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GENERALIZED SOLUTIONS OF DEGENERATE INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. In this paper, we present a technique for constructing generalized solutions of the Cauchy problem for abstract integro-differential equations with degeneration in Banach spaces. A generalized solution is constructed as the convolution of the fundamental operator function (fundamental solution, influence function) of the integro-differential operator of the equation with a generalized function of a special form, which involves all input data of the original problem. Based on the analysis of the representation for the generalized solution, we obtain sufficient solvability conditions for the original Cauchy problem in the class of functions of finite smoothness. Under these sufficient conditions, the generalized solution constructed turns out to be a classical solution with the required smoothness. The abstract results obtained in the paper are applied to the study of applied initial-boundary-value problems from the theory of oscillations in viscoelastic media.

Keywords and phrases: Banach space, Fredholm operator, generalized function, fundamental solution, convolution, resolvent, Cauchy–Dirichlet problem.

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1. Statement of the problem. In this paper, we examine the Cauchy problem

$$Bu^{(N)}(t) - Au(t) - \int_0^t k(t-s)u(s)ds = f(t), \tag{1}$$

$$u(0) = u_0, \quad u'(0) = u_1, \quad \ldots, \quad u^{(N-1)}(0) = u_{N-1}, \tag{2}$$

where $N \geq 2$, B, A, and $k(t)$ are closed linear operators with dense domains acting from a Banach space E_1 into a Banach space E_2 (B is a Fredholm operator) (see [11]), and $f(t)$ is a sufficiently smooth function taking values in E_2.

It is well known that the Cauchy problem (1)–(2) with a noninvertible operator B is solvable in the class $C^N(t \geq 0, E_1)$ only under some strong conditions imposed on the initial conditions (2) and the right-hand side of Eq. (1). For the solvability of this problem in the class of distributions (generalized function), such conditions are not required and, therefore, it seems natural to construct generalized solutions in the class $K'_+(E_1)$ of generalized functions with left-bounded support. The class $K'_+(E_1)$ is natural for such constructions for a whole host of reasons. First, a solution of the Cauchy problem (1)–(2) is constructed for "positive" time ($t \geq 0$) and the supports of distributions from the class $K'_+(E_1)$ also lie on the ray $t \geq 0$. Second, for constructing generalized solutions, one must apply the convolution operation repeatedly; this operation always exists in the class $K'_+(E_1)$ but this is not the case in general. Third, in the class $K'_+(E_1)$ the convolution operation is associative in contrast with the general case (see [12]). A generalized solution of the class $K'_+(E_1)$ is constructed as the convolution of the fundamental operator function $\mathcal{E}_N(t)$ of the integro-differential operator $\big(B\delta^{(N)}(t) - A\delta(t) - k(t)\theta(t)\big)$ corresponding to Eq. (1) and a generalized function of the form

$$F(t) = f(t)\theta(t) + Bu_{N-1}\delta(t) + Bu_{N-2}\delta'(t) + \cdots + Bu_1\delta^{(N-2)}(t) + Bu_0\delta^{(N-1)}(t), \tag{3}$$

which involves all "input data" of the problem, i.e., the initial conditions (2) and the right-hand side of Eq. (1); here and below, $\delta(t)$ is the Dirac delta function and $\theta(t)$ is the Heaviside function (see [12]).

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Earlier, in [10], for the case $N = 1$, under the condition that the operator B possesses a complete Jordan set with respect to the operator function

$$A + \int_0^t k(s)ds,$$

a fundamental operator function for the integro-differential operator $\big(B\delta'(t) - A\delta(t) - k(t)\theta(t)\big)$ was constructed. Similar problems for a "complete" second-order integro-differential operator of the form $\big(B\delta''(t) - A_1\delta'(t) - A_0\delta(t) - k(t)\theta(t)\big)$ were solved in [10] and for a "complete" (i.e., of all operator coefficients are nonzero) order-N integro-differential operator of the form

$$B\delta^N(t) - A_{N-1}\delta^{(N-1)}(t) - A_{N-2}\delta^{(N-2)}(t) - \cdots - A_0\delta(t) - k(t)\theta(t)$$

in [7]; in this case, the operator B was required to have a complete generalized Jordan set with respect to the operator function

$$A_{N-1} + A_{N-2}t + \cdots + A_0\frac{t^{N-1}}{(N-1)!} + \int_0^t \frac{(t-s)^{N-1}}{(N-1)!}k(s)ds.$$

This approach assumes that all operator coefficients are nonzero, and this condition was used significantly in what follows; this, in turn, does not allow using some theorems from [7, 10] to examine Eq. (1) with the initial condition (2). However, some initial-boundary-value problems of the theory of oscillations in viscoelastic media and the theory of electrical circuits are reduced to problems of the form (1)–(2); this makes the research presented relevant not only from the theoretical point of view, but also from practical.

In a number of works devoted to the Cauchy problem (1)–(2), studies were carried out under conditions of "direct subordination" of the properties of the operator kernel $k(t)$ to the Jordan structure of the operator pencil $(B - \lambda A)$ (see [6]). In particular, according to the form of the A-Jordan set of the operator B (see [11]), the following conditions for the kernel $k(t)$ were imposed:

(1) the point $t = 0$ is a zero of some order;
(2) A-adjoint elements (functionals) belong to the kernel $N(k(0))$ (or $N(k^*(0))$; see [3, 5]);
(3) the sets of values $R(k(0))$ and $R(k^*(0))$ belong to the corresponding subspaces (see [4]);
(4) the kernel $k(t)$ can be represented as the linear combination $\alpha(t)A + \beta(t)B$ (see [2]).

Similar effects were also observed in the study of operators with kernels of loaded linear evolution equations with degeneracy by methods of semigroup theory (see [8, Theorem 2]). In this work, such restrictions are removed.

2. Basic notation and auxiliary assertions. Below, we assume that the following conditions for the operator coefficients of Eq. (1) are fulfilled:

(A) $\underline{D(B)} \subset D(A)$, $D(k(t)) = D(k) \subset D(B)$, $D(k)$ is independent of time t, $\overline{D(A)} = \overline{D(B)} = \overline{D(k)} = E_1$, $\overline{R(B)} = R(B)$, and $\dim N(B) = \dim N(B^*) = n \geq 1$.

Let $\{\varphi_i\} \in E_1$ be a basis of the kernel of the operator B and $\{\psi_i\} \in E_2^*$ be a basis of the kernel of the adjoint operator B^*, $i = 1, \ldots, n$. Let $\{\gamma_i\} \in E_1^*$ and $\{z_i\} \in E_2$ be the corresponding biorthogonal systems, i.e., $\langle \varphi_i, \gamma_j \rangle = \langle z_i, \psi_j \rangle = \delta_{ij}$, where δ_{ij} is the Kronecker delta. In this case, there exists a bounded operator of the form

$$\Gamma = (\tilde{B})^{-1} = \left(B + \sum_{i=1}^n \langle \cdot, \gamma_i \rangle z_i\right)^{-1} \in \mathcal{L}(E_2, E_1)$$

called the Trenogin–Schmidt operator (see [11]).

Introduce the following notation:

$$\mathcal{U}_N(A\Gamma t) = \sum_{i=1}^{\infty} (A\Gamma)^{i-1} \frac{t^{iN-1}}{(iN-1)!}$$

is a bounded operator function on the space E_2; $\mathcal{R}(t)$ is the resolvent of the convolution operator kernel $k(t)\Gamma\theta(t) * \mathcal{U}_N(A\Gamma t)\theta(t)$;

$$Q = \sum_{i=1}^{n} Q_i = \sum_{i=1}^{n} \langle \cdot, \psi_i \rangle z_i$$

is a projector in the space E_2;

$$\mathcal{M}(t) = \mathcal{R}(t)\theta(t) + A\Gamma\mathcal{U}_N(A\Gamma t)\theta(t) * \big(I\delta(t) + \mathcal{R}(t)\theta(t)\big)$$

is an operator function on the space E_2. In this notation, the following convolution equality for generalized operator functions holds:

$$\begin{aligned} G(t) &= \big(\mathcal{U}_N(A\Gamma t)\theta(t)\big)^{(N)} * \big(I\delta(t) + \mathcal{R}(t)\theta(t)\big) \\ &= \big(I\delta(t) + A\Gamma\mathcal{U}_N(A\Gamma t)\theta(t)\big) * \big(I\delta(t) + \mathcal{R}(t)\theta(t)\big) = I\delta(t) + \mathcal{M}(t)\theta(t). \end{aligned} \tag{4}$$

Lemma 1. *The generalized operator function*

$$\tilde{\mathcal{E}}_N(t) = \mathcal{U}_N(A\Gamma t)\theta(t) * \big(I\delta(t) + \mathcal{R}(t)\theta(t)\big)$$

is a fundamental function for the integro-differential operator

$$\tilde{\mathcal{L}}_N(\delta(t)) = I\delta^{(N)}(t) - A\Gamma\delta(t) - k(t)\Gamma\theta(t).$$

Proof. According to the definition of the fundamental operator function (see [10]), we must verify the following double equality:

$$\tilde{\mathcal{E}}_N(t) * \tilde{\mathcal{L}}_N(\delta(t)) = \tilde{\mathcal{L}}_N(\delta(t)) * \tilde{\mathcal{E}}_N(t) = I_2\delta(t),$$

where I_2 is the identity operator in the space E_2. Indeed, on one hand,

$$\begin{aligned} &\tilde{\mathcal{L}}_N(\delta(t)) * \tilde{\mathcal{E}}_N(t) \\ &= \Big(I\delta(t) + A\Gamma\mathcal{U}_N(A\Gamma t)\theta(t) - A\Gamma\mathcal{U}_N(A\Gamma t)\theta(t) - k(t)\Gamma\theta(t) * \mathcal{U}_N(A\Gamma t)\theta(t)\Big) * \Big(I\delta(t) + \mathcal{R}(t)\theta(t)\Big) \\ &= \Big(I\delta(t) - k(t)\Gamma\theta(t) * \mathcal{U}_N(A\Gamma t)\theta(t)\Big) * \Big(I\delta(t) + \mathcal{R}(t)\theta(t)\Big) = I_2\delta(t). \end{aligned}$$

On the other hand,

$$\begin{aligned} \tilde{\mathcal{E}}_N(t) * \tilde{\mathcal{L}}_N(\delta(t)) &= \Big(\mathcal{U}_N(A\Gamma t)\theta(t)\Big)^{(N)} * \Big(I\delta(t) + \mathcal{R}(t)\theta(t)\Big) - \tilde{\mathcal{E}}_N(t) * \Big(A\Gamma\delta(t) + k(t)\Gamma\theta(t)\Big) \\ &= I_2\delta(t) + A\Gamma\mathcal{U}_N(A\Gamma t)\theta(t) + \mathcal{U}_N(A\Gamma t)\theta(t) * \Big(\mathcal{R}(t)\theta(t)\Big)^{(N)} \\ &\quad - \mathcal{U}_N(A\Gamma t)\theta(t) * \left(A\Gamma\delta(t) + k(t)\Gamma\theta(t) + \mathcal{R}(t)\theta(t) * \Big(A\Gamma\delta(t) + k(t)\Gamma\theta(t)\Big)\right) \\ &= I_2\delta(t) + \mathcal{U}_N(A\Gamma t)\theta(t) * \left(\Big(\mathcal{R}(t)\theta(t)\Big)^{(N)} - k(t)\Gamma\theta(t) - \mathcal{R}(t)\theta(t) * \Big(A\Gamma\delta(t) + k(t)\Gamma\theta(t)\Big)\right) = I_2\delta(t), \end{aligned}$$

since

$$\begin{aligned} \Big(\mathcal{R}(t)\theta(t)\Big)^{(N)} &= \Big(I\delta(t) + \mathcal{R}(t)\theta(t)\Big) * k(t)\Gamma\theta(t) * \Big(\mathcal{U}_N(A\Gamma t)\theta(t)\Big)^{(N)} \\ &= \Big(I\delta(t) + \mathcal{R}(t)\theta(t)\Big) * k(t)\Gamma\theta(t) * \Big(I\delta(t) + A\Gamma\mathcal{U}_N(A\Gamma t)\theta(t)\Big) \end{aligned}$$

$$= k(t)\Gamma\theta(t) + \mathcal{R}(t)\theta(t) * k(t)\Gamma\theta(t) + \mathcal{R}(t)\theta(t) * A\Gamma\delta(t)$$
$$= k(t)\Gamma\theta(t) + \mathcal{R}(t)\theta(t) * \Big(A\Gamma\delta(t) + k(t)\Gamma\theta(t)\Big).$$

Lemma 1 is proved. □

Remark 1. It is easy to see that $\tilde{\mathcal{E}}_N^{(N)}(t) = G(t)$.

For each element $\{z_i\} \in E_2$, $i = 1, \ldots, n$, introduce the notation $l_k(\varphi_i) = \mathcal{M}^{(k)}(0)z_i$; moreover,

$$\langle l_k(\varphi_i), \psi_j\rangle = 0, \quad j = 1, \ldots, n,\ k = 0, 1, \ldots, p_i - 1,$$

however not all number $\langle l_{p_i}(\varphi_i), \psi_j\rangle$, $j = 1, \ldots, n$, are equal to zero. Since $\mathcal{M}^{(k)}(0) = 0$, $k = 0, 1, \ldots, N-2$, we have $p_i \geq 1$.

Remark 2. By the Fredholm alternative (see [11]), these conditions mean the solvability of the equations

$$B\varphi_i^{(k+1)} = \mathcal{M}^{(k)}(0)z_i, \quad k = 0, 1, \ldots, p_i - 1.$$

For these equations, particular solutions of the form $\varphi_i^{(k+1)} = \Gamma\mathcal{M}^{(k)}(0)z_i$ are called *formally* $(k+1)$-*adjoint elements* for φ_i; the set of elements $\{\varphi_i^{(k)}\}$, $i = 1, \ldots, n$, $k = 1, \ldots, p_i$, where $\varphi_i^{(k+1)} = \varphi_i$ for $\mathcal{M}^{(k)}(0)z_i = 0$, is called the *set of formally adjoint element*, the family $\{\varphi_i^{(k)}\}$, $k = 1, \ldots, p_i$, is called the *chain of formally adjoint elements*, and the parameter p_i is called the *length* of the chain of formally adjoint elements.

Below, for definiteness we assume that the basis elements of the kernel $\{\varphi_i\} \in N(B)$ are numbered in the order of increasing lengths of chains of formally adjoint elements, i.e., $1 \leq p_1 \leq p_2 \leq \ldots \leq p_n$. The family of formally adjoint element $\{\varphi_i^{(k)}\}$ is said to be *complete* if the condition $\det \big\|\langle l_{p_i}(\varphi_i), \psi_j\rangle\big\| \neq 0$ is fulfilled; in this case, the basis of the kernel of the adjoint operator $\{\psi_i\} \in N(B^*)$ can be reconstructed (see [11]) so that the equalities $\langle l_{p_i}(\varphi_i), \psi_j\rangle = \delta_{ij}$ will hold. Therefore, we can assume that such reconstruction of the basis $\{\psi_i\}$ have already been performed.

Thus, we assume that the following conditions are fulfilled:

(B)

$$\langle l_k(\varphi_i), \psi_j\rangle = \begin{cases} 0, & k = 0, 1, \ldots, p_i - 1, \\ \delta_{ij}, & k = p_i, \end{cases} \qquad i, j = 1, \ldots, n,$$

and

$$\langle l_{p_i+k}(\varphi_i), \psi_j\rangle = 0, \quad i = 1, \ldots, n-1,\ j = 1, \ldots, n,\ k = 1, \ldots, p_n - p_i.$$

Remark 3. The condition **(B)** is a generalization of the notion of a complete Jordan set of a Fredholm operator B with respect to the operator function of the form

$$A + \int_0^t k(s)ds$$

(see [9]) and exactly coincides with it in the case $N = 1$ (see [10]).

Remark 4. Due to the condition $1 \leq p_1 \leq p_2 \leq \ldots \leq p_n$, the elements of the basis $\{\varphi_i\} \in N(B)$ can be grouped according to the following principle: the first group consists of the first l_1 element of the basis whose lengths of chain of formally adjoint elements are minimal and are equal to p_1; the second group consists of the following l_2 elements of the basis whose lengths of chains of formally adjoint elements are equal to $p_{l_1+1} > p_1$, etc.; the last group consists of elements of the basis whose lengths of chains of formally adjoint element are maximal and are equal to p_n. We denote the number of such groups by m; then, obviously, $l_1 + l_2 + \ldots + l_m = n$ and the projector Q has the following natural direct decomposition:

$$Q = \sum_{i=1}^{n} Q_i = \sum_{j=1}^{m} \tilde{Q}_j, \quad \text{where} \quad \tilde{Q}_j = \sum_{i=l_1+l_2+\ldots+l_{j-1}+1}^{i=l_1+l_2+\ldots+l_j} Q_i;$$

due to the condition **(B)**, these projectors satisfy the equalities

$$\tilde{Q}_j \mathcal{M}^{(k)}(0) \tilde{Q}_i = \begin{cases} \delta_{ij} \tilde{Q}_i, & k = p_i, \\ 0, & k \neq p_i. \end{cases} \tag{5}$$

Remark 5. In the sequel, we use the notation p_j for the length of the chain of formally adjoint elements for the basis vectors (elements) of the kernel $\{\varphi_i\} \in N(B)$ lying in the same group with the number j.

Introduce the operator function $\mathcal{M}_1(t)$, which is the resolvent of the operator kernel of the form

$$-\sum_{i=1}^{n} Q_i \mathcal{M}^{(p_i+1)}(t)\theta(t) = -\sum_{j=1}^{m} \tilde{Q}_j \mathcal{M}^{(p_j+1)}(t)\theta(t).$$

By Remark 1 and the rule of differentiating convolutions of generalized functions (see [12]), we obtain the following property of pseudo-commutation.

Lemma 2. *If the conditions **(A)** and **(B)** are fulfilled, then*

$$\tilde{\mathcal{E}}_N(t) * \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t) \Big) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) * G(t)$$
$$= G(t) * \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t) \Big) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) * \tilde{\mathcal{E}}_N(t).$$

Remark 6. Obviously, due to the construction of the resolvent $\mathcal{M}_1(t)$, the following equality hold:

$$Q\delta(t) * \mathcal{M}_1(t) = \mathcal{M}_1(t).$$

We prove some auxiliary equalities for operator convolutions, which will be needed below.

Lemma 3. *If the conditions **(A)** and **(B)** are fulfilled, then*

$$\Big(I\delta(t) + \mathcal{M}_1(t)\theta(t) \Big) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) * \mathcal{M}(t)\theta(t) * Q\delta(t) = Q\delta(t); \tag{6}$$

$$Q\delta(t) + \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t) \Big) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) * \Big\{ Q\delta(t) + G(t) * (I - Q)\delta(t) \Big\}$$
$$= \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t) \Big) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) * G(t); \tag{7}$$

$$\mathcal{F}_k(t) = Q\delta(t) * \Big[\mathcal{M}^{(p_k+1)}(t)\theta(t) + \Big(\mathcal{M}(t)\theta(t) \Big)^{(p_k+1)} * Q\delta(t) * \mathcal{M}_1(t)\theta(t) \Big] * \tilde{Q}_k \delta(t) \equiv 0, \tag{8}$$

where $k = 1, \ldots, m$.

Proof. By the condition **(B)** and Eq. (5),

$$\sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) * \mathcal{M}(t)\theta(t) * Q\delta(t) = \sum_{j=1}^{m} \Big(\tilde{Q}_j \delta(t) + \tilde{Q}_j \mathcal{M}^{(p_j+1)}(t)\theta(t) * Q\delta(t) \Big)$$
$$= Q\delta(t) + \sum_{j=1}^{m} \tilde{Q}_j \mathcal{M}^{(p_j+1)}(t)\theta(t) * Q\delta(t).$$

This implies the validity of Eq. (6). Indeed,

$$\Big(I\delta(t)+\mathcal{M}_1(t)\theta(t)\Big)*\sum_{j=1}^{m}\tilde{Q}_j\delta^{(p_j+1)}(t)*\mathcal{M}(t)\theta(t)*Q\delta(t)$$
$$=\Big(I\delta(t)+\mathcal{M}_1(t)\theta(t)\Big)*\Big(Q\delta(t)+\sum_{j=1}^{m}\tilde{Q}_j\mathcal{M}^{(p_j+1)}(t)\theta(t)*Q\delta(t)\Big)$$
$$=Q\delta(t)+\mathcal{M}_1(t)\theta(t)*Q\delta(t)-\mathcal{M}_1(t)\theta(t)*Q\delta(t)=Q\delta(t).$$

The relations (4) and (6) imply the following chain of identities:

$$Q\delta(t)+\Big(I\delta(t)+\mathcal{M}_1(t)\theta(t)\Big)*\sum_{j=1}^{m}\tilde{Q}_j\delta^{(p_j+1)}(t)*\Big\{I\delta(t)-G(t)\Big\}*Q\delta(t)$$
$$=Q\delta(t)-\Big(I\delta(t)+\mathcal{M}_1(t)\theta(t)\Big)*\sum_{j=1}^{m}\tilde{Q}_j\delta^{(p_j+1)}(t)*\mathcal{M}(t)\theta(t)*Q\delta(t)=Q\delta(t)-Q\delta(t)\equiv 0,$$

which proves Eq. (7).

Now we prove Eq. (8). By the formula of differentiating generalized functions, we have

$$\mathcal{F}_k(t)=Q\delta(t)*\Big[\mathcal{M}^{(p_k+1)}(t)\theta(t)+\Big(\mathcal{M}(t)\theta(t)\Big)^{(p_k+1)}*Q\delta(t)*\mathcal{M}_1(t)\theta(t)\Big]*\tilde{Q}_k\delta(t)$$
$$=\Big[Q\mathcal{M}^{(p_k+1)}(t)\theta(t)-\sum_{j=1}^{m}Q\delta(t)*\Big(\mathcal{M}^{(p_k+1)}(t)\theta(t)+\mathcal{M}^{(p_k)}(0)\delta(t)$$
$$+\mathcal{M}^{(p_k-1)}(0)\delta'(t)+\ldots+\mathcal{M}(0)\delta^{(p_k)}(t)\Big)*\tilde{Q}_j\mathcal{M}^{(p_j+1)}(t)\theta(t)*\Big(I\delta(t)+\mathcal{M}_1(t)\theta(t)\Big)\Big]*\tilde{Q}_k\delta(t).$$

Thus, due to the relations (5), we obtain

$$\mathcal{F}_k(t)=\Big[Q\mathcal{M}^{(p_k+1)}(t)\theta(t)*\Big(I\delta(t)+\mathcal{M}_1(t)\theta(t)\Big)$$
$$-\sum_{i=1}^{k}\Big(\tilde{Q}_i\mathcal{M}^{(p_i+1)}(t)\theta(t)\Big)^{(p_k-p_i)}*\Big(I\delta(t)+\mathcal{M}_1(t)\theta(t)\Big)\Big]*\tilde{Q}_k\delta(t).$$

Calculating all generalized derivatives $\big(\tilde{Q}_i\mathcal{M}^{(p_i+1)}(t)\theta(t)\big)^{(p_k-p_i)}$ and simplifying, we arrive at the formula

$$\mathcal{F}_k(t)=\Bigg[\sum_{j=k+1}^{m}\tilde{Q}_j\mathcal{M}^{(p_k+1)}(t)\theta(t)*\Big(I\delta(t)+\mathcal{M}_1(t)\theta(t)\Big)$$
$$-\sum_{i=1}^{k-1}\tilde{Q}_i\delta(t)*\Big(\mathcal{M}^{(p_k)}(0)\delta(t)+\mathcal{M}^{(p_k-1)}(0)\delta'(t)+\ldots+\mathcal{M}^{(p_i+1)}(0)\delta^{(p_k-p_i-1)}(t)\Big)$$
$$*\left(I\delta(t)-\sum_{j=1}^{m}\tilde{Q}_j\mathcal{M}^{(p_j+1)}(t)\theta(t)*\Big(I\delta(t)+\mathcal{M}_1(t)\theta(t)\Big)\right)\Bigg]*\tilde{Q}_k\delta(t).$$

Again, due to (5) we have

$$\mathcal{F}_k(t) = \sum_{j=k+1}^{m} \tilde{Q}_j \mathcal{M}^{(p_k+1)}(t)\theta(t) * \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big) * \tilde{Q}_k\delta(t)$$

$$= \sum_{j=k+1}^{m} \tilde{Q}_j \frac{t^{p_j-p_k}}{(p_j-p_k)!}\theta(t) * \frac{d^{p_j-p_k}}{dt^{p_j-p_k}} \Big(\tilde{Q}_j\mathcal{M}^{(p_k+1)}(t)\theta(t)\Big) * \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big) * \tilde{Q}_k\delta(t)$$

$$= \sum_{j=k+1}^{m} \tilde{Q}_j \frac{t^{p_j-p_k}}{(p_j-p_k)!}\theta(t) * \tilde{Q}_j\delta(t)$$

$$* \Big(\mathcal{M}^{(p_j+1)}(t)\theta(t) + \mathcal{M}^{(p_j)}(0)\delta(t) + M^{(p_j-1)}(0)\delta'(t) + \ldots + \mathcal{M}^{(p_k+1)}(0)\delta^{(p_j-p_k-1)}(t)\Big)$$

$$* \left(I\delta(t) - \sum_{i=1}^{m} \tilde{Q}_i\mathcal{M}^{(p_i+1)}(t)\theta(t) * \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big)\right) * \tilde{Q}_k\delta(t).$$

From this representation, due to the relations (5), we finally obtain

$$\mathcal{F}_k(t) = \sum_{j=k+1}^{m} \tilde{Q}_j \frac{t^{p_j-p_k}}{(p_j-p_k)!}\theta(t) * \Bigg[\tilde{Q}_j\mathcal{M}^{(p_j+1)}(t)\theta(t) * \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big)$$

$$- \tilde{Q}_j\mathcal{M}^{(p_j+1)}(t)\theta(t) * \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big)\Bigg] * \tilde{Q}_k\delta(t) \equiv 0.$$

Lemma 3 is proved. □

3. Theorem on the form of the fundamental operator function.

Theorem 1. *If the conditions **(A)** and **B)** are fulfilled, then the integro-differential operator* $\mathcal{L}_N(\delta(t)) = B\delta^{(N)}(t) - A\delta(t) - k(t)\theta(t)$ *on the class of distributions with left-bounded supports* $K'_+(E_1)$ *has the fundamental operator function of the form*

$$\mathcal{E}_N(t) = \Gamma\delta(t) * \tilde{\mathcal{E}}_N(t) * \left[I\delta(t) - \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big) * \sum_{j=1}^{m} \tilde{Q}_j\delta^{(p_j+1)}(t) * G(t)\right] \tag{9}$$

or

$$\mathcal{E}_N(t) = \Gamma\delta(t) * \left[I\delta(t) - G(t) * \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big) * \sum_{j=1}^{m} \tilde{Q}_j\delta^{(p_j+1)}(t)\right] * \tilde{\mathcal{E}}_N(t). \tag{10}$$

Remark 7. The equivalence of these two representations for the fundamental operator function $\mathcal{E}_N(t)$ follows from Lemma 2.

As a consequence of the relation (7) of Lemma 3, we obtain another representation for $\mathcal{E}_N(t)$.

Theorem 2. *If the conditions **(A)** and **(B)** are fulfilled, then the integro-differential operator* $\mathcal{L}_N(\delta(t)) = B\delta^{(N)}(t) - A\delta(t) - k(t)\theta(t)$ *on the class of distributions with left-bounded support* $K'_+(E_1)$ *has the fundamental operator function of the form*

$$\mathcal{E}_N(t) = \Gamma\delta(t) * \tilde{\mathcal{E}}_N(t) * \Bigg[\Big(I - Q\Big)\delta(t)$$

$$- \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big) * \sum_{j=1}^{m} \tilde{Q}_j\delta^{(p_i+1)}(t) * \Big\{Q\delta(t) + G(t) * (I - Q)\delta(t)\Big\}\Bigg].$$

Proof of Theorem 1. In accordance with the definition of the fundamental operator function (see [10]), we prove that the following two convolution equalities hold:

$$\mathcal{E}_N(t) * \Big(B\delta^{(N)}(t) - A\delta(t) - k(t)\theta(t)\Big) * u(t) = u(t) \quad \forall u(t) \in K'_+(E_1), \tag{11}$$

$$\Big(B\delta^{(N)}(t) - A\delta(t) - k(t)\theta(t)\Big) * \mathcal{E}_N(t) * v(t) = v(t) \quad \forall v(t) \in K'_+(E_2). \tag{12}$$

To prove the identity (11), we use the representation (10) for $\mathcal{E}_N(t)$:

$$\mathcal{E}_N(t) * \mathcal{L}_N(\delta(t)) = \mathcal{E}_N(t) * \mathcal{L}_N(\delta(t)) * \Gamma\delta(t) * \tilde{B}\delta(t) = \mathcal{E}_N(t) * \Big(\tilde{\mathcal{L}}_N(\delta(t)) - Q\delta^{(N)}(t)\Big) * \tilde{B}\delta(t)$$

$$= \Gamma\delta(t) * \left[I\delta(t) - G(t) * \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t)\right] * \Big[I\delta(t) - G(t) * Q\delta(t)\Big] * \tilde{B}\delta(t).$$

Having prove the equality

$$\mathcal{F}(t) = \left[I\delta(t) - G(t) * \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t)\right] * \Big[I\delta(t) - G(t) * Q\delta(t)\Big] = I\delta(t),$$

we obtain the required assertion. Indeed,

$$\mathcal{F}(t) = I\delta(t) - G(t) * Q\delta(t)$$
$$- G(t) * \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) * \Big[I\delta(t) - G(t)\Big] * Q\delta(t);$$

here we used the properties of the projectors:

$$\sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) * \Big[I\delta(t) - G(t) * Q\delta(t)\Big]$$
$$= \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) * Q\delta(t) * \Big[I\delta(t) - G(t) * Q\delta(t)\Big]$$
$$= \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) * \Big[Q^2\delta(t) - Q\delta(t) * G(t) * Q\delta(t)\Big]$$
$$= \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) * Q\delta(t) * \Big[I\delta(t) - G(t)\Big] * Q\delta(t)$$
$$= \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) * \Big[I\delta(t) - G(t)\Big] * Q\delta(t).$$

By the relation (4) we have

$$\mathcal{F}(t) = I\delta(t) - G(t) * Q\delta(t) + G(t) * \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) * \mathcal{M}(t)\theta(t) * Q\delta(t).$$

It remains to verify the relation

$$\mathcal{G}(t) = \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) * \mathcal{M}(t)\theta(t) * Q\delta(t) = Q\delta(t).$$

By the rules of differentiating generalized functions and the relations (5), we have

$$\mathcal{G}(t) = \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big) * \sum_{j=1}^{m} \Big(\tilde{Q}_j \mathcal{M}(t) Q\theta(t)\Big)^{(p_j+1)}$$

$$= \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big) * \sum_{j=1}^{m} \Big(\tilde{Q}_j \delta(t) + \tilde{Q}_j \mathcal{M}^{(p_j+1)}(t)\theta(t) * Q\delta(t)\Big)$$

$$= \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big) * \left(Q\delta(t) + \sum_{j=1}^{m} \tilde{Q}_j \mathcal{M}^{(p_j+1)}(t)\theta(t) * Q\delta(t)\right)$$

$$= \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big) * \left(I\delta(t) + \sum_{j=1}^{m} \tilde{Q}_j \mathcal{M}^{(p_j+1)}(t)\theta(t)\right) * Q\delta(t) = Q\delta(t).$$

Thus, Eq. (11) is proved.

To prove the identity (12), we use the representation (9) for $\mathcal{E}_N(t)$:

$$\mathcal{L}_N(\delta(t)) * \mathcal{E}_N(t) = \Big(\tilde{\mathcal{L}}_N(\delta(t)) - Q\delta^{(N)}(t)\Big) * \tilde{\mathcal{E}}_N(t)$$

$$* \left[I\delta(t) - \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) * G(t)\right]$$

$$= \Big(I\delta(t) - Q\delta(t) * G(t)\Big) * \left[I\delta(t) - \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) * G(t)\right]$$

$$= I\delta(t) - \left[Q\delta(t) + \Big(I\delta(t) - Q\delta(t) * G(t)\Big) * \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t)\right] * G(t).$$

To complete the proof of Eq. (12) we must show that

$$\mathcal{N}(t) = Q\delta(t) + \Big(I\delta(t) - Q\delta(t) * G(t)\Big) * \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) \equiv 0.$$

By the properties of projectors, Remark 6, and the relation (4), we have

$$\Big(I\delta(t) - Q\delta(t) * G(t)\Big) * \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t)$$

$$= \Big(I\delta(t) - Q\delta(t) * G(t)\Big) * \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big) * Q\delta(t) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t)$$

$$= \Big(I\delta(t) - Q\delta(t) * G(t)\Big) * \Big(Q^2\delta(t) + Q\delta(t) * \mathcal{M}_1(t)\theta(t) * Q\delta(t)\Big) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t)$$

$$= \Big(I\delta(t) - Q\delta(t) * G(t)\Big) * Q\delta(t) * \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big) * Q\delta(t) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t)$$

$$= \Big(Q^2\delta(t) - Q\delta(t) * G(t) * Q\delta(t)\Big) * \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t)$$

$$= Q\delta(t) * \Big(I\delta(t) - G(t)\Big) * Q\delta(t) * \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t)$$

$$= -Q\delta(t) * \mathcal{M}(t)\theta(t) * Q\delta(t) * \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t)$$

$$= -\sum_{j=1}^{m} \Big(Q\mathcal{M}(t)\tilde{Q}_j\theta(t)\Big)^{(p_j+1)} - \sum_{j=1}^{m} Q\mathcal{M}(t)Q\theta(t) * \mathcal{M}_1(t)\theta(t) * \tilde{Q}_j\delta^{(p_j+1)}(t).$$

By the relations (5) and the rules of differentiating generalized functions

$$\sum_{j=1}^{m} \Big(Q\mathcal{M}(t)\tilde{Q}_j\theta(t)\Big)^{(p_j+1)} = \sum_{j=1}^{m} \Big(\tilde{Q}_j\delta(t) + Q\mathcal{M}^{(p_j+1)}(t)\tilde{Q}_j\theta(t)\Big)$$

$$= Q\delta(t) + \sum_{j=1}^{m} Q\mathcal{M}^{(p_j+1)}(t)\theta(t) * \tilde{Q}_j\delta(t).$$

Thus, in the notation of Lemma 3 (see the relation (8)),

$$\mathcal{N}(t) = Q\delta(t) - Q\delta(t)$$
$$- Q\delta(t) * \sum_{j=1}^{m} \left(\mathcal{M}^{(p_j+1)}(t)\theta(t) + \Big(\mathcal{M}(t)\theta(t)\Big)^{(p_j+1)} * Q\delta(t) * \mathcal{M}_1(t)\theta(t)\right) * \tilde{Q}_j\delta = -\sum_{j=1}^{m} \mathcal{F}_j(t) \equiv 0.$$

Theorem 1 is proved. □

Remark 8. The method used for the proof of Theorem 1 can be adapted to the proof of the theorem on the form of the fundamental operator function for complete differential and integro-differential operators from [7, 10]; note that this method is more convenient than methods used in [7, 10].

As was noted above, using the fundamental operator function one can perform a compete analysis of the Cauchy problem (1)–(2). In generalized functions, the problem (1)–(2) can be rewritten as follows:

$$\mathcal{L}_N(\delta(t)) * \tilde{u}(t) = F(t), \quad \text{where } F(t) \text{ is defined in (3).}$$

Then the generalized function

$$\tilde{u}(t) = \mathcal{E}_N(t) * F(t) = \mathcal{E}_N(t) * \Big(f(t)\theta(t) + Bu_{N-1}\delta(t) + Bu_{N-2}\delta'(t) + \dots$$
$$+ Bu_1\delta^{(N-2)}(t) + Bu_0\delta^{(N-1)}(t)\Big) \in K'_+(E_1)$$

is a solution of this equation (see (12)); due to (11), it is unique in the class $K'_+(E_1)$.

We obtain the most compact representation for $\mathcal{E}_N(t)$ if all $p_i = N-1$ in the condition **(B)**; this means that the Fredholm operator B has no A-adjoint elements (see [11]). In this case, the fundamental operator function can be rewritten in the form

$$\mathcal{E}_N(t) = \Gamma\delta(t) * \Big[\tilde{\mathcal{E}}_N(t) - G(t) * \Big(I\delta(t) + \mathcal{M}_1(t)\theta(t)\Big) * Q\delta(t) * G(t)\Big].$$

Taking into account Remark 1 (i.e., the representation $G(t) = I\delta(t) + \mathcal{M}(t)\theta(t)$), analyzing the structure of the generalized solution $\tilde{u}(t) = \mathcal{E}_N(t) * F(t) \in K'_+(E_1)$, we conclude that this is a regular generalized function. In this case, the function $\tilde{u}(t) = \mathcal{E}_N(t) * F(t)$ possesses a required smoothness and satisfies Eq. (1). If we assume that it satisfies the initial conditions (2), we arrive at the following theorem.

Theorem 3. *Let the conditions* ***(A)*** *and* ***(B)*** *be fulfilled and* $p_i = N-1$ *for all* i. *The Cauchy problem* (1)–(2) *has a unique solution of the class* $C^N(t \geq 0, E_1)$ *if and only if the following conditions hold*:

$$\Big\langle Au_i + f^{(i)}(0) + k(0)u_{i-1} + k'(0)u_{i-2} + \dots + k^{(i-2)}(0)u_1 + k^{(i-1)}(0)u_0,\ \psi_j\Big\rangle = 0,$$
$$i = 0, 1, \dots, N-1, \quad j = 1, \dots, n.$$

4. Applications. Partial integro-differential equations are effective tools of mathematical modeling of the evolution of physical processes whose current state is influenced by the entire prehistory of observations. In this class of problems, a special role is played by initial-boundary-value problems with noninvertible operators with the highest time derivative.

Consider several analogs of initial-boundary-value problems from the theory of viscoelasticity (see [1]):

$$(\lambda - \Delta)\, u_{tt} - (\mu - \Delta)\, u - \int_0^t g(t-\tau)\Delta^2 u(\tau, \bar{x})d\tau = f(t, \bar{x}), \tag{13}$$

$$(\lambda - \Delta)\, u_{tt} - \left(\mu - \Delta^2\right) u - \int_0^t g(t-\tau)\left(\lambda^2 - \Delta^2\right) u(\tau, \bar{x})d\tau = f(t, \bar{x}), \tag{14}$$

$$(\lambda - \Delta)\, u_{tt} + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau, \bar{x})d\tau = f(t, \bar{x}), \tag{15}$$

$$(\lambda - \Delta)\, u_{tt} - (\mu - \Delta)\, u - \int_0^t g(t-\tau)\,(\gamma - \Delta)\, u(\tau, \bar{x})d\tau = f(t, \bar{x}), \tag{16}$$

$$u\Big|_{t=0} = u_0(\bar{x}), \quad u_t\Big|_{t=0} = u_1(\bar{x}), \quad \bar{x} \in \Omega; \qquad u\Big|_{\bar{x}\in\partial\Omega} = 0, \quad t \geq 0, \tag{17}$$

where $g(t)$, $f(t, \bar{x})$ are given functions, $u = u(t, \bar{x})$ is the unknown function, $\bar{x} \in \Omega \subset \mathbb{R}^m$ is a bounded domain with infinitely smooth boundary $\partial\Omega$, Δ is the Laplace operator, and the function $u = u(t, \bar{x})$ is defined on the cylinder $\mathbb{R}_+ \times \Omega$, $\lambda \in \sigma(\Delta)$.

For the Cauchy–Dirichlet problems (13)–(17), (14)–(17), and (15)–(17), where $\mu \neq \lambda$ ($\mu \neq \lambda^2$), we consider the following Banach spaces and the operator B:

$$E_1 \equiv \left\{ v(\bar{x}) \in W_2^4(\Omega) : \ v\big|_{\partial\Omega} = 0 \right\}, \quad E_2 \equiv W_2(\Omega), \quad B = \lambda - \Delta, \quad \lambda \in \sigma(\Delta),$$

where $W_2^4(\Omega)$ and $W_2(\Omega)$ are the Sobolev spaces; the operator A has the form $(\mu - \Delta)$ in the problem (13)–(17), $(\mu - \Delta^2)$ in the problem (14)–(17), and $(-\Delta^2)$ in the problem (15)–(17). Let $\varphi_i(\bar{x})$, $i = 1, \ldots, n$, be an orthonormal basis of the space of solutions of the homogeneous problem

$$\lambda\varphi_i = \Delta\varphi_i, \quad \varphi_i\big|_{\bar{x}\in\partial\Omega} = 0.$$

In the initial-boundary-value problems (13)–(17), (14)–(17), and (15)–(17), we have $N = 2$ and, moreover,

(i) $\langle A\varphi_i, \varphi_j \rangle = (\mu - \lambda)\delta_{ij}$ in the problem (13)–(17) ,

(ii) $\langle A\varphi_i, \varphi_j \rangle = (\mu - \lambda^2)\delta_{ij}$ in the problem (14)–(17) ,

(iii) $\langle A\varphi_i, \varphi_j \rangle = -\lambda^2\delta_{ij}$ in the problem (15)–(17);

in terms of Theorem 3 this means that all $p_i = 1$ and, therefore, the following three assertions hold.

Theorem 4. *The Cauchy–Dirichlet problem* (13)–(17) *is uniquely solvable in the functional class* $C^2(t \geq 0, E_1)$ *if and only of the initial conditions* (17) *and the function* $f(t, \bar{x})$ *satisfy the relations*

$$\Big\langle (\mu - \lambda)u_1(\bar{x}) + f_t'(0, \bar{x}) + g(0)\lambda^2 u_0(\bar{x}), \ \varphi_i(\bar{x}) \Big\rangle = 0,$$

$$\Big\langle (\mu - \lambda)u_0(\bar{x}) + f(0, \bar{x}), \ \varphi_i(\bar{x}) \Big\rangle = 0, \quad i = 1, \ldots, n.$$

Theorem 5. *The Cauchy–Dirichlet problem* (14)–(17) *is uniquely solvable in the functional class* $C^2(t \geq 0, E_1)$ *if and only if the initial conditions* (17) *and the function* $f(t, \bar{x})$ *satisfy the relations*

$$\Big\langle (\mu - \lambda^2)u_1(\bar{x}) + f_t'(0, \bar{x}), \ \varphi_i(\bar{x}) \Big\rangle = 0,$$

$$\Big\langle (\mu-\lambda^2)u_0(\bar{x})+f(0,\bar{x}),\ \varphi_i(\bar{x})\Big\rangle=0,\quad i=1,\dots,n.$$

Theorem 6. *The Cauchy–Dirichlet problem* (15)–(17) *is uniquely solvable in the functional class* $C^2(t\ge 0,E_1)$ *if and only if the initial conditions* (17) *and the function* $f(t,\bar{x})$ *satisfy the relations*

$$\Big\langle -\lambda^2 u_1(\bar{x})+f'_t(0,\bar{x})+g(0)\lambda^2 u_0(\bar{x}),\ \varphi_i(\bar{x})\Big\rangle=0,$$
$$\Big\langle -\lambda^2 u_0(\bar{x})+f(0,\bar{x}),\ \varphi_i(\bar{x})\Big\rangle=0,\quad i=1,\dots,n.$$

For the initial-boundary-value problem (16)–(17), where $\mu\ne\lambda$, we set

$$E_1\equiv\Big\{v(\bar{x})\in W_2^2(\Omega):\ v\big|_{\partial\Omega}=0\Big\},\quad E_2\equiv W_2(\Omega),\quad B=\lambda-\Delta,\quad A=\mu-\Delta,\quad \lambda\in\sigma(\Delta);$$

then the kernel of the integral term is a linear combination of the operators A and B, i.e., it can be represented in the form

$$g(t)(\gamma-\Delta)=\alpha(t)A+\beta(t)B,\quad \text{where}\quad \alpha(t)=\frac{\gamma-\lambda}{\mu-\lambda}g(t),\quad \beta(t)=\frac{\mu-\gamma}{\mu-\lambda}g(t).$$

Then due to Theorem 3, the following assertion holds.

Theorem 7. *The Cauchy–Dirichlet problem* (16)–(17) *is uniquely solvable in the functional class* $C^2(t\ge 0,E_1)$ *if and only if the initial and boundary conditions* (16) *and the function* $f(t,\bar{x})$ *satisfy the relations*

$$\Big\langle (\mu-\lambda)u_1(\bar{x})+f'_t(0,\bar{x})+(\gamma-\lambda)g(0)u_0(\bar{x}),\ \varphi_i(\bar{x})\Big\rangle=0,$$
$$\Big\langle (\mu-\lambda)u_0(\bar{x})+f(0,\bar{x}),\ \varphi_i(\bar{x})\Big\rangle=0,\quad i=1,\dots,n.$$

Remark 9. The solvability conditions for the Cauchy–Dirichlet problems (16)–(17) can be rewritten in the following equivalent form (see [2]):

$$\Big\langle (\mu-\lambda)^2u_1(\bar{x})+(\mu-\lambda)f'_t(0,\bar{x})-(\gamma-\lambda)g(0)f(0,\bar{x}),\ \varphi_i(\bar{x})\Big\rangle=0,$$
$$\Big\langle (\mu-\lambda)u_0(\bar{x})+f(0,\bar{x}),\ \varphi_i(\bar{x})\Big\rangle=0,\quad i=1,\dots,n.$$

According to a similar scheme, we can analyze the Cauchy–Dirichlet problem of the form

$$(\lambda-\Delta)u_t^{(N)}+\Delta^2u-\int\limits_0^t g(t-\tau)\Delta^2u(\tau,\bar{x})d\tau=f(t,\bar{x}),\tag{18}$$

$$u\big|_{t=0}=u_0(\bar{x}),\ u_t\big|_{t=0}=u_1(\bar{x}),\ \dots,\ u_t^{(N-1)}\big|_{t=0}=u_{N-1}(\bar{x}),\ \bar{x}\in\Omega;\quad u\big|_{\bar{x}\in\partial\Omega}=0,\ t\ge 0,\tag{19}$$

and obtain (in the notation of Theorem 6) the following result.

Theorem 8. *The Cauchy–Dirichlet problem* (18)–(19) *is uniquely solvable in the class* $C^N(t\ge 0,E_1)$ *if and only if the initial and boundary conditions* (19) *and the function* $f(t,\bar{x})$ *satisfy the relations*

$$\Big\langle -u_i(\bar{x})+\frac{1}{\lambda^2}f_t^{(i)}(0,\bar{x})+g(0)u_{i-1}(\bar{x})+g'(0)u_{i-2}(\bar{x})+\dots+g^{(i-1)}(0)u_0(\bar{x}),\ \varphi_j(\bar{x})\Big\rangle=0,$$
$$i=0,1,\dots,N-1,\quad j=1,\dots,n.$$

5. Conclusion. The method proposed in this paper allows one to construct unique generalized solutions using the appropriate formulas. Based on the analysis of the representation for the generalized solution, it is possible to obtain unique solutions to Cauchy problems of the required smoothness and conditions for their existence. The abstract results presented are applicable to the study of other mathematical models of the theory of vibrations in viscoelastic media or in the theory of electrical circuits.

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COMPLIANCE WITH ETHICAL STANDARDS

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