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CONVERGENCE THEORY OF ADAPTIVE MIXED FINITE ELEMENT METHODS FOR THE STOKES PROBLEM

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We establish a conditional optimality result for an adaptive mixed finite element method for the stationary Stokes problem discretized by the standard Taylor–Hood elements under the assumption of the so-called general quasiorthogonality. Optimality is measured in terms of a modified approximation class defined through the total error. We prove that the modified approximation class coincides with the standard approximation class, modulo the assumption that the data is regular enough in an appropriate scale of Besov spaces. Bibliography: 35 titles. Illustrations: 2 figures.

1 Introduction

We consider adaptive mixed finite element methods for the stationary Stokes problem

$$\begin{aligned} -\Delta u + \nabla p &= f, \\ \nabla \cdot u &= 0, \end{aligned} \tag{1.1}$$

discretized by the standard Taylor–Hood elements. Here, $u : \Omega \to \mathbb{R}^n$ is the unknown velocity field, $p : \Omega \to \mathbb{R}$ is the unknown pressure field, $f \in L^2(\Omega, \mathbb{R}^n)$ is the given data, and $\Omega \subset \mathbb{R}^n$ is a bounded polyhedral domain with Lipschitz boundary. One can think of the space dimension to be $n = 2$ or $n = 3$. We impose the no slip boundary condition $u|_{\partial\Omega} = 0$ on the velocity field, and in order to ensure uniqueness, we require that the pressure field is of vanishing mean.

Convergence theory of adaptive finite element methods has been an active field of research especially since the influential paper [1]. A near complete understanding was achieved for Poisson type problems [2]–[5]. There is a growing body of literature on adaptive discretization of saddle point problems such as (1.1), but the question of convergence rate for adaptive mixed finite element methods with standard Taylor–Hood elements is entirely open.

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Building on the pioneering works [6, 7], optimal convergence rates were established for certain elliptic reformulations of the Stokes problem in [8]. Moreover, for nonconforming discretizations of the Stokes problem the same question was investigated in [9]– [11]. On the other hand, adaptive mixed finite element methods for the Poisson problem were treated in [12]– [14].

Getting back to the discussion of adaptive mixed finite element methods with standard Taylor–Hood elements, the first proof of convergence for such a method was published in [15], where the standard *a posteriori* error estimator from [16] was modified. This proof was improved in [17] to incorporate the *a posteriori* error estimator from [16] into analysis. We note that the aforementioned results do not provide information about the rate of convergence.

In this paper, under the assumption of the so-called general quasiorthogonality, we establish bounds on the convergence rates of adaptive mixed finite element methods for the Stokes problem discretized by standard Taylor–Hood elements, and show that these bounds are in a certain sense optimal. This was motivated by the conceptual understanding of the role played by the general quasiorthogonality in the analysis of adaptive methods, cf. [18]. What the concept of general quasiorthogonality provides is a framework to potentially exploit the Galerkin orthogonality for noncoercive or strongly nonsymmetric problems, a bottleneck that has been faced by researchers for some time.

In order to discuss our other results, we need to fix some notation and terminologies. The solution $(u,p) \in H^1(\Omega,\mathbb{R}^n)\times L^2(\Omega)$ of (1.1) is said to be a member of the *standard approximation class* $\mathscr{A}^s$ if there exists a sequence of conforming triangulations $P_1, P_2, \ldots$ of Ω, obtained from a fixed initial triangulation P_0 by applications of newest vertex bisections, such that $\#P_N - \#P_0 \leqslant N$ and

$$\|u-u_N\|_{H^1(\Omega,\mathbb{R}^n)} + \|p-p_N\|_{L^2(\Omega)} \leqslant CN^{-s} \quad \forall\ N, \tag{1.2}$$

where (u_N,p_N) is the Galerkin approximation of (u,p) from the Taylor–Hood finite element space (of degree d for the velocity field and degree $d-1$ for the pressure field) defined over the triangulation P_N, and C is a constant independent of N. Suppose that we have an adaptive algorithm (based on the same Taylor–Hood spaces and newest vertex bisections) that takes the data $f \in L^2(\Omega,\mathbb{R}^n)$ and the initial triangulation P_0 of Ω as its input, and produces the sequence of triangulations $P_1, P_2, \ldots$, and the corresponding Galerkin solutions (u_k,p_k) for $k=0,1,\ldots$. Then it stands to reason to say that the algorithm converges at the *optimal rate* if

$$\|u-u_k\|_{H^1(\Omega,\mathbb{R}^n)} + \|p-p_k\|_{L^2(\Omega)} \leqslant C(\#P_k - \#P_0)^{-s} \quad \forall\ k, \tag{1.3}$$

whenever $(u,p) \in \mathscr{A}^s$ for some $s>0$. For the Poisson equation $\Delta u = g$, such an optimality result was obtained in [4], for an adaptive algorithm that uses an inner loop for resolving the data g. It was later discovered in [5] that if we do not add an inner iteration, the algorithm remains optimal *provided* that we modify the approximation class to include in its definition a measure of resolution of g. The analogue of this modification in our setting is as follows. The solution $(u,p) \in H^1(\Omega,\mathbb{R}^n)\times L^2(\Omega)$ of (1.1) is said to be a member of the *modified approximation class* $\mathscr{A}^s_*$ if there exists a sequence of conforming triangulations $P_1, P_2, \ldots$ of Ω, obtained from a fixed initial triangulation P_0 by applications of newest vertex bisections, such that $\#P_N - \#P_0 \leqslant N$ and

$$\|u-u_N\|_{H^1(\Omega,\mathbb{R}^n)} + \|p-p_N\|_{L^2(\Omega)} + \operatorname{osc}_N(f) \leqslant CN^{-s} \quad \forall\ N, \tag{1.4}$$

where (u_N,p_N) is the Galerkin approximation of (u,p) from the Taylor–Hood finite element space defined over the triangulation P_N, and $\operatorname{osc}_N(f)$ is the so-called *oscillation* term, which depends only on P_N and f. Note that $\mathscr{A}^s_*$ is indeed a space of pairs (u,p), because f is completely determined by u and p. Note also that $\mathscr{A}^s_* \subset \mathscr{A}^s$, since the membership of $\mathscr{A}^s_*$ has the extra requirement that the oscillation $\operatorname{osc}_N(f)$ is suitably reduced as N grows. Although reducing the oscillation is by no means among our initial ambitions, it turns out that $\mathscr{A}^s_*$ is completely natural from the perspective of adaptive methods. In particular, the quantity on the left-hand side of (1.4), called the *total error*, is equivalent to the error estimator, and so an algorithm that only "sees" the error estimator will have to reduce the oscillation anyway. With respect to the approximation classes $\mathscr{A}^s_*$, we have the following *conditional* optimality result, which will be proved in Section 4. The aforementioned general quasiorthogonality assumption appears here in (1.5).

Main result 1. Let $(u,p) \in \mathscr{A}^s_*$ for some $s > 0$, and let $P_0, P_1, \ldots$ be the sequence of triangulations generated by the adaptive algorithm defined in Section 4 with the Galerkin solutions (u_k, p_k), $k = 0, 1, \ldots$. In addition, we assume that there exists a constant $c > 0$ such that

$$\begin{aligned}&\sum_{k=\ell}^{\infty}(\|u_k - u_{k+1}\|^2_{H^1(\Omega,\mathbb{R}^n)} + \|p_k - p_{k+1}\|^2_{L^2(\Omega)})\\ &\leqslant c(\|u - u_\ell\|^2_{H^1(\Omega,\mathbb{R}^n)} + \|p - p_\ell\|^2_{L^2(\Omega)})\end{aligned} \tag{1.5}$$

for any integer ℓ. Then

$$\|u - u_k\|_{H^1(\Omega,\mathbb{R}^n)} + \|p - p_k\|_{L^2(\Omega)} + \operatorname{osc}_k(f) \leqslant C(\#P_k - \#P_0)^{-s} \quad \forall\, k, \tag{1.6}$$

where $\operatorname{osc}_k(f)$ denotes the oscillation on the mesh P_k.

We have this conditional optimality result for an algorithm that uses the *a posteriori* error estimator from [16]. As a theoretical tool to be employed in the analysis, we introduce a supposedly new *a posteriori* error estimator, which also yields optimal algorithms. For algorithms that use the modified estimator from [15], we establish geometric error reduction, but we were unable to prove optimal convergence rates, because of the apparently nonlocal character of the estimator.

There is a remark to be made on the nature of the constant C that appears in (1.6). From the experience with Poisson-type problems, one would expect that the constant C must be of the form $C = c|(u,p)|_{\mathscr{A}^s_*}$, where c does not depend on (u,p), and $|(u,p)|_{\mathscr{A}^s_*}$ is the norm of (u,p) in the space $\mathscr{A}^s_*$. In Main result 1, however, we do not rule out the possibility that c depends on (u,p).

The final part of the current paper is independent of optimality results, and concerns interrelations between the modified approximation classes $\mathscr{A}^s_*$ and the standard approximation classes $\mathscr{A}^s$. The pairs (u,p) contained in the gap $\mathscr{A}^s \setminus \mathscr{A}^s_*$ can in principle be approximated with the rate s by some approximation procedure, but Main result 1 cannot guarantee the convergence of the adaptive finite element methods with the same rate. Our approach to this problem is to show that (u,p) does not lie in the gap as long as the data f have some regularity in terms of a scale of Besov spaces. Namely, the following result will be proved in Section 5.

Main result 2. Let $f \in B^{\alpha}_{q,q}(\Omega,\mathbb{R}^n)$ for some $0 < q < \infty$ and $\alpha \geqslant n/q - n/2$ satisfying $0 < \alpha < d - 1 + \max\{0, 1/q - 1\}$. Then $(u,p) \in \mathscr{A}^s$ implies $(u,p) \in \mathscr{A}^s_*$ with $s = (\alpha+1)/n$.

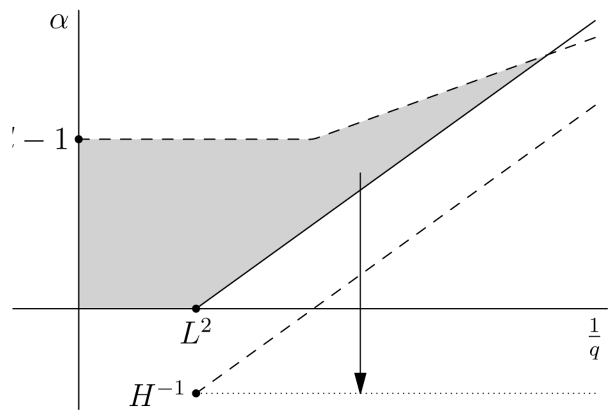

Figure 1. The shaded region represents the pairs $(1/q, \alpha)$ allowed in Main result 2. If the function $f \in B^{\alpha}_{q,q}$ was being adaptively approximated in the H^{-1} norm and $B^{\alpha}_{q,q} \subset H^{-1}$, then we would expect the rate of convergence to be determined by the vertical offset of the point $(1/q, \alpha)$ in relation to the dotted line. Main result 2 basically says that the same convergence rate is restored for the total error as long as $B^{\alpha}_{q,q}$ is embedded into L^2 and (u, p) is regular enough (to be in $\mathscr{A}^s$).

To prove this result, we adapt the techniques from [19] and [20], where the said techniques have been used to show embeddings of the form $B^{\alpha}_{q,q} \subset \mathscr{A}^s$. Earlier influential works in the same vein include [21] and [22]. We also establish embeddings of the form $B^{\alpha}_{q,q} \subset \mathscr{A}^s$ in the context of the Stokes problem, see Theorem 5.2. Note that the regularity of the Stokes problem in the same Besov scale has been studied in [23].

This paper is organized as follows. In the next section, we discuss assumptions on the triangulations, the refinement procedures, and the finite element spaces for discretizing the Stokes problem. Then in Section 3, we introduce three kinds of *a posteriori* error estimators, and establish some of their properties. The conditional optimality result mentioned above, together with a theorem on geometric error reduction are proved in Section 4. In Section 5, we deal with interplays between the approximation classes and Besov spaces.

2 Discretization of the Stokes Problem

Let $\Omega \subset \mathbb{R}^n$ be a polyhedral domain with Lipschitz boundary, where $n = 2$ or $n = 3$. We call a collection P of triangles (or tetrahedra) a *partition* of Ω if $\overline{\Omega} = \bigcup_{\tau \in P} \overline{\tau}$, and $\tau \cap \sigma = \varnothing$ for any two different $\tau, \sigma \in P$. For refining the partitions we use the so called *newest vertex bisection* algorithm; details can be found in [3, 24]. A partition P' is called a *refinement* of P and denoted $P \preceq P'$ if P' can be obtained by replacing zero or more $\tau \in P$ by its children, or by a recursive application of this procedure. Throughout this paper, we only consider *conforming* partitions that are refinements of some *fixed* conforming partition P_0 of Ω. The newest vertex bisection procedure produces *shape regular* partitions, meaning that

$$\sigma_{\mathrm{s}} = \sup\left\{\frac{(\mathrm{diam}\ \tau)^n}{\mathrm{vol}(\tau)} : \tau \in P,\, P \in \mathrm{conf}(P_0)\right\} < \infty, \tag{2.1}$$

where $\mathrm{conf}(P_0)$ denotes the family of all conforming partitions that are refinements of P_0. This family is *graded* (or *locally quasi-uniform*), in the sense that

$$\sigma_{\mathrm{g}} = \sup\left\{\frac{\mathrm{diam}\ \sigma}{\mathrm{diam}\ \tau} : \sigma, \tau \in P, \overline{\sigma} \cap \overline{\tau} \neq \varnothing, P \in \mathrm{conf}(P_0)\right\} < \infty. \tag{2.2}$$

Note that the shape regularity and gradedness together imply *local finiteness*, meaning that the number of triangles meeting at any given point is bounded by a constant that depends only on σ_{s}, σ_{g}, and n.

In general, a naive refinement of a conforming partition would produce a nonconforming partition, so, in order to ensure conformity, one must perform additional refinements. This procedure is called *completion*, and a quite satisfactory theory of completion has been developed in [3, 24]. We consider the whole process of obtaining a conforming partition from an initial conforming partition as a single refinement step that works in the category of conforming partitions. Given a partition $P \in \mathrm{conf}(P_0)$ and a set $R \subset P$ of its triangles, the refinement step produces $P' \in \mathrm{conf}(P_0)$, such that $P \setminus P' \supseteq R$, i.e., the triangles in R are refined at least once. Let us denote it by $P' = \mathrm{refine}(P, R)$. We have the following on its efficiency: If $\{P_k\} \subset \mathrm{conf}(P_0)$ and $\{R_k\}$ are sequences such that $P_{k+1} = \mathrm{refine}(P_k, R_k)$ and $R_k \subset P_k$ for $k = 0, 1, \ldots$, then

$$\#P_\ell - \#P_0 \leqslant C_c \sum_{k=0}^{\ell-1} \#R_k, \quad \ell = 1, 2, \ldots, \tag{2.3}$$

where $C_c > 0$ is a constant.

Another notion we need is that of *overlay* of partitions: We assume that there is an operation $\oplus : \mathrm{conf}(P_0) \times \mathrm{conf}(P_0) \to \mathrm{conf}(P_0)$ satisfying

$$P \oplus Q \succeq P, \quad P \oplus Q \succeq Q, \quad \#(P \oplus Q) \leqslant \#P + \#Q - \#P_0 \tag{2.4}$$

for $P, Q \in \mathrm{conf}(P_0)$. This assumption is verified in [4, 5], where $P \oplus Q$ is taken to be the smallest and common conforming refinement of P and Q.

Let $V = H_0^1(\Omega, \mathbb{R}^n)$ and $Q = L^2(\Omega)/\mathbb{R}$, the latter being the space of L^2 functions with vanishing mean, and let $X = V \times Q$ be the Hilbert space equipped with the norm

$$\|(v, q)\|_{V \times Q} = \left(\|v\|_V^2 + \|q\|_Q^2\right)^{\frac{1}{2}}. \tag{2.5}$$

We consider the following weak formulation of the Stokes problem (1.1): Find $(u, p) \in X$ satisfying

$$a(u, v) - b(v, p) - b(u, q) = \langle f, v\rangle_{L^2} \quad \forall\ (v, q) \in X, \tag{2.6}$$

where the bilinear forms $a : V \times V \to \mathbb{R}$ and $b : V \times Q \to \mathbb{R}$ are defined respectively by

$$a(u, v) = \int\limits_\Omega \sum_{i,k} \partial_i u_k \partial_i v_k, \quad b(u, q) = \int\limits_\Omega q \sum_i \partial_i u_i, \tag{2.7}$$

and $\langle \cdot, \cdot \rangle_{L^2}$ denotes the inner L^2 product. It is known that for any $f \in L^2(\Omega, \mathbb{R}^n)$ the problem (2.6) admits a unique solution $(u, p) \in X$ (see, for example, [25]). In fact, the operator $A : X \to X'$ defined by

$$\langle A(u, p), (v, q)\rangle = a(u, v) - b(v, p) - b(u, q), \quad (u, p), (v, q) \in X, \tag{2.8}$$

is invertible, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X' and X. It is easy to see that A is linear, bounded, and self-adjoint. In particular, by the Banach bounded inverse theorem, the inverse $A^{-1}: X' \to X$ is bounded as well. In terms of the operator A, the Stokes problem (2.6) can be written as

$$A(u,p) = F, \tag{2.9}$$

where the linear functional $F \in X'$ is defined by

$$\langle F, (v,q)\rangle = \langle f, v\rangle_{L^2} \quad \forall\ (v,q) \in X. \tag{2.10}$$

In the discretization of (2.6), we use the classical finite element spaces introduced by [26]. Given a partition $P \in \text{conf}(P_0)$, we define the discontinuous piecewise polynomial space S_P^d by

$$S_P^d = \{u \in L^\infty(\Omega) : u|_\tau \in \mathbb{P}_d\, \forall \tau \in P\}, \tag{2.11}$$

where $\mathbb{P}_d$ denotes the set of polynomials of degree less than or equal to d. Then the Taylor–Hood finite element spaces are $V_P = V \cap (S_P^d)^n$ and $Q_P = Q \cap C(\Omega) \cap S_P^{d-1}$, and the *Galerkin approximation* $(u_P, p_P) \in V_P \times Q_P$ of (u,p) from $V_P \times Q_P$ is characterized by

$$a(u_P, v) - b(v, p_P) - b(u_P, q) = \langle f, v\rangle_{L^2} \quad \forall\ (v,q) \in V_P \times Q_P. \tag{2.12}$$

It is proved in [27, 28] that for any $d \geqslant 2$ and for $n \in \{2,3\}$ the pair (V_P, Q_P) satisfies the stability property

$$\|q\|_Q \leqslant C_s \sup_{v \in V_P} \frac{b(v,q)}{\|v\|_V}, \quad q \in Q_P, \tag{2.13}$$

with C_s depending only on the initial partition P_0, under the sufficient condition that P_0 contains at least three simplices and each simplex has at least one vertex in Ω. Earlier works on the stability of the Taylor–Hood elements include [25] and [29]– [32].

Throughout the paper, we assume that the stability (2.13) holds with C_s depending only on P_0. This assumption implies the well-posedness of the discrete problem (2.12), as well as the *a priori* error estimate

$$\|(u - u_P, p - p_P)\|_{V\times Q} \leqslant C_s' \inf_{(v,q)\in V_P\times Q_P} \|(u - v, p - q)\|_{V\times Q}, \tag{2.14}$$

where the constant C_s' depends only on C_s and the geometry of the domain Ω. Moreover, for any $(w,r) \in V_P \times Q_P$ we have

$$\|(w,r)\|_{V\times Q} \leqslant C_s' \sup_{(v,q)\in V_P\times Q_P} \frac{a(w,v) - b(w,q) - b(v,r)}{\|(v,q)\|_{V\times Q}}. \tag{2.15}$$

We close this section by remarking that the Galerkin problem (2.12) can be given in a convenient operator formulation. Let $X_P = V_P \times Q_P$, and let $j_P : X_P \to X$ be the natural injection. Then (2.12) is simply

$$j_P' A j_P(u_P, p_P) = j_P' F, \tag{2.16}$$

where $j_P' : X' \to X_P$ is the dual of j_P and A and F are defined in (2.8) and (2.10) respectively. The significance of the stability assumption in this formulation is that (2.15) gives not only the invertibility of the operator $A_P = j_P' A j_P : X_P \to X_P$, but also it implies the bound $\|A_P^{-1}\| \leqslant C_s'$.

3 A Posteriori Error Estimators

For $P \in \text{conf}(P_0)$ we denote by E_P the set of interior edges (or faces if $n = 3$) in the partition P. Let $h_\tau = |\tau|^{1/n}$, and let $h_e = |e|^{1/(n-1)}$, where $|\tau|$ and $|e|$ are the n- and $(n-1)$-dimensional volumes of $\tau \in P$ and $e \in E_P$ respectively. Moreover, for $Q \subset P$, we denote by E_Q the set of interior edges of Q, i.e., the set of edges that are adjacent to two triangles from Q. Then for $Q \subset P$ we introduce the residual based *a posteriori* error estimators

$$\eta_0(P,Q) = \sum_{\tau \in Q} h_\tau^2 \|f + \Delta u_P - \nabla p_P\|_{L^2(\tau)}^2 + \sum_{e \in E_Q} h_e \|[\partial_\nu u_P]\|_{L^2(e)}^2, \tag{3.1}$$

$$\eta_1(P,Q) = \eta_0(P,Q) + \sum_{\tau \in Q} \|\nabla \cdot u_P\|_{L^2(\tau)}^2, \tag{3.2}$$

$$\eta_2(P,Q) = \eta_0(P,Q) + \sum_{\tau \in Q} h_\tau \|\nabla \cdot u_P|_\tau\|_{L^2(\partial\tau)}^2, \tag{3.3}$$

where $[\partial_\nu u_P]$ is the jump in the normal derivative of u_P across the edge e. It is understood that the differential operators Δ and ∂_ν act on vector functions such as u_P component-wise. The estimator η_1 was introduced in [16], and the estimator η_0 was proposed in [15] as a variation on η_1. The estimator η_2 seems to be new.

As shown in [16], we have the equivalence

$$\|u - u_P\|_V^2 + \|p - p_P\|_Q^2 \lesssim \eta(P,P) \lesssim \|u - u_P\|_V^2 + \|p - p_P\|_Q^2 + \text{osc}(P), \tag{3.4}$$

for the error estimator $\eta = \eta_1$, where the oscillation is defined by

$$\text{osc}(P) = \min_{g \in (S_P^{d-2})^n} \sum_{\tau \in P} h_\tau^2 \|f - g\|_{L^2(\tau)}^2. \tag{3.5}$$

Hereinafter, we often dispense with giving explicit names to constants and use the Vinogradov style notation $X \lesssim Y$ which means $X \leqslant C \cdot Y$ with some constant C that is allowed to depend only on P_0 and (the geometry of) the domain Ω. Moreover, even when we give names to constants, we will not explicitly mention that the constants can depend on P_0 and Ω, and this dependence will always be implicitly assumed.

The equivalence (3.4) also holds for $\eta = \eta_0$ because

$$\|\nabla \cdot u_P\|_{L^2(\Omega)}^2 \lesssim \sum_{e \in E_P} h_e \|[\partial_\nu u_P]\|_{L^2(e)}^2 \tag{3.6}$$

and, consequently ([15, §3.3] and [7, Proposition 5.4]),

$$\eta_0(P,P) \leqslant \eta_1(P,P) \lesssim \eta_0(P,P). \tag{3.7}$$

Now, that we have (3.4) for both η_0 and η_1, we get it also for η_2 because

$$\eta_0(P,P) \leqslant \eta_2(P,P) \lesssim \eta_1(P,P), \tag{3.8}$$

where the second inequality follows from $\|\nabla \cdot u_P|_\tau\|_{L^2(\partial\tau)} \lesssim h_\tau^{-1/2} \|\nabla \cdot u_P\|_{L^2(\tau)}$. To reiterate, the global equivalence (3.4) holds for all three estimators η_0, η_1, and η_2.

A convenient fact is that each estimator dominates the oscillation in the sense that

$$\operatorname{osc}(P) \leqslant \eta_0(P,P), \quad P \in \operatorname{conf}(P_0). \tag{3.9}$$

This is immediate because $\Delta u_P - \nabla p_P \in (S_P^{d-2})^n$ in (3.1), while (3.5) involves the minimization over the space $(S_P^{d-2})^n$.

By standard arguments, we easily get local discrete upper bounds for η_1 and η_2, which we record in the next lemma. In the statement of the lemma, we note that $P \setminus P'$ is the set of triangles in P that are refined as one goes from P to P'.

Lemma 3.1. *For $P, P' \in \operatorname{conf}(P_0)$ with $P \preceq P'$ the local discrete upper bound holds*

$$\|u_{P'} - u_P\|_V^2 + \|p_{P'} - p_P\|_Q^2 \lesssim \eta_1(P, P \setminus P'). \tag{3.10}$$

Moreover, the local equivalence takes place

$$\alpha\eta_1(P,Q) \leqslant \eta_2(P,Q) \leqslant \beta\eta_1(P,Q), \quad Q \subset P, \tag{3.11}$$

where $\alpha > 0$ and β are constants.

Proof. For any $(v,q) \in (V_{P'}, Q_{P'})$ and $(v_P, q_P) \in (V_P, Q_P)$ Equation (2.12) and integration by parts give

$$\begin{aligned}
&a(u_{P'} - u_P, v) - b(v, p_{P'} - p_P) - b(u_{P'} - u_P, q) \\
&= a(u_{P'} - u_P, v - v_P) - b(v - v_P, p_{P'} - p_P) - b(u_{P'} - u_P, q - q_P) \\
&= \langle f, v - v_P \rangle - a(u_P, v - v_P) + b(v - v_P, p_P) + b(u_P, q - q_P) \\
&= \langle f, v - v_P \rangle + \sum_{\tau \in P} \Bigg(\int_\tau \Delta u \cdot (v - v_P) - \int_{\partial\tau} \partial_\nu u_P \cdot (v - v_P) \Bigg) \\
&\quad - \int_\Omega (v - v_P) \cdot \nabla p_P + \int_\Omega (q - q_P) \nabla \cdot u_P.
\end{aligned} \tag{3.12}$$

Let $\omega = \operatorname{int} \bigcup_{\tau \in P \setminus P'} \overline{\tau}$, i.e., let ω be the interior of the region covered by the refined triangles. Then we set (v_P, q_P) to be equal to (v,q) in $\Omega \setminus \omega$ and to the Scott–Zhang interpolator of (v,q) in ω. In doing so, we choose the Scott–Zhang interpolator to be adapted to the boundary of ω, thus ensuring that $v_P \in V_P$ and $q_P \in Q_P$ (see [33]). With this preparation, we can continue the chain of reasoning as follows:

$$\begin{aligned}
&a(u_{P'} - u_P, v) - b(v, p_{P'} - p_P) - b(u_{P'} - u_P, q) \\
&= \sum_{\tau \in P} \int_\tau (f + \Delta u - \nabla p_P) \cdot (v - v_P) - \sum_{e \in E_P} \int_e [\partial_\nu u_P] \cdot (v - v_P) + \int_\Omega (q - q_P) \nabla \cdot u_P \\
&\leqslant \sum_{\tau \in P \setminus P'} \|f + \Delta u - \nabla p_P\|_{L^2(\tau)} \|v - v_P\|_{L^2(\tau)} \\
&\quad + \sum_{e \in E_P \setminus E_{P'}} \|[\partial_\nu u_P]\|_{L^2(\tau)} \|v - v_P\|_{L^2(\tau)} + \sum_{\tau \in P \setminus P'} \|\nabla \cdot u_P\|_{L^2(\tau)} \|q - q_P\|_{L^2(\tau)}.
\end{aligned} \tag{3.13}$$

Since $(u_{P'} - u_P, p_{P'} - p_P) \in (V_{P'}, Q_{P'})$, we use the stability (2.15), in combination with standard estimates for the Scott–Zhang interpolator and local finiteness to establish (3.10).

The second inequality in (3.11), namely, $\eta_2(P,Q) \lesssim \eta_1(P,Q)$ follows from the inverse estimate $\|\nabla \cdot u_P|_\tau\|_{L^2(\partial\tau)} \lesssim h_\tau^{-1/2}\|\nabla \cdot u_P\|_{L^2(\tau)}$. In order to prove the other inequality $\eta_1(P,Q) \lesssim \eta_2(P,Q)$, we localize the argument from [7, 15]. Putting $v = 0$ in (2.12), we see that $\nabla \cdot u_P$ is L^2-orthogonal to the pressure space Q_P. Since

$$\int\limits_\Omega \nabla \cdot u_P = 0$$

by the divergence theorem, this means that $\nabla \cdot u_P$ is L^2-orthogonal to the full space $C(\Omega) \cap S_P^{d-1}$. In particular, $(\nabla \cdot u_P)|_\omega$ is L^2-orthogonal to $H_0^1(\omega) \cap S_P^{d-1}$ on ω, where $\omega = \operatorname{int} \bigcup\limits_{\tau \in Q} \overline{\tau}$, meaning that

$$\|\nabla \cdot u_P\|_{L^2(\omega)} \leqslant \|\nabla \cdot u_P - q\|_{L^2(\omega)} \tag{3.14}$$

for any $q \in H_0^1(\omega) \cap S_P^{d-1}$. Let $S^* = \{g \in L^2(\omega) \cap S_P^{d-1} : g \perp_{L^2(\omega)} H_0^1(\omega) \cap S_P^{d-1}\}$. Then we claim that

$$\|g\|_{L^2(\omega)}^2 \lesssim \sum_{e \in E_Q} h_e \|[g]\|_{L^2(e)}^2 + \sum_{\{e \in E_P : e \subset \partial\omega\}} h_e \|g\|_{L^2(e)}^2 \tag{3.15}$$

for any $g \in S^*$, where $[g]$ denotes the jump in g across e. Indeed, the right-hand side defines a (squared) norm on S^* since the vanishing of this quantity implies that $g \in C(\omega) \cap S_P^{d-1}$ and $g|_{\partial\omega} = 0$, which means that $g = 0$ by $g \perp_{L^2(\omega)} H_0^1(\omega) \cap S_P^{d-1}$. The local scaling by h_e can be deduced by a local homogeneity argument. Finally, plugging in $g = \nabla \cdot u_P$ and using straightforward bounds, we complete the proof. □

We end this section with the following standard result.

Lemma 3.2. (a) *Let $P, P' \in \operatorname{conf}(P_0)$ be such that $P \preceq P'$, and let*

$$\eta_2(P, P \setminus P') \geqslant \theta\eta_2(P,P) \tag{3.16}$$

for some $0 < \theta \leqslant 1$. Then

$$\eta_2(P', P') \leqslant \mu\eta_2(P,P) + \gamma\|(u_P - u_{P'}, p_P - p_{P'})\|_{V \times Q}^2, \tag{3.17}$$

with $\mu < 1$ and γ depending only on θ.

(b) *Let $P, P' \in \operatorname{conf}(P_0)$ be such that $P \preceq P'$, and let for some $0 < \mu < 1/2$*

$$\eta_2(P', P') \leqslant \mu\eta_2(P,P). \tag{3.18}$$

Then

$$\eta_2(P, P \setminus P') \geqslant \theta^*(1 - 2\mu)\eta_2(P,P), \tag{3.19}$$

where $\theta^* > 0$ is a constant independent of μ.

Proof. (a) By standard arguments, one can prove that

$$\eta_2(P', P') \leqslant (1+\delta)\eta_2(P,P) - \lambda(1+\delta)\eta_2(P, P \setminus P') + C_\delta\|(u_P - u_{P'}, p_P - p_{P'})\|_{V \times Q}^2 \tag{3.20}$$

for any $P, P' \in \mathrm{conf}(P_0)$ with $P \preceq P'$ and any $\delta > 0$, where C_δ can depend on δ, and $\lambda > 0$ is independent of δ. Upon using (3.16), this gives

$$\eta_2(P', P') \leqslant (1+\delta)(1-\lambda\theta)\eta_2(P, P) + C_\delta \|(u_P - u_{P'}, p_P - p_{P'})\|^2_{V\times Q}. \tag{3.21}$$

Choosing $\delta > 0$ small enough we get (3.17).

(b) Observe that

$$\begin{aligned}\eta_2(P, P\cap P') \leqslant 2\eta_2(P', P\cap P') &+ \sum_{\tau\in P\cap P'} 2h_\tau^2 \|\Delta(u_P - u_{P'}) - \nabla(p_P - p_{P'})\|^2_{L^2(\tau)} \\ &+ \sum_{e\in E_P\cap E_{P'}} 2h_e \|[\partial_\nu(u_P - u_{P'})]\|^2_{L^2(e)} + \sum_{\tau\in P\cap P'} 2h_\tau \|\nabla\cdot(u_P - u_{P'})|_\tau\|^2_{L^2(\partial\tau)} \\ &\leqslant 2\eta_2(P', P\cap P') + C\|(u_P - u_{P'}, p_P - p_{P'})\|^2_{V\times Q}. \end{aligned} \tag{3.22}$$

By this observation, the property (3.18), and the local upper bound from Lemma 3.1, we infer

$$\begin{aligned}(1-2\mu)\eta_2(P, P) &\leqslant \eta_2(P, P\setminus P') + \eta_2(P, P\cap P') - 2\eta_2(P', P') \\ &\leqslant \eta_2(P, P\setminus P') + C\|(u_P - u_{P'}, p_P - p_{P'})\|^2_{V\times Q} \lesssim \eta_2(P, P\setminus P'),\end{aligned} \tag{3.23}$$

which concludes the proof. □

4 Convergence Rate

We are ready to start our discussion of adaptive algorithms and their convergence rates. A template of an adaptive finite element method is displayed in Figure 2. The *a posteriori* error estimator η can be chosen to be η_0 from (3.1), η_1 from (3.2), or η_2 from (3.3). For theoretical purposes, we think of the algorithm as generating an infinite sequence of triples $\{(P_k, u_k, p_k)\}$, where $P_k \in \mathrm{conf}(P_0)$ and $(u_k, p_k) \in V_{P_k} \times Q_{P_k}$ for all $k \in \mathbb{N}_0$. The heart of the method is, after computing the Galerkin solution (u_k, p_k) on the mesh P_k, to identify a minimal (up to a constant factor) set $R_k \subset P_k$ of triangles satisfying the so-called *Dörfler property*

$$\eta(P_k, R_k) \geqslant \theta\eta(P_k, P_k), \tag{4.1}$$

where $0 < \theta \leqslant 1$ is a global parameter. Then the next mesh is obtained as $P_{k+1} = \mathrm{refine}(P_k, R_k)$.

---

input : conforming partition P_0, and $0 < \theta \leqslant 1$
output: $P_k \in \mathrm{conf}(P_0)$ and $(u_k, p_k) \in V_{P_k} \times Q_{P_k}$ for all $k \in \mathbb{N}_0$

for $k = 0, 1, \dots$ **do**
  Compute $(u_k, p_k) \in V_{P_k} \times Q_{P_k}$ as the Galerkin approximation of (u, p);
  Identify a minimal (up to a constant factor) set $R_k \subset P_k$ of triangles satisfying

$$\eta(P_k, R_k) \geqslant \theta\eta(P_k, P_k);$$

  Set $P_{k+1} = \mathrm{refine}(P_k, R_k)$;
endfor

Figure 2. Adaptive FEM.

The first important question is if and how fast the approximations (u_k, p_k) converge to the exact solution (u, p) as $k \to \infty$. The following plain convergence result was obtained in [15, §3.3] for $\eta = \eta_1$ and in [17, §4.4] for $\eta = \eta_0$. Following [34], we give here a slightly different proof.

Lemma 4.1. *In the context of* Adaptive FEM *in Figure* 2*, let η be one of η_0, η_1, and η_2. Then $(u_k, p_k) \to (u, p)$ in $V \times Q$ as $k \to \infty$.*

Proof. By [15, Lemma 4.2], we have $(u_k, p_k) \to (u_\infty, p_\infty)$ in $V \times Q$ as $k \to \infty$ for some $(u_\infty, p_\infty) \in V \times Q$.

On the other hand, since $\eta_0(P_k, R_k) \leqslant \eta_2(P_k, R_k)$ and $\eta_2(P_k, P_k) \lesssim \eta_0(P_k, P_k)$ from the global equivalences (3.7) and (3.8), the Dörfler property (4.1) for $\eta = \eta_0$ implies the same for $\eta = \eta_2$ with possibly a different constant $\theta > 0$. Similarly, by the equivalence (3.11), the Dörfler property (4.1) for $\eta = \eta_1$ implies the same for $\eta = \eta_2$ with possibly a different constant $\theta > 0$. The latter argument runs also in the other direction since (3.11) is a local equivalence. To conclude, we can assume the Dörfler property (4.1) for both $\eta = \eta_1$ and $\eta = \eta_2$ with possibly different constants $\theta > 0$.

In any case, by Lemma 3.2 (a), there exist constants $\mu < 1$ and $\gamma \geqslant 0$ such that

$$\eta_2(P_{k+1}, P_{k+1}) \leqslant \mu \eta_2(P_k, P_k) + \gamma \|(u_k - u_{k+1}, p_k - p_{k+1})\|^2_{V \times Q} \tag{4.2}$$

for all $k \in \mathbb{N}$. The last term converges to 0 as $k \to \infty$ since (u_k, p_k) is convergent. Thus, introducing the abbreviation $e_k = \eta_2(P_k, P_k)$, we have

$$e_{k+1} \leqslant \mu e_k + \alpha_k, \tag{4.3}$$

with $\alpha_k \to 0$. Let $\varepsilon > 0$, and let k be such that $\alpha_{k+m} \leqslant \varepsilon$ for all $m \geqslant 0$. Then we have

$$e_{k+m} \leqslant \mu^m e_k + \varepsilon(1 + \mu + \ldots + \mu^{m-1}) \leqslant \mu^m e_k + \frac{\varepsilon}{1 - \mu} \tag{4.4}$$

for all $m \geqslant 0$. This shows that $\limsup_{k \to \infty} e_k \leqslant \varepsilon/(1 - \mu)$. Since $\varepsilon > 0$ is arbitrary and $e_k \geqslant 0$, we conclude that $\lim_{k \to \infty} e_k = 0$. Finally, the global upper bound in (3.4) implies the convergence $(u_k, p_k) \to (u, p)$ in $V \times Q$ as $k \to \infty$. □

We have the following conditional error reduction theorem for all three estimators.

Theorem 4.1. *In the context of* Adaptive FEM *in Figure* 2*, let η be one of η_0, η_1, and η_2. Assume that there exists a constant $c > 0$ such that*

$$\sum_{k=\ell}^{N} \|(u_k - u_{k+1}, p_k - p_{k+1})\|^2_{V \times Q} \leqslant c \|(u - u_\ell, p - p_\ell)\|^2_{V \times Q} \tag{4.5}$$

for any integers ℓ and N. Then there are constants $\rho < 1$ and $C > 0$ such that

$$\eta(P_k, P_k) \leqslant C \rho^{k-\ell} \eta(P_\ell, P_\ell) \tag{4.6}$$

for all $k \geqslant \ell \geqslant 0$. In particular,

$$\|(u - u_k, p - p_k)\|_{V \times Q} \leqslant C' \rho^k \tag{4.7}$$

for some constant C'.

Proof. As we have discussed in the proof of Lemma 4.1, we can assume the Dörfler property (4.1) for both $\eta = \eta_1$ and $\eta = \eta_2$, with possibly different constants $\theta > 0$. Hence, by Lemma 3.2 (a), there exist constants $\mu < 1$ and $\gamma \geqslant 0$ such that

$$\eta_2(P_{k+1}, P_{k+1}) \leqslant \mu\eta_2(P_k, P_k) + \gamma\|(u_k - u_{k+1}, p_k - p_{k+1})\|^2_{V\times Q} \tag{4.8}$$

for all $k \in \mathbb{N}$. Then, by the assumption (4.5) and the global upper bound from (3.4), we get

$$\begin{aligned}\sum_{k=\ell}^{N} \eta_2(P_{k+1}, P_{k+1}) &\leqslant \mu \sum_{k=\ell}^{N} \eta_2(P_k, P_k) + \gamma \sum_{k=\ell}^{N} \|(u_k - u_{k+1}, p_k - p_{k+1})\|^2_{V\times Q} \\ &\leqslant \mu \sum_{k=\ell}^{N} \eta_2(P_k, P_k) + \gamma c\|(u - u_\ell, p - p_\ell)\|^2_{V\times Q} \leqslant \mu \sum_{k=\ell}^{N} \eta_2(P_k, P_k) + C\eta_2(P_\ell, P_\ell)\end{aligned} \tag{4.9}$$

for any $\ell \in \mathbb{N}$ and $N \geqslant \ell$. Since $\mu < 1$, this implies the convergence of the series $\sum_k \eta_2(P_k, P_k)$ and, consequently,

$$\alpha_\ell \leqslant \mu\alpha_\ell + (1 + C)\eta_2(P_\ell, P_\ell), \quad \alpha_\ell \leqslant \frac{1 + C}{1 - \mu}\eta_2(P_\ell, P_\ell), \tag{4.10}$$

where $\eta_2(P_\ell, P_\ell)$ is added to both sides of (4.9) and

$$\alpha_\ell = \sum_{k=\ell}^{\infty} \eta_2(P_k, P_k) \tag{4.11}$$

is introduced. As a consequence,

$$\alpha_{\ell+1} = \alpha_\ell - \eta_2(P_\ell, P_\ell) \leqslant \frac{\mu + C}{1 + C}\alpha_\ell, \tag{4.12}$$

which means that α_ℓ decays geometrically. This yields

$$\eta_2(P_k, P_k) \leqslant \alpha_k \leqslant \Big(\frac{\mu + C}{1 + C}\Big)^{k-\ell} \frac{1 + C}{1 - \mu}\eta_2(P_\ell, P_\ell), \tag{4.13}$$

where we took into account (4.10). Finally, the same geometric decay for both η_0 and η_1 follows from the equivalences (3.7) and (3.8). □

Now, we address the question of convergence rate. We start by defining

$$E_P(u, p) = \inf_{(v,q)\in V_P\times Q_P} \|(u - v, p - q)\|_{V\times Q} \tag{4.14}$$

for $P \in \operatorname{conf}(P_0)$ and

$$\sigma^*_N(u, p) = \inf_{P\in\mathscr{P}_N} \big(E_P(u, p)^2 + \operatorname{osc}(P)\big)^{\frac{1}{2}} \tag{4.15}$$

for $N \in \mathbb{N}$, where $\mathscr{P}_N = \{P \in \operatorname{conf}(P_0) : \#P - \#P_0 \leqslant N\}$. Note that the oscillation depends on u and p implicitly through Equation (2.6). Following [5], we then define the *modified approximation class*

$$\mathscr{A}^s_* = \{(u, p) \in V \times Q : |(u, p)|_{\mathscr{A}^s_*} \equiv \sup_{N\in\mathbb{N}} N^s\sigma^*_N(u, p) < \infty\} \tag{4.16}$$

for $s > 0$. Thus, $(u,p) \in \mathscr{A}_*^s$ if and only if for each $N \in \mathbb{N}$ there exists a partition $P \in \operatorname{conf}(P_0)$ with $\#P - \#P_0 \leqslant N$ such that

$$E_P(u,p)^2 + \operatorname{osc}(P) \leqslant c(\#P - \#P_0)^{-2s}, \tag{4.17}$$

where the constant $c = c(u,p)$ is independent of P. The greatest lower bound for such constants c is the quantity $|(u,p)|^2_{\mathscr{A}_*^s}$.

The following is one of our main results alluded to in the Introduction.

Theorem 4.2. *In the context of* Adaptive FEM *in Figure* 2, *let η be either η_1 or η_2, and let $\theta > 0$ be small enough. A sufficient condition is $\theta < \theta^*$ for $\eta = \eta_2$ and $\theta < \frac{\alpha}{\beta}\theta^*$ for $\eta = \eta_1$. Suppose that $f \in L^2(\Omega, \mathbb{R}^n)$ and $(u,p) \in \mathscr{A}_*^s$ for some $s > 0$. In addition, assume* (4.5) *for the solution sequence $\{(u_k, p_k)\}$. Then there exists a constant $c > 0$ such that*

$$\|(u - u_k, p - p_k)\|^2_{V\times Q} + \operatorname{osc}(P_k) \leqslant c|(u,p)|^2_{\mathscr{A}_*^s}(\#P_k - \#P_0)^{-2s}. \tag{4.18}$$

Proof. It suffices to consider $\eta = \eta_1$ since this case is slightly nonstandard. Our strategy is to use the local equivalence (3.11) to relate η_1 with η_2 and use standard arguments for η_2. For $P \in \operatorname{conf}(P_0)$, we set

$$e(P) = \|(u - u_P, p - p_P)\|^2_{V\times Q} + \operatorname{osc}(P). \tag{4.19}$$

Note that from the *a priori* estimate (2.14) and the definition of oscillation (3.5) we have the weak monotonicity

$$e(P') \lesssim e(P) \tag{4.20}$$

for any refinement $P' \in \operatorname{conf}(P_0)$ of P.

By the definition of $\mathscr{A}_*^s$, there exists a partition $P \in \operatorname{conf}(P_0)$ such that

$$\#P - \#P_0 \leqslant \varepsilon_k^{-1/s}|(u,p)|^{1/s}_{\mathscr{A}_*^s}, \quad E_P(u,p)^2 + \operatorname{osc}(P) \leqslant \varepsilon_k^2, \tag{4.21}$$

where $\varepsilon_k = \delta\eta_2(P_k, P_k)$ and $\delta > 0$ is a small constant. Let $P' = P \oplus P_k$. Then the global lower bound (3.4) and the *a priori* estimate (2.14) yield

$$\eta_2(P', P') \lesssim e(P') \lesssim e(P) \lesssim \varepsilon_k^2 = \delta\eta_2(P_k, P_k). \tag{4.22}$$

Upon choosing $\delta > 0$ small enough, this implies $\eta_2(P', P') \leqslant \mu\eta_2(P_k, P_k)$ with

$$\mu = \frac{1}{2}\Big(1 - \frac{\beta\theta}{\alpha\theta^*}\Big).$$

By Lemma 3.2, we have

$$\eta_2(P_k, P_k \setminus P') \geqslant \frac{\beta\theta}{\alpha}\eta_2(P_k, P_k).$$

Under the local equivalence (3.11), it becomes $\eta_1(P_k, P_k \setminus P') \geqslant \theta\eta_1(P_k, P_k)$. Since, by construction, $R_k \subset P_k$ is a minimal (up to a constant factor) set satisfying $\eta_1(P_k, R_k) \geqslant \theta\eta_1(P_k, P_k)$, we infer $\#R_k \lesssim \#(P_k \setminus P')$. Hence

$$\#R_k \lesssim \#P_k - \#P' \leqslant \#P - \#P_0 \leqslant \varepsilon_k^{-1/s}|(u,p)|^{1/s}_{\mathscr{A}_*^s}, \tag{4.23}$$

where we used the property (2.4) of overlays. Now, we invoke (2.3) and the geometric decay (4.6) to conclude

$$\#P_\ell - \#P_0 \lesssim \sum_{k=0}^{\ell-1}\#R_k \lesssim |(u,p)|^{1/s}_{\mathscr{A}_*^s}\sum_{k=0}^{\ell-1}\varepsilon_k^{-1/s} \lesssim \varepsilon_\ell^{-1/s}|(u,p)|^{1/s}_{\mathscr{A}_*^s}. \tag{4.24}$$

Recalling that the estimators dominate oscillation, we complete the proof. □

5 Approximation Classes

In Section 4, we established optimal convergence rates with respect to the modified approximation classes (4.16). Ideally, however, one would like to have optimality with respect to the *standard approximation classes*

$$\mathscr{A}^s = \{(u,p) \in V \times Q : |(u,p)|_{\mathscr{A}^s} \equiv \sup_{N\in\mathbb{N}} N^s \inf_{P\in\mathscr{P}_N} E_P(u,p) < \infty\}, \tag{5.1}$$

where we recall $\mathscr{P}_N = \{P \in \mathrm{conf}(P_0) : \#P - \#P_0 \leqslant N\}$. Perhaps, a more practical goal is to know interrelations between $\mathscr{A}^s$ and $\mathscr{A}^s_*$. In this section, we study the interrelations in terms of Besov space memberships of the data f.

It is convenient to define the *oscillation classes*

$$\mathscr{O}^s = \{f \in L^2(\Omega,\mathbb{R}^n) : |f|_{\mathscr{O}^s} \equiv \sup_{N\in\mathbb{N}} N^s \inf_{P\in\mathscr{P}_N} \mathrm{osc}(P)^{\frac{1}{2}} < \infty\}. \tag{5.2}$$

It is obvious that $\mathscr{A}^s_* \subset \mathscr{A}^s$ for all $s > 0$. In the converse direction, we have the following well-known result.

Lemma 5.1. *Let $(u,p) \in \mathscr{A}^s$, and let $f \in \mathscr{O}^s$ with $s > 0$, where u, p, and f satisfy Equation* (2.6)*. Then $(u,p) \in \mathscr{A}^s_*$ with $|(u,p)|_{\mathscr{A}^s_*} \lesssim |(u,p)|_{\mathscr{A}^s} + |f|_{\mathscr{O}^s}$.*

Proof. Let $N \in \mathbb{N}$ be an arbitrary number. By the definition of $\mathscr{A}^s$, there exists a partition $P' \in \mathrm{conf}(P_0)$ such that

$$E_{P'}(u,p) \leqslant 2N^{-s}|(u,p)|_{\mathscr{A}^s}, \quad \#P' - \#P_0 \leqslant N. \tag{5.3}$$

Similarly, by the definition of $\mathscr{O}^s$, there exists a partition $P'' \in \mathrm{conf}(P_0)$ such that

$$\mathrm{osc}(P'') \leqslant 2N^{-2s}|f|^2_{\mathscr{O}^s}, \quad \#P'' - \#P_0 \leqslant N. \tag{5.4}$$

Then for $P = P' \oplus P''$ we have $\#P - \#P_0 \leqslant 2N$ by (2.4). Moreover, the monotonicity arguments guarantee that

$$E_P(u,p)^2 + \mathrm{osc}(P) \leqslant E_{P'}(u,p)^2 + \mathrm{osc}(P'') \lesssim N^{-2s}(|(u,p)|^2_{\mathscr{A}^s} + |f|^2_{\mathscr{O}^s}), \tag{5.5}$$

which completes the proof. □

Lemma 5.1 makes us wonder how regular f must be in order for it to be a member of $\mathscr{O}^s$. Using quasiuniform partitions, one can show that $H^\alpha(\Omega,\mathbb{R}^n) \subset \mathscr{O}^s$ for $s = (\alpha+1)/n$ and $\alpha \geqslant 0$. For instance, if we want to recover the optimal convergence rates of the lowest order Taylor–Hood elements ($d = 2$), then this would require $f \in H^1(\Omega,\mathbb{R}^n)$, which appears to be a bit excessive. As it is natural in the current setting, we would like to investigate the question in terms of the Besov regularity of f. Let us make precise what we mean by Besov spaces. For $0 < q \leqslant \infty$, the mth order *L^q-modulus of smoothness* is

$$\omega_m(u,t,\Omega)_q = \sup_{|h|\leqslant t} \|\Delta^m_h u\|_{L^q(\Omega_{rh})}, \tag{5.6}$$

where $\Omega_{mh} = \{x \in \Omega : [x + mh] \subset \Omega\}$ and Δ^m_h is the mth order forward difference operator defined recursively by $[\Delta^1_h u](x) = u(x+h) - u(x)$ and $\Delta^k_h u = \Delta^1_h(\Delta^{k-1}_h)u$, i.e.,

$$\Delta^m_h u(x) = \sum_{k=0}^{m} (-1)^{m+k} \binom{m}{k} u(x+kh). \tag{5.7}$$

Then for $0 < q, r \leqslant \infty$ and $\alpha \geqslant 0$ with $m > \alpha - \max\{0, 1/q - 1\}$ being an integer, the *Besov space* $B^\alpha_{q,r}(\Omega)$ consists of those $u \in L^q(\Omega)$ for which

$$|u|_{B^\alpha_{q,r}(\Omega)} = \|t \mapsto t^{-\alpha-1/r}\omega_m(u,t,\Omega)_q\|_{L^r((0,\infty))}, \tag{5.8}$$

is finite. Since Ω is bounded, being in a Besov space is a statement about the size of $\omega_m(u,t,\Omega)_q$ only for small t. From this it is easy to derive the useful equivalence

$$|u|_{B^\alpha_{q,r}(\Omega)} \eqsim \|(\lambda^{j\alpha}\omega_m(u,\lambda^{-j},\Omega)_q)_{j\geqslant 0}\|_{\ell^r} \tag{5.9}$$

for any constant $\lambda > 1$. The mapping $\|\cdot\|_{B^\alpha_{q,r}(\Omega)} = \|\cdot\|_{L^q(\Omega)} + |\cdot|_{B^\alpha_{q,r}(\Omega)}$ defines a norm when $q, r \geqslant 1$ and only a quasinorm otherwise. So long as $m > \alpha - \max\{0, 1/q - 1\}$, different choices of m result in (quasi-) norms that are equivalent to each other. On the other hand, if we took $m < \alpha - \max\{0, 1/q - 1\}$, then the space $B^\alpha_{q,r}$ would have been trivial in the sense that $B^\alpha_{q,r} = \mathbb{P}_{m-1}$.

We have the subadditivity property

$$\sum_{\tau\in P} |f|^q_{B^\alpha_{q,q}(\tau)} \lesssim |f|^q_{B^\alpha_{q,q}(\Omega)}, \quad f \in B^\alpha_{q,q}(\Omega), \tag{5.10}$$

for $P \in \operatorname{conf}(P_0)$ and $0 < q < \infty$. A slightly stronger form of this is also true. Let $\{\tau_k\}$ be a finite collection of disjoint triangles with each $\tau_k \in P_k$ for some $P_k \in \operatorname{conf}(P_0)$. Let $\widehat{\tau}_k$ denote the star around τ_k with respect to P_k, i.e., let $\widehat{\tau}_k$ be the interior of $\bigcup\{\overline{\sigma} \in P_k : \overline{\sigma} \cap \overline{\tau}_k \neq \varnothing\}$. Then

$$\sum_{k} |f|^q_{B^\alpha_{q,q}(\widehat{\tau}_k)} \lesssim |f|^q_{B^\alpha_{q,q}(\Omega)}, \quad f \in B^\alpha_{q,q}(\Omega). \tag{5.11}$$

Now, we describe various embedding relationships among the Besov and Sobolev spaces. Since Ω is bounded, it is clear that $B^\alpha_{q,r}(\Omega) \hookrightarrow B^\alpha_{q',r}(\Omega)$ for any $\alpha \geqslant 0$, $0 < r \leqslant \infty$ and $\infty \geqslant q > q' > 0$. From the equivalence (5.9) we have the lexicographical ordering $B^\alpha_{q,r}(\Omega) \hookrightarrow B^{\alpha'}_{q,r'}(\Omega)$ for $\alpha > \alpha'$ with any $0 < r, r' \leqslant \infty$ and $B^\alpha_{q,r}(\Omega) \hookrightarrow B^\alpha_{q,r'}(\Omega)$ for $0 < r < r' \leqslant \infty$. Nontrivial embeddings are $B^\alpha_{q,r}(\Omega) \hookrightarrow B^{\alpha'}_{q',r}(\Omega)$ for $(\alpha - \alpha')/n = 1/q - 1/q' > 0$, and

$$B^\alpha_{q,q}(\Omega) \hookrightarrow L^r(\Omega), \quad \frac{\alpha}{n} = \frac{1}{q} - \frac{1}{r} > 0. \tag{5.12}$$

Finally, we recall the fact that $B^\alpha_{2,2}(\Omega) = H^\alpha(\Omega)$ for all $\alpha > 0$.

We need the *Whitney estimate*

$$\inf_{g\in\mathbb{P}_m} \|f - g\|_{L^q(G)} \lesssim \omega_{m+1}(f, \operatorname{diam} G, G)_q, \quad f \in L^q(G), \tag{5.13}$$

valid for any convex domain $G \subset \mathbb{R}^n$ with an implicit constant depending only on n, m, and q (see [35]). The same estimate is also true when G is the star around $\tau \in P$ for some partition $P \in \operatorname{conf}(P_0)$ with the implicit constant additionally depending on the shape regularity constant of $\operatorname{conf}(P_0)$ (see [20]).

In the ensuing discussions, we often need vector-valued versions of the Besov and other function spaces that should, strictly speaking, be denoted by $B^\alpha_{p,q}(\Omega, \mathbb{R}^n)$ or $L^p(\Omega)^n$ etc. However, for simplicity of the notation we write $B^\alpha_{p,q}(\Omega)$ or $L^p(\Omega)$ etc. to mean the same things.

Theorem 5.1. *We have* $B^{\alpha}_{q,q}(\Omega,\mathbb{R}^n)\subset\mathscr{O}^s$ *with* $s=(\alpha+1)/n$ *as long as* $0<q<\infty$, $\alpha/n\geqslant 1/q-1/2$, *and* $\alpha<d-1+\max\{0,1/q-1\}$.

Proof. In this proof, it is understood that all Besov seminorms are defined by using ω_{d-1}. Hence, in particular, we have $|g|_{B^{\alpha}_{q,q}}=0$ for $g\in\mathbb{P}_{d-2}$. For any $g\in(\mathbb{P}_{d-2})^n$ and convex domain $G\subset\mathbb{R}^n$

$$\|f-g\|_{L^2(G)}\lesssim\|f-g\|_{L^q(G)}+|f|_{B^{\alpha}_{q,q}(G)}\tag{5.14}$$

by the embedding (5.12) and

$$\|f-g\|_{L^q(G)}\lesssim\omega_{d-1}(f,\operatorname{diam}G,G)_q\lesssim|f|_{B^{\alpha}_{q,q}(G)}\tag{5.15}$$

by the Whitney estimate (5.13). Since $h_\tau=|\tau|^{1/n}$, the homogeneity argument gives

$$\operatorname{osc}(P)=\min_{g\in(S_P^{d-2})^n}\sum_{\tau\in P}h_\tau^2\|f-g\|^2_{L^2(\tau)}\lesssim\sum_{\tau\in P}|\tau|^{2\delta}|f|^2_{B^{\alpha}_{q,q}(\tau)}\tag{5.16}$$

for $P\in\operatorname{conf}(P_0)$, where $\delta=(\alpha+1)/n+1/2-1/q\geqslant 1/n$.

The rest of the proof follows that of Proposition 5.2 in [19]; we include it here for the sake of completeness. Let

$$e(\tau,P)=|\tau|^{2\delta}|u|^2_{B^{\alpha}_{q,q}(\tau)}\tag{5.17}$$

for $\tau\in P$ and $P\in\operatorname{conf}(P_0)$. Then for any given $\varepsilon>0$ we below specify a procedure to generate a partition $P\in\operatorname{conf}(P_0)$ satisfying

$$\sum_{\tau\in P}e(\tau,P)\lesssim(\#P)\varepsilon\tag{5.18}$$

and

$$\#P-\#P_0\leqslant c\varepsilon^{-1/(1+2s)}|f|^{2/(1+2s)}_{B^{\alpha}_{q,q}(\Omega)},\tag{5.19}$$

where $s=(\alpha+1)/n$. Then for any given $N>0$, choosing

$$\varepsilon=(c/N)^{1+2s}|f|^2_{B^{\alpha}_{q,q}(\Omega)},\tag{5.20}$$

we would be able to guarantee a partition $P\in\operatorname{conf}(P_0)$ satisfying $\#P\leqslant\#P_0+N$ and

$$\operatorname{osc}(P)\lesssim\sum_{\tau\in P}e(\tau,P)\lesssim N^{-2s}|f|^2_{B^{\alpha}_{q,q}(\Omega)}.\tag{5.21}$$

Let $\varepsilon>0$. We then recursively define $R_k=\{\tau\in P_k:e(\tau,P_k)>\varepsilon\}$ and $P_{k+1}=\operatorname{refine}(P_k,R_k)$ for $k=0,1,\ldots$. For all sufficiently large k we have $R_k=\varnothing$ since $|f|_{B^{\alpha}_{q,q}(\tau)}\lesssim|f|_{B^{\alpha}_{q,q}(\Omega)}$ and $|\tau|$ is halved at each refinement. Let $P=P_k$, where k marks the first occurrence of $R_k=\varnothing$. Since $e(\tau,P_k)\leqslant\varepsilon$ for $\tau\in P_k$, (5.18) is immediate.

In order to get a bound on $\#P$, we estimate the cardinality of $R=R_0\cup R_1\cup\ldots\cup R_{k-1}$ and use (2.3). Let $\Lambda_j=\{\tau\in R:2^{-j-1}\leqslant|\tau|<2^{-j}\}$ for $j\in\mathbb{Z}$, and let $m_j=\#\Lambda_j$. Note that the elements of Λ_j (for any fixed j) are disjoint since if any two elements intersect, then they must come from different R_k's since each R_k consists of disjoint elements and hence the ratio between the measures of the two elements must lie outside $(1/2,2)$. This gives the trivial bound

$$m_j\leqslant 2^{j+1}|\Omega|.\tag{5.22}$$

On the other hand, we have $e(\tau, P_k) > \varepsilon$ for $\tau \in \Lambda_j$ with some k, which means

$$\varepsilon < |\tau|^{2\delta} |f|^2_{B^\alpha_{q,q}(\tau)} < 2^{-2j\delta} |f|^2_{B^\alpha_{q,q}(\tau)}. \tag{5.23}$$

Summing over $\tau \in \Lambda_j$, we get

$$m_j \varepsilon^{q/2} \leqslant 2^{-jq\delta} \sum_{\tau \in \Lambda_j} |f|^q_{B^\alpha_{q,q}(\tau)} \lesssim 2^{-jq\delta} |f|^q_{B^\alpha_{q,q}(\Omega)}, \tag{5.24}$$

where we used (5.10). Finally, summing over j, we find

$$\#R \leqslant \sum_{j=-\infty}^{\infty} m_j \lesssim \sum_{j=-\infty}^{\infty} \min\{2^j, \varepsilon^{-q/2} 2^{-jq\delta'} |f|^q_{B^\alpha_{q,q}(\Omega)}\} \lesssim \varepsilon^{-q/(2+2q\delta)} |f|^{q/(1+q\delta)}_{B^\alpha_{q,q}(\Omega)}, \tag{5.25}$$

which, in view of (2.3) and $q/(1+q\delta) = 2/(1+2s)$, establishes the bound (5.19). □

In light of Lemma 5.1, we immediately get the following corollary, which is one of our main results mentioned in the Introduction.

Corollary 5.1. *Let $f \in B^\alpha_{q,q}(\Omega, \mathbb{R}^n)$ for some $0 < q < \infty$ and $\alpha \geqslant n/q - n/2$ satisfying $0 < \alpha < d - 1 + \max\{0, 1/q - 1\}$. Then $(u,p) \in \mathscr{A}^s$ implies $(u,p) \in \mathscr{A}^s_*$ with $s = (\alpha+1)/n$.*

For sake of completeness we include the following result which establishes a (one-sided) characterization of the standard approximation classes $\mathscr{A}^s$ in terms of Besov spaces.

Theorem 5.2. *We have $(B^{1+ns}_{q,q}(\Omega, \mathbb{R}^n) \cap V) \times (B^{ns}_{q,q}(\Omega) \cap Q) \hookrightarrow \mathscr{A}^s$ as long as $0 < q < \infty$, $s > 1/q - 1/2$, and $0 < ns < d + \max\{0, 1/q - 1\}$.*

Proof. Assume that the Besov space seminorms related to the velocity variable are defined through ω_{d+1} and the seminorms related to the pressure variable are defined through ω_d. For $P \in \mathrm{conf}(P_0)$, let $Z_P : V \to V_P$ be a Scott–Zhang quasiinterpolation operator preserving the Dirichlet condition on $\partial\Omega$. Let $u \in B^{1+ns}_{q,q}(\Omega, \mathbb{R}^n) \cap V$, and let $P \in \mathrm{conf}(P_0)$. Then for any $\tau \in P$ and $v \in (\mathbb{P}_d)^n$

$$\begin{aligned} \|u - Z_P u\|_{H^1(\tau)} &\leqslant \|u - v\|_{H^1(\tau)} + \|Z_P(u - v)\|_{H^1(\tau)} \\ &\lesssim |\tau|^{-1/n} \|u - v\|_{L^2(\widehat{\tau})} + |u - v|_{H^1(\widehat{\tau})}, \end{aligned} \tag{5.26}$$

where $\widehat{\tau}$ is the star around τ. Now, we shift to the reference situation where $\mathrm{diam}\,\widehat{\tau} = 1$. By the embedding $B^{1+ns}_{q,q} \hookrightarrow H^1$, we can bound the last term as

$$|u - v|_{H^1(\widehat{\tau})} \lesssim \|u - v\|_{L^q(\widehat{\tau})} + |u - v|_{B^{1+ns}_{q,q}(\widehat{\tau})} = \|u - v\|_{L^q(\widehat{\tau})} + |u|_{B^{1+ns}_{q,q}(\widehat{\tau})}. \tag{5.27}$$

For the other term we have

$$\|u - v\|_{L^2(\widehat{\tau})} \lesssim \|u - v\|_{L^q(\widehat{\tau})} + |u|_{B^{1+ns}_{q,q}(\widehat{\tau})}, \tag{5.28}$$

this time using the embedding $B^{1+ns}_{q,q} \hookrightarrow L^2$. Finally, the Whitney estimate gives

$$\|u - p\|_{L^q(\widehat{\tau})} \lesssim \omega_{d+1}(u, \widehat{\tau})_q \lesssim |u|_{B^{1+ns}_{q,q}(\widehat{\tau})}. \tag{5.29}$$

Leaving the reference situation by homogeneity and combining the result, we have

$$\|u - Z_P u\|^2_{H^1(\Omega)} \lesssim \sum_{\tau \in P} |\tau|^{2\delta} |u|^2_{B^{1+ns}_{q,q}(\widehat{\tau})}, \tag{5.30}$$

with $\delta = s + 1/2 - 1/q > 0$. Similarly, one can derive

$$\|p - \Pi_P p\|^2_{L^2(\Omega)} \lesssim \sum_{\tau \in P} |\tau|^{2\delta} |p|^2_{B^{ns}_{q,q}(\tau)}, \quad p \in B^{ns}_{q,q}(\Omega) \cap Q, \tag{5.31}$$

where $\Pi_P : Q \to Q_P$ is the L^2-orthogonal projector. The rest of the proof can be completed in the same way as that of Theorem 5.1. □

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Declarations

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References

1. W. Dörfler, "A convergent adaptive algorithm for Poisson's equation," *SIAM J. Numer. Anal.* **33**, No. 3, 1106–1124 (1996).

2. P. Morin, R. H. Nochetto, and K. G. Siebert, "Data oscillation and convergence of adaptive FEM," *SIAM J. Numer. Anal.* **38**, No. 2, 466–488 (2000).

3. P. Binev, W. Dahmen, and R. DeVore, "Adaptive finite element methods with convergence rates," *Numer. Math.* **97**, No. 2, 219–268 (2004).

4. R. Stevenson, "Optimality of a standard adaptive finite element method," *Found. Comput. Math.*, **7**, No. 2, 245–269 (2007).

5. J. M. Cascon, Ch. Kreuzer, R. H. Nochetto, and K.G. Siebert, "Quasi-optimal convergence rate for an adaptive finite element method," *SIAM J. Numer. Anal.* **46**, No. 5, 2524–2550 (2008).

6. S. Dahlke, R. Hochmuth, and K. Urban, “Convergent adaptive wavelet methods for the Stokes problem,” *Lect. Notes Comput. Sci. Eng.* **14**, 66–72. (2000).

7. E. Bänsch, P. Morin, and R. H. Nochetto, “An adaptive Uzawa FEM for the Stokes problem: convergence without the inf-sup condition,” *SIAM J. Numer. Anal.* **40**, No. 4, 1207–1229 (2002).

8. Y. Kondratyuk and R. Stevenson, “An optimal adaptive finite element method for the Stokes problem,” *SIAM J. Numer. Anal.* **46**, No. 2, 747–775 (2008).

9. R. Becker and S. Mao, “Quasi-optimality of adaptive nonconforming finite element methods for the Stokes equations,” *SIAM J. Numer. Anal.* **49**, No. 3, 970–991 (2011).

10. J. Hu and J. Xu, “Convergence and optimality of the adaptive nonconforming linear element method for the Stokes problem,” *J. Sci. Comput.* **55**, No. 1, 125–148 (2013).

11. C. Carstensen, D. Peterseim, and H. Rabus, “Optimal adaptive nonconforming FEM for the Stokes problem,” *Numer. Math.* **123**, No. 2, 291–308 (2013).

12. C. Carstensen and R. H.W. Hoppe, “Error reduction and convergence for an adaptive mixed finite element method,” *Math. Comput.* **75**, No. 255, 1033–1042 (2006).

13. R. Becker and S. Mao, “An optimally convergent adaptive mixed finite element method,” *Numer. Math.* **111**, No. 1, 35–54 (2008).

14. L. Chen, M. Holst, and J. Xu, “Convergence and optimality of adaptive mixed finite element methods,” *Math. Comput.* **78**, No. 265, 35–53 (2009).

15. P. Morin, K. G. Siebert, and A. Veeser, “A basic convergence result for conforming adaptive finite elements,” *Math. Models Methods Appl. Sci.* **18**, No. 5, 707–737 (2008).

16. R. Verfürth, “A posteriori error estimators for the Stokes equations,” *Numer. Math.* **55**, No. 3, 309–325 (1989).

17. K. G. Siebert, “A convergence proof for adaptive finite elements without lower bound,” *IMA J. Numer. Anal.* **31**, No. 3, 947–970 (2011).

18. C. Carstensen, M. Feischl, M. Page, and D. Praetorius, “Axioms of adaptivity,” *Comput. Math. Appl.* **67**, No. 6, 1195–1253 (2014).

19. P. Binev, W. Dahmen, R. DeVore, and P. Petrushev, “Approximation classes for adaptive methods,” *Serdica Math. J.* **28**, No. 4, 391–416 (2002).

20. F. D. Gaspoz and P. Morin, “Approximation classes for adaptive higher order finite element approximation,” *Math. Comput.* **83**, No. 289, 2127–2160 (2014).

21. M. Sh. Birman and M. Z. Solomyak, “Piecewise polynomial approximations of functions of classes W_p^α,” *Math. USSR, Sb.*, **2**, No. 3, 295–317 (1967).

22. P. Oswald, “On the degree of nonlinear spline approximation in Besov-Sobolev spaces,” *J. Approx. Theory* **61**, No. 2, 131–157 (1990).

23. S. Dahlke, “Besov regularity for the Stokes problem,” *Math. Res.* **107**, 129–138 (1999).

24. R. Stevenson, “The completion of locally refined simplicial partitions created by bisection,” *Math. Comput.* **77**, No. 261, 227–241 (2008).

25. F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer, New York etc. (1991).

26. C. Taylor and P. Hood, “A numerical solution of the Navier–Stokes equations using the finite element technique,” *Comput. Fluids* **1**, No. 1, 73–100 (1973).

27. D. Boffi, “Stability of higher order triangular Hood-Taylor methods for the stationary Stokes equations,” *Math. Models Methods Appl. Sci.* **4**, No. 2, 223–235 (1994).

28. D. Boffi, “Three-dimensional finite element methods for the Stokes problem,” *SIAM J. Numer. Anal.* **34**, No. 2, 664–670 (1997).

29. M. Bercovier and O. Pironneau, “Error estimates for finite element method solution of the Stokes problem in the primitive variables,” *Numer. Math.* **33**, No. 2, 211–224 (1979).

30. R. Verfürth, “Error estimates for a mixed finite element approximation of the Stokes equations,” *RAIRO, Anal. Numér.* **18**, No. 2, 175–182 (1984).

31. R. Stenberg, “On some three-dimensional finite elements for incompressible media,” *Comput. Methods Appl. Mech. Eng.* **63**, No. 3, 261–269 (1987).

32. F. Brezzi and R. S. Falk, “Stability of higher order Hood-Taylor methods,” *SIAM J. Numer. Anal.* **28**, No. 3, 581–590 (1991).

33. L. R. Scott and S. Zhang, “Finite element interpolation of nonsmooth functions satisfying boundary conditions,” *Math. Comput.* **54**, No. 190, 483–493 (1990).

34. M. Aurada, S. Ferraz-Leite, and D. Praetorius, “Estimator reduction and convergence of adaptive BEM,” *Appl. Numer. Math.* **62**, No. 6, 787–801 (2012).

35. S. Dekel and D. Leviatan, “Whitney estimates for convex domains with applications to multivariate piecewise polynomial approximation,” *Found. Comput. Math.*, **4**, No. 4, 345–368 (2004).

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