DOI 10.1007/s10958-024-07041-z
Journal of Mathematical Sciences, Vol. 279, No. 5, March, 2024

FIXED-POINT METHODS IN OPTIMIZATION PROBLEMS FOR CONTROL SYSTEMS

A. S. Buldaev UDC 517.977

Abstract. In this paper, we consider a new approach to optimization of nonlinear control systems based on the representation of optimality conditions and improvement of the control in the form of special fixed-point problems for control operators. We propose algorithms for approximate solution of optimal control problems based on iterative methods for finding fixed points. The effectiveness of the optimization methods proposed is illustrated by model and test problems.

***Keywords and phrases*:** control system, conditions for improvement and optimality of control, fixed-point problem, numerical optimization algorithms.

***AMS Subject Classification*:** 49Kxx, 37Cxx

The methods proposed generalize nonlocal methods for improving and optimizing controls originally developed in the classes of dynamical systems that are linear (see [6]) or polynomial (see [1]) with respect to the state.

1. Fixed-point problems. We illustrate the fixed point approach considered below within the framework of the following class of optimal control problems:

$$\Phi(\sigma) = \varphi(x(t_1), \omega) + \int_T F(x(t), u(t), \omega, t)dt \to \inf_{\sigma\in\Omega}, \tag{1}$$

$$\dot{x}(t) = f\big(x(t), u(t), \omega, t\big), \quad x(t_0) = a, \quad u(t) \in U, \omega \in W, \quad a \in A, \quad t \in T = [t_0, t_1], \tag{2}$$

where $x(t) = (x_1(t), \dots, x_n(t))$ is the state vector, $u(t) = (u_1(t), \dots, u_m(t))$ is the vector of control functions, and $\omega = (\omega_1, \dots, \omega_l)$ and $a = (a_1, \dots, a_n)$ are the vectors of control parameters. We assume that the following conditions are fulfilled. The sets $U \subseteq \mathbb{R}^m$, $W \subseteq \mathbb{R}^l$, and $A \subseteq \mathbb{R}^n$ are closed and convex. The interval T is fixed. The set V of admissible control functions consists of U-valued functions that are piecewise continuous on T. Let $\sigma = (u, \omega, a)$ be an admissible control with values in the set $\Omega = V \times W \times A$. The function $\varphi(x, \omega)$ is continuously differentiable on $\mathbb{R}^n \times W$ and the functions $F(x, u, \omega, t)$ and $f(x, u, \omega, t)$ and their partial derivatives with respect to x, u, and ω are continuous on the set $\mathbb{R}^n \times U \times W \times T$. The function $f(x, u, \omega, t)$ satisfies the Lipschitz condition with respect to x in $\mathbb{R}^n \times U \times W \times T$ with a constant $L > 0$:

$$\big\|f(x, u, \omega, t) - f(y, u, \omega, t)\big\| \le L\|x - y\|.$$

These conditions guarantee the existence and uniqueness of a solution $x(t, \sigma)$, $t \in T$, of the system (2) for any admissible control $\sigma \in \Omega$.

The Pontryagin function with the adjoint variable $\psi \in \mathbb{R}^n$ and the standard adjoint system have the form

$$H(\psi, x, u, \omega, t) = \big\langle \psi, f(x, u, \omega, t)\big\rangle - F(x, u, \omega, t),$$

$$\dot{\psi}(t) = -H_x\big(\psi(t), x(t), u(t), \omega, t\big), \quad t \in T, \quad \psi(t_1) = -\varphi_x(x(t_1), \omega). \tag{3}$$

For an admissible control $\sigma \in \Omega$, we denote by $\psi(t, \sigma)$, $t \in T$, a solution of the standard adjoint system (3) for $x(t) = x(t, \sigma)$ and the arguments u and ω corresponding to the components of the

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematicheskie Obzory, Vol. 183, Differential Equations and Optimal Control, 2020.

control σ. We denote the partial increment of an arbitrary vector-valued function $g(y_1, \dots, y_l)$ with respect to the variables y_{s_1}, y_{s_2} as follows:

$$\Delta_{y_{s_1}+\Delta y_{s_1}, y_{s_2}+\Delta y_{s_2}} g(y_1, \dots, y_l) = g\Big(y_1, \dots, y_{s_1} + \Delta y_{s_1}, \dots, y_{s_2} + \Delta y_{s_2}, \dots, y_l\Big) - g(y_1, \dots, y_l).$$

In addition, introduce the notation

$$\Delta x(t) = x(t, \sigma) - x(t, \sigma^{\mathrm{I}}), \quad \Delta u(t) = u(t) - u^{\mathrm{I}}(t), \quad \Delta\omega = \omega - \omega^{\mathrm{I}}, \quad \Delta a = a - a^{\mathrm{I}}.$$

Let P_Y be the projector onto the set $Y \subset \mathbb{R}^k$ in the Euclidean norm:

$$P_Y(z) = \arg\min_{y \in Y} \big(\|y - z\|\big), \quad z \in \mathbb{R}^k.$$

In [2], necessary optimality conditions for the problem (1), (2) were obtained in the form of the maximum principle:

$$u^{\mathrm{I}}(t) = \arg\max_{\tilde{u} \in U} H\Big(\psi(t, \sigma^{\mathrm{I}}),\ x(t, \sigma^{\mathrm{I}}),\ \tilde{u},\ \omega^{\mathrm{I}}, t\Big), \quad t \in T, \tag{4}$$

$$\omega^{\mathrm{I}} = \arg\max_{\tilde{\omega} \in W} \left\langle -\varphi_\omega\big(x(t_1, \sigma^{\mathrm{I}}), \omega^{\mathrm{I}}\big) + \int_T H_\omega\Big(\psi(t, \sigma^{\mathrm{I}}), x(t, \sigma^{\mathrm{I}}), u^{\mathrm{I}}(t), \omega^{\mathrm{I}}, t\Big) dt,\ \tilde{\omega} \right\rangle, \tag{5}$$

$$a^{\mathrm{I}} = \arg\max_{\tilde{a} \in A} \left\langle \psi(t_0, \sigma^{\mathrm{I}}),\ \tilde{a} \right\rangle. \tag{6}$$

The conditions (4)–(6) imply the weakened necessary conditions in the form of the differential maximum principle:

$$u^{\mathrm{I}}(t) = \arg\max_{\tilde{u} \in U} \left\langle H_u\Big(\psi(t, \sigma^{\mathrm{I}}), x(t, \sigma^{\mathrm{I}}), u^{\mathrm{I}}(t), \omega^{\mathrm{I}}, t\Big), \tilde{u} \right\rangle, \quad t \in T,$$

$$\omega^{\mathrm{I}} = \arg\max_{\tilde{\omega} \in W} \left\langle -\varphi_\omega\big(x(t_1, \sigma^{\mathrm{I}}), \omega^{\mathrm{I}}\big) + \int_T H_\omega\Big(\psi(t, \sigma^{\mathrm{I}}),\ x(t, \sigma^{\mathrm{I}}),\ u^{\mathrm{I}}(t),\ \omega^{\mathrm{I}}, t\Big) dt,\ \tilde{\omega} \right\rangle,$$

$$a^{\mathrm{I}} = \arg\max_{\tilde{a} \in A} \langle \psi(t_0, \sigma^{\mathrm{I}}), \tilde{a} \rangle,$$

which can be represented in the projection form with a parameter $\alpha > 0$:

$$u^{\mathrm{I}}(t) = P_U\Big(u^{\mathrm{I}}(t) + \alpha H_u\Big(\psi(t, \sigma^{\mathrm{I}}), x(t, \sigma^{\mathrm{I}}), u^{\mathrm{I}}(t), \omega^{\mathrm{I}}, t\Big)\Big), \quad t \in T, \tag{7}$$

$$\omega^{\mathrm{I}} = P_W\Big(\omega^{\mathrm{I}} + \alpha\Big(- \varphi_\omega\big(x(t_1, \sigma^{\mathrm{I}}), \omega^{\mathrm{I}}\big) + \int_T H_\omega\big(\psi(t, \sigma^{\mathrm{I}}),\ x(t, \sigma^{\mathrm{I}}),\ u^{\mathrm{I}}(t),\ \omega^{\mathrm{I}}, t\big) dt\Big)\Big), \tag{8}$$

$$a^{\mathrm{I}} = P_A\big(a^{\mathrm{I}} + \alpha\psi(t_0, \sigma^{\mathrm{I}})\big). \tag{9}$$

We consider the following general statement of the problem of improving a control: for a given control $\sigma^{\mathrm{I}} \in \Omega$, find a control $\sigma \in \Omega$ satisfying the condition

$$\Delta_\sigma \Phi(\sigma^{\mathrm{I}}) = \Phi(\sigma) - \Phi(\sigma^{\mathrm{I}}) \le 0.$$

Consider the modified differential algebraic adjoint system

$$\dot{p}(t) = -H_x\big(p(t), x(t), u(t), \omega, t\big) - r(t), \tag{10}$$

$$\Big\langle H_x\big(p(t), x(t), u(t), \omega, t\big) + r(t),\ y(t) - x(t) \Big\rangle = \Delta_{y(t)} H\big(p(t), x(t), u(t), \omega, t\big) \tag{11}$$

with the boundary condition

$$p(t_1) = -\varphi_x\big(x(t_1), \omega\big) - q, \tag{12}$$

$$\Big\langle \varphi_x\big(x(t_1), \omega\big) + q,\ y(t_1) - x(t_1) \Big\rangle = \Delta_{y(t_1)} \varphi(x(t_1), \omega); \tag{13}$$

by definition, we set here $r(t)=0$ and $q=0$ in the case where the functions φ, F, and f are linear with respect to x (the state-linear problem (1)), (2)) and also in the case where $y(t)=x(t)$ for corresponding $t\in T$.

In the state-linear problem (1), (2), the modified adjoint system (10)–(13) coincides with the standard adjoint system (3) by definition.

In state-nonlinear problems (1), (2), the algebraic equations (11) and (13) can be solved with respect to $r(t)$ and q (perhaps, not uniquely). Thus, the differential algebraic adjoint system (10)–(13) can be reduced (perhaps, not uniquely) to the differential adjoint system in which $r(t)$ and q are defined uniquely.

For admissible controls $\sigma\in\Omega$ and $\sigma^{\mathrm{I}}\in\Omega$, we denote by $p(t,\sigma^{\mathrm{I}},\sigma)$, $t\in T$, a solution of the modified adjoint system (10)–(13) for $x(t)=x(t,\sigma^{\mathrm{I}})$, $y(t)=x(t,\sigma)$, $u(t)=u^{\mathrm{I}}(t)$, and $\omega=\omega^{\mathrm{I}}$. From the definition, we obtain the following obvious equality:

$$p(t,\sigma,\sigma)=\psi(t,\sigma),\quad t\in T.$$

It was proved in [3] that to solve the problem of improving a given control $\sigma^{\mathrm{I}}\in\Omega$, it suffices to solve the following system with respect to $\sigma=(u,\omega,a)$ for $\alpha>0$:

$$u(t)=P_U\Big(u^{\mathrm{I}}(t)+\alpha\Big(H_u\Big(p\big(t,\sigma^{\mathrm{I}},\sigma\big),\ x(t,\sigma),\ u^{\mathrm{I}}(t),\ \omega^{\mathrm{I}},t\Big)+s^u(t)\Big)\Big),\quad t\in T,\tag{14}$$

$$\begin{aligned}&\Delta_{u(t)}H\Big(p\big(t,\sigma^{\mathrm{I}},\sigma\big),\ x(t,\sigma),\ u^{\mathrm{I}}(t),\ \omega^{\mathrm{I}},t\Big)\\&\qquad=\Big\langle H_u\Big(p\big(t,\sigma^{\mathrm{I}},\sigma\big),\ x(t,\sigma),\ u^{\mathrm{I}}(t),\omega^{\mathrm{I}},t\Big)+s^u(t),\ u(t)-u^{\mathrm{I}}(t)\Big\rangle,\end{aligned}\tag{15}$$

$$\omega=P_W(\omega^{\mathrm{I}}+\alpha(-\varphi_\omega(x(t_1,\sigma),\omega^{\mathrm{I}})+\int_T H_\omega(p(t,\sigma^{\mathrm{I}},\sigma),x(t,\sigma),u(t),\omega^{\mathrm{I}},t)dt+s^\omega)),\tag{16}$$

$$\begin{aligned}&\Delta_\omega\left\{-\varphi\big(x(t_1,\sigma),\omega^{\mathrm{I}}\big)+\int_T H\Big(p\big(t,\sigma^{\mathrm{I}},\sigma\big),x(t,\sigma),u(t),\omega^{\mathrm{I}},t\Big)dt\right\}\\&\qquad=\left\langle-\varphi_\omega\big(x(t_1,\sigma),\omega^{\mathrm{I}}\big)+\int_T H_\omega\Big(p\big(t,\sigma^{\mathrm{I}},\sigma\big),x(t,\sigma),u(t),\omega^{\mathrm{I}},t\Big)dt+s^\omega,\ \omega-\omega^{\mathrm{I}}\right\rangle,\end{aligned}\tag{17}$$

$$a=P_A\Big(a^{\mathrm{I}}+\alpha p(t_0,\ \sigma^{\mathrm{I}},\sigma)\Big).\tag{18}$$

In the case where the functions F and f are linear with respect to u (the problem (1), (2) is said to be linear in the control function u) and in the case where $u(t)=u^{\mathrm{I}}(t)$ for corresponding $t\in T$, we set $s^u(t)=0$ in Eq. (15) by definition. Similarly, we set $s^\omega=0$ in Eq. (17) by definition in the case where the functions F and f are linear with respect to ω (the problem (1), (2) is said to be linear in the parameter ω) and in the case where $\omega=\omega^{\mathrm{I}}$.

In the problem (1), (2) that is nonlinear with respect to the control function u and the parameter ω, Eqs. (15) and (17) can be solved with respect to $s^u(t)$ and s^ω (perhaps, not uniquely). Thus, the system (14)–(18) can always be reduced (perhaps, not uniquely) to a system with the functions $s^u(t)$ and s^ω defined uniquely.

Let the system of conditions (14)–(18) have a solution $\sigma^{\mathrm{II}}=(u^{\mathrm{II}},\omega^{\mathrm{II}},a^{\mathrm{II}})$ (perhaps, not unique) and let the control u^{II} be piecewise continuous. Then the following estimate for the improved functional holds:

$$\Delta_{\sigma^{\mathrm{II}}}\Phi(\sigma^{\mathrm{I}})\le-\frac{1}{\alpha}\int_T\big\|u^{\mathrm{II}}(t)-u^{\mathrm{I}}(t)\big\|^2dt-\frac{1}{\alpha}\big\|\omega^{\mathrm{II}}-\omega^{\mathrm{I}}\big\|^2-\frac{1}{\alpha}\big\|a^{\mathrm{II}}-a^{\mathrm{I}}\big\|^2.$$

The structure of the conditions for optimality and improvement of control constructed above and the notation system for solutions of the phase and adjoint systems in the form of an explicit dependence on control allows us to interpret the systems (4)–(6), (7)–(9), and (14)–(18) as fixed-point problems for special control operators. This allows us to apply the theory developed and fixed-point methods (FPM) to the effective search for extremal and improving controls.

2. Iteration algorithms. In the class of problems (1), (2), the methods proposed can be illustrated by the following two optimization algorithms.

1. For numerical solution of the fixed-point problem (14)–(18) on improving a given control σ^{I}, we apply the following iteration process with $k \geq 0$:

$$u^{k+1}(t) = P_U\Big(u^{\mathrm{I}}(t) + \alpha\Big(H_u\big(p(t,\sigma^{\mathrm{I}},\sigma^k), x(t,\sigma^k), u^{\mathrm{I}}(t), \omega^{\mathrm{I}}, t\big) + s^u(t)\Big)\Big), \quad t \in T,$$

$$\Delta_{u^k(t)} H\Big(p\big(t,\sigma^{\mathrm{I}},\sigma^k\big), x(t,\sigma^k), u^{\mathrm{I}}(t), \omega^{\mathrm{I}}, t\Big) = \Big\langle H_u\Big(p\big(t,\sigma^{\mathrm{I}},\sigma^k\big), x(t,\sigma^k), u^{\mathrm{I}}(t), \omega^{\mathrm{I}}, t\Big) + s^u(t), u^k(t) - u^{\mathrm{I}}(t)\Big\rangle,$$

$$\omega^{k+1} = P_W\Big(\omega^{\mathrm{I}} + \alpha\Big(-\varphi_\omega\big(x(t_1,\sigma^k),\omega^{\mathrm{I}}\big) + \int_T H_\omega\Big(p\big(t,\sigma^{\mathrm{I}},\sigma^k\big), x(t,\sigma^k), u^k(t), \omega^{\mathrm{I}}, t\Big)dt + s^\omega\Big)\Big),$$

$$\Delta_{\omega^k}\Big\{-\varphi\big(x(t_1,\sigma^k),\omega^{\mathrm{I}}\big) + \int_T H\Big(p\big(t,\sigma^{\mathrm{I}},\sigma^k\big), x(t,\sigma^k), u^k(t), \omega^{\mathrm{I}}, t\Big)dt\Big\}$$
$$= \left\langle -\varphi_\omega\big(x(t_1,\sigma^k),\omega^{\mathrm{I}}\big) + \int_T H_\omega\Big(p\big(t,\sigma^{\mathrm{I}},\sigma^k\big), x(t,\sigma^k), u^k(t), \omega^{\mathrm{I}}, t\Big)dt + s^\omega,\ \omega^k - \omega^{\mathrm{I}}\right\rangle,$$

$$a^{k+1} = P_A\big(a^{\mathrm{I}} + \alpha p(t_0,\sigma^{\mathrm{I}},\sigma^k)\big).$$

An initial approximation $\sigma^0 \in \Omega$ of the iteration process is given for $k = 0$.

Computations for the fixed-point problem (14)–(18) are performed until the initial control σ^{I} improves. Then one must construct a new improvement problem for the control σ^{II} and continue computations. One performs iteration until the following condition holds:

$$\big|\Phi(\sigma^{\mathrm{II}}) - \Phi(\sigma^{\mathrm{I}})\big| \leq \varepsilon\big|\Phi(\sigma^{\mathrm{I}})\big|,$$

where $\varepsilon > 0$ is the given accuracy.

2. The fixed-point problem (7)–(9) for the differential maximum principle can also be solved by the following iteration process with the initial control $\sigma^0 \in \Omega$ for $k \geq 0$:

$$u^{k+1}(t) = P_U\Big(u^k(t) + \alpha H_u\Big(\psi\big(t,\sigma^k\big), x(t,\sigma^k), u^k(t), \omega^k, t\Big)\Big), \quad t \in T,$$

$$\omega^{k+1} = P_W\Big(\omega^k + \alpha\Big(-\varphi_\omega\big(x(t_1,\sigma^k),\omega^k\big) + \int_T H_\omega\big(\psi(t,\sigma^k), x(t,\sigma^k), u^k(t), \omega^k, t\big)dt\Big)\Big),$$

$$a^{k+1} = P_A\big(a^k + \alpha\psi(t_0,\sigma^k)\big).$$

Iterations are performed until the following condition holds:

$$\big|\Phi(\sigma^{k+1}) - \Phi(\sigma^k)\big| \leq \varepsilon\big|\Phi(\sigma^k)\big|,$$

where $\varepsilon > 0$ is the given accuracy.

To analyze the convergence of these iteration processes to the solutions of the fixed-point problems for sufficiently small values of the projection parameter $\alpha > 0$, one can use the well-known contraction mapping principle (see [1, 2, 5]).

The fixed-point projection methods constructed above are characterized by the fact that improving and extremal controls are determined by solutions of the corresponding fixed-point problems for any value of the projection parameter $\alpha > 0$; in particular, sufficiently small values of $\alpha > 0$ ensure the convergence of the iteration processes to solutions of fixed-point problems.

The effectiveness of the algorithms proposed is demonstrated and analyzed by model and test simulations.

3. Examples. Many optimal control problems with restrictions, including problems with nonfixed time can be reduced to problems of the form (1), (2). We demonstrate the fixed-point approach by two well-known examples.

Example 1. In [4], the time-optimal problem (Zermelo's problem) was reduced to the following optimal-control problem with nonfixed time by the penalty method:

$$\begin{cases} \dot{x}_1(t) = \cos x_3(t), & x_1(0) = 0, \\ \dot{x}_2(t) = \sin x_3(t), & x_2(0) = 0, \\ \dot{x}_3(t) = u(t), & x_3(0) = 0, \end{cases}$$

$$t \in [0, t_1], \quad \big|u(t)\big| \le 0.5,$$

$$G(u, t_1) = t_1 + 1000\Big(\big(x_1(t_1) - 4\big)^2 + \big(x_2(t_1) - 3\big)^2\Big) \to \min .$$

Using the substitution of time

$$t(\tau) = t_0 + \big(t_1 - t_0\big)\tau = \omega\tau, \quad \tau \in [0, 1], \quad \omega = t_1 \ge 0,$$

$$v(\tau) = u\big(t(\tau)\big), \quad y(\tau) = x\big(t(\tau)\big), \quad \sigma = (v, w),$$

one can further reduce this problem to the following problem with fixed time:

$$\begin{cases} \dot{y}_1(\tau) = \omega \cos y_3(\tau), & y_1(0) = 0, \\ \dot{y}_2(\tau) = \omega \sin y_3(\tau), & y_2(0) = 0, \\ \dot{y}_3(\tau) = \omega v(\tau), & y_3(0) = 0, \end{cases}$$

$$\Phi(\sigma) = \omega + 1000\Big(\big(y_1(1) - 4\big)^2 + \big(y_2(1) - 3\big)^2\Big) \to \min,$$

$$T = [0, 1], \quad U = \big\{u : \ |u| \le 0.5\big\}, \quad W = \big\{\omega : \ \omega \ge 0\big\}.$$

The Pontryagin function with derivatives and the standard adjoint system for this problem are as follows:

$$H\big(\psi, y, v, \omega, \tau\big) = \omega\big(\psi_1 \cos y_3 + \psi_2 \sin y_3 + \psi_3 v\big),$$

$$\begin{cases} \dot{\psi}_1(\tau) = 0, & \psi_1(1) = -2000(y_1(1) - 4), \\ \dot{\psi}_2(\tau) = 0, & \psi_2(1) = -2000(y_2(1) - 3), \\ \dot{\psi}_3(\tau) = \omega\psi_1 \sin y_3(\tau) - \omega\psi_2 \cos y_3(\tau), & \psi_3(1) = 0. \end{cases}$$

For an admissible control $\sigma = (v, \omega) \in \Omega$, we denote by $\psi(\tau, \sigma)$, $\tau \in [0, 1]$, the solution of the adjoint system for $y(\tau) = y(\tau, \sigma)$ and the components of v and ω corresponding to σ.

The problem obtained is a problem with fixed time; its linearity with respect to$v$ and ω substantially simplifies the structure of the fixed-point problem. In particular, due to the formulas (7)–(9), the fixed-point problem for the differential maximum principle in the projection form with respect to the extremal control $\sigma = (v, \omega) \in \Omega$ is as follows:

$$v(\tau) = P_U\Big(v(\tau) + \alpha\omega\psi_3(\tau, \sigma)\Big), \quad \tau \in [0, 1],$$

Table 1.

Method	Φ^*	N
1	5.85	100010
2	5.26	100125
3	5.13	100002
4	5.18	100019
5	5.26	100036
6	6.41	100008
7	5.33	100015
8	5.11	100012
FPM	5.117	70619

Table 2.

α	v^0	ω^0	Φ^*	N
10^{-5}	**0.5**	**5**	**5.117**	**70619**
$> 10^{-5}$	0.5	5	does not converge	—
10^{-6}	0.5	5	5.117	> 200000
10^{-5}	0	5	5.117	103999
$> 10^{-5}$	0	5	does not converge	—
10^{-6}	0	5	5.117	> 200000
10^{-5}	−0.5	5	5.117	141853
$> 10^{-5}$	−0.5	5	does not converge	—
10^{-6}	−0.5	5	7.00	200001
10^{-5}	0.5	10	5.117	> 200000
$> 10^{-5}$	0.5	10	does not converge	—
10^{-6}	0.5	10	5.117	> 200000
10^{-5}	0	10	5.117	> 200000
$> 10^{-5}$	0	10	does not converge	—
10^{-6}	0	10	7.00	7715
10^{-5}	−0.5	10	does not converge	—
$> 10^{-5}$	−0.5	10	does not converge	—
10^{-5}	0.5	1	5.117	> 200000
$> 10^{-5}$	0.5	1	does not converge	—
10^{-6}	0.5	1	5.117	> 200000
10^{-5}	0	1	does not converge	
$> 10^{-5}$	0	1	does not converge	—
10^{-6}	0	1	5.117	> 200000
10^{-5}	−0.5	1	does not converge	—
$>10^{-5}$	−0.5	1	does not converge	—

$$\omega = P_W\Bigg(\omega + \alpha\Bigg(-1 + \int_0^1 \Big(\psi_1(\tau,\sigma)\cos y_3(\tau,\sigma) + \psi_2(\tau,\sigma)\sin y_3(\tau,\sigma) + \psi_3(\tau,\sigma)v(\tau)\Big)d\tau\Bigg)\Bigg), \quad \alpha > 0.$$

To solve this fixed-point problem, consider the following explicit iteration process for $k \geq 0$ with a given initial approximation $\sigma^0 = (v^0, \omega^0) \in \Omega$:

$$v^{k+1}(\tau) = P_U\Big(v^k(\tau) + \alpha\omega\psi_3\big(\tau,\sigma^k\big)\Big), \quad \tau \in [0,1],$$

$$\omega^{k+1} = P_W\Bigg(\omega^k + \alpha\Bigg(-1 + \int_0^1 \Big(\psi_1\big(\tau,\sigma^k\big)\cos y_3\big(\tau,\sigma^k\big) + \psi_2\big(\tau,\sigma^k\big)\sin y_3\big(\tau,\sigma^k\big) + \psi_3\big(\tau,\sigma^k\big)v^k(\tau)\Big)d\tau\Bigg)\Bigg), \quad \alpha > 0.$$

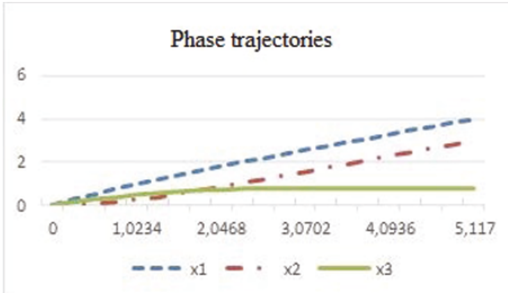

Fig. 1. Computed optimal trajectory $x(t)$.

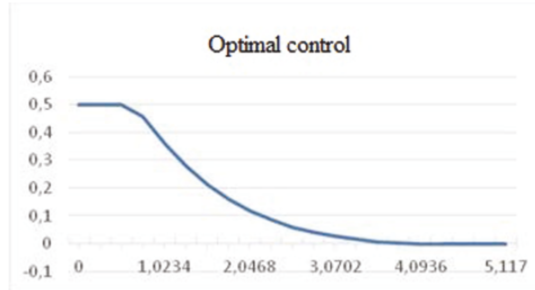

Fig. 2. Computed optimal control $u(t)$.

The numerical solution of the phase and adjoint Cauchy problems was performed by the Runge–Kutta–Werner method of the fifth or sixth order using the IMSL library of Fortran PowerStation 4.0 language. The values of the control, phase, and adjoint variables were stored at the nodes of a fixed uniform grid T_h with a sampling step $h = 10^{-3}$. On the intervals between neighboring grid nodes T_h, the value of the control function was taken constant and equal to the value at the left node. The computations were performed until the following condition was fulfilled:

$$\left|\Phi(\sigma^{k+1}) - \Phi(\sigma^k)\right| \le \left|\Phi(\sigma^k)\right| \cdot \varepsilon,$$

where $\varepsilon > 0$ is the given accuracy. As $\varepsilon > 0$, the value of "computer epsilon" (see [7]) was taken; for double precision computations, it is equal to 10^{-16}.

Table 1 shows the comparative results of computing the Zermelo problem using the proposed fixed-point method with the methods from [4] denoted by the numbers from 1 to 8. The table shows the computed values of the functional Φ^* and the number N of phase and adjoint Cauchy problems.

Table 2 shows the results of simulation by the fixed-point algorithm for various values of the projection parameter $\alpha > 0$ and the admissible initial approximation $\sigma^0 = (v^0, \omega^0) \in \Omega$.

Computations show that for $\alpha > 10^{-5}$ the algorithm does not converge. For $\alpha < 10^{-5}$, the convergence of the algorithm slows down significantly and the number of the Cauchy problems increases. The computational efficiency of the algorithm estimated by the number of the Cauchy problems significantly depends on the choice of the initial approximation.

Figures 1 and 2 show the computed optimal trajectories of the phase variables and the computed optimal control strategy.

The projection fixed-points method for simulation of the model Zermelo problem is characterized by relatively high computational efficiency, a fairly wide range of convergence of the iteration algorithm in the initial approximation, and ease of adjustment of the projection parameter, which determines the rate of convergence of the iteration process.

Example 2. Consider the time-optimal problem on the orientation of an aircraft, which can be reduced to the following problem by the method of penalty functionals (see [4]):

$$\begin{cases} \dot{x}_1(t) = x_3(t), & x_1(0) = 10, \\ \dot{x}_2(t) = x_4(t), & x_2(0) = 0, \\ \dot{x}_3(t) = -x_4(t) + u_1(t)\sin u_2(t), & x_3(0) = 0, \\ \dot{x}_4(t) = x_3(t) + u_1(t)\cos u_2(t), & x_4(0) = 0, \end{cases}$$

$$t \in [0, t_1], \quad 0 \le u_1(t) \le 1, \quad -\pi \le u_2(t) \le \pi, \quad u = (u_1, u_2),$$

$$G(u, t_1) = t_1 + 1000\Big(x_1^2(t_1) + x_2^2(t_1) + x_3^2(t_1) + x_4^2(t_1)\Big) \to \min .$$

Using the substitution of time

$$t(\tau) = t_0 + (t_1 - t_0)\tau = \omega\tau, \quad t_0 = 0, \quad \tau \in [0,1], \quad \omega = t_1 \ge 0,$$
$$v(\tau) = u\big(t(\tau)\big), \quad y(\tau) = x\big(t(\tau)\big), \quad \sigma = \big(v, \omega\big)$$

one can reduce this problem to the following problem with fixed time and mixed control functions and parameters, which is nonlinear with respect to the control:

$$\begin{cases} \dot y_1(\tau) = \omega y_3(\tau), & y_1(0) = 10, \\ \dot y_2(\tau) = \omega y_4(\tau), & y_2(0) = 0, \\ \dot y_3(\tau) = -\omega y_4(\tau) + \omega v_1(\tau)\sin v_2(\tau), & y_3(0) = 0, \\ \dot y_4(\tau) = \omega y_3(\tau) + \omega v_1(\tau)\cos v_2(\tau), & y_4(0) = 0, \end{cases}$$
$$\Phi(\sigma) = \omega + 1000\Big(y_1^2(1) + y_2^2(1) + y_3^2(1) + y_4^2(1)\Big) \to \min,$$
$$T = [0,1], \quad U = \big\{u = (u_1, u_2) : \ 0 \le u_1 \le 1, \ -\pi \le u_2 \le \pi\big\}, \quad W = \big\{\omega : \ \omega \ge 0\big\}.$$

The Pontryagin function for this problem is

$$H\big(p, y, v, \omega, \tau\big) = \omega\Big((p_1 + p_4)y_3 + (p_2 - p_3)y_4 + \big(p_3 \sin v_2 + p_4 \cos v_2\big)v_1\Big).$$

Due to the linearity of the Pontryagin function with respect to the variable $y(\tau)$, according to (10)–(13), the modified differential algebraic adjoint system takes the following form:

$$\begin{cases} \dot p_1(\tau) = 0, & p_1(1) = -2000y_1(1) - q_1, \\ \dot p_2(\tau) = 0, & p_2(1) = -2000y_2(1) - q_2, \\ \dot p_3(\tau) = -\omega\big(p_1(\tau) + p_4(\tau)\big), & p_3(1) = -2000y_3(1) - q_3, \\ \dot p_4(\tau) = -\omega\big(p_2(\tau) - p_3(\tau)\big), & p_4(1) = -2000y_4(1) - q_4, \end{cases}$$

where $q = \big(q_1, q_2, q_3, q_4\big)$ is defined by the equation

$$\begin{aligned} &1000\Big(z_1^2(1) - y_1^2(1) + z_2^2(1) - y_2^2(1) + z_3^2(1) - y_3^2(1) + z_4^2(1) - y_4^2(1)\Big) \\ &\quad = \big(2000y_1(1) + q_1\big)\big(z_1(1) - y_1(1)\big) + \big(2000y_2(1) + q_2\big)\big(z_2(1) - y_2(1)\big) \\ &\qquad + \big(2000y_3(1) + q_3\big)\big(z_3(1) - y_3(1)\big) + \big(2000y_4(1) + q_4\big)\big(z_4(1) - y_4(1)\big). \end{aligned}$$

We choose $q = (q_1, q_2, q_3, q_4)$ as follows:

1. If $z_1(1) \ne y_1(1)$, then $q_2 = 0$, $q_3 = 0$, $q_4 = 0$, and

$$\begin{aligned} q_1 = {}& \frac{1000\Big(z_1^2(1) - y_1^2(1) + z_2^2(1) - y_2^2(1) + z_3^2(1) - y_3^2(1) + z_4^2(1) - y_4^2(1)\Big)}{z_1(1) - y_1(1)} \\ & - \frac{2000y_2(1)(z_2(1) - y_2(1))}{z_1(1) - y_1(1)} - \frac{2000y_3(1)(z_3(1) - y_3(1))}{z_1(1) - y_1(1)} - \frac{2000y_4(1)(z_4(1) - y_4(1))}{z_1(1) - y_1(1)} \\ & - 2000y_1(1); \end{aligned}$$

2. If $z_1(1) = y_1(1)$, then:

2.1. If $z_2(1) \ne y_2(1)$, then $q_1 = 0$, $q_3 = 0$, $q_4 = 0$, and

$$\begin{aligned} q_2 = {}& \frac{1000\Big(z_1^2(1) - y_1^2(1) + z_2^2(1) - y_2^2(1) + z_3^2(1) - y_3^2(1) + z_4^2(1) - y_4^2(1)\Big)}{z_2(1) - y_2(1)} \\ & - \frac{2000y_3(1)(z_3(1) - y_3(1))}{z_2(1) - y_2(1)} - \frac{2000y_4(1)(z_4(1) - y_4(1))}{z_2(1) - y_2(1)} - 2000y_2(1); \end{aligned}$$

2.2. If $z_2(1) = y_2(1)$, then:

2.2.1. If $z_3(1) \neq y_3(1)$, then $q_1 = 0$, $q_2 = 0$, $q_4 = 0$, and

$$q_3 = \frac{1000\Big(z_1^2(1) - y_1^2(1) + z_2^2(1) - y_2^2(1) + z_3^2(1) - y_3^2(1) + z_4^2(1) - y_4^2(1)\Big)}{z_3(1) - y_3(1)} - \frac{2000 y_4(1)(z_4(1) - y_4(1))}{z_3(1) - y_3(1)} - 2000 y_3(1);$$

2.2.2. If $z_3(1) = y_3(1)$, then:

2.2.2.1. If $z_4(1) \neq y_4(1)$, then $q_1 = 0$, $q_2 = 0$, $q_3 = 0$, and

$$q_4 = \frac{1000\Big(z_1^2(1) - y_1^2(1) + z_2^2(1) - y_2^2(1) + z_3^2(1) - y_3^2(1) + z_4^2(1) - y_4^2(1)\Big)}{z_4(1) - y_4(1)} - 2000 y_4(1);$$

2.2.2.2. If $z_4(1) = y_4(1)$, then $q_1 = 0$, $q_2 = 0$, $q_3 = 0$, and $q_4 = 0$.

For admissible controls $\sigma \in \Omega$ and $\sigma^{\mathrm{I}} \in \Omega$, we denote by $p(\tau, \sigma^{\mathrm{I}}, \sigma)$, $\tau \in [0,1]$, the solution of the modified adjoint system for $y(\tau) = y(\tau, \sigma^{\mathrm{I}})$, $z(\tau) = y(\tau, \sigma)$, $v(\tau) = v^{\mathrm{I}}(\tau)$, and $\omega = \omega^{\mathrm{I}}$.

Due to (14)–(18), the system of conditions for improvement of an admissible control $\sigma^{\mathrm{I}} \in \Omega$ for $\alpha > 0$ in the form of a fixed-point problem is a follows:

$$\begin{aligned}\big(v_1(\tau), v_2(\tau)\big) = P_U\Bigg(& v_1^{\mathrm{I}}(\tau) + \alpha\Big(\omega^{\mathrm{I}}\Big(p_3\big(\tau, \sigma^{\mathrm{I}}, \sigma\big) \sin v_2^{\mathrm{I}}(\tau) + p_4\big(\tau, \sigma^{\mathrm{I}}, \sigma\big) \cos v_2^{\mathrm{I}}(\tau)\Big) + s_1(\tau)\Big),\\ & v_2^{\mathrm{I}}(\tau) + \alpha\Big(\omega^{\mathrm{I}} v_1^{\mathrm{I}}(\tau)\Big(p_3\big(\tau, \sigma^{\mathrm{I}}, \sigma\big) \cos v_2^{\mathrm{I}}(\tau) - p_4\big(\tau, \sigma^{\mathrm{I}}, \sigma\big) \sin v_2^{\mathrm{I}}(\tau)\Big) + s_2(\tau)\Big)\Bigg) \quad \tau \in [0,1],\end{aligned}$$

$$\begin{aligned}\omega = P_W\Bigg(\omega^{\mathrm{I}} + \alpha\Bigg(-1 + \int_T \Bigg(& \Big(p_1\big(\tau, \sigma^{\mathrm{I}}, \sigma\big) + p_4\big(\tau, \sigma^{\mathrm{I}}, \sigma\big)\Big) y_3(\tau, \sigma)\\ & + \Big(p_2\big(\tau, \sigma^{\mathrm{I}}, \sigma\big) - p_3\big(\tau, \sigma^{\mathrm{I}}, \sigma\big)\Big) y_4(\tau, \sigma)\\ & + v_1(\tau)\Big(p_3\big(\tau, \sigma^{\mathrm{I}}, \sigma\big) \sin v_2(\tau) + p_4\big(\tau, \sigma^{\mathrm{I}}, \sigma\big) \cos v_2(\tau)\Big)\Bigg) d\tau\Bigg)\Bigg);\end{aligned}$$

here $s(\tau) = \big(s_1(\tau), s_2(\tau)\big)$ is defined by the algebraic equation

$$\begin{aligned}& \omega^{\mathrm{I}} v_1(\tau)\Big(p_3\big(\tau, \sigma^{\mathrm{I}}, \sigma\big) \sin v_2(\tau) + p_4\big(\tau, \sigma^{\mathrm{I}}, \sigma\big) \cos v_2(\tau)\Big)\\ & \qquad - \omega^{\mathrm{I}} v_1^{\mathrm{I}}(\tau)\Big(p_3\big(\tau, \sigma^{\mathrm{I}}, \sigma\big) \sin v_2^{\mathrm{I}}(\tau) + p_4\big(\tau, \sigma^{\mathrm{I}}, \sigma\big) \cos v_2^{\mathrm{I}}(\tau)\Big)\\ & = \Big(\omega^{\mathrm{I}}\Big(p_3\big(\tau, \sigma^{\mathrm{I}}, \sigma\big) \sin v_2^{\mathrm{I}}(\tau) + p_4\big(\tau, \sigma^{\mathrm{I}}, \sigma\big) \cos v_2^{\mathrm{I}}(\tau)\Big) + s_1(\tau)\Big)\big(v_1(\tau) - v_1^{\mathrm{I}}(\tau)\big)\\ & \qquad + \Big(\omega^{\mathrm{I}} v_1^{\mathrm{I}}(\tau)\Big(p_3\big(\tau, \sigma^{\mathrm{I}}, \sigma\big) \cos v_2^{\mathrm{I}}(\tau) - p_4\big(\tau, \sigma^{\mathrm{I}}, \sigma\big) \sin v_2^{\mathrm{I}}(\tau)\Big) + s_2(\tau)\Big)\big(v_2(\tau) - v_2^{\mathrm{I}}(\tau)\big).\end{aligned}$$

We use the following method of searching for the value $s(\tau) = (s_1(\tau), s_2(\tau))$ from the algebraic equation:

1. If $v_1(\tau) \neq v_1^{\mathrm{I}}(\tau)$, then $s_2(\tau) = 0$ and

$$s_1(\tau) = \frac{\omega^{\mathrm{I}} v_1(\tau)\Big(p_3\big(\tau, \sigma^{\mathrm{I}}, \sigma\big)\sin v_2(\tau) + p_4\big(\tau, \sigma^{\mathrm{I}}, \sigma\big)\cos v_2(\tau)\Big)}{v_1(\tau) - v_1^{\mathrm{I}}(\tau)} - \frac{\omega^{\mathrm{I}} v_1^{\mathrm{I}}(\tau)\Big(p_3\big(\tau, \sigma^{\mathrm{I}}, \sigma\big)\sin v_2^{\mathrm{I}}(\tau) + p_4\big(\tau, \sigma^{\mathrm{I}}, \sigma\big)\cos v_2^{\mathrm{I}}(\tau)\Big)}{v_1(\tau) - v_1^{\mathrm{I}}(\tau)} - \frac{\omega^{\mathrm{I}} v_1^{\mathrm{I}}(\tau)\Big(p_3\big(\tau, \sigma^{\mathrm{I}}, \sigma\big)\cos v_2^{\mathrm{I}}(\tau) - p_4\big(\tau, \sigma^{\mathrm{I}}, \sigma\big)\sin v_2^{\mathrm{I}}(\tau)\Big)\big(v_2(\tau) - v_2^{\mathrm{I}}(\tau)\big)}{v_1(\tau) - v_1^{\mathrm{I}}(\tau)} - \omega^{\mathrm{I}}\Big(p_3\big(\tau, \sigma^{\mathrm{I}}, \sigma\big)\sin v_2^{\mathrm{I}}(\tau) + p_4\big(\tau, \sigma^{\mathrm{I}}, \sigma\big)\cos v_2^{\mathrm{I}}(\tau)\Big);$$

2. If $v_1(\tau) = v_1^{\mathrm{I}}(\tau)$, then

2.1. If $v_2(\tau) \neq v_2^{\mathrm{I}}(\tau)$, then $s_1(\tau) = 0$ and

$$s_2(\tau) = \frac{\omega^{\mathrm{I}} v_1(\tau)\Big(p_3\big(\tau, \sigma^{\mathrm{I}}, \sigma\big)\sin v_2(\tau) + p_4\big(\tau, \sigma^{\mathrm{I}}, \sigma\big)\cos v_2(\tau)\Big)}{v_2(\tau) - v_2^{\mathrm{I}}(\tau)} - \frac{\omega^{\mathrm{I}} v_1^{\mathrm{I}}(\tau)\Big(p_3\big(\tau, \sigma^{\mathrm{I}}, \sigma\big)\sin v_2^{\mathrm{I}}(\tau) + p_4\big(\tau, \sigma^{\mathrm{I}}, \sigma\big)\cos v_2^{\mathrm{I}}(\tau)\Big)}{v_2(\tau) - v_2^{\mathrm{I}}(\tau)} - \omega^{\mathrm{I}} v_1^{\mathrm{I}}(\tau)\Big(p_3\big(\tau, \sigma^{\mathrm{I}}, \sigma\big)\cos v_2^{\mathrm{I}}(\tau) - p_4\big(\tau, \sigma^{\mathrm{I}}, \sigma\big)\sin v_2^{\mathrm{I}}(\tau)\Big);$$

2.2. If $v_2(\tau) = v_2^{\mathrm{I}}(\tau)$, then $s_1(\tau) = 0$ and $s_2(\tau) = 0$.

This fixed-point problem can be used by the explicit iterative algorithm:

$$\big(v_1^{k+1}(\tau), v_2^{k+1}(\tau)\big) = P_U\Bigg(v_1^{\mathrm{I}}(\tau) + \alpha\Big(\omega^{\mathrm{I}}\Big(p_3\big(\tau, \sigma^{\mathrm{I}}, \sigma^k\big)\sin v_2^{\mathrm{I}}(\tau) + p_4\big(\tau, \sigma^{\mathrm{I}}, \sigma^k\big)\cos v_2^{\mathrm{I}}(\tau)) + s_1^k(\tau)\Big), \quad v_2^{\mathrm{I}}(\tau) + \alpha\Big(\omega^{\mathrm{I}} v_1^{\mathrm{I}}(\tau)\Big(p_3\big(\tau, \sigma^{\mathrm{I}}, \sigma^k\big)\cos v_2^{\mathrm{I}}(\tau) - p_4\big(\tau, \sigma^{\mathrm{I}}, \sigma^k\big)\sin v_2^{\mathrm{I}}(\tau)\Big) + s_2^k(\tau)\Big)\Bigg), \quad \tau \in [0,1],$$

$$\omega^{k+1} = P_W\Bigg(\omega^{\mathrm{I}} + \alpha\bigg(-1 + \int_T \bigg(\Big(p_1\big(\tau, \sigma^{\mathrm{I}}, \sigma^k\big) + p_4\big(\tau, \sigma^{\mathrm{I}}, \sigma^k\big)\Big)y_3\big(\tau, \sigma^k\big) + \Big(p_2\big(\tau, \sigma^{\mathrm{I}}, \sigma^k\big) - p_3\big(\tau, \sigma^{\mathrm{I}}, \sigma^k\big)\Big)y_4\big(\tau, \sigma^k\big) + v_1^k(\tau)\Big(p_3\big(\tau, \sigma^{\mathrm{I}}, \sigma^k\big)\sin v_2^k(\tau) + p_4\big(\tau, \sigma^{\mathrm{I}}, \sigma^k\big)\cos v_2^k(\tau)\Big)\bigg)d\tau\bigg)\Bigg),$$

where $s^k(\tau) = (s_1^k(\tau), s_2^k(\tau))$ is defined by the corresponding algebraic equation

$$\omega^{\mathrm{I}} v_1^k(\tau)\Big(p_3\big(\tau, \sigma^{\mathrm{I}}, \sigma^k\big)\sin v_2^k(\tau) + p_4\big(\tau, \sigma^{\mathrm{I}}, \sigma^k\big)\cos v_2^k(\tau)\Big) - \omega^{\mathrm{I}} v_1^{\mathrm{I}}(\tau)\Big(p_3\big(\tau, \sigma^{\mathrm{I}}, \sigma^k\big)\sin v_2^{\mathrm{I}}(\tau) + p_4\big(\tau, \sigma^{\mathrm{I}}, \sigma^k\big)\cos v_2^{\mathrm{I}}(\tau)\Big)$$
$$= \Big(\omega^{\mathrm{I}}\Big(p_3\big(\tau, \sigma^{\mathrm{I}}, \sigma^k\big)\sin v_2^{\mathrm{I}}(\tau) + p_4\big(\tau, \sigma^{\mathrm{I}}, \sigma^k\big)\cos v_2^{\mathrm{I}}(\tau)\Big) + s_1^k(\tau)\Big)\big(v_1^k(\tau) - v_1^{\mathrm{I}}(\tau)\big) + \Big(\omega^{\mathrm{I}} v_1^{\mathrm{I}}(\tau)\Big(p_3\big(\tau, \sigma^{\mathrm{I}}, \sigma^k\big)\cos v_2^{\mathrm{I}}(\tau) - p_4\big(\tau, \sigma^{\mathrm{I}}, \sigma^k\big)\sin v_2^{\mathrm{I}}(\tau)\Big) + s_2^k(\tau)\Big)\big(v_2^k(\tau) - v_2^{\mathrm{I}}(\tau)\big),$$

which is similar to the rule for $s(\tau)$ indicated above. The projector $P_U(z)$ onto the set $U = \{u = (u_1, u_2): \ 0 \le u_1 \le 1, \ -\pi \le u_2 \le \pi\}$ is implemented analytically in the form of a simple conditional formula.

If in the solution of the fixed-point problem, the condition of the first improvement of the control $\sigma^{\mathrm{I}} \in \Omega$ is fulfilled,

$$\Phi(\sigma^{k+1}) + \varepsilon_1 < \Phi(\sigma^{\mathrm{I}}),$$

where $\varepsilon_1 \ge 0$ is a given accuracy of the improvement of the control, then one constructs a subsequent fixed-point problem for improving the control obtained and repeats the iterative algorithm.

If an improvement does not occur, then one repeats the numerical solution of the fixed-point problem until the following condition was fulfilled:

$$\left\|\sigma^{k+1} - \sigma^k\right\|_{T_h} \le \varepsilon_2,$$

where $\varepsilon_2 > 0$ is a given accuracy for the fixed-point problem. This completes the process of construction and computation of consecutive problems for control improvement.

For comparing the results of computing the original problem using the fixed-point approach proposed with the methods used [4], similar computation conditions were chosen: $h = 10^{-2}$, $\varepsilon_1 = 10^{-7}$, $\varepsilon_2 = 10^{-10}$,

$$\left\|\sigma^{k+1} - \sigma^k\right\|_{T_h} = \max\left\{\left|\omega^{k+1} - \omega^k\right|, \ \left|v^{k+1}(t) - v^k(t)\right|, \ t \in T_h\right\}.$$

In Table 3, we compare results obtained by approaches based on the fixed-point algorithm with multi-method iterative technologies developed in [4].

in Table 3, we use the notation $\Delta\Phi^* = \left|\Phi^* - 10.285456\right|$, where Φ^* is the computed value of the functional and N is the number of phase and adjoint Cauchy problems.

Table 4 shows the results obtained by the fixed-point method for $\alpha = 10^{-6}$ and various initial approximations. At $N \ge 400000$, the computation is stopped.

Analysis of the proposed fixed-point algorithm demonstrates patterns similar to the first model example depending on the convergence of the algorithm on the projection parameter $\alpha > 0$ and initial approximations.

Experiments on model time-optimal problems show that the computational and qualitative efficiency of the proposed fixed-point approach is acceptable for practice in comparison with known methods (see [4]). The developed nonlocal approach to searching for approximate optimal solutions has a fairly wide range of convergence in the initial approximation and is characterized by the convenience and simplicity of experimental adjustment of the scalar projection parameter, which regulates the quality and rate of convergence of the iterative process considered. Approximate optimal solutions to time-optimal problems obtained by using the approach proposed can be considered as acceptable initial approximations for further iterative refinement by other methods.

4. Conclusion. The developed fixed-point algorithms for searching for extremal controls do not guarantee relaxation with respect to the objective functional, unlike gradient methods, but compensate for this property by the nonlocality of successive control approximations due to the fixedness of the parameter; absence of the operation of varying the control in a neighborhood of the current approximation. The constructed fixed-point algorithms for improving control in the considered class of optimization problems possess the nonlocality property and the absence of a procedure for varying the improving control in a sufficiently small neighborhood of the control being improved, characteristic of standard gradient methods, as well as the possibility of strictly improving nonoptimal extremal controls. This possibility appears in the case of nonunique solution to the fixed-point problem. Gradient methods do not have this capability.

In general, optimization of controls based on the calculation of constructed fixed-point problems by using the iterative methods of successive approximations is reduced to the alternating solution of Cauchy problems for phase and adjoint variables.

Table 3.

Method	$\Delta\Phi^*$	N
1	$7.0-1$	100002
2	$7.1-1$	103191
3	$2.4-2$	100146
4	$5.3+1$	100069
5	$6.7-1$	100159
6	$6.8+2$	4978
7	$6.6-1$	100003
8	does not converge	—

Table 4.

ω^0	v_1^0	v_2^0	$\Delta\Phi^*$	N
1	1	1	does not converge	—
5	1	1	$1.4-1$	12181
10	1	1	$3.9-3$	11805
11	1	1	$7.0-1$	125925
15	1	1	4.7	137377
1	0.5	-1	$1.4-1$	12713
5	0.5	-1	$1.4-1$	12201
10	0.5	-1	$3.9-3$	11583
11	0.5	-1	$6.7-1$	206623
15	0.5	-1	4.5	400000
1	0.5	1	does not converge	—
5	0.5	1	$1.4-1$	12207
10	0.5	1	$5.1-3$	11117
11	0.5	1	$7.0-1$	177871
15	0.5	1	4.5	400000
1	0.5	0.5	does not converge	—
5	0.5	0.5	$1.4-1$	12183
10	0.5	0.5	$1.1-2$	8907
11	0.5	0.5	$6.5-1$	260473
15	0.5	0.5	4.5	400000
1	0.5	-0.5	$1.4-1$	12713
5	0.5	-0.5	$1.4-1$	12197
10	0.5	-0.5	$3.9-3$	11727
11	0.5	-0.5	$7.1-1$	87367
15	0.5	-0.5	4.5	400000

The indicated properties of the fixed-point methods proposed are important factors in increasing the computational and qualitative efficiency of solving optimal control problems; they determine the promising direction of development of optimization methods for nonlinear dynamic systems.

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COMPLIANCE WITH ETHICAL STANDARDS

Conflict of interests. The author declares no conflict of interest.

Funding. This work was supported by the Russian Foundation for Basic Research (project No. 18-41-030005ra) and the Ministry of Education and Science of the Russian Federation (project No. 1.5049.2017/BC)..

Financial and non-financial interests. The author has no relevant financial or non-financial interests to disclose.

A. S. Buldaev
D. Banzarov Buryat State University, Ulan-Ude
E-mail: buldaev@mail.ru