative (1.2). The constant A must be calculated from condition (1.3). We have

$$A = 1 - \frac{2(\mu+\gamma)}{\sqrt{2\lambda+\mu^2}+\mu+\gamma}.$$

Thus, for the Laplace transform of the transition density, we derive the formula

$$\begin{aligned} \frac{d}{dz}\mathbf{P}_x(W_\mu^\circ(\tau) < z) &= e^{\mu(z-x)} \frac{\lambda}{\sqrt{2\lambda+\mu^2}} \\ &\times \Big\{ e^{-|z-x|\sqrt{2\lambda+\eta^2}} + \Big(1 - \frac{2(\mu+\gamma)}{\sqrt{2\lambda+\mu^2}+\gamma+\mu}\Big) e^{-(z+x)\sqrt{2\lambda+\mu^2}} \Big\}, \qquad x \ge 0. \end{aligned} \tag{1.5}$$

This expression coincides with the expression for the Green function $G_\lambda(x,z)$ calculated in [1, Sec. 19, Appendix 1] with respect to the speed measure $m(dz) = 2e^{2\mu z}dz$.

There is an absolute continuity of measures,

$$\frac{d\mathbf{P}_x^\circ}{d\mathbf{P}_x^+}\Big|_{\mathcal{F}_t} = \exp\Big(\mu(W_+(t)-x) - \frac{\mu+\gamma}{2}\,\ell_+(t,0) - \tfrac{1}{2}\mu^2 t\Big) \qquad \mathbf{P}_x\text{-a.s.}, \tag{1.6}$$

where $\mathbf{P}_x^\circ$ and $\mathbf{P}_x^+$ are the measures with respect to Brownian motion on the positive half-line with linear drift and elastically killed at zero, and reflecting Brownian motion, respectively, $\mathcal{F}_t$ is the σ-algebra generated by the Brownian motion up to moment t, and $\ell_+(t,0)$ is the local time of the reflecting Brownian motion with respect to Lebesgue measures, i.e.,

$$\ell_+(t,0) := \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{[0,\varepsilon)}(W_+(s))\,ds.$$

For $\gamma = 0$, the process W_μ° turns into a Brownian motion on $[0,\infty)$ with linear drift μ reflecting at zero. A brief description of this process is given in [1, Sec. 16, Appendix 1]. In this case, (1.6) turns into a result by G. N. Kinkladze in [7].

Since the Brownian motion with linear drift and elastically killed at zero, and reflecting Brownian motion are homogeneous Markov processes, to establish the absolute continuity of measures (1.6), it suffices to prove the following equality for the Laplace transforms of transition densities with respect to time:

$$\frac{d}{dz}\mathbf{P}_x\big(W_\mu^\circ(\tau) < z\big) = e^{\mu(z-x)} \frac{d}{dz}\mathbf{E}_x\Big\{ \exp\Big(- \frac{\mu+\gamma}{2}\ell_+(\tau,0) - \frac{\mu^2\tau}{2}\Big); W_+(\tau) < z\Big\}. \tag{1.7}$$

Here and below, in order to simplify formulas, we put $\mathbf{E}\{\xi; A\} := \mathbf{E}\{\xi \mathbb{1}_A\}$.

The proof of a similar statement can be found in [8].

Let us verify (1.7). According to formula 3.1.3.5 in [1], for $r = 0$, we have

$$\frac{d}{dz}\mathbf{E}_x\Big\{\exp\Big(-\frac{\mu+\gamma}{2}\ell_+(\tau,0)\Big); W_+(\tau) < z\Big\} = \frac{\sqrt{\lambda}}{\sqrt{2}}\Big\{e^{-|z-x|\sqrt{2\lambda}} + \Big(1 - \frac{2(\mu+\gamma)}{\sqrt{2\lambda}+\mu+\gamma}\Big)e^{-(z+x)\sqrt{2\lambda}}\Big\}. \tag{1.8}$$

Adding $-\frac{\mu^2\tau}{2}$ to the exponent on the left-hand side leads to the transformation of the Laplace transform with respect to time, which implies the replacements in (1.8): $\sqrt{2\lambda}$ by $\sqrt{2\lambda+\mu^2}$

and the factor $\frac{\sqrt{\lambda}}{\sqrt{2}}$ by the factor $\frac{\lambda}{\sqrt{2\lambda+\mu^2}}$. Together with (1.5), formula (1.8) after such a transformation proves (1.7) and hence also (1.6).

By virtue of the absolute continuity of measures, the process $W^\circ_\mu(s)$, $s \ge 0$, a.s. has the local time

$$\ell^\circ_\mu(t,y) := \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{[y,y+\varepsilon)}(W^\circ_\mu(s))\,ds, \qquad y \in [0,\infty), \tag{1.9}$$

since it exists for the reflecting Brownian motion.

From formula (1.6), it follows that for any bounded measurable functional $\wp(X(s),\ 0\le s\le t)$, we have

$$\begin{aligned} &\mathbf{E}_x \wp\big(W^\circ_\mu(s), 0 \le s \le t\big) \\ &\quad = \mathbf{E}_x\Big\{\wp(W_+(s), 0 \le s \le t) \exp\Big(\mu(W_+(t)-x) - \frac{\gamma+\mu}{2}\ell_+(t,0) - \frac{\mu^2 t}{2}\Big)\Big\}. \end{aligned} \tag{1.10}$$

The Laplace transform with respect to t in this equality leads to the fact that it holds for τ instead of t, and hence

$$\begin{aligned} &\mathbf{E}_x\big\{\wp\big(W^\circ_\mu(s), 0 \le s \le \tau\big); W^\circ_\mu(\tau) \in dz\big\} \\ &\quad = e^{\mu(z-x)} \mathbf{E}_x\Big\{\wp(W_+(s), 0 \le s \le \tau) \exp\Big(\frac{\gamma+\mu}{2}\ell_+(\tau,0) - \frac{\mu^2\tau}{2}\Big); W_+(\tau) \in dz\Big\} \\ &\quad = \frac{2\lambda e^{\mu(z-x)}}{2\lambda+\mu^2} \mathbf{E}_x\Big\{\wp(W_+(s), 0 \le s \le \widetilde{\tau}) \exp\Big(\frac{\gamma+\mu}{2}\ell_+(\widetilde{\tau},0)\Big); W_+(\widetilde{\tau}) \in dz\Big\}. \end{aligned} \tag{1.11}$$

We used the change of time in the Laplace transform. Here and below, $\widetilde{\tau}$ is a random variable exponentially distributed with parameter $\lambda + \frac{\mu^2}{2}$ and independent of other processes.

We focus on integral functionals with respect to spatial variable of local time.

2. Distributions of functionals of local time

We consider the following question: how to calculate the distributions of functionals of local time? An integral functional of local time $\ell^\circ_\mu(t,y)$ with respect to space variable has the form

$$B^\circ_\mu(t) := \int_0^\infty f(\ell^\circ_\mu(t,y))\,dy, \tag{2.1}$$

where $f(v)$, $v \in [0,\infty)$, is some nonnegative piecewise continuous function. For the Laplace transform of the distribution of such a functional, we will obtain explicit formulas expressed in terms of solutions of second-order differential equations satisfying some boundary conditions. Having expressions for the Laplace transforms of distributions of nonnegative integral functionals of the process, one can calculate distributions of supremum-type functionals. Thus, for example, to calculate the supremum of an arbitrary continuous process $X(y)$, we can use the relation

$$\mathbf{P}_x\Big(\sup_{0\le y\le b} X(y) \le h\Big) = \lim_{\gamma\to\infty} \mathbf{E}_x \exp\Big(-\gamma \int_0^b \mathbb{1}_{(h,\infty)}(X(y))\,dy\Big), \tag{2.2}$$

see [4, Chap. III, Sec. 2]. In many cases, if

$$\mathbf{E}_x \exp\Big(-\gamma \int_0^b \mathbb{1}_{[h,\infty)}(X(y))\,dy\Big) \tag{2.3}$$

is expressed by solutions of some differential equations, then it is not necessary to calculate the mathematical expectation explicitly and then find the limit. Instead, one can to prove only that the limit value for this mathematical expectation is also expressed via solutions of equations with some boundary conditions. Such an approach simplifies the calculations substantially. It has already been used by us in the proofs of [4, Chap. III, Theorem 2.1] and [4, Chap. IV, Theorem 4.2], and also in [2] and [3]. In this section, we obtain results allowing us to calculate the joint distribution of the functional $B^\circ_\mu(\tau)$ and variables $\sup\limits_{y\in[0,\infty)} \ell^\circ_\mu(\tau,y)$.

Calculation of distribution of these functionals for a fixed time t is reduced to calculating the inverse Laplace transforms with respect to λ of the distribution of the same functionals stopped at random time τ.

Theorem 2.1. *Let $f(v), v\in[0,h]$, be a nonnegative piecewise continuous function satisfying the condition $f(0)=0$. Then*

$$\mathbf{E}_0\Big[\exp\Big(-\int_0^\infty f(\ell^\circ_\mu(\tau,y))\,dy\Big);\ \sup_{y\in[0,\infty)}\ell_\mu(\tau,y)\le h\Big]=\lambda\int_0^h e^{-(\mu+\gamma)v/2}Q(v)\,dv, \tag{2.4}$$

where the function $Q(v)$, $v\in[0,h]$, is the bounded continuous solution to the problem

$$2vQ''(v)+2Q'(v)-\Big(\big(\lambda+\tfrac{\mu^2}{2}\big)v-\mu+f(v)\Big)Q(v)=-R(v), \tag{2.5}$$

$$Q(h)=0, \tag{2.6}$$

and the function $R(v)$, $v\in[0,h]$, is the unique bounded continuous solution to the problem

$$2vR''(v)-\Big(\big(\lambda+\tfrac{\mu^2}{2}\big)v+f(v)\Big)R(v)=0, \tag{2.7}$$

$$R(0)=1, \qquad R(h)=0. \tag{2.8}$$

Remark 2.1. For a piecewise continuous function f equations (2.5), (2.7) should be interpreted as follows: they hold at all points of continuity of the function f, and at points of discontinuity of the function f, their solutions are continuous together with the first derivative.

Proof of Theorem 2.1. First assume that $h=\infty$ and f is bounded twice continuously differentiable function with bounded first and second derivatives. Since in the calculations below, all the integrands are positive, and the left-hand side of the equalities is bounded, all the integrals converge.

Using (1.11), we find

$$\begin{aligned}\mathbf{E}_0\exp\Big(-\int_0^\infty f(\ell^\circ_\mu(\tau,y))\,dy\Big)&=\int_0^\infty\mathbf{E}_0\Big\{\exp\Big(-\int_0^\infty f(\ell^\circ_\mu(\tau,y))\,dy\Big);W^\circ_\mu(\tau)\in dz\Big\}\\&=\frac{2\lambda}{2\lambda+\mu^2}\int_0^\infty e^{\mu z}\mathbf{E}_0\Big\{\exp\Big(-\int_0^\infty f(\ell_+(\widetilde\tau,y))\,dy-\frac{\mu+\gamma}{2}\ell_+(\widetilde\tau,0)\Big);W_+(\widetilde\tau)\in dz\Big\}\\&=\frac{2\lambda}{\sqrt{2\lambda+\mu^2}}\int_0^\infty e^{(\mu-\sqrt{2\lambda+\mu^2})z}\mathbf{E}_0\Big\{\exp\Big(-\int_0^\infty f(\ell_+(\widetilde\tau,y))\,dy\\&\quad-\frac{\mu+\gamma}{2}\ell_+(\widetilde\tau,0)\Big)\Big|W_+(\widetilde\tau)=z\Big\}\,dz=\frac{2\lambda}{\sqrt{2\lambda+\mu^2}}\int_0^\infty e^{z(\mu-\sqrt{2\lambda+\mu^2})}I(z)\,dz,\end{aligned} \tag{2.9}$$

where

$$I(z) := \mathbf{E}_0^z \exp\Big(-\int_0^\infty f(\ell_+(\widetilde{\tau},y))\,dy - \frac{\mu+\gamma}{2}\ell_+(\widetilde{\tau},0)\Big).$$

In this representation, we used a new probability space that is generated by conditional distributions

$$\mathbf{P}_0^z(B) = \mathbf{P}_0(B|W_+(\widetilde{\tau}) = z).$$

Probability and mathematical expectation related to this space will be supplied with superscribed index z and subscribed index 0.

Using the expression for the distribution density of local time of the reflecting Brownian motion in a new probability space (formula 3.1.3.6 in [1] for $x = 0$, $r = 0$), we obtain

$$I(z) = \frac{\sqrt{2\lambda+\mu^2}}{2}\int_0^\infty e^{-v\sqrt{2\lambda+\mu^2}/2}e^{-(\mu+\gamma)v/2}\,\mathbf{E}_0^z \exp\Big(-\int_0^\infty f(\ell_+(\widetilde{\tau},y))\,dy\Big)\Big|\ell_+(\widetilde{\tau},0) = v\Big\}\,dv.$$

Now we make use of the description of the local time $\ell_+(\widetilde{\tau},y)$, $y \ge 0$, as a Markov process in the probability space with measure $\mathbf{P}_0^z$. Such a description follows, for example, from paper [2] for $\beta = 1$. Our notation are consistent with those of paper [2] as follows: $\ell_+(\widetilde{\tau},y) = \ell_1(\widetilde{\tau},y)$ if $y > 0$ and $\ell_+(\widetilde{\tau},0) = \ell_1(\widetilde{\tau},0+) = 2\ell_1(\widetilde{\tau},0)$.

As a result, the probability space with measure $\mathbf{P}_0^z$ admits the following representation:

$$\ell_+(\widetilde{\tau},y) = \begin{cases} V_1(y-z) & \text{if} \quad y \ge z, \\ V_2(y) & \text{if} \quad 0 \le y \le z, \end{cases}$$

where $V_k(h)$, $h \ge 0$, $k = 1,2$, are nonnegative homogeneous diffusion processes independent for fixed initial values. They have the same initial values $V_1(0) = V_2(z)$ and

$$\frac{d}{dv}\mathbf{P}(V_2(0) < v) = \frac{\sqrt{2\lambda+\mu^2}}{2}\,e^{-v\sqrt{2\lambda+\mu^2}/2}, \qquad v > 0,$$

and the generating operators have the form

$$\mathbf{L}_1 = 2v\Big(\frac{d^2}{dv^2} - \sqrt{2\lambda+\mu^2}\frac{d}{dv}\Big), \quad \mathbf{L}_2 = 2v\Big(\frac{d^2}{dv^2} - \sqrt{2\lambda+\mu^2}\frac{d}{dv}\Big) + 2\frac{d}{dv},$$

respectively.

Applying the Markov property of local time in the new probability space, we get

$$I(z) = \frac{\sqrt{2\lambda+\mu^2}}{2}\int_0^\infty e^{-v\sqrt{2\lambda+\mu^2}/2}\,e^{-(\mu+\gamma)v/2}\bar{q}(z,v)\,dv, \tag{2.10}$$

where

$$\begin{aligned}\bar{q}(z,v) &:= \mathbf{E}\Big\{\exp\Big(-\int_0^\infty f(V_1(h)\,dh - \int_0^z f(V_2(h))\,dh\Big)\Big|V_2(0) = v\Big\}\,dv \\ &= \int_0^\infty \mathbf{E}_v\Big\{\exp\Big(-\int_0^\infty f(V_1(h)\,dh - \int_0^z f(V_2(h))\,dh\Big)\Big|V_2(z) = g\Big\}\mathbf{P}_v\big(V_2(z) \in dg\big),\end{aligned}$$

and the subscript v means that the expectation and probability are calculated for the process V_2 with initial value $V_2(0) = v$. From (2.9) and (2.10), it follows that

$$\mathbf{E}_0 \exp\Big(-\int_0^\infty f(\ell_\mu^\circ(\tau,y))\,dy\Big) = \lambda\int_0^\infty dv e^{-(\mu+\gamma)v/2}e^{-v\sqrt{2\lambda+\mu^2}/2}\int_0^\infty e^{z(\mu-\sqrt{2\lambda+\mu^2})}\bar{q}(v,z)\,dz. \tag{2.11}$$

Let us continue the calculation of the function $\bar{q}(v,z)$. Using independence of the processes V_1 and V_2 for fixed initial values, and condition $V_1(0)=V_2(z)$, we get

$$\bar{q}(z,v)=\int_0^\infty \mathbf{E}\Big\{\exp\Big(-\int_0^\infty f(V_1(h))\,dh\Big)\Big|V_1(0)=g\Big\}$$

$$\times\mathbf{E}_v\Big\{\exp\Big(-\int_0^z f(V_2(h))\,dh\Big)\Big|V_2(z)=g\Big\}\mathbf{P}_v\big(V_2(z)\in dg\big),$$

$$=\int_0^\infty \bar{R}(g)\mathbf{E}_v\Big\{\exp\Big(-\int_0^z f(V_2(h))\,dh\Big);V_2(z)\in dg\Big\}$$

$$=\mathbf{E}\Big\{\bar{R}(V_2(z))\exp\Big(-\int_0^z f(V_2(h))\,dh\Big)\Big|V_2(0)=v\Big\}.$$

Here,

$$\bar{R}(g):=\mathbf{E}\Big\{\exp\Big(-\int_0^\infty f(V_1(h))\,dh\Big)\Big|V_1(0)=g\Big\}.$$

Using the expression for the generating operator of the process V_1 and applying [4, Chap. II, Theorem 12.5], we conclude that the function $\bar{R}(v)$, $v\in(0,\infty)$, is a bounded solution of the homogeneous equation

$$2v\big(\bar{R}''(v)-\sqrt{2\lambda+\mu^2}\,\bar{R}'(v)\big)-f(v)\bar{R}(v)=0. \tag{2.12}$$

According to [4, Chap. V, Proposition 2.1], the process $V_1(h)$, $h\ge 0$, is expressed as the square of a 0-dimensional Bessel process. It is known that if a 0-dimensional Bessel process hits zero or starting from zero, then it never leaves zero, i.e., it stays at zero. In the description of the process V_1, a similar statement is true for it. Since $f(0)=0$, this implies that $\bar{R}(0)=1$.

Let us apply [4, Chap. II, Theorem 13.2]. Then the function $\bar{q}(z,v)$, $(z,v)\in[0,\infty)\times[0,\infty)$, is the solution to the problem

$$\frac{\partial}{\partial z}\bar{q}(z,v)=2v\Big(\frac{\partial^2}{\partial v^2}\bar{q}(z,v)-\sqrt{2\lambda+\mu^2}\frac{\partial}{\partial v}\bar{q}(z,v)\Big)+2\frac{\partial}{\partial v}\bar{q}(z,v)-f(v)\bar{q}(z,v), \tag{2.13}$$

$$\bar{q}(0,v)=\bar{R}(v). \tag{2.14}$$

The specific of application of Theorems 12.5 and 13.2 in [4, Chap. II] is that the processes V_1 and V_2 are nonnegative and their diffusion coefficient $\sigma^2(v)=v$ degenerates at zero.

The replacement $q(z,v)=e^{-v\sqrt{2\lambda+\mu^2}/2}\bar{q}(z,v)$ leads to to the problem

$$\frac{\partial}{\partial z}q(z,v)=2v\frac{\partial^2}{\partial v^2}q(z,v)+2\frac{\partial}{\partial v}q(z,v)-\Big(\Big(\lambda+\frac{\mu^2}{2}\Big)v-\sqrt{2\lambda+\mu^2}+f(v)\Big)q(z,v), \tag{2.15}$$

$$q(0,v)=R(v), \tag{2.16}$$

where the substitution $R(v):=e^{-v\sqrt{2\lambda+\mu^2}/2}\bar{R}(v)$ leads to the problem

$$2vR''(v)-\big(\big(\lambda+\frac{\mu^2}{2}\big)v+f(v)\big)R(v)=0,\qquad R(0)=1. \tag{2.17}$$

Put

$$Q(v):=\int_0^\infty e^{z(\mu-\sqrt{2\lambda+\mu^2})}q(z,v)\,dz.$$

Then from (2.15) and (2.16), it follows that the function $Q(v)$ satisfies equation (2.5) for $h \in (0, \infty)$. Now, by virtue of (2.11), we have

$$\mathbf{E}_0 \exp\Big(-\int_0^\infty f(\ell_\mu^\circ(\tau, y))\, dy \Big) = \lambda \int_0^\infty e^{-(\mu+\gamma)v/2} Q(v)\, dv.$$

This is the same as (2.4) for the case $h = \infty$ and f is bounded twice continuously differentiable function with bounded first and second derivatives.

As in the proof of Theorem 4.1 in [4, Chap. IV], the assertion for piecewise continuous functions f is proved via approximating f by continuously differentiable functions. The proof of Theorem 2.1 for $h < \infty$ is based on an obvious variant of relation (2.2):

$$\begin{aligned} E_\gamma &:= \mathbf{E}\Big[\exp\Big(-\int_0^\infty f(\ell_\mu^\circ(\widetilde{\tau}, y))\, dy \Big);\ \sup_{y\in[0,\infty)} \ell_\mu^\circ(\widetilde{\tau}, y) \le h \Big] \\ &= \lim_{\gamma\to\infty} \mathbf{E}\Big[\exp\Big(-\int_0^\infty (f(\ell_\mu^\circ(\widetilde{\tau}, y)) + \gamma \mathbb{1}_{(h,\infty)}(\ell_\mu^\circ(\widetilde{\tau}, y)))\, dy \Big) \Big]. \end{aligned} \tag{2.18}$$

Similar calculations are described in more detail in [4, Chap. V, Sec, 5]. □

3. Distribution of the supremum of local time

Consider an example of application of Theorem 2.1. Consider a Brownian motion on $[0, \infty)$ with linear drift μ reflecting at zero, i.e., the case $\gamma = 0$. For $\gamma = 0$, we calculate an explicit form of the distribution of the supremum of the local time $\ell_\mu^+(\tau, y) := \ell_\mu^\circ(\tau, y)$ with respect to the variable $y \in [0, \infty)$. In [3], the distribution of the supremum of the local time of Brownian motion with discontinuous drift was calculated.

We use the standard notation: the functions $I_l(x)$, $x \in \mathbf{R}$, are modified Bessel functions of order l, functions $M_{n,m}(x)$, $W_{n,m}(x)$, $x \in (0, \infty)$, are the Whittaker functions (see [1, Appendix 2] or [9, Chap. 13]).

Theorem 3.1. *For $h \ge 0$,*

$$\mathbf{P}_0\Big(\sup_{y\in[0,\infty)} \ell_\mu^+(\tau, y) > h \Big) = \frac{\sqrt{\theta}\sqrt{h}}{\operatorname{sh}(h\sqrt{\theta}) M_{\mu/4\sqrt{\theta},0}(2h\sqrt{\theta})} \int_0^h \frac{e^{-\mu v/2}}{\sqrt{v}} M_{\mu/4\sqrt{\theta},0}(2v\sqrt{\theta})\, dv, \tag{3.1}$$

where $\theta := \frac{\lambda}{2} + \frac{\mu^2}{4}$.

Remark 3.1. For $\mu = 0$, we have $W_\mu^\circ(s) = W_+(s)$ and

$$\mathbf{P}\Big(\sup_{y\in[0,\infty)} \ell_+(\tau, y) > h \Big) = \frac{\sqrt{\lambda/2}}{\operatorname{sh}(h\sqrt{\lambda/2}) I_0(h\sqrt{\lambda/2})} \int_0^h I_0(v\sqrt{\lambda/2})\, dv, \tag{3.2}$$

which coincides with formula 3.1.11.2 in [1].

Indeed, in this case $M_{0,0}(2x) = \sqrt{2x} I_0(x)$ (see [1, Appendix 2] or [9, Chap. 13]).

Proof of Theorem 3.1. We apply Theorem 2.1 for $f = 0$. Let $\theta := \frac{\lambda}{2} + \frac{\mu^2}{4}$. A solution to problem (2.7), (2.8) for $f \equiv 0$ has the form

$$R(v) = \frac{\operatorname{sh}((h - v)\sqrt{\theta})}{\operatorname{sh}(h\sqrt{\theta})}, \qquad 0 \le v \le h. \tag{3.3}$$

Linearly independent solutions of the homogeneous equation

$$Y''(v) + \frac{1}{v}Y'(v) - \big(\theta - \frac{\mu}{2v}\big)Y(v) = 0, \quad v > 0,$$

have (see Eq. (16) in [1, Appendix 4]) the form

$$\psi(v) = \frac{1}{\sqrt{v}} M_{\mu/4\sqrt{\theta},0}(2v\sqrt{\theta}), \qquad \varphi(v) = \frac{1}{\sqrt{v}} W_{\mu/4\sqrt{\theta},0}(2v\sqrt{\theta}),$$

and their Wronskian is $\omega(v) = \dfrac{2\sqrt{\theta}}{v\Gamma(1/2 - \mu/4\sqrt{\theta})}$. Moreover, ψ is a nonnegative increasing solution and φ is a nonnegative decreasing solution. It is important here that $\frac{1}{2} - \frac{\mu}{4\sqrt{\theta}} > 0$ for any $\mu \in \mathbf{R}$. From handbook [9, Chap. 13], one can extract that $\varphi(v) \asymp -\ln v$, as $v \downarrow 0$, i.e., φ is unbounded at zero.

A partial solution of equation (2.5) for $R(v) = \operatorname{sh}((h - v)\sqrt{\theta}) / \operatorname{sh}(h\sqrt{\theta})$ has the form

$$Q_0(v) = \frac{2\sqrt{\theta}\operatorname{ch}((h - v)\sqrt{\theta}) + \mu \operatorname{sh}((h - v)\sqrt{\theta})}{2\lambda \operatorname{sh}(h\sqrt{\theta})}, \qquad 0 \le v \le h.$$

Then the bounded solution to the problem (2.5), (2.6) for $f \equiv 0$ has the form

$$Q(v) = Q_0(v) - \frac{Q_0(h)\psi(v)}{\psi(h)}, \qquad 0 \le v \le h, \tag{3.4}$$

i.e.,

$$Q(v) = \frac{2\sqrt{\theta}\operatorname{ch}((h - v)\sqrt{\theta}) + \mu \operatorname{sh}((h - v)\sqrt{\theta})}{2\lambda \operatorname{sh}(h\sqrt{\theta})} - \frac{\sqrt{\theta}\sqrt{h} M_{\mu/4\sqrt{\theta},0}(2v\sqrt{\theta})}{\lambda\sqrt{v}\operatorname{sh}(h\sqrt{\theta}) M_{\mu/4\sqrt{\theta},0}(2h\sqrt{\theta})}, \qquad 0 \le v \le h. \tag{3.5}$$

It is easy to check that

$$\lambda \int_0^h e^{-\mu v/2} Q_0(v)\, dv = 1.$$

Now from (2.4) with $f \equiv 0$, $\gamma = 0$, and from (3.5), we get (3.1). □

Translated by A. N. Borodin

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Declarations

Data availability This manuscript has no associated data.
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