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SOLUTION OF THE PROBLEM OF STRESSED STATE FOR A CLOSED ELASTOPLASTIC CYLINDRICAL SHELL CONTAINING A CRACK IN THE COMPLEX FORM

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To study the stressed state and limit equilibrium of a closed elastoplastic cylindrical shell containing a plane longitudinal internal crack of any configuration, we use an analog of the δ_c-model and represent the resolving system of equations of the problem in the complex form. The obtained system of equations is reduced to a system of nonlinear singular integral equations whose solution is constructed by the method of mechanical quadratures with regard for the conditions of plasticity of thin shells, the conditions of boundedness of stresses, and the conditions of uniqueness of displacements. We also perform the numerical analysis of the dependences of the crack opening displacements and the sizes of plastic zones on the boundary conditions imposed on the shell edges, on the configuration of the crack, and on the geometric and mechanical parameters.

Keywords: stressed state, closed elastoplastic cylindrical shell, δ_c-model, complex form of equations, parabolic crack, fracture criterion.

Introduction

The analysis of the available literature shows that the stressed state and limit equilibrium of the shells containing cracks are usually investigated on the basis of equations of the theory of thin shells in displacements. At the same time, it proves to be more reasonable to construct the solutions of a series of problems of the linear theory of shells by the complex method proposed by Novozhilov [13] and developed by Chernykh in [18]. These equations are more compact than the equations of the theory of shells in forces and moments or displacements. Moreover, a twofold decrease in the order of resolving equations as a result of the complex transformation significantly simplifies the procedure of construction of analytic solutions of the problems and reduces the time required for the numerical analyses of specific problems. This, in particular, was demonstrated in [1, 2, 12].

By analyzing the works dealing with the theory of shells with cracks, it is possible to conclude that the accumulated results on the stressed state and limit equilibrium of shells were, as a rule, obtained for the case of infinite shells. As an exception, we can mention the works [16, 19] in which closed shells were investigated in the elastic and elastoplastic statements.

In the present work, we study the limit equilibrium of an isotropic elastoplastic closed shell.

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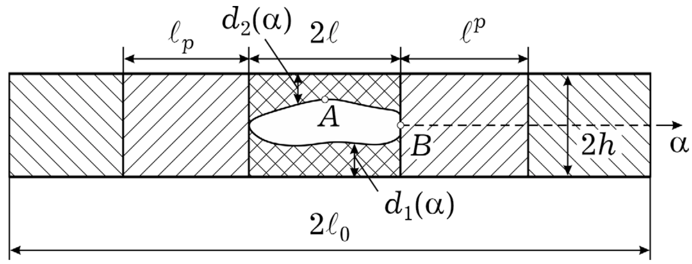

Fig. 1

1. Formulation of the Problem

Consider a closed elastoplastic cylindrical shell with a thickness $2h$ and a length $2\ell_0$.

The shell is referred to a triorthogonal coordinate system $O\alpha\beta\gamma$. Assume that the shell is weakened by a longitudinal internal crack of length 2ℓ located in the cross section $\beta = 0$ and bounded by continuous curves $d_1(\alpha)$ and $d_2(\alpha)$ (see Fig. 1) that establish the distance from the crack boundary to the outer and inner surfaces of the shell. It is assumed that the shell and the crack faces are loaded solely by forces and moments symmetric about the crack plane. In the process of deformation, the crack faces do not contact. We restrict ourselves to the case of sufficiently deep cracks: $d_3 = d_1 + d_2 \le 0.6h$. The crack sizes, the level of external loading, and the properties of the material are assumed to be such that, in the vicinity of the crack, plastic deformations develop within a narrow band over the entire thickness of the shell. Thus, according to an analog of the δ_c-model [3, 4, 20], we replace the zones of plastic strains by the surfaces of discontinuity of elastic displacements and the angles of rotation. Moreover, the response of the material of the plastic zone to the elastic one is replaced by the corresponding forces and moments.

We also assume that constant stresses $\sigma^0 = (\sigma_B + \sigma_T)/2$, where σ_B and σ_T are, respectively, the ultimate strength and the yield strength of the material, act on the continuation of the crack in depth toward the outer and inner surfaces of the shell, i.e., in the region $\alpha \in (-\alpha_0, \alpha_0)$, $\alpha_0 = \ell_1/R$, $\gamma \in [-h, -h + 2d_1] \cup [h - 2d_2, h]$ (R is the radius of the middle surface of the shell). Further, suppose that unknown normal forces $N^{(i)}$ and bending moments $M^{(i)}$, $i = 1, 2$, act in the plastic zones on the continuation of the crack along its length, i.e., in the domains $\gamma \in [-h, h]$, $\alpha \in (-\alpha_p, -\alpha_0) \cup (\alpha_0, \alpha^p)$ (here, $\alpha_p = \ell_p/R$, $\alpha^p = \ell^p/R$, and ℓ_p and ℓ^p are the lengths of the plastic zones located to the left and to the right of the crack) and that, for perfectly elastoplastic materials, these forces and moments satisfy one of the Tresca plasticity conditions:

either the condition of plasticity of the surface layer

$$\frac{N}{2h\sigma_T} + \frac{3|M|}{2h^2\sigma_T} = 1, \tag{1}$$

or the condition of plastic hinge [6, 7]

$$\left(\frac{N}{2h\sigma_T}\right)^2 + \frac{|M|}{h^2\sigma_T} = 1. \tag{2}$$

Thus, within the framework of the accepted analog of the δ_c-model, the three-dimensional elastoplastic problem for an internal crack of length 2ℓ is replaced by a two-dimensional elastic problem for a fictitious through crack of unknown length $2\ell_1$ $(2\ell_1 = 2\ell + \ell^{\mathrm{p}} + \ell_{\mathrm{p}})$ with the following conditions imposed on its faces:

$$N_2(\alpha) = \begin{cases} N_2^{(1)} - N^{\ell} - N_2^0, & |\alpha| < \alpha_0, \\ N - N_2^0, & -\alpha_0 < \alpha < -\alpha_1, \end{cases}$$

$$M_2(\alpha) = \begin{cases} M_2^{(1)} - M^{\ell} - M_2^0, & |\alpha| < \alpha_0, \\ M - M_2^0, & -\alpha_0 < \alpha < -\alpha_1. \end{cases} \tag{3}$$

Here, N^{ℓ} and M^{ℓ} are the normal force and bending moment, which describe the response of the material to breaking of the internal bonds over and under the crack. According to the accepted assumptions about stresses acting in these zones, these characteristics are determined by the formulas

$$N^{\ell} = 2d_3\sigma^0, \qquad M^{\ell} = 2\sigma^0(h - d_3)(d_2 - d_1),$$

where $N_2^{(1)}$ and $M_2^{(1)}$ are, respectively, the forces and moments applied to the crack faces and N_2^0 and M_2^0 are the same parameters for the main stressed state (the shell without cracks).

2. Main Relations for the Cylindrical Shell Weakened by a Crack in the Complex Form

To present the basic relations of the linear theory of thin cylindrical shells that take into account the presence of cracks, we use the concept of complex static-geometric analogy [13, 14, 18] and consider the following complex forces and moments:

$$\tilde{N}_1 = N_1 - iD_0 c\chi_{22}^s, \qquad \tilde{M}_1 = M_1 + iD_0 c\varepsilon_{22}^s,$$

$$\tilde{N}_2 = N_2 - iD_0 c\chi_{11}^s, \qquad \tilde{M}_2 = M_2 + iD_0 c\varepsilon_{11}^s, \tag{4}$$

$$\tilde{S} = S + iD_0 c\chi_{12}^s, \qquad \tilde{H} = H - iD_0 c\frac{\varepsilon_{12}^s}{2},$$

where $D_0 = 2Eh$, $c = h/\sqrt{3(1-\nu^2)}$, $i = \sqrt{-1}$, and ε_{ij}^s, χ_{ij}^s, $i, j = 1, 2$, are the components of elastic strains.

Multiplying the equations of continuity of strains by $-iD_0c$ and adding them to the equilibrium equations, we arrive at the balance equations for a thin cylindrical shell in complex forces and moments:

$$L_i\{\tilde{N}_2, \tilde{N}_1, \tilde{S}, \tilde{M}_2, \tilde{M}_1, \tilde{H}\} = \tilde{q}_i, \quad i = 1, 2, 3. \tag{5}$$

Here,

$$\tilde{q}_\ell = -Rq_\ell + iD_0 c L_\ell\left\{\chi_{11}^0,\ldots,\varepsilon_{12}^0/2\right\}, \quad \ell = 1,2,$$
$$\tilde{q}_3 = Rq_3 + iD_0 c L_3\left\{\chi_{11}^0,\ldots,\varepsilon_{12}^0/2\right\}, \tag{6}$$

ε_{ij}^0 and χ_{ij}^0 are the components of stress-free strains [15] such that the components of small strains are given by the formulas $\varepsilon_{ij} = \varepsilon_{ij}^s + \varepsilon_{ij}^0$ and $\chi_{ij} = \chi_{ij}^s + \chi_{ij}^0$.

It follows from the first three relations in (4) that

$$N_\ell = \operatorname{Re}\tilde{N}_\ell, \quad \ell = 1,2, \qquad S = \operatorname{Re}\tilde{S},$$
$$\chi_{ii}^s = -\frac{1}{D_0}\operatorname{Im}\tilde{N}_j, \quad i \neq j = 1,2, \qquad \chi_{12}^s = -\frac{1}{D_0}\operatorname{Im}\tilde{S}. \tag{7}$$

By using expressions (7) and the relations of Hooke's law

$$\varepsilon_{ii}^s = \frac{1}{D_0}[N_i - \nu N_j], \qquad \chi_{ii}^s = \frac{3}{D_1}[M_i - \nu M_j], \quad i \neq j = 2,$$
$$\varepsilon_{12}^s = \frac{2(1+\nu)}{D_0}S, \qquad \chi_{12}^s = \frac{3(1+\nu)}{D_1}H, \tag{8}$$

we arrive at the following equations:

$$M_i = -c\operatorname{Im}[\tilde{N}_j + \nu\tilde{N}_i], \qquad \varepsilon_{ii}^s = \frac{1}{D_0}\operatorname{Re}[\tilde{N}_i - \nu\tilde{N}_j], \quad i \neq j = 1,2,$$
$$H = c(1 - \nu\operatorname{Im}\tilde{S}), \qquad \varepsilon_{12}^s = \frac{1+\nu}{D_0}\operatorname{Re}\tilde{S}. \tag{9}$$

Substituting (9) in the last three equations in (4), we find

$$\tilde{M}_1 = ic\left[\tilde{N}_2 - \nu\bar{\tilde{N}}_1\right], \qquad \tilde{M}_2 = ic\left[\tilde{N}_1 - \nu\bar{\tilde{N}}_2\right], \qquad \tilde{H} = -ic\left[\tilde{S} - \nu\bar{\tilde{S}}\right]. \tag{10}$$

Here and in what follows, the overbar above the tilde denotes the operation of conjugation of the corresponding complex quantities.

Thus, replacing the complex moments in Eqs. (5) by their expressions from (10), we get the following system of equations in complex forces:

$$L_\ell\left\{\tilde{N}_2, \tilde{N}_1, \tilde{S}, ic\left[\tilde{N}_1 - \nu\bar{\tilde{N}}_2\right], ic\left[\tilde{N}_2 - \nu\bar{\tilde{N}}_1\right], -ic\left[\tilde{S}_2 + \nu\tilde{S}_1\right]\right\} = \tilde{q}_\ell, \quad \ell = 1,2,3. \tag{11}$$

Equations (11) are regarded as “exact” because the "real" equations of the theory of thin shells are exact. As a disadvantage, we can mention the presence of the operation of conjugation. However, it is possible to show that the terms appearing in equations with the sign of conjugation are small and, hence, can be neglected. The other possibility is to somewhat modify the introduced complex quantities.

We now introduce the Novozhilov complex function $\tilde{N} = \tilde{N}_1 + \tilde{N}_2$. Then, after necessary transformations, relations (10) can be rewritten in the form

$$\frac{\partial \tilde{N}_1}{\partial \alpha} + \frac{\partial \tilde{S}}{\partial \beta} = \tilde{q}_1,$$

$$\frac{\partial \tilde{N}_2}{\partial \beta} + \frac{\partial \tilde{S}}{\partial \alpha} + \frac{ic}{R}\frac{\tilde{N}}{\partial \beta} = \tilde{q}_2, \tag{12}$$

$$\tilde{N}_2 - \frac{ic}{R}\Delta\tilde{N} = \tilde{q}_{30},$$

where

$$\tilde{q}_{30} = \tilde{q}_3 - \frac{ic(1-\nu)}{R}\left[\frac{\partial}{\partial \alpha}\tilde{q}_1 + \frac{\partial}{\partial \beta}\tilde{q}_2\right], \quad \Delta = \partial_1^2 + \partial_2^2, \quad \partial_1 = \frac{\partial}{\partial \alpha}, \quad \partial_2 = \frac{\partial}{\partial \beta}.$$

Eliminating the unknown $\tilde{S}$ from the balance equations, we get

$$\Delta\tilde{N}_2 = \frac{\partial^2 \tilde{N}}{\partial \alpha^2} + \frac{ic}{R}\frac{\partial^2 \tilde{N}}{\partial \beta^2} = \frac{\partial \tilde{q}_2}{\partial \beta} - \frac{\partial \tilde{q}_1}{\partial \alpha},$$

$$\tilde{N}_2 - \frac{ic}{R}\Delta\tilde{N} = \tilde{q}_{30}. \tag{13}$$

Substituting $\tilde{N}_2$ from the second equation in (13) in the first equation, we get the differential equation of the fourth order with one unknown complex function $\tilde{N}$ for a cylindrical shell subjected to the action of a field of distortions

$$\left(\Delta\Delta + \underline{\partial_2^2} + 2ib^2\partial_1^2\right)\tilde{N} = D_0\left\{L_{11}\varepsilon_{11}^0 + L_{12}\varepsilon_{12}^0 + L_{22}\varepsilon_{22}^0 + P_{11}\chi_{11}^0 + P_{12}\chi_{12}^0 + P_{22}\chi_{22}^0\right\}, \tag{14}$$

where

$$L_{11} = -\partial_2^2(\Delta + \underline{1 + \Delta\mu}), \quad L_{22} = -\Delta\partial_1^2, \quad L_{12} = -\partial_1\partial_2(\Delta + \underline{1 + \Delta\mu}),$$

$$P_{11} = -R\partial_1^2 + \mu R\Delta\partial_2^2, \quad P_{22} = -R\partial_1^2 + \mu R\Delta\partial_1^2, \quad P_{12} = 2\mu\Delta\partial_1\partial_2,$$

$$\mu = \frac{i(1-\nu)}{2b^2}, \qquad 2b^2 = \frac{R}{c}.$$

The complex forces $\tilde{N}_1$ and $\tilde{N}_2$ are expressed via the main complex function $\tilde{N}$ by the formulas

$$\tilde{N}_2 = \frac{i}{2b^2}\Delta\tilde{N} + \frac{i}{2b^2}D_0\{C_{11}\varepsilon_{11}^0 + C_{12}\varepsilon_{12}^0 + C_{22}\varepsilon_{22}^0 + D_{11}\chi_{11}^0 + D_{12}\chi_{12}^0 + D_{22}\chi_{22}^0\},$$
$$\tilde{N}_1 = \tilde{N} - \tilde{N}_2. \tag{15}$$

Here,

$$C_{11} = \partial_2^2(1+\mu), \qquad C_{12} = \partial_1\partial_2(1-\mu), \qquad C_{22} = \partial_1^2,$$
$$D_{11} = R(1+\mu\partial_2^2), \qquad D_{12} = 2\mu R\partial_1\partial_2, \qquad D_{22} = -R\mu\partial_1^2. \tag{16}$$

The system of complex equations (12) and the main key equation (14) are written for the case of the general moment theory of shells. Within the framework of the technical theory [8], the underlined terms in the system of equations (12) are omitted.

Since the shell is subjected to the action of external loads symmetric about the crack surface and the forces and moments with the same magnitude but opposite directions are applied to the opposite crack faces, in view of the fact that, in passing through the crack, the forces and moments remain continuous functions, whereas the displacements v and angles of rotation of the normal θ have discontinuities of the first kind, i.e., are generalized functions, they can be represented in the form

$$\varepsilon_{22}^0 = \frac{[v(\alpha)]\,\delta(\beta)}{R}, \qquad \chi_{22}^0 = \frac{[\theta_2(\alpha)]\,\delta(\beta)}{R} - \frac{[\omega(\alpha)]\,\delta(\beta)}{R^2}, \tag{17}$$

where

$$[v(\alpha)] = v^+(\alpha) - v^-(\alpha), \quad [\theta(\alpha)] = \theta^+(\alpha) - \theta^-(\alpha), \quad \text{and} \quad [\omega(\alpha)] = \omega^+(\alpha) - \omega^-(\alpha).$$

We restrict ourselves to the investigation of the perturbed stressed state of a cylindrical shell under the action of symmetric forces within the framework of the technical theory of thin shells. As the source system, we use the system of balance equations in the complex form. Reducing this system to as single key equation for the complex function $\tilde{\Phi}$ related to the Novozhilov function by the formula

$$\tilde{N} = -\Delta\tilde{\Phi} + iD_0c(\chi_{22}^0 + \chi^0), \qquad \chi^0 = -\frac{R\chi_{22}^0}{R+ic}, \tag{18}$$

we get the following key equation:

$$L^{\mathrm{T}}\tilde{\Phi} = D_0\left\{\frac{iR}{2b^2}\left(\partial_2^2+\nu\partial_1^2\right)\chi_{22}^0+\partial_1^2\varepsilon_{22}^0\right\}. \tag{19}$$

Here, the operator $L^{\mathrm{T}} = \partial_1^4+2\partial_1^2\partial_2^2+\partial_2^4+2ib^2\partial_1^2$.

Assume that, the following homogeneous boundary conditions are specified on the end faces of the shell $\alpha_0 = \pm\ell_0/R$:

$$P_i(\tilde{\Phi}) = 0, \quad i = 1,\ldots,4, \tag{20}$$

where restrictions accepted in the theory of thin shells [13, 14, 17] are imposed on the operators P_i.

We construct the solution of Eq. (19) in the form of the sum of the general solution of the homogeneous equation and a partial solution of the inhomogeneous equation. The indicated partial solution is represented in the following form:

$$\tilde{\Phi}^{\mathrm{pt}}(\alpha,\beta) = D_0\int_{-\alpha_0}^{\alpha_0}\left\{\frac{1}{R}\partial_1^2[v(\xi)] - \frac{i}{2b^2}\left(\nu\partial_2^2+\partial_1^2\right)[\theta_2(\xi)]\right\}\frac{k}{\pi}\sum_{n=0}^{\infty}\Phi_n(\alpha-\xi)\cos n\beta\, d\xi, \tag{21}$$

where $\Phi_n(\alpha-\xi)$ is the fundamental solution of Eq. (19).

We seek the solution of the homogeneous equation (19) with regard for the conditions of cyclic symmetry in the form

$$\tilde{\Phi}^0(\alpha,\beta) = \sum_{n=0}^{\infty}\Phi_n^0(\alpha)\cos n\beta, \tag{22}$$

where $\Phi_n^0(\alpha) = \sum_{j=0}^{2}\tilde{C}_{jn}\cosh\ell_{jn}\alpha$, $\ell_{jn} = x_{jn} - iy_{jn}$, $\tilde{C}_{jn}$ are arbitrary complex variables $\left(\tilde{C}_{1n} = C_{1n} + iC_{2n}\right.$ and $\left.\tilde{C}_{2n} = C_{3n} + iC_{4n}\right)$, and the values x_{jn} and iy_{jn}, $j = 1,2$, are the real and imaginary parts of the roots of the characteristic equation, respectively,

$$\lambda^4 + 2\lambda^2\left(n^2 - ib^2\right) + n^4 = 0.$$

Thus, in view of (21) and (22), we can write

$$\tilde{N}_i = \tilde{N}_i^{\mathrm{pt}} + \tilde{N}_i^0, \quad i = 1,2, \qquad \tilde{S} = \tilde{S}^{\mathrm{pt}} + \tilde{S}^0, \tag{23}$$

where

$$\tilde{N}_1^{\mathrm{pt}} = -\partial_2^2\tilde{\Phi}^{\mathrm{pt}} + iD_0c\chi_{22}^0, \qquad \tilde{N}_2^{\mathrm{pt}} = -\partial_1^2\tilde{\Phi}^{\mathrm{pt}}, \qquad \tilde{S}^{\mathrm{pt}} = \partial_1\partial_2\tilde{\Phi}^{\mathrm{pt}} + iD_0c\chi_{12}^0, \tag{24}$$

$$\tilde{N}_1^0 = -\partial_2^2\tilde{\Phi}^0, \qquad \tilde{N}_2^0 = -\partial_1^2\tilde{\Phi}^0, \qquad \tilde{S}^0 = \partial_1\partial_2\tilde{\Phi}^0. \tag{25}$$

By using relations (23)–(25) and satisfying the boundary conditions (20), for the determination of the unknown constants C_{jn}, $j = 1, \ldots, 4$, we obtain a system of algebraic equations of the form

$$\mathbf{A}_n \cdot \mathbf{C}_n = \mathbf{B}_n, \quad n = 1, 2, \ldots, \tag{26}$$

where $\mathbf{A}_n$ is the matrix of coefficients, $\mathbf{B}_n$ is the column vector of free terms, and $\mathbf{C}_n$ is the column vector of unknowns C_{jn}.

Substituting the obtained constants C_{jn}, $j = 1, \ldots, 4$, in (22), and using relations (23)–(25), we obtain the distributions of forces and moments at any point of the shell. Satisfying the boundary conditions on the crack faces, we reduce the problem of determination of the stressed state of a closed cylindrical shell with the boundary conditions (20) imposed on the end faces to a system of singular integral equations.

As an example, we consider the following condition of fastening of the shell:

$$v = 0, \quad w = 0, \quad \frac{dw}{d\alpha} = 0, \quad N_1 = 0 \quad \text{for} \quad \alpha = \pm\alpha_1, \quad \alpha_1 = \ell_0/R, \tag{27}$$

where u, v, and w are the displacements of the middle surface of the shell and N_1 is a normal axial force.

Conditions (27) correspond to the reinforcement of the shell on the end faces by rigid frames freely moving in the axial direction. It is known [5, 10] that these frames are used in the course of operation of the main pipelines with an aim to suppress the process of crack growth.

To determine the unknown constants C_{jn}, $j = 1, \ldots, 4$, it is necessary to satisfy the boundary conditions (27). For this purpose, we use the representation of displacements via the forces and moments

$$u = \frac{R}{D_0} \int \mathrm{Re}\,[\tilde{N}_1 - \nu \tilde{N}_2]\, d\beta + C,$$

$$v = -\int w\, d\beta + \frac{R}{D_0} \int \mathrm{Re}\,[\tilde{N}_2 - \nu \tilde{N}_1]\, d\beta + C_3, \tag{28}$$

$$w = \frac{2b^2 R}{D_0} \iint \mathrm{Im}\, \tilde{N}_2 \, d\alpha\, d\beta + C_1 \alpha + C_2.$$

It follows from the given boundary conditions (27) that, in relations (28), the constants $C_1 = C_2 = C_3 = 0$ and the displacement u determines the state of the shell with an accuracy to within the unknown constant C, which corresponds to the rigid displacements of the shell.

By using (28) and (23)–(25) and satisfying the boundary conditions (27), for the evaluation of the unknown constants C_{jn}, $j = 1, \ldots, 4$, we arrive at the following system of algebraic equations:

$$\mathbf{A}_n \cdot \mathbf{C}_n = \mathbf{B}_n, \quad n = 1, 2, \ldots . \tag{29}$$

Here, $\mathbf{A}_n$ is the matrix of the coefficients

$$\mathbf{A}_n = \begin{pmatrix} u_{1n} & u_{2n} & u_{3n} & u_{4n} \\ z_{1n} & z_{2n} & z_{3n} & z_{4n} \\ v_{1n} & v_{2n} & v_{3n} & v_{4n} \\ w_{1n} & w_{2n} & w_{3n} & w_{4n} \end{pmatrix}, \tag{30}$$

$\mathbf{C}_n$ is the column vector of the unknowns C_{jn}, $j=1,\ldots,4$, and $\mathbf{B}_n$ is the column vector of free terms

$$\mathbf{B}_n = \frac{D_0 k}{\pi} \int_{-\alpha_0}^{\alpha_0} \left\{ \frac{1}{R}\frac{d}{d\xi}[v(\xi)] \begin{bmatrix} v_n^{\varepsilon}(\alpha-\xi) \\ w_n^{\varepsilon}(\alpha-\xi) \\ \theta_n^{\varepsilon}(\alpha-\xi) \\ N_{1n}^{\varepsilon}(\alpha-\xi) \end{bmatrix} + c\frac{d}{d\xi}[\theta(\xi)] \begin{bmatrix} v_n^{\chi}(\alpha-\xi) \\ w_n^{\chi}(\alpha-\xi) \\ \theta_n^{\chi}(\alpha-\xi) \\ N_{1n}^{\chi}(\alpha-\xi) \end{bmatrix} \right\} d\xi. \tag{31}$$

Thus, by using relations (22)–(25), we can determine the stressed state of a cylindrical shell with boundary conditions imposed on the end faces, which is induced by an arbitrary distribution of the jumps of displacements and the angles of rotation along a fictitious crack. If we now satisfy the boundary conditions (3) on the faces of the fictitious crack, then we get a system of singular integral equations of the problem. Thus, in the case of free crack faces, under a load symmetric about the crack surface, the corresponding system takes the form

$$\sum_{i=1,2} \int_{-\alpha_1}^{\alpha_1} F_i(u) \left\{ \frac{a_{mi}}{u-s} + \alpha_0 \mathcal{K}_{mi}^0[\alpha_0(s-u)] \right\} du = \pi f_{m0}(s), \quad |s|<1, \quad m=1,2. \tag{32}$$

Its solution satisfies the conditions

$$\int_{-\alpha_1}^{\alpha_1} F_i(u)\, du = 0, \quad i=1,2, \tag{33}$$

where

$$F_1(u) = \frac{1}{R}\frac{d}{du}[v(u)], \qquad F_2(u) = -c\frac{d}{du}[\theta_2(u)],$$

$$f_{10}(s) = N_s(\alpha), \qquad f_{20}(s) = M_s(\alpha), \quad s = \alpha/\alpha_0.$$

The kernels of the system of singular integral equations (32) have the form

$$\mathcal{K}_{11} = \mathcal{K}_{11}^0 - \mathcal{K}_{11}^1, \quad \mathcal{K}_{12} = \mathcal{K}_{12}^0 - \mathcal{K}_{12}^1, \quad \mathcal{K}_{21} = \mathcal{K}_{12}^0 - \mathcal{K}_{21}^1, \quad \mathcal{K}_{22} = \mathcal{K}_{22}^0 - \mathcal{K}_{22}^1.$$

The values of $\mathcal{K}_{mi}^0$, $i,m=1,2$, were presented in [11] and the components of regular kernels $\mathcal{K}_{mi}^1$ characterizing the influence of boundary conditions imposed on the end faces of the shell were presented in [10]. The kernels of the obtained system of singular integral equations are continuous for the entire collection of the real values of s and u.

In the system of singular integral equations, the limits of integration α_1 are unknown because we do not know the lengths of the plastic zones ℓ_p and ℓ^p. Moreover, the right-hand sides $f_m(\alpha)$ are discontinuous functions, which contain the unknown forces N^i and moments M^i. This is why the system of integral equations should be supplemented by one of the Tresca plasticity conditions [(1) or (2)] and the conditions of boundedness of the forces and moments near the tips of fictitious crack. To do this, it suffices to assume that the intensity factors of the normal force K_N and of the bending moment K_M are equal to zero:

$$K_N(-\alpha_0-\alpha_p) = K_N(\alpha_0+\alpha^p) = K_M(-\alpha_0-\alpha_p) = K_M(\alpha_0+\alpha^p) = 0. \tag{34}$$

As a result, we get the complete system of equations for the evaluation of the jumps of displacements and the angle of rotation, the lengths of the plastic zones, and the forces and moments acting in these zones.

Integrating the obtained solution, we determine the crack opening displacements $\delta(\gamma,\alpha)$ at any point of the crack by the formula

$$\delta(\gamma,\alpha) = [v(\alpha/\alpha_1)] + \gamma[\gamma(\alpha/\alpha_1)], \quad |\alpha|<\alpha_1, \quad |\gamma|=h. \tag{35}$$

Equating the right-hand side of relation (35) to the critical value δ_{cr} of the crack-front opening displacement for the investigated material, we obtain a criterial relation connecting the ultimate load with the admissible crack sizes.

3. Numerical Results

An algorithm for the numerical solution of the analyzed nonlinear systems together with the additional conditions (1) or (2), conditions of uniqueness of displacements (33), and conditions (34) was presented in [11]. We perform the numerical analysis of the considered problem for a shell made of perfectly elastoplastic material $(\sigma_B=\sigma_T)$, fixed at both ends according to conditions (27), and weakened by an internal parabolic crack

$$\begin{aligned} d_1(\alpha) &= \frac{1}{\tau_0^2}\left(h-d_1'-d_2'\right)\alpha^2 - h + d_1', \\ d_2(\alpha) &= \frac{1}{\tau_0^2}\left(h-d_1'-d_2'\right)\alpha^2 + h - d_2', \end{aligned} \tag{36}$$

where d_1' and d_2' are the distances from the vertex of the corresponding parabola to the inner and outer surfaces of the shell, $\tau_0=\ell_0/\ell_1$, $d_1'/h=0.15$, and $d_2'/h=0.25$. Clearly, in this case, $\ell_p=\ell^p$ and, hence, $N^1=N^2$ and $M^1=M^2$.

In view (36), we determine the force $N^1(\alpha)$ and the moment $M^1(\alpha)$ as follows:

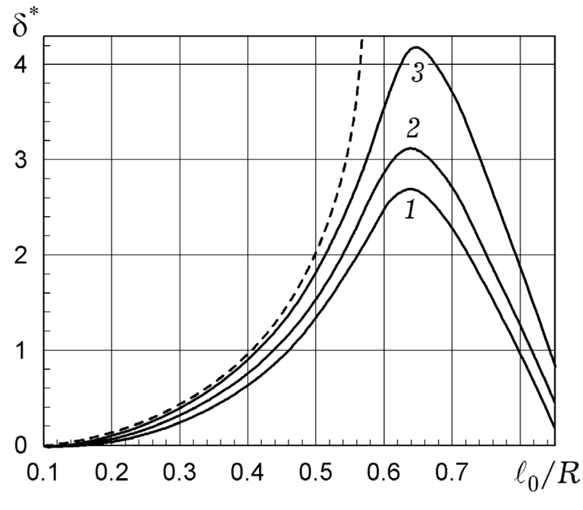

Fig. 2

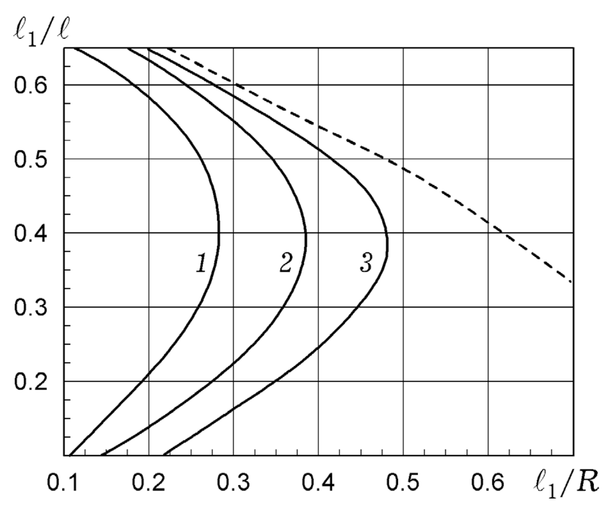

Fig. 3

$$N^1(\alpha) = \sigma_\tau(d_1+d_2) + \sigma_\tau(h-d_1-d_2)\frac{\alpha^2}{(\tau^0)^2} = F_1^1 + k_1^1\frac{\alpha^2}{(\tau^0)^2},$$

$$M^1(\alpha) = \frac{\sigma_\tau}{2}(h-d_1-d_2)(d_1-d_2)\left(1-\frac{\alpha^2}{(\tau^0)^2}\right) = F_3^1 + k_3^1\frac{\alpha^2}{(\tau^0)^2},$$

where

$$F_1^1 = N^1 = \sigma_\tau(d_1+d_2), \quad F_3^1 = M^1 = \sigma_\tau(h-d_1-d_2)(d_1-d_2)/2,$$

$$k_1^1 = \sigma_\tau(h-d_1-d_2), \quad k_3^1 = -M^1.$$

In the numerical calculations aimed at the investigation of the behavior of the crack opening displacements and the lengths of the plastic zones on the length of the parabolic crack, the sizes of the shell, and various physical and geometric parameters, we choose the following values: $R = 0.15$ m, $h = 0.15 \cdot 10^{-2}$ m, $\nu = 0.3$, and $\ell_0 = 0.15$ m.

In Fig. 2, we present the plots of dependences of the relative crack opening displacement $\delta^* = \delta(0, \ell/R)E/(\ell\sigma_T)$ on the relative length of the actual crack ℓ_0/R and the parameters of the parabolic crack d_1' and d_2'. The dashed curve corresponds to the case of an infinite shell and curves ***1–3*** correspond to the values $d_1' = 0.15, 0.20, 0.30$.

In Fig. 3, we show the plots of dependence of the relative length of plastic zones ℓ_1/ℓ on the same parameters. It is easy to see that first the relative crack opening displacements δ^* in the closed shell (curves ***1–3***) behave as in the case of an infinite shell (dashed line). However, as the parameter $\tau_0 = \ell_0 / \ell_1$ increases, they tend to zero. The same picture is observed in the case of variations of the length of plastic zones.

CONCLUSIONS

If the criterion of critical crack opening displacement is used as a fracture criterion, then, for the chosen parameters, the fracture process in the bounded shell containing an internal parabolic crack starts at the point ***A***, i.e., at the point closest to the outer surface (see Fig. 1). In the case where the internal crack has the form of a rectangle with inscribed parabolic crack, its opening displacement is also maximum at the point ***A*** but its value is 8–10 times higher.

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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