DOI 10.1007/s10958-024-07055-7
Journal of Mathematical Sciences, Vol. 279, No. 6, March, 2024

ON GAP FUNCTIONS FOR QUASI-EQUILIBRIUM PROBLEMS VIA DUALITY

Lkhamsuren Altangerel
German-Mongolian Institute for Resources and Technology
Nalaikh district, Ulaanbaatar 12800, Mongolia
altangerel@gmit.edu.mn

We extend gap functions to quasi-equilibrium problems by using the duality results. In particular, we obtain new results for quasi-equilibrium problems known earlier for equilibrium problems and mixed quasi-variational inequalities. Bibliography: 12 *titles.*

1 Introduction

Let $K : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping, and let $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bifunction such that $f(x, x) = 0$ for all $x \in \mathbb{R}^n$. The quasi-equilibrium problem consists in finding $x \in K(x)$ such that

$$\text{(QEP)} \qquad f(x, y) \geqslant 0 \quad \forall\, y \in K(x).$$

If $K(x) = K$ for all $x \in \mathbb{R}^n$, then Problem (QEP) becomes the classical equilibrium problem (EP) introduced by Blum and Oettli [1]. In recent years, Problem (QEP) has attracted the attention of authors who contribute to such areas as the existence results [2, 3], gap functions [4], and development of algorithms [5]. Moreover, for $f(x, y) = \langle F(x), y - x\rangle$ with $F : \mathbb{R}^n \to \mathbb{R}^n$ Problem (QEP) reduces to the quasi-variational inequality (QVI) (see [6]–[8]), where $\langle\cdot, \cdot\rangle$ denotes the Euclidean inner product.

Since Problems (QEP) and (QVI) allow us to unify problems such as optimization problems, Nash equilibrium problems, complementarity problems, fixed point problems, variational inequalities, and generalized Nash equilibrium problems in an appropriate way, it is important to study quasi-equilibrium problems from both theoretical and practical points of view.

Unlike the existing results dealing with gap functions for quasi-variational inequalities based on duality [6], we consider quasi-equilibrium problems. The paper is organized as follows. In Section 2, we recall definitions and duality results from [9]. In Section 3, we study gap functions for quasi-equilibrium problems. As particular cases of the results of Section 3, we obtain the existing results for equilibrium problems and mixed quasi-variational inequalities.

International Mathematical Schools **7**. Mathematical Schools in Mongolia. In Honor of Academician T. Zhanlav

1072-3374/24/2796-0730 © 2024 Springer Nature Switzerland AG

2 Preliminaries

For a nonempty set $C \subseteq \mathbb{R}^n$ we introduce the indicator function $\delta_C : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$,

$$\delta_C(x) = \begin{cases} 0, & x \in C, \\ +\infty & \text{otherwise.} \end{cases}$$

For a function $h : \mathbb{R}^n \to \overline{\mathbb{R}}$, we denote by $\text{dom}\ h = \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$ its effective domain. A function is said to be *proper* if $\text{dom}\ h \neq \varnothing$. The (Fenchel–Moreau) conjugate function $h^* : \mathbb{R}^n \to \overline{\mathbb{R}}$ of h is defined by

$$h^*(p) = \sup_{x \in \mathbb{R}^n} [\langle p, x\rangle - h(x)].$$

We consider the optimization problem

$$\text{(P)} \qquad \inf_{0 \in F(x)} h(x),$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is a given function and $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping such that $\text{dom}\ h \cap F^{-1}(0) \neq \varnothing$. The corresponding dual problem takes the form

$$\text{(D)} \qquad \sup_{p \in \mathbb{R}^n} \inf_{x \in \mathbb{R}^n} [h(x) + s_F(x, p)],$$

where $s_F : \mathbb{R}^n \to \overline{\mathbb{R}}$ is the lower support function associated to F by

$$s_F(x, p) = \inf_{y \in F(x)} \langle p, y\rangle,$$

Proposition 2.1 ([9]). *Let $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper convex function, and let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a convex set-valued mapping. If the constraint qualification*

$$\text{(CQ)} \qquad \exists \overline{x} \in \text{ri}\,(\text{dom}\ h) \cap \text{ri}\,(\text{dom}\ F), \quad 0 \in \text{ri}\,(F(\overline{x})),$$

is fulfilled, then for (P) *and* (D) *strong duality holds, i.e., there exists $\overline{p} \in \mathbb{R}^n$ such that*

$$\inf_{0 \in F(x)} = \sup_{p \in \mathbb{R}^n} \inf_{x \in \mathbb{R}^n} [h(x) + s_F(x, p)] = \inf_{x \in \mathbb{R}^n} [h(x) + s_F(x, \overline{p})],$$

where $\text{ri}\,(C)$ *is the relative interior of a given set $C \subseteq \mathbb{R}^n$.*

3 Gap Functions for Quasi-Equilibrium Problems

One of the approaches to solving the equilibrium problem is to reduce it to an optimization problem by using a gap function. A function $\gamma : \mathbb{R}^n \to \overline{\mathbb{R}}$ is called a *gap function* for Problem (QEP) if

(i) $\gamma(y) \geqslant 0$ for all $y \in K(x)$,

(ii) $\gamma(\overline{x}) = 0$ if and only if $\overline{x}$ solves Problem (QEP).

We recall the approach based on the conjugate duality [10] (see also [11]). For a fixed $x \in \mathbb{R}^n$ Problem (QEP) can be reduced to the optimization problem

$$(\mathrm{P}^{\mathrm{QEP}}; \mathrm{x}) \qquad \inf_{y \in K(x)} f(x, y)$$

or, equivalently,

$$(\mathrm{P}^{\mathrm{QEP}}; \mathrm{x}) \qquad \inf_{0 \in K(x) - y} f(x, y).$$

The corresponding dual problem takes the form

$$(\mathrm{D}^{\mathrm{QEP}}; \mathrm{x}) \qquad \sup_{p \in \mathbb{R}^n} \inf_{y \in \mathbb{R}^n} [f(x, y) + s_{K(x)-\mathrm{id}}(y, p)],$$

where id denotes the identity mapping. For $x \in \mathbb{R}^n$ we introduce the function

$$\gamma^{\mathrm{QEP}}(x) := -\sup_{p \in \mathbb{R}^n} \inf_{y \in \mathbb{R}^n} [f(x, y) + s_{K(x)-\mathrm{id}}(y, p)],$$

where $v(\mathrm{D}^{\mathrm{QEP}}; x)$ denotes the optimal objective value of Problem $(\mathrm{D}^{\mathrm{QEP}}; x)$.

Lemma 3.1 ([6]). *Let $K : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping, and let $p \in \mathbb{R}^n$ be fixed. Then for any $x \in \mathbb{R}^n$*

$$s_{K(x)-\mathrm{id}}(y, p) = s_K(x, p) - \langle p, y \rangle.$$

By Lemma 3.1, γ^{QEP} can be written as

$$\gamma^{\mathrm{QEP}}(x) = \inf_{p \in \mathbb{R}^n} \sup_{y \in \mathbb{R}^n} [-f(x, y) + s_K(x, p) - \langle p, y \rangle]$$

$$= \inf_{p \in \mathbb{R}^n} \{s_K(x, p) + \sup_{y \in \mathbb{R}^n} [\langle -p, y \rangle - f(x, y)]\} = \inf_{p \in \mathbb{R}^n} \{s_K(x, p) + f_y^*(x; -p)\},$$

where $f_y^*(x; -p) := \sup_{y \in \mathbb{R}^n} [\langle -p, y \rangle - f(x, y)]$.

Theorem 3.1. *Let $K : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping such that $K(x)$ is a nonempty closed convex set for each $x \in \mathbb{R}^n$. If for each $x \in \mathbb{R}^n$*

$$(\mathrm{CQ}; x) \qquad \exists \overline{x} \in \mathrm{ri}\,(\mathrm{dom}\ K(x)), \quad 0 \in \mathrm{ri}\,(K(x) - \overline{x}),$$

then γ^{QEP} is a gap function for Problem (QEP).

Proof. 1. Let $x \in \mathbb{R}^n$ be fixed. By the weak duality,

$$v(\mathrm{D}^{\mathrm{QEP}}; x) \leqslant v(\mathrm{P}^{\mathrm{QEP}}; x) \leqslant 0.$$

Consequently,

$$\gamma^{\mathrm{QEP}}(x) = -v(\mathrm{D}^{\mathrm{QEP}}; x) \geqslant 0.$$

2. Let $\gamma^{\mathrm{QEP}}(\overline{x}) = 0$. Then

$$0 = v(\mathrm{D}^{\mathrm{QEP}}; \overline{x}) \leqslant v(\mathrm{P}^{\mathrm{QEP}}; \overline{x}) \leqslant 0.$$

Consequently, $v(\mathrm{P}^{\mathrm{QEP}}; \overline{x}) = 0$, which means that $\overline{x}$ is a solution to Problem (QEP). Conversely, if $\overline{x}$ is a solution to Problem (QEP), then $v(\mathrm{P}^{\mathrm{QEP}}; \overline{x}) = 0$. By Proposition 2.1, we have $\gamma^{\mathrm{QEP}}(\overline{x}) = -v(\mathrm{D}^{\mathrm{QEP}}; \overline{x}) = v(\mathrm{P}^{\mathrm{QEP}}; \overline{x}) = 0$. ◻

Example 3.1. Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be a vector-valued function, and let $C \subseteq \mathbb{R}^n$ be a closed convex set. If $K(x) = h(x) + C$, then Problem (QEP) is reduced to a special case with moving set (see [12]). In this case,

$$s_K(x,p) = \inf_{y\in K(x)} \langle p, y\rangle = \inf_{y\in h(x)+C} \langle p, y\rangle = \inf_{y-h(x)\in C} \langle p, y\rangle$$
$$= \inf_{z\in C} \langle p, z + h(x)\rangle = \langle p, h(x)\rangle + \inf_{z\in C} \langle p, z\rangle,$$

where $z := y - h(x)$.

We consider Example 2.6 of [5]. For Problem (QEP) with the bifunction $f(x, y) = x(y - x)$ and moving set $K(x) = [2x - 1, 2x] = [-1, 0] + 2x$ the function γ^{QEP} has the form

$$\gamma^{\mathrm{QEP}}(x) = \inf_{p\in\mathbb{R}} \{2px + \inf_{z\in[-1,0]} pz + \sup_{y\in\mathbb{R}}[-py - xy + x^2]\}$$
$$= x^2 + \inf_{p\in\mathbb{R}} \{2px + \inf_{z\in[-1,0]} pz + \sup_{y\in\mathbb{R}}[-(p + x)y]\}.$$

Taking into account that

$$\sup_{y\in\mathbb{R}}[-(p + x)y] = \begin{cases} 0, & p + x = 0, \\ +\infty & \text{otherwise}, \end{cases}$$

we have

$$\gamma^{\mathrm{QEP}}(x) = x^2 - 2x^2 + \inf_{z\in[-1,0]}(-xz) = \begin{cases} x - x^2, & x < 0, \\ -x^2, & x \geqslant 0. \end{cases}$$

The associated Minty quasi-equilibrium problem (see [2]) of Problem (QEP) consists in finding $x \in K(x)$ such that

(DMEP) $$f(y, x) \leqslant 0 \quad \forall\, y \in K(x).$$

We denote by K^{QEP} and K^{QMEP} the sets of solutions to Problems (QEP) and (QMEP) respectively and recall some definitions and facts.

Definition 3.1. A bifunction $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is said to be

— *monotone* on a subset $C \subseteq \mathbb{R}^n$ if $f(x, y) + f(y, x) \leqslant 0$ for all $x, y \in C$,

— *pseudomonotone* on a subset $C \subseteq \mathbb{R}^n$ if $f(x, y) \geqslant 0$ implies $f(y, x) \leqslant 0$ for all $x, y \in C$.

Definition 3.2. We say that a bifunction $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ possesses the *upper sign property* at $x \in C \subseteq \mathbb{R}^n$ if for all $y \in C$

$$[f((1 - \lambda)x + \lambda y) \leqslant 0,\ \forall\lambda \in (0, 1)] \ \Rightarrow\ f(x, y) \geqslant 0.$$

Proposition 3.1 ([2])**.** (i) *If f is pseudomonotone on $K(x)$ for any $x \in \mathbb{R}^n$, then*

$$K^{\mathrm{QEP}} \subseteq K^{\mathrm{QMEP}}.$$

(ii) *If f has the upper sign property on $\mathbb{R}^n$ and for any $x \in \mathbb{R}^n$ the set $K(x)$ is convex, then*

$$K^{\mathrm{QMEP}} \subseteq K^{\mathrm{QEP}}.$$

Using Problem (QMEP) and arguing as above, we introduce a new gap function for Problem (QEP). Let us note that $x \in K(x)$ is a solution to Problem (QMEP), which is equivalent to that x is a solution to the optimization problem

$$(\mathrm{P}^{\mathrm{QMEP}}; x) \qquad \inf_{y \in K(x)} [-f(y, x)].$$

The corresponding dual problem of Problem $(\mathrm{P}^{\mathrm{QMEP}}; x)$ has the form

$$(\mathrm{D}^{\mathrm{QMEP}}; x) \qquad \sup_{p \in \mathbb{R}^n} \inf_{y \in \mathbb{R}^n} [-f(y, x) + s_{K(x)-\mathrm{id}}(y, p)].$$

Let us define the function

$$\begin{aligned} \gamma_M^{\mathrm{QEP}}(x) &:= -v(\mathrm{D}^{\mathrm{QMEP}}; x) = -\sup_{p \in \mathbb{R}^n} \inf_{y \in \mathbb{R}^n} [-f(y, x) + s_{K(x)-\mathrm{id}}(y, p)] \\ &= \inf_{p \in \mathbb{R}^n} \{ s_K(x, p) + \sup_{y \in \mathbb{R}^n} [f(y, x) - \langle p, y \rangle] \}. \end{aligned}$$

Proposition 3.2. *Let f be a monotone bifunction on $\mathbb{R}^n$. Then*

$$\gamma_M^{\mathrm{QEP}}(x) \leqslant \gamma^{\mathrm{QEP}}(x) \quad \forall\, x \in \mathbb{R}^n.$$

Proof. Let $x \in \mathbb{R}^n$. Since f is monotone, $f(x, y) + f(y, x) \leqslant 0$ implies $f(y, x) \leqslant -f(x, y)$ for all $x, y \in \mathbb{R}^n$. Adding $-\langle p, y \rangle$ to both sides and taking the supremum over all $y \in \mathbb{R}^n$, we find

$$\sup_{y \in \mathbb{R}^n} [f(y, x) - \langle p, y \rangle] \leqslant \sup_{y \in \mathbb{R}^n} [-f(x, y) - \langle p, y \rangle].$$

We obtain the desired conclusion by adding $s_K(x, p)$ to both sides and taking the infimum over $p \in \mathbb{R}^n$. □

Theorem 3.2. *Let the assumptions of Theorem* 3.1*, Proposition* 3.1 (ii)*, and Proposition* 3.2 *be fulfilled. Then γ_M^{QEP} is a gap function for Problem* (QEP)*.*

Proof. The property (i) of the definition of a gap function follows from the weak duality.

Let $\overline{x}$ be a solution to Problem (QEP). By Theorem 3.1, $\overline{x}$ is a solution to Problem (QEP) if and only if $\gamma^{\mathrm{QEP}}(\overline{x}) = 0$. By Proposition 3.2,

$$0 \leqslant \gamma_M^{\mathrm{QEP}}(\overline{x}) \leqslant \gamma^{\mathrm{QEP}}(\overline{x}) = 0.$$

Hence $\gamma_M^{\mathrm{QEP}}(\overline{x}) = 0$. Conversely, let us assume that $\gamma_M^{\mathrm{QEP}}(\overline{x}) = 0$. By the weak duality,

$$0 = v(\mathrm{D}^{\mathrm{QMEP}}; \overline{x}) \leqslant v(\mathrm{P}^{\mathrm{QMEP}}; \overline{x}) \leqslant 0.$$

Hence $v(\mathrm{P}^{\mathrm{QMEP}}; \overline{x}) = 0$ which implies $\overline{x} \in K^{\mathrm{QMEP}}$. By Proposition 3.1 (ii), $\overline{x} \in K^{\mathrm{QEP}}$. □

4 Particular Cases

4.1. Equilibrium problem. Let $K \subseteq \mathbb{R}^n$ be a nonempty convex subset. Then the equilibrium problem consists in finding $x \in K$ such that

$$\text{(EP)} \qquad f(x,y) \geqslant 0 \quad \forall\, y \in K,$$

where $f : K \times K \to \mathbb{R}$ is a given function such that $f(x,x) = 0$ for all $x \in K$.

Since $K(x) = K$, we have $s_K(x,p) = -\delta_K^*(-p)$, where δ_K denotes the indicator function of the set K. Consequently,

$$\gamma^{EP}(x) = -\sup_{p\in\mathbb{R}^n}\inf_{y\in\mathbb{R}^n}[f(x,y) - \delta_K^*(-p) - \langle p,y\rangle]$$

$$= \inf_{p\in\mathbb{R}^n}\{\sup_{y\in\mathbb{R}^n}[\langle p,y\rangle - f(x,y)] + \delta_K^*(-p)\} = \inf_{p\in\mathbb{R}^n}\{f_y^*(x,p) + \delta_K^*(-p)\},$$

which was studied in [10].

3.2. Mixed quasi-variational inequality. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector-valued function, and let $K : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping such that $K(x)$ is a nonempty closed convex set for each $x \in \mathbb{R}^n$. Let $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a given function. Then the mixed quasi-variational inequality problem consists in finding a vector $x \in K(x)$ such that

$$\text{(MQVI)} \qquad \langle T(x), y - x\rangle + \varphi(y) - \varphi(x) \geqslant 0 \quad \forall\, y \in K(x).$$

Since $f(x,y) = \langle T(x), y-x\rangle + \varphi(y) - \varphi(x)$, we have

$$\gamma^{\text{MQVI}}(x) = -\sup_{p\in\mathbb{R}^n}\inf_{y\in\mathbb{R}^n}[\langle T(x), y-x\rangle + \varphi(y) - \varphi(x) + s_K(x,p) - \langle p,y\rangle]$$

$$= \langle T(x), x\rangle + \varphi(x) + \inf_{p\in\mathbb{R}^n}\{\sup_{y\in\mathbb{R}^n}[\langle p - T(x), y\rangle - \varphi(y)] - s_K(x,p)\}$$

$$= \langle T(x), x\rangle + \varphi(x) + \inf_{p\in\mathbb{R}^n}\{\varphi^*(p - T(x))] - s_K(x,p)\}.$$

We note that the gap function $\gamma^{MQVI}(x) = \langle T(x), x\rangle + \varphi(x) + \inf_{p\in\mathbb{R}^n}\{\varphi^*(p-T(x))] - s_K(x,p)\}$ was studied in [6].

4.3. Quasi-optimization problems. Let $h : C \to \mathbb{R}$, $C \subseteq \mathbb{R}^n$, be a given real-valued function, and let $K : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping such that $K(x)$ is a nonempty closed convex set for each $x \in \mathbb{R}^n$. The quasi-optimization problem is to find $x \in K(x)$ such that

$$\text{(QOP)} \qquad \min_{y\in K(x)} h(y) = h(x).$$

The existence of a solution to Problem (QOP) was established in [3]. For $f(x,y) = h(y) - h(x)$ we have

$$\gamma^{\text{QOP}}(x) = -\sup_{p\in\mathbb{R}^n}\inf_{y\in\mathbb{R}^n}[h(y) - h(x) + s_K(x,p) - \langle p,y\rangle]$$

$$= \inf_{p\in\mathbb{R}^n}\sup_{y\in\mathbb{R}^n}[-h(y) + h(x) - s_K(x,p) + \langle p,y\rangle]$$

$$= h(x) + \inf_{p\in\mathbb{R}^n}\{\sup_{y\in\mathbb{R}^n}[\langle p,y\rangle - h(y)] - s_K(x,p)\} = h(x) + \inf_{p\in\mathbb{R}^n}\{h^*(p) - s_K(x,p)\}.$$

Remark 4.1. If $K(x) = K$, then $s_K(x, p) = -\delta_K^*(-p)$. Consequently,

$$\gamma^{\rm QOP}(x) = h(x) + \inf_{p\in\mathbb{R}^n}\{h^*(p) + \delta_K^*(-p)\}.$$

Since $\gamma^{\rm QOP}(x) \geqslant 0$, it follows that

$$\inf_{x\in K} h(x) \geqslant \sup_{p\in\mathbb{R}^n}\{-h^*(p) - \delta_K^*(-p)\},$$

which can be interpreted as the weak duality in optimization.

Declarations

Data availability This manuscript has no associated data.

Ethical Conduct Not applicable.

Conflicts of interest The authors declare that there is no conflict of interest.

References

1. E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *Math Stud.* **63**, No. 1–4, 123–145 (1994).
2. D. Aussel and J. Cotrina, "An existence result for quasi-equilibrium problems," *J. Convex Anal.* **24**, No. 1, 55–66 (2017).
3. J. Cotrina and J. Zúñiga, "A note on quasi-equilibrium problems," textitOper. Res. Lett. **46**, No. 1, 138–140 (2018).
4. G. Bigi and M. Passacantando, "Gap functions for quasi-equilibria," *J. Glob. Optim.* **66**, No. 4, 791–810 (2016).
5. P. J. S. Santos, P. S. M. Santos, and S. Scheimberg, "A Newton-type method for quasi-equilibrium problems and applications," *Optimization* **71**, No. 1, 7–32 (2022).
6. L. Altangerel, "Gap functions for quasi-variational inequalities via duality," *J. Inequal. Appl.* Paper No. 13 (2018).
7. M. Fukushima, "A class of gap functions for quasi-variational inequality problems," *J. Ind. Manag. Optim.* **3**, No. 2, 165–171 (2007).
8. K. Taji, "On gap functions for quasi-variational inequalities," *Abstr. Appl. Anal.* **2008**, Article ID 631361 (2008).
9. R. I. Boţ and E. R. Csetnek, "Conjugate duality and the control of linear discrete systems," *J. Optim. Theory Appl.* **159**, No. 3, 576–589 (2013).
10. L. Altangerel, R. I. Boţ, and G. Wanka, "On gap functions for equilibrium problems via Fenchel duality," *Pac. J. Optim.* **2**, No. 3, 667–678 (2006).
11. L. Altangerel, R. I. Boţ, and G. Wanka, "On the construction of gap functions for variational inequalities via conjugate duality," *Asia-Pac. J. Oper. Res.* **24**, No. 3, 353–371 (2007).

12. L. F. Bueno, G. Haeser, F. Lara, and F. N. Rojas, “An augmented Lagrangian method for quasi-equilibrium problems,” *Comput. Optim. Appl.* **76**, No. 3, 737–766 (2020).

Submitted on December 3, 2023

Publisher’s note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.