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DIFFERENTIAL EQUATIONS IN BANACH SPACES WITH AN NONINVERTIBLE OPERATOR IN THE PRINCIPAL PART AND NONCLASSICAL INITIAL CONDITIONS

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Abstract. In this paper, we examine differential equations with nonclassical initial conditions and noninvertible operators in their principal parts. We find necessary and sufficient conditions for the existence of unbounded solutions with a pole of order p at points where the operator in the principal part of the differential equation is noninvertible. Based on the alternative Lyapunov–Schmidt method and Laurent expansions, we propose a two-stage method for constructing expansion coefficients of the solution in a neighborhood of a pole. We develop the techniques of skeleton chains of linear operators in Banach spaces and discuss its applications to the statement of initial conditions for differential equations. The results obtained develop the theory of degenerate differential equations. Illustrative examples are given.

Keywords and phrases: Fredholm operator, Laurent series, collapsing solution, skeleton chain, initial-value problem.

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1. Introduction. The differential equation

$$B(t)\frac{\partial u}{\partial t} = A(t,u) \tag{1}$$

with an noninvertible operator $B(t)$ attracted the attention of many mathematicians. Equation (1), which is a Sobolev-type equation (see [18]), has recently found many new applications in modern mathematical physics, mathematical modeling in technology, energy industry, and economics. Important general results were obtained in the case of an noninvertible stationary operator $B : D \subset X \to Y$, where X and Y are Banach space. In [1–4, 6, 7, 9–11, 13–15, 17, 21], a wide range of methods from the theory of differential operator equations, functional analysis, and asymptotic methods were used for similar problems. Theorems of existence, convergence, and stability of solutions were proved, approximate methods were developed for solving initial boundary-value problems, including solutions in neighborhoods of branch points. In the linear case, for some classes, an analytical technique for constructing exact solutions was proposed. Nevertheless, at present, the problem of constructive description of the structure of a solution in a neighborhood of a point where the nonstationary operator $B(t)$ has no inverse remains unsolved. The results of numerical experiments in neighborhoods of such points often turn out to be unstable.

We draw the reader's attention to the fact that the equation $\boldsymbol{B}\dfrac{\partial u}{\partial t} = A(t,u)$ with a noninvertible operator $\boldsymbol{B}$ and the Cauchy condition $u(t_0) = u_0$ requires a priori information $A(t_0, u_0)$ in the range of the operator $\boldsymbol{B}$. Analysis of analytical structures of collapsing solutions is especially difficult. Therefore, the development of methods for asymptotic analysis in neighborhoods of singular points of the operator $B(t)$ and the statement of well-posed initial-valued problems in these neighborhoods is very important from the theoretical and applied points of view. A promising approach here is a combination of modification of subtle methods of the analytical theory of differential equations (e.g., the Fuchs–Frobenius and Puiseux–Bruno expansions) with methods of the spectral theory of linear operators, functional and group methods.

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In this paper, we give necessary and sufficient analytical conditions for the existence of collapsing and bounded solutions in neighborhoods of points t_0 at which the operator $B(t_0)$ is noninvertible analyze the asymptotic behavior of solutions with nonclassical initial conditions.

The structure of the paper is as follows. In Sec. 2, we state the problem of constructing an unbounded solution as a special limit initial-value problem in a certain class of functions. In Sec. 3, necessary conditions for fulfilling the initial condition (3) of the problem are given. In Sec. 4, a theorem on the existence of a unique solution of the initial-value problem is proved and examples are given. In Sec. 5, we discuss skeletal chains of linear operators and some nonclassical initial-value problems for linear degenerate differential equations.

2. Statement of the initial-value problem.

Consider the linear equation

$$B(t)\frac{\partial u}{\partial t} = A(t)u + f(t) \tag{2}$$

with the initial condition

$$\lim_{t\to t_0} B(t)u(t) = y_0, \tag{3}$$

which is a generalization of the Cauchy initial condition $u(t_0) = y_0$. We assume that the Fredholm operator $u(t_0)$ is noninvertible and the element y_0 belongs to the range of the operator $B(t_0)$.

The condition (3) is called the limit initial condition. We need to obtain conditions of the existence of an unbounded solution of the problem (2)–(3). Without loss of generality, we assume that $t_0 = 0$.

Let the following conditions hold:

I. $B(t) = B_0 + B_1 t + \ldots$, $A(t) = A_0 + A_1 t + \ldots$, $f(t) = f_0 + f_1 t + \ldots$ for $|t| < \rho$, $B_0 : D \subset X \to Y$ is a Fredholm operator, $\bar{D} = X$, $B_1, B_2, \ldots$, $A_0, A_1, \ldots$ are linear bounded operators from X into Y, where X and Y are Banach spaces, and f_i are elements of Y.

We search for a solution $u(t)$ in the class X_p of X-valued functions defined for $0 < |t| < \rho$ and having a pole of order p:

$$u(t) = \frac{u_{-p}}{t^p} + \cdots + \frac{u_{-1}}{t} + u_0 + u_1 t + \ldots. \tag{4}$$

The class X_0 corresponds to the case where the condition (3) turns into the condition $B(0)u(0) = y_0$ used earlier in a number of works.

3. Necessary condition of the solvability of the problem (2)–(3) in the class X_p, $p \geq 1$.

Introduce the vectors $w = \big(u_{-p}, u_{-p+1}, \ldots, u_{-1}, u_0\big)^T$ and $\beta = \big(0, \ldots, 0, y_0\big)^T$ and the following block operator of dimension $(p+1) \times (p+1)$:

$$\Phi_p = \begin{bmatrix} B_0 & 0 & 0 & \ldots & 0 \\ B_1 & B_0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_p & B_{p-1} & \cdots & B_1 & B_0 \end{bmatrix}.$$

Lemma 1. *If the problem* (2)–(3) *has a solution in the class* X_p, $p \in \mathbb{N}$, *then the system*

$$\Phi_p w = \beta \tag{5}$$

is solvable.

The proof is obvious: substituting the series (4) into the initial condition (3) and applying the method of indefinite coefficients, we arrive at the system (5).

Introduce the following condition:

II. $\{\phi_1, \ldots, \phi_n\}$ is a basis in $N(B_0)$, $\{\psi_1, \ldots, \psi_n\}$ is a basis in $N(B_0^*)$, $B_i\phi = 0$, $i = \overline{1,p}$, for all $\phi \in N(B_0)$,

$$\langle \psi_i, \phi_j \rangle = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}, \qquad \langle \psi_i, z_j \rangle = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}, \qquad \langle \psi_i, y_0 \rangle = 0,\ i = \overline{1,n}.$$

Lemma 2. *If the condition* II *is fulfilled, then the set of elements*

$$\begin{cases} u_k = (c_k, \phi), \quad k = -p, -p+1, \ldots, -1, \\ u_0 = (c_0, \phi) + \left(B_0 + \sum_{i=1}^{n} \langle \cdot,\ \gamma_i \rangle z_i \right)^{-1} y_0, \end{cases} \tag{6}$$

where

$$(c_k, \phi) = \sum_{i=1}^{n} c_{k_i} \phi_i$$

is a general solution of the system (5) *and* c_{k_i} *are arbitrary constants. Moreover, the corresponding expansion* (4) *satisfies the initial condition* (3) *for all* c_{k_i} *and* $u_1, u_2, \ldots$.

4. Sufficient conditions of the existence of a unique solution of the problem (2), (3) **in the class** X_p, $p \geq 1$**.** Assume that the conditions I and II are fulfilled and the coefficients $u_{-p}, \ldots, u_{-1}, u_0$ are defined by the formulas (6). Moreover, let the following conditions be fulfilled:

III. $A_i\phi = 0$, $i = \overline{0, p-1}$, for all $\phi \in N(B_0)$.

Substituting the expansion (4) into Eq. (2) and applying the method of indefinite coefficients for calculating the coefficients $u_1, u_2, \ldots$, we arrive at the following recurrent sequence of equations:

$$B_0 u_1 = (pB_{p+1} + A_p) u_{-p} + f_0 \triangleq L_1(c_{-p}), \tag{7}$$

$$mB_0 u_m = ((p+1-m)B_{p+1} + A_p) u_{m-p-1} - \sum_{i+l=m} iB_l u_i + \sum_{i+l=m-1} A_l u_i + f_{m-1} \triangleq L_m(c_{m-p-1}), \quad m = 2, 3 \ldots. \tag{8}$$

Introduce the projector P and the invertible operator $\hat{B}_0$ as follows:

$$P = \sum_{j=1}^{n} \langle \cdot, \gamma_j \rangle \phi_j, \quad \hat{B}_0 = B_0 + \sum_{j=1}^{n} \langle \cdot, \gamma_j \rangle z_j.$$

Using the projector P, we can represent the coefficients u_m of the expansion (4) in the form

$$u_m = Pu_m + (I-P)u_m, \quad m = -p,\ -p+1,\ \ldots,$$

or, in a more convenient notation, $u_m = (c_m, \phi) + \hat{u}_m$, where $c_m \in \mathbb{R}^n$ and $\phi = (\phi_1, \ldots, \phi_n)^T$.

We find the vectors c_m consecutively from the solvability conditions for the linear equations (7) and (8):

$$\langle \phi_i, L_m(c_{m-p-1}) \rangle = 0, \quad i = \overline{1,n}. \tag{9}$$

Lemma 2 and Eqs. (7) and (8) imply that the projections $\hat{u}_m$ are defined by the rules

$$\hat{u}_m = \begin{cases} 0 & \text{for } m = -p,\ -p+1,\ \ldots,\ -1, \\ \hat{B}_0^{-1} y_0 & \text{for } m = 0, \text{ if } \langle \psi_i, y_0 \rangle = 0,\ i = \overline{1,n}, \\ \frac{1}{j} \hat{B}_0^{-1} L_m(C_{m-p-1}) & \text{for } m = 1, 2, \end{cases}$$

where C_{m-p-1} satisfies the system of linear algebraic equations (SLAE) (9).

Indeed, the orthogonality conditions (9) are the solvability conditions for Eqs. (7) and (8); they allow constructing the systems of equations for calculating c_{-p}, c_{-p+1}, ... The element y_0 in the initial condition belong to the range of the operator $\boldsymbol{B}_0$.

To solve the system (9), we introduce the pencil M_m of square $(n \times n)$-matrices by the formula

$$M_m = D + (p + 1 - m)\bar{B}, \quad m = 1, 2, \ldots,$$

where

$$D \triangleq \left[\langle \psi_i, A_p \phi_j \rangle\right]_{i,j=\overline{1,n}}, \quad \bar{B} \triangleq \left[\langle \psi_i, B_{p+1} \phi_j \rangle\right]_{i,j=\overline{1,n}}.$$

Then the condition (9) can be represented as the following sequence of systems of linear algebraic equations:

$$M_m C_{-p+m-1} + \beta_{-p+m-1} = 0, \quad m = 1, 2, \ldots, \tag{10}$$

where the vector $\beta_{m-p-1} \in \mathbb{R}^n$ depends for each m on the projections of the coefficients of the series (4) calculated earlier.

To provide the uniqueness of coefficients of the projections Pu_m (i.e., the solutions of the system (10)), we impose the following condition:

IV $\det M_m \neq 0$ for $m \in \mathbb{N}$.

Remark 1. If $\det \bar{B} \neq 0$ and $\det M_m \neq 0$ for $m = 1, 2, \ldots, N$, where $N = \max\{p+2, \|\bar{B}^{-1}D\|\}$, then the condition IV holds for all m due to the theorem on inverse operators.

Thus, we arrive at the following assertion.

Theorem 1. *Let B_0 be a Fredholm operator and let the conditions* I–IV *be fulfilled. Then the coefficients of the Laurent expansion* (4) *of a solution of the initial-value problem* (2), (3) *are defined uniquely. Namely, the coefficients C_j of the projections Pu_j, $j = -p, -p+1$, are defined from the regular systems of linear algebraic equations* (10)*. The projections $(I - P)u_j$ are calculated by the formula*

$$(I - p)u_j = \begin{cases} 0 & \text{for } j = -p,\ -p+1,\ \ldots,\ -1, \\ \hat{B}_0^{-1} y_0 & \text{for } j = 0, \\ \frac{1}{j}\hat{B}_0^{-1} L_j(C^{\star}_{-p+j-1}) & \text{for } j = 1, 2, \ldots. \end{cases}$$

Using the majorant method, it is possible to estimate the radius of convergence of the regular part of the Laurent series (4) in the analytical case.

If the input data are sufficiently smooth and the operator $B(t)$ is invertible in a punctured neighborhood $0 < |t| < p$, one can proof an analog of Theorem 1 in the nonanalytic case.

Remark 2. The order of the pole of a solution of the initial-value problem (2)–(3) depends on the function $f(t)$. If $\langle \psi, f(0) \rangle = 0$ for all $\psi \in N(B(0))$, then $u_{-p} = 0$. In particular, under the conditions of the theorem, the point $t = 0$ becomes a removable singular point for $p = 1$. Therefore, the set of points t_0 at which the operator $B(t)$ has no bounded inverse operator can be classified as the set of quasi-movable singular points of the initial-value problem (2)–(3).

Example 1. Consider the integro-differential equation

$$\left(\frac{\partial^2}{\partial x^2} + 1\right)\frac{\partial u(x,t)}{\partial t} + \int_0^{\pi} B(x,s,t)\frac{\partial u(s,t)}{\partial t} ds = \int_0^{\pi} A(x,s,t)u(s,t)ds + f(x,t) \tag{11}$$

with the boundary conditions

$$u|_{x=0} = 0, \quad u|_{x=\pi} = 0 \tag{12}$$

and the initial condition

$$\lim_{t \to 0}\left(\frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t) + \int_0^{\pi} B(x,s,t)u(s,t)ds\right) = y_0(x). \tag{13}$$

Let

$$\int_0^\pi y_0(x)\sin x\,dx = 0$$

and the following expansions are fulfilled for $|t| < \rho$:

$$B(x,s,t) = \sum_{i=1}^{\infty} B_i(x,s)t^i, \quad A(x,s,t) = \sum_{i=0}^{\infty} A_i(x,s)t^i, \quad f(x,t) = \sum_{i=0}^{\infty} f_i(x)t^i$$

where $B_i(x,s)$, $A_i(x,s)$, and $f_i(x)$ are continuous functions.

In this example, $X = Y = C_{[0,\pi]}$, the operator $B_0 = \partial^2/\partial x^2 + 1$ satisfying the conditions (2) is a Fredholm operator, and $\phi(x) = \psi(x) = \sin x$. Moreover, let

$$\int_0^\pi B_i(x,s)\sin s\,ds = 0, \quad i = \overline{1,p}, \qquad \int_0^\pi A_i(x,s)\sin s\,ds = 0, \quad i = \overline{0,p-1},$$

$$\int_0^\pi\int_0^\pi B_{p+1}(x,s)\sin x\sin s\,dx\,ds = \beta \neq 0, \qquad \int_0^\pi\int_0^\pi A_p(x,s)\sin x\sin s\,dx\,ds = d \neq 0.$$

Due to the condition IV, we introduce the algebraic equation $(p+1-m)\beta + d = 0$ for m. If it has no natural solutions, then all conditions of Theorem 1 are fulfilled and the problem (11)–(3) has a solution of the form (4) with a pole at the point $t = 0$ of order $\leq p$. If

$$\int_0^\pi f(x,0)\sin x\,dx \neq 0,$$

then $t = 0$ is a pole of order p, i.e.,

$$u(x,t) \sim \frac{\sin x}{t^p}c$$

in a punctured neighborhood $0 < |t| < p$ for

$$c = -(p\beta + d)^{-1}\int_0^\pi f(x,0)\sin x\,dx.$$

The following example shows that different components of the vector u can have poles of different orders.

Example 2. Consider a system of the form (2) consisting of two ordinary differential equations with

$$B(t) = \begin{bmatrix} 1+t+2t^2 & 3t^2 \\ t+4t^2 & 5t^2 \end{bmatrix}, \quad A(t) = \begin{bmatrix} 1+t & 0 \\ t+2t & t \end{bmatrix}, \quad f(t) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Here $X = Y = \mathbb{R}^2$ and $\phi = \psi = [0,1]^T$. Introduce the initial condition

$$\lim_{t\to 0} B(t)u(t) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}.$$

Obviously, the vector' $[-2,0]^T$ lies in the range of the matrix B_0. In this example,

$$B_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -2 & 3 \\ 4 & 5 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix},$$

$$B_1\phi = A_0\phi = 0, \quad (B_2\phi,\phi) = 5, \quad (A_1\phi,\phi) = 1.$$

In the representation (4) of the solution we set $p = 1$. Since $1 + (2 - m)5 \neq 0$ for $m \in \mathbb{N}$, all conditions of Theorem 1 are fulfilled and we can construct a solution in the form (4) for $p = 1$. Following the algorithm described in the theorem, we obtain the following solution:

$$u_1(t) \sim 2 + t + t^2 + \frac{t^3}{12} + \frac{15}{24}t^4 + \ldots, \quad u_2(t) \sim -\frac{2}{3t} - 4 - \frac{3}{4}t - \frac{25}{36}t^2 + \ldots;$$

here the component $u_2(t)$ has a pole of the first order.

Using the method described, it is possible to construct collapsing solutions of

$$B(t)\frac{du}{dt} = A(t)u + F(u,t),$$

where $\|F(u,t)\| = O(1)$ for all $u \in X_p$ and $t \in (-p, \rho)$, with the initial condition (3). The initial condition (3) is a generalization of the Showalter–Sidorov conditions (see [3, 19]); it allows one to construct collapsing solutions in Sobolev-type models by using two-stage iterative procedures developed in [1–4, 6, 6, 8–13, 15, 17, 21] and the present paper.

5. Skeletal chains of linear operators and their applications to nonclassical initial-valued problems. Let $\boldsymbol{B} \in \mathcal{L}(E \to E)$ and $\boldsymbol{B} = \boldsymbol{A}_1\boldsymbol{A}_2$, where $\boldsymbol{A}_2 \in \mathcal{L}(E \to E_1)$, $\boldsymbol{A}_1 \in \mathcal{L}(E_1 \to E)$, and E_1 and E be linear normed spaces. The representation $\boldsymbol{B} = \boldsymbol{A}_1\boldsymbol{A}_2$ is called the *skeletal decomposition* of the operator $\boldsymbol{B}$. Introduce the linear operator $\boldsymbol{B}_1 = \boldsymbol{A}_2\boldsymbol{A}_1$. Obviously, $\boldsymbol{B}_1 \in \mathcal{L}(E_1 \to E_1)$. If the operator $\boldsymbol{B}_1$ has a bounded inverse operator or $\boldsymbol{B}_1$ is the zero operator from E_1 into E_1, then we say that $\boldsymbol{B}$ generates a *skeletal chain* $\{\boldsymbol{B}_1\}$ of length 1; in this case, the operator $\boldsymbol{B}_1$ is called the *skeletal adjoint operator* for the operator $\boldsymbol{B}$. A skeletal chain is said to be *degenerate* if $\boldsymbol{B}_1 = 0$ and *nondegenerate* if $\boldsymbol{B}_1$ is an invertible operator. If $\boldsymbol{B}_1$ is a noninvertible nonzero operator, we similarly admit the skeletal decomposition $\boldsymbol{B}_1 = \boldsymbol{A}_3\boldsymbol{A}_4$, where $\boldsymbol{A}_4 \in \mathcal{L}(E_1 \to E_2)$, $\boldsymbol{A}_3 \in \mathcal{L}(E_2 \to E_1)$, and E_2 is another linear normed space. Obviously, $\boldsymbol{A}_2\boldsymbol{A}_1 = \boldsymbol{A}_3\boldsymbol{A}_4$, and we can introduce the operator $\boldsymbol{B}_2 = \boldsymbol{A}_4\boldsymbol{A}_3 \in \mathcal{L}(E_2 \to E_2)$. If $\boldsymbol{B}_2$ has a bounded inverse operator or $\boldsymbol{B}_2 = 0$, then we say that the operator $\boldsymbol{B}$ has the skeletal chain $\{\boldsymbol{B}_1, \boldsymbol{B}_2\}$ of length 2. The chain $\{\boldsymbol{B}_1, \boldsymbol{B}_2\}$ is said to be degenerate if $\boldsymbol{B}_2 = 0$ and nondegenerate if $\boldsymbol{B}_2$ is an invertible operator. In the third case where $\boldsymbol{B}_2$ is a noninvertible nonzero operator, the length of the chain is greater than 2 and one can continue the construction.

This process can be continued for some classes of linear operators. For this purpose, we need to introduce linear normed spaces E_i, $i = 1, \ldots, p$, and to construct bounded operators $\boldsymbol{A}_{2i} \in \mathcal{L}(E_{i-1} \to E_i)$ and $\boldsymbol{A}_{2i-1} \in \mathcal{L}(E_i \to E_{i-1})$ satisfying the equalities

$$\boldsymbol{A}_{2i}\boldsymbol{A}_{2i-1} = \boldsymbol{A}_{2i+1}\boldsymbol{A}_{2i+2}, \quad i = 1, 2, \ldots, p-1. \tag{14}$$

According to (14), consider the sequence of linear operators $\{\boldsymbol{B}_1, \ldots, \boldsymbol{B}_p\}$ defined by the formulas

$$\boldsymbol{B}_i = \boldsymbol{A}_{2i}\boldsymbol{A}_{2i-1}, \quad i = 1, 2, \ldots, p. \tag{15}$$

Obviously, $\boldsymbol{B}_i \in \mathcal{L}(E_i \to E_i)$. Here we assume that the last operator $\boldsymbol{B}_p$ either has a bounded inverse operator or $\boldsymbol{B}_p$ is the zero operator from E_p into E_p. We introduce the following notion related to this construction.

Definition 1. Let $\boldsymbol{B} = \boldsymbol{A}_1\boldsymbol{A}_2$ and the operators $\{\boldsymbol{A}_i\}_{i=1}^{2p}$ satisfy Eqs. (14). Assume that the operators $\{\boldsymbol{B}_1, \ldots, \boldsymbol{B}_p\}$ are defined by the formulas (15), the nonzero operators $\{\boldsymbol{B}_1, \ldots, \boldsymbol{B}_{p-1}\}$ are noninvertible, and the operator $\boldsymbol{B}_p$ either has a bounded inverse operator or is the zero operator from E_p into E_p. Then we say that the operator $\boldsymbol{B}$ generates the skeletal chain $\{\boldsymbol{B}_1, \ldots, \boldsymbol{B}_p\}$ of linear operators of length p; this skeletal chain is said to be *nondegenerate* if the operator $\boldsymbol{B}_p \neq 0$ is invertible and *degenerate* if $\boldsymbol{B}_p = 0$. The operators $\{\boldsymbol{B}_1, \ldots, \boldsymbol{B}_p\}$ are called the *skeletal adjoint* operators for the operator $\boldsymbol{B}$.

We give examples of linear operators that generate skeletal chains of finite length.

Example 3. Let $E = \mathbb{R}^m$. Obviously, a degenerate square matrix $B : \mathbb{R}^m \to \mathbb{R}^m$ has a skeletal chain consisting of a finite number of degenerate square matrices of decreasing dimensions. Moreover, the last matrix of the chain is either nondegenerate or zero.

Example 4. Let E be an infinite-dimensional normed space. Similarly to Example 3, the finite-dimensional operator

$$\boldsymbol{B} = \sum_{i=1}^{n} \langle \cdot, \gamma_i \rangle z_i,$$

where $\{z_i\} \in E$ and $\gamma_i \in E^*$, has a skeletal chain of finite length consisting of matrices $\{\boldsymbol{B}_1, \dots, \boldsymbol{B}_p\}$ of decreasing dimensions. Moreover, $\boldsymbol{B}_1 = \big\| \langle z_i, \gamma_j \rangle \big\|_{i,j=1}^{n}$ is the first skeletal adjoint element of the chain.

In Example 4, according to Definition 1, the length of the chain is $p = 1$ if $\det \big[\langle z_i, \gamma_j \rangle \big]_{i,j=1}^{n} \neq 0$ or $\langle z_i, \gamma_j \rangle = 0$, $i, j = 1, 2, \dots, n$. In the general case, as in Example 3, the chain consists of a finite number of matrices of decreasing dimensions.

Taking into account the formulas (14) and (15) and Definition 1, we arrive at the following result.

Lemma 3. *If an operator $\boldsymbol{B}$ has a skeletal chain of length p, then its powers $\boldsymbol{B}^n$, $n = 2, 3, \dots, p+1$, can be represented as follows*:

$$\boldsymbol{B}^n = \boldsymbol{A}_1 \boldsymbol{A}_3 \dots \boldsymbol{A}_{2n-1} \boldsymbol{B}_{n-1} \boldsymbol{A}_{2n-2} \boldsymbol{A}_{2n-4} \dots \boldsymbol{A}_2; \tag{16}$$

here $\boldsymbol{B}_1, \boldsymbol{B}_2, \dots, \boldsymbol{B}_p$ are the elements of the skeletal chain of the operator $\boldsymbol{B}$.

Corollary 1. *If an operator $\boldsymbol{B}$ has a degenerate skeletal chain of length p, then $\boldsymbol{B}$ is a nilpotent operator of index $p + 1$.*

To prove the Corollary, it suffices to set $n = p + 1$ in the formula (16) and to verify that $\boldsymbol{B}^{p+1}$ is the zero operator since the representation of $\boldsymbol{B}^{p+1}$ contains the zero operator $\boldsymbol{B}_p$ due to the fact that the skeletal chain is degenerate.

Based on the notion of skeletal chains, it is possible to reduce an irregular systems to a sequence of regular problems. This was shown in [5, 16] by examples of classical equations of the form

$$\boldsymbol{B} L(t) u = L_1(t) u + f(t),$$

where $\boldsymbol{B}$ is a continuous operator and $L(t)$ and $L_1(t)$ are partial differential operators.

Below we discuss applications of skeletal chains to the analysis of the simplest irregular differential equation

$$\boldsymbol{B} \frac{du(t)}{dt} = u(t) + f(t), \tag{17}$$

where $f(t) : [0, \infty) \to E$ and $\boldsymbol{B} \in \mathcal{L}(E \to E)$ is a Fredholm operator. Let $\{\boldsymbol{B}_1, \dots, \boldsymbol{B}_p\}$ be the skeletal chain of the operator $\boldsymbol{B}$, $\boldsymbol{B}_i = \boldsymbol{A}_{2i} \boldsymbol{A}_{2i-1}$, $i = 1, \dots, p$.

Theorem 2. *Let $\{\boldsymbol{B}_1, \dots, \boldsymbol{B}_p\}$ be the regular skeletal chain of the operator $\boldsymbol{B}$ and let the function $f(t)$ be differentiable $p - 1$ times. Then Eq.* (17) *with the initial condition*

$$\prod_{j=1}^{p} A_{2j} u(t) \big|_{t=0} = c_0, \quad c_0 \in E_p \tag{18}$$

has a unique classical solution

$$u(t, c_0) = -f(t) + \boldsymbol{A}_1 \frac{du_1}{dt}, \tag{19}$$

where the function $u_1(t, c_0)$ is defined uniquely.

We describe the scheme of constructing the function $u_1(t, c_0)$ in the solution (19) based on the definition of the skeletal chain of the operator $\boldsymbol{B}$:

(1) if $p = 1$, then $u_1(t, c_0)$ satisfies the regular Cauchy problem

$$\begin{cases} \boldsymbol{B}_1 \dfrac{du_1}{dt} = u_1 + \boldsymbol{A}_2 f(t), \\ u_1(0) = c_0; \end{cases}$$

(2) if $p \geq 2$, then $u_1(t, c_0)$ can be constructed recursively as follows:

$$\begin{cases} \boldsymbol{B}_p \dfrac{du_p}{dt} = u_p + \prod\limits_{j=1}^{p} \boldsymbol{A}_{2j} f(t), \\ u_p(0) = c_0, \end{cases}$$

$$u_i(t, c_0) = \boldsymbol{A}_{2i+1} \frac{du_{i+1}(t, c_0)}{dt} - \prod_{j=1}^{i} \boldsymbol{A}_{2j} f(t), \quad i = p-1,\ p-2,\ \ldots,\ 1.$$

Theorem 3. *Let* $\{\boldsymbol{B}_1, \ldots, \boldsymbol{B}_{p-1}, 0\}$ *be the singular chain of length* $p \geq 1$ *and* 0 *be the zero operator acting from* E_p *into* E_p. *Then* $\boldsymbol{B}$ *is a nilpotent operator and the homogeneous equation*

$$\boldsymbol{B} \frac{du}{dt} = u$$

has only trivial solution. Moreover, if the function $f(t)$ *is differentiable* p *times, then the inhomogeneous equation* (17) *has a unique classical solution, which can be constructed iteratively*:

$$u_n(t) = -f(t) + \boldsymbol{B} \frac{d}{dt} u_{n-1}(t)$$

for $u_0(t) = -f(t)$, $n = 1, 2, \ldots, p$.

Thus, under the conditions of Theorem 3, the iteration $u_p(t)$ is a unique classical solution of the inhomogeneous equation (17).

To illustrate Theorems 2 and 3, we consider two simple examples in the space $E = \mathbb{R}^3$.

Example 5. Let

$$\boldsymbol{B} = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{aligned} f(t) &= \big(f_1(t), f_2(t), f_3(t)\big)^T, \\ u(t) &= \big(u_1(t), u_2(t), u_3(t)\big)^T \end{aligned}$$

in Eq. (17). Obviously, $\boldsymbol{B} = \boldsymbol{A}_1 \boldsymbol{A}_2$, where $\boldsymbol{A}_1 = [2, 1, 0]^T$ and $\boldsymbol{A}_2 = (a, \cdot)$ (scalar product), $a = [0, 1, 1]^T$. Therefore, $\boldsymbol{B}_1 = \boldsymbol{A}_2 \boldsymbol{A}_1 = 1$, and the matrix $\boldsymbol{B}$ generates a regular skeletal chain of length $p = 1$. Using Theorem 2, introduce the following system:

$$\begin{cases} \dfrac{dv}{dt} = v + f_2(t) + f_3(t), \\ v(0) = 0, \\ u(t) = -f(t) + \boldsymbol{A}_1 \dfrac{dv}{dt}, \end{cases}$$

where $v = \boldsymbol{A}_2 u = u_2(t) + u_3(t)$. The initial condition $u_2(0) + u_3(0) = 0$ follows from the condition $v(0) = 0$. Let $f_i(t)$ be continuous functions. Then

$$v(t) = \int_0^t e^{t-s} \big(f_2(s) - f_3(s)\big)\, ds$$

and the system (17) with the matrix $\boldsymbol{B}$ and the initial condition $u_2(0) + u_3(0) = 0$ has a unique classical solution

$$u(t) = -f(t) + \big(2v(t) + 2f_2(t) + 2f_3(t),\ v(t) + f_2(t) + f_3(t), 0\big)^T.$$

Example 6. As in Example 5, let $\boldsymbol{B} = \boldsymbol{A}_1\boldsymbol{A}_2$ but

$$\boldsymbol{B} = \begin{bmatrix} 0 & 2 & -2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \quad \boldsymbol{A}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad a = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

$\boldsymbol{A}_2 = (a, \cdot)$, and B be a nilpotent matrix. Then $\boldsymbol{B}_1 = \boldsymbol{A}_2\boldsymbol{A}_1 = 0$. Thus, in Example 6, $\{\boldsymbol{B}_1\}$ is a singular skeletal chain of length $p = 1$ corresponding to the nilpotent matrix $\boldsymbol{B}$. Therefore, due to Theorem 3, we can construct a unique solution of the system (17) with the matrix $\boldsymbol{B}$ by the iterations

$$u_n(t) = -f(t) + \boldsymbol{B}\frac{d}{dt}u_{n-1}(t), \quad n = 1, 2, \dots p, \quad u_0(t) = -f(t).$$

Since $p = 1$, the unique solution of the inhomogeneous system (17) has the form

$$u(t) = -f(t) + \Big(2\big(f_2(t) - f_3(t)\big)', \ \big(f_2(t) - f_3(t)\big)', \ \big(f_2(t) - f_3(t)\big)'\Big)^T.$$

If $f(t)$ is a differentiable vector-valued function, then we obtain a classical solution. If $f_i(t)$, $i = 2, 3$, are piecewise, absolutely continuous functions with discontinuity points of first kind and piecewise continuous derivatives, then we obtain a solution on the space of distributions K'. Thus, the method proposed allows also constructing generated solutions of singular ordinary differential equations.

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COMPLIANCE WITH ETHICAL STANDARDS

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