DOI 10.1007/s10958-024-06977-6
Journal of Mathematical Sciences, Vol. 278, No. 6, February, 2024

THE TWO-DIMENSIONAL BOOLE-TYPE TRANSFORM AND ITS ERGODICITY

A. K. Prykarpatski[1,2] **and A. A. Balinsky**[3] UDC 517.9

Dedicated to the commemoration of untimely passed away outstanding Ukrainian mathematician Prof. A. M. Samoilenko

Based on the Schweiger's smooth fibered approach and the related Bernoulli shift transformation scheme, we prove the ergodicity of the two-dimensional Boole-type transformations. New multidimensional Boole-type transformations invariant with respect to the Lebesgue measure and their ergodicity properties are also discussed.

1. Introduction

With its origins going back for several centuries, the discrete analysis now becomes an increasingly central methodology for many mathematical problems related to discrete dynamical systems and algorithms widely applied in modern science. Our theme related to the study of topological and measure-theoretical ergodicity aspects of Boole-type discrete dynamical systems [1–12] is of great interest in numerous branches of modern science and technology [11–25], especially in the statistical mechanics, discrete mathematics, numerical analysis, chaos theory, statistics, and probability theory, as well as in the electric and electronic engineering. From this viewpoint, the investigated topic belongs to a much more general realm of mathematics, namely, to calculus, differential equations, and differential geometry due to the remarkable analogy of the subject, especially for these branches of mathematics. Nevertheless, although the topic is discrete, our approach to treating topological and measure-theoretical ergodicity and the related arithmetic properties of generalized Boole-type discrete dynamical systems is completely analytic, which results in the proof of ergodicity of the two-dimensional Boole-type transformation.

2. Ergodicity and Bernoulli-Type Transformations

We consider a class of mappings called [5, 12, 23, 26, 27] *smooth fibered multidimensional mappings* $\varphi: X \to X$ if the following conditions are satisfied:

(a) there is an invariant Lebesgue equivalent probability measure $\mu: \mathcal{B} \to \mathbb{R}_+$ for which there exist positive constants $c_1, c_2 \in \mathbb{R}_+$ such that

$$c_1\lambda(E) \le \mu(E) \le c_2\lambda(E)$$

for every Borel set $E \subset X$;

(b) there is a family of finite or countable infinite digit sets D_j, $j = \overline{1, N}$;

[1] Kraków University of Technology, Warszawska str., 24, Kraków, 31-155, Poland; Institute of Applied Mathematics and Fundamental Science, Lviv Polytechnic National University, Mytropolyta Andreia str., 5, Lviv, 79013, Ukraine; e-mail: pryk.anat@cybergal.com.
[2] Corresponding author.
[3] Mathematical Institute, Cardiff University, Senghennydd Rd., Cardiff CF24 4AG, UK; email: BalinskyA@cardiff.ac.uk.

Published in Neliniini Kolyvannya, Vol. 26, No. 1, pp. 113–131, January–March, 2023. Original article submitted September 11, 2022.

(c) there is a mapping $k\colon X \to D$, where $D := D_1 \times D_2 \times \ldots \times D_N$, such that the sets

$$X_i := k^{-1}\{i\} = \{x \in X : k(x) = i\}, \quad i \in D,$$

are measurable and form a partition of the space X, i.e., the sets $\sqcup_{i\in D} X_i = X$;

(d) the restrictions $\varphi|_{X_i}\colon X_i \to X$, $i \in D$, are injective and smooth maps.

It is easy to see that the mapping $\varphi\colon X \to X$ is equivalent to the Bernoulli shifts mapping $T_\varphi\colon D^\infty \to D^\infty$, where

$$T_\varphi\colon (k_1, k_2, k_3, \ldots, k_n, \ldots) \to (k_2, k_3, \ldots, k_n, \ldots) \tag{2.1}$$

with respect to the isomorphism $\psi\colon X \ni x \to (k_2, k_3, \ldots, k_n, \ldots) \in D^\infty$, and

$$X(k_1, k_2, k_3, \ldots, k_n; x) \Longleftrightarrow (k_1, k_2, k_3, \ldots, k_n, \ldots), \tag{2.2}$$

determined for the rank-n cylinder sets $X_n(k_1, k_2, k_3, \ldots, k_n) \subset X$, $n \in \mathbb{N}$:

$$X_n(k_1, k_2, k_3, \ldots, k_n) := \cap_{j=\overline{1,n}} X_{k_j}. \tag{2.3}$$

The sequence $(k_1, k_2, k_3, \ldots, k_n) \in D^n$ is called *admissible* if there exists a point $x \in X$ such that

$$X_n(k_1, k_2, k_3, \ldots, k_n; x) \subset \cap_{j=\overline{1,n}} X_{k_j}, \qquad n \in \mathbb{N}.$$

In numerous concrete cases, the ergodicity of a mapping $\varphi\colon X \to X$ can be formulated more efficiently by using standard-measure theoretical calculations. In particular, based on the construction presented above, we can propose a slightly alternative to [12, 28, 29] approach to proving ergodicity by using the following two lemmas from the classical measure theory [30, 31]:

Lemma 2.1 (Hahn–Caratheodory–Kolmogorov extension theorem)**.** *Let $\mathcal{A}$ be an algebra of subsets of X and let $\mathcal{B}(\mathcal{A})$ denote the σ-algebra generated by $\mathcal{A}$. Suppose that a mapping $\mu\colon \mathcal{A} \to [0,1]$ satisfies the conditions:*

(a) $\mu(\varnothing) = 0$;

(b) if $A_n \in \mathcal{A}$, $n \in \mathbb{N}$, are pairwise disjoint and if $\sqcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$, then

$$\mu\big(\sqcup_{n\in\mathbb{N}} A_n\big) = \sum_{n\in\mathbb{N}} \mu(A_n).$$

Then there is a unique probability measure $\mu\colon \mathcal{B}(\mathcal{A}) \to [0,1]$, which is an extension of the mapping $\mu\colon \mathcal{A} \to [0,1]$.

Lemma 2.2. *Let $(X, \mathcal{B}, \mu)$ be a probability space and let $\mathcal{A} \subset \mathcal{B}$ be an algebra that generates $\mathcal{B}$, i.e., $\mathcal{B} = \mathcal{B}(\mathcal{A})$. Suppose that there exists $C > 0$ such that, for a fixed $B \in \mathcal{B}$, the inequality*

$$\mu(B)\mu(I) \le C\mu(B \cap I) \tag{2.4}$$

holds for all $I \in \mathcal{A}$. Then the measure $\mu(B)\mu(B^c) = 0$, where $B^c := X \backslash B \in \mathcal{B}$ denotes the complement of the set $B \in \mathcal{B}$.

Owing to the special properties of *smooth fibered* multidimensional mappings $\varphi\colon X \to X$ equivalent to the Bernoulli shifts [27, 28, 32] mapping (2.1), we can formulate the following important theorem:

Theorem 2.1. *Let the cylinder sets of a smooth fibered multidimensional mapping $\varphi X \to X$ satisfy the conditions of Lemma 2.2 with respect to the Lebesgue measure λ on X absolute continuous to the invariant measure μ on X. Then the mapping $\varphi\colon X \to X$ is ergodic with respect to its invariant probability measure μ on X.*

Example 2.1. A simple example is given by the following doubling mapping:

$$\varphi\colon [0,1) \ni x \to \{2x\} \in [0,1), \tag{2.5}$$

where $k\colon [0,1) \ni x \to \lfloor 2x \rfloor \in \{0,1\} := D$.

It is ergodic [28, 29] with respect to the finite Lebesgue measure $d\lambda(x) = dx$, $x \in [0,1)$, and admits a generating partition

$$\xi = \{X_0 = [0,1/2), X_1 = [1/2,1)\}, \quad X_0 \sqcup X_1 = [0,1) = X.$$

As for the ergodicity of the doubling mapping (2.5), it can be easily demonstrated if we represent any number $x \in [0,1)$ in the form of a binary expansion

$$x := (\cdot x_0 x_1 x_2 \ldots x_n \ldots) = \sum_{j \in \mathbb{Z}_+} x_j 2^{-(j+1)}, \tag{2.6}$$

where $x \in \{0,1\} = D$. For convenience, we denote the set of all expansions of this kind by

$$Y = \{(\cdot x_0 x_1 x_2 \ldots x_n \ldots)\colon x_j \in \{0,1\}\} \simeq \{0,1\}^{\mathbb{Z}_+}.$$

It is easy to see that mapping (2.5) is equivalent to the left Bernoulli-type shift

$$T_\varphi(\cdot x_0 x_1 x_2 \ldots x_n \ldots) = (\cdot x_1 x_2 \ldots x_n \ldots) \tag{2.7}$$

for any element $(\cdot x_0 x_1 x_2 \ldots x_n \ldots) \in Y$. We can now introduce the so-called dyadic intervals or cylinder sets as the sets

$$I(k_0, k_1, \ldots, k_{n-1}) = \left\{x \in [0,1)\colon x_j = k_j,\ j = \overline{1, n-1}\right\}, \tag{2.8}$$

where, for instance, $I(0) = [0,1/2)$, $I(1) = [1/2,1)$, $I(0,0) = [0,1/4)$, $I(0,1) = [1.4,1/2)$, etc. If $\mathcal{A}$ denotes the algebra of finite unions of cylinders of this kind, then it is easy to see that it generates the ordinary Borel σ-algebra $\mathcal{B}$ of the interval $[0,1)$. Moreover, if we take two separate points $x \neq y \in [0,1)$, then their expansions are different at a certain place $n \in \mathbb{Z}_+$ of the 2-expansions, which means that these numbers belong to different disjoint cylinders. We now define the following mappings inverse to (2.5): $\sigma_0\colon [0,1) \to [0,1/2)$ and $\sigma_1\colon [0,1) \to [1/2,1)$, where

$$\begin{aligned} \sigma_0(x) &= \Big\{x/2, \quad \text{if} \quad x \in [0,1/2), \\ \sigma_1(x) &= \Big\{(1+x)/2, \quad \text{if} \quad x \in [1/2,1), \end{aligned} \tag{2.9}$$

where, in turn, $\varphi \circ \sigma_j(x) = x$, $j = \overline{0, 1}$, for any $x \in [0, 1)$, whose actions upon the elements of the set Y are the corresponding right shifts:

$$\begin{aligned} \sigma_0(\ldots x_0 x_1 x_2 \ldots x_n \ldots) &= (\ldots 0 x_0 x_1 x_2 \ldots x_n \ldots), \\ \sigma_1(\ldots x_0 x_1 x_2 \ldots x_n \ldots) &= (\ldots 1 x_0 x_1 x_2 \ldots x_n \ldots). \end{aligned} \tag{2.10}$$

Based on the definitions of the cylinder sets (2.8) and actions (2.10), we observe that

$$I_n := I(k_0, k_1, \ldots, k_n) = \sigma_{k_0} \circ \sigma_{k_1} \circ \sigma_{k_2} \circ \ldots \circ \sigma_{k_n}([0, 1)) \tag{2.11a}$$

whose Lebesgue measure can be easily found as follows:

$$\lambda(I_n) = 2^{-(n+1)} \sum_{j \in \mathbb{Z}_+} 2^{-j} = 2^{-n} \tag{2.12}$$

for any $n \in \mathbb{N}$.

We are now in the position to apply Lemmas 2.1 and 2.2. Assume that a measurable set $B \subset [0, 1)$ is invariant: $B = \varphi^{-1} B = \varphi^{-n} B$, $n \in \mathbb{N}$. The Lebesgue measure is calculated as follows:

$$\begin{aligned} \lambda(B \cap I_n) &= \int\limits_{[0,1)} \chi_{B \cap I_n}(x)\, dx = \int\limits_{[0,1)} \chi_B(z) \chi_{I_n}(x)\, dx = \int\limits_{I_n} \chi_B(x) dx \\ &= \int\limits_{[0,1)} \chi_B(\sigma_{k_0} \circ \sigma_{k_1} \circ \sigma_{k_2} \circ \ldots \circ \sigma_{k_n}(x)) d \left(\sigma_{k_0} \circ \sigma_{k_1} \circ \sigma_{k_2} \circ \ldots \circ \sigma_{k_n}(x)\right) \\ &= \int\limits_{[0,1)} \chi_{\tilde{\varphi}^{-n} B} \left(\sigma_{k_0} \circ \sigma_{k_1} \circ \sigma_{k_2} \circ \ldots \circ \sigma_{k_n}(x)\right) d \left(\sigma_{k_0} \circ \sigma_{k_1} \circ \sigma_{k_2} \circ \ldots \circ \sigma_{k_n}(x)\right) \\ &= \int\limits_{[0,1)} \chi_B \left(\tilde{\varphi}^n \circ \sigma_{k_0} \circ \sigma_{k_1} \circ \sigma_{k_2} \circ \ldots \circ \sigma_{k_n}(x)\right) d \left(\sigma_{k_0} \circ \sigma_{k_1} \circ \sigma_{k_2} \circ \ldots \circ \sigma_{k_n}(x)\right) \\ &= \int\limits_{[0,1)} \chi_B(x) d \left(\sigma_{k_0} \circ \sigma_{k_1} \circ \sigma_{k_2} \circ \ldots \circ \sigma_{k_n}(x)\right) \\ &= \int\limits_{[0,1)} \chi_B(x) \sigma'_{k_0} \sigma'_{k_1} \sigma'_{k_2} \ldots \sigma'_{k_n}(x) dx = 2^{-n} \lambda(B) = \lambda(I_n) \lambda(B). \end{aligned} \tag{2.13}$$

This means that

$$\lambda(I_n)\lambda(B) = \lambda(B \cap I_n) \le C \lambda(B \cap I_n),$$

where $C = 1$. Thus, either the Lebesgue measure $\lambda(B) = 1$ or $\lambda(B) = 0$, which means that the doubling mapping is ergodic (2.5).

Example 2.2. A very interesting example is given by the classical continued-fraction expansion via the Gauss ergodic mapping

$$\varphi\colon [0,1) \ni x \to \{1/x\} \in [0,1) \tag{2.14}$$

whose fibering is defined by the mapping $k : [0,1) \ni x \to \lfloor 1/x \rfloor \in \mathbb{N} := D$ and the generating partition is given by the sets $X_i = (1/(i+1), 1/i]$, $i \in \mathbb{N}$, $X = \sqcup_{i\in\mathbb{N}} X_i$.

The invariant measure is the well-known Gauss measure $d\mu(x) = d\lambda(x)/[(1+x)\ln 2]$, where $d\lambda(x) := dx$, $x \in [0,1)$. The ergodicity of mapping (2.14) can be easily stated by reducing it [29] via the continuum-fraction expansion to a Bernoulli shifting and applying Lemmas 2.1 and 2.2.

Namely, we take a number $x \in [0,1)$ and denote by $[x_0, x_1, \ldots, x_n, \ldots]$ its continuous-fraction expansion:

$$x = \frac{1}{x_0+}\frac{1}{x_2+\ldots}\frac{1}{x_n+\ldots}, \tag{2.15}$$

where $x_i \in \mathbb{Z}_+$ for all indices $i \in \mathbb{Z}_+$. Observe that the induced continuous-fraction mapping acts by left shifting as $T_\varphi[x_0, x_1, \ldots, x_n, \ldots] = [x_1, \ldots, x_n, \ldots]$ for any expansion (2.15). This expansion $[x_0, x_1, \ldots, x_n, \ldots]$ can be reduced to the nth order by introducing, for every $t \in [0,1)$, a rational t-fraction

$$[x_0, x_1, \ldots, x_{n-1} + t] := \frac{P_n(x_0, x_1, \ldots, x_{n-1}; t)}{Q_n(x_0, x_1, \ldots, x_{n-1}; t)}, \tag{2.16}$$

where, by definition,

$$P_n(x_0, x_1, \ldots, x_{n-1}; t)$$

and

$$Q_n(x_0, x_1, \ldots, x_{n-1}; t)$$

are coprime polynomials in the variables $x_0, x_1, \ldots, x_{n-1} \in \mathbb{Z}_+$ and $t \in [0,1)$ for all $n \in \mathbb{N}$. If we define the nth order polynomials $P_n = P_n(x_0, x_1, \ldots, x_{n-1}) := P_n(x_0, x_1, \ldots, x_{n-1}; 0)$ and $Q_n = Q_n(x_0, x_1, \ldots, x_{n-1}) := Q_n(x_0, x_1, \ldots, x_{n-1}; 0)$, then we can easily observe that the following iterative expressions hold:

$$\begin{gathered} P_n(x_0, x_1, \ldots, x_{n-1}; t) = P_n + tP_{n-1}, \\ Q_n(x_0, x_1, \ldots, x_{n-1}; t) = Q_n + tQ_{n-1}, \\ P_n(x_0, x_1, \ldots, x_{n-1}) = Q_{n-1}(x_1, \ldots, x_{n-1}), \\ P_{n+1}(x_0, x_1, \ldots, x_{n-1}, x_n; t) = x_n P_n + P_{n-1} + tP_n, \\ Q_{n+1}(x_0, x_1, \ldots, x_{n-1}, x_n; t) = x_n Q_n + Q_{n-1} + tQ_n, \end{gathered} \tag{2.17}$$

for any $t \in [0,1)$ and arbitrary $n \in \mathbb{N}$. By using (2.17) and setting the parameter $t = 0$, we also derive the following important iterative relationships for all $n \in \mathbb{N}$:

$$P_{n+1} = x_n P_n + P_{n-1}, \qquad Q_{n+1} = x_n Q_n + Q_{n-1} \tag{2.18}$$

with the initial conditions $P_0 = 0$, $P_1 = 1$ and $Q_0 = 1$, $Q_1 = x_0 \in \mathbb{Z}_+$. In particular, the following invariant condition $Q_n\ P_{n-1} - P_n Q_{n-1} = (-1)^n$ and the inequality $Q_{n-1} \leq Q_n$ readily follow from (2.18) for all $n \in \mathbb{N}$. Now let the indices $k_0, k_1, \ldots, k_{n-1} \in \mathbb{N}$ for any $n \in \mathbb{N}$. We define cylindrical intervals $I_n \subset [0,1)$ as the corresponding collection of rational t-fractions:

$$I_n = I_n(k_0, k_1, \ldots, k_{n-1}) := \big\{[k_0, k_1, \ldots, k_{n-1} + t] \colon t \in [0,1)\big\}. \tag{2.19}$$

If, in addition, we define the inverse mappings

$$[0,1) \ni x \to \frac{k}{k+x} \in I_1(k) \subset [0,1), \quad k \in \mathbb{N},$$

then we can easily show that the composition

$$\sigma_{k_0} \circ \sigma_{k_1} \circ \ldots \circ \sigma_{k_{n-1}} \colon [0,1) \to I_n(k_0, k_1, \ldots, k_{n-1}) \subset [0,1) \tag{2.20}$$

for every $n \in \mathbb{N}$. Moreover, the condition $\varphi^n \circ \sigma_{k_0} \circ \sigma_{k_1} \circ \ldots \circ \sigma_{k_{n-1}}(x) = x$ holds for every $x \in [0,1)$ and $n \in \mathbb{N}$. We now take any $t \in [0,1)$, indicate that

$$\sigma_{k_0} \circ \sigma_{k_1} \circ \ldots \circ \sigma_{k_{n-1}}(t) = [k_0, k_1, \ldots, k_{n-1} + t] = \frac{P_n + tP_{n-1}}{Q_n + tQ_{n-1}}, \tag{2.21}$$

and estimate the Lebesgue measure of the interval (2.19):

$$\lambda(I_n) := \int\limits_{[0,1)} \chi_{I_n(t)} dt = \int\limits_{I_n} dt = \int\limits_{[0,1)} \left| J_{\sigma_{k_0} \circ \sigma_{k_1} \circ \ldots \circ \sigma_{k_{n-1}}}(t) \right| dt$$

$$= \int\limits_{[0,1)} \left| \frac{d}{dt} \left(\frac{P_n + tP_{n-1}}{Q_n + tQ_{n-1}} \right) \right| dt = \int\limits_{[0,1)} \frac{dt}{|Q_n + tQ_{n-1}|^2} \in \left[\frac{1}{4Q_n^2}, \frac{1}{Q_n^2} \right], \tag{2.22}$$

where we have taken into account that $0 < Q_{n-1} \leq Q_n$ for all $n \in \mathbb{N}$.

We are now in the position to estimate the Lebesgue measure $\lambda(B \cap I_n)$ of the intersection $B \cap I_n$ of an invariant set $B = \varphi^{-1} B = \varphi^{-n} B \subset [0,1)$ with an arbitrary cylindrical interval $I_n \subset [0,1)$, $n \in \mathbb{N}$:

$$\lambda(B \cap I_n) = \int\limits_{I_n} \chi_B(x) dx$$

$$= \int\limits_{[0,1)} \chi_B \left(\sigma_{k_0} \circ \sigma_{k_1} \circ \ldots \circ \sigma_{k_{n-1}}(x) \right) dx$$

$$= \int\limits_{[0,1)} \chi_{\varphi^{-n} B} \left(\sigma_{k_0} \circ \sigma_{k_1} \circ \ldots \circ \sigma_{k_{n-1}}(x) \right) \left| J_{\sigma_{k_0} \circ \sigma_{k_1} \circ \ldots \circ \sigma_{k_{n-1}}}(x) \right| dx$$

$$= \int_{[0,1)} \chi_B \left(\varphi^n \circ \sigma_{k_0} \circ \sigma_{k_1} \circ \ldots \circ \sigma_{k_{n-1}} x\right) \left| J_{\sigma_{k_0} \circ \sigma_{k_1} \circ \ldots \circ \sigma_{k_{n-1}}}(x) \right| dx$$

$$= \int_{[0,1)} \chi_B(x) \left| J_{\sigma_{k_0} \circ \sigma_{k_1} \circ \ldots \circ \sigma_{k_{n-1}}}(x) \right| dx = \int_{[0,1)} \chi_B(x) \frac{dt}{|Q_n + x Q_{n-1}|^2}$$

$$\geq \frac{1}{4Q_n^2} \lambda(B) \geq \frac{1}{4} \lambda(I_n) \lambda(B), \tag{2.23}$$

which completely fits the conditions of Lemma 2.2 with constant $C = 4$. Thus, as a consequence of estimation (2.23), we conclude that either the measure $\lambda(B) = 1$ or $\lambda(B) = 0$, which proves the ergodicity both of the Lebesgue measure $d\lambda(x)$, $x \in [0, 1)$, and the invariant Gauss measure

$$d\mu(x) = \frac{dx}{(1 + x)\ln 2}, \quad x \in [0, 1),$$

on the unit interval $[0, 1)$.

3. One-Dimensional Boole-Type Mappings and Invariant Ergodic Measures

The classical one-dimensional Boole [4] mapping is defined as follows:

$$\varphi\colon \mathbb{R}\backslash\{0\} \ni x \to x - 1/x \in \mathbb{R}. \tag{3.1}$$

As shown by Adler and Weiss in [33], the Boole mapping (3.1) is ergodic with respect to the invariant σ-finite Lebesgue measure $d\lambda(x) := dx$, $x \in \mathbb{R} := X$. Their proof of ergodicity was strongly based on the measure-theoretic reduction of mapping (3.1) to the corresponding *induced* [28, 29, 32] transformation $\varphi_A\colon [-1, 1] \in [-1, 1] \subset \mathbb{R}$ followed by the proof of its ergodicity. The φ-invariance of the Lebesgue measure $d\lambda(x) := dx$, $x \in \mathbb{R}$, can be easily checked by using the Perron–Frobenius condition: For the preimages $u_\pm := u_\pm(x) \in \mathbb{R}$, $x \in \mathbb{R}$, where $\varphi(u_\pm(x)) = x$, $u_+ + u_- = x$, $u_- u_+ = -1$, we can directly verify that the preimage measure

$$\sum_\pm du_\pm(x) = \sum_\pm \left| \frac{du_\pm}{dx} \right| dx = \sum_\pm \frac{dx}{|J_\varphi(u_\pm)|} = \sum_\pm \frac{dx}{(1 + u_\pm^{-2})}$$

$$= \sum_\pm \frac{u_\pm^2 dx}{(1 + u_\pm^2)} = \frac{(u_+^2 + 2 + u_-^2) dx}{1 + (u_+ u_-)^2 + u_+^2 + u_-^2} = \frac{\left(u_+^2 + 2 + u_-^2\right) dx}{2 + u_+^2 + u_-^2} = dx \tag{3.2}$$

exactly coincides with the Lebesgue measure on the axis $\mathbb{R}$.

In what follows, we present a modified proof of ergodicity of the Boole transformation (3.1). Namely, the approach discussed above and applied to the Boole transformation (3.1) proves to be successful and allows one to obtain a new proof of the Adler–Weiss [33] result on the ergodicity of this Boole transformation.

Theorem 3.1. *The one-dimensional Boole transformation (3.1) is ergodic with respect to the invariant Lebesgue measure λ on $\mathbb{R}$.*

Proof. As already indicated, the proof can be reduced to Theorem 2.1, by applying its relation (3.4) mentioned above to the doubling mapping $T_\varphi\colon [0,1) \ni s \to \{2s\} \in [0,1)$. Moreover, as shown in [12, 23, 34], the Boole transformation (3.1) can be related to the doubling mapping $T_\varphi\colon [0,1) \ni s \to \{2s\} \in [0,1)$ via the commutative diagram

$$\begin{array}{ccccc} [0,1) & \overset{\cot(\pi\circ)}{\to} & \mathbb{R} & \overset{\varphi}{\to} & \mathbb{R} \\ \downarrow\uparrow Id & & & & \downarrow \pi^{-1}\cot^{-1}, \\ [0,1) & \overset{T_\varphi}{\to} & [0,1) & \overset{\alpha^{-1}(\pi\circ)}{\to} & [0,1) \end{array} \tag{3.3}$$

where $\alpha^{-1}\colon [0,1) \to [0,1)$ is a diffeomorphism defined by the expression $\alpha(s) = \pi^{-1}\operatorname{arccot}(\pi s/2)$, $s \in [0,1)$, related to mapping (3.1) as follows:

$$\varphi = \cot\pi \circ \alpha^{-1} \circ T_\varphi\left(\pi^{-1} \circ \cot^{-1}\right). \tag{3.4}$$

Now let $\tilde{\varphi}\colon [0,1) \to [0,1)$, $\tilde{\varphi} := \alpha^{-1} \circ T_\varphi$, be the mapping equivalent to (3.4), where

$$\tilde{\varphi}(s) = \pi^{-1}\cot^{-1}(\varphi(\cot(\pi s)))$$

for any $s \in [0,1)$. Since every number $a \in [0,1)$ has a binary expansion

$$a := (\cdot k_0 k_1 k_2 \ldots k_n \ldots) = \sum_{j\in\mathbb{Z}_+} k_j 2^{-(j+1)}, \tag{3.5}$$

we can define the so-called proper [12, 34] cylindrical sets $I_n := I_n(k_0, k_{1,}, \ldots, k_n) \subset [0,1)$, $n \in \mathbb{Z}_+$, as follows:

$$I_n = \left\{(\sigma_{k_{n-1}} \circ \alpha) \circ (\sigma_{k_{n-1}} \circ \alpha) \circ \ldots \circ (\sigma_{k_0} \circ \alpha)(t)\colon t \in [0,1)\right\}, \tag{3.6}$$

where $\sigma_0(s) = s/2$ if $s \in [0,1/2)$ and $\sigma_1(s) = (1+s)/2$ if $s \in [1/2,1)$. We also note that $\tilde{\varphi} \circ \left(\sigma_{k_j} \circ \alpha\right)(s) = s$ for every $s \in [0,1)$, $k_j \in \{0,1\}$, $j = \overline{0, n-1}$. The Lebesgue measure of interval (3.6) can be easily estimated as follows: By definition, we get

$$\lambda(I_n) = \int_{I_n} dx = \int_{\mathbb{R}} \chi_{I_n}(x)dx\Big|_{x=\cot(\pi t)} = \int_{[0,1)} \left|J_{(\sigma_{k_n}\circ\alpha)\circ(\sigma_{k_{n-1}}\circ\alpha)\circ\ldots\circ(\sigma_{k_0}\circ\alpha))}(t)\right| dt$$

$$= \int_{[0,1)} \left(\sigma'_{k_{n-1}}\alpha'(t_{n-1})\right)\left(\sigma'_{k_{n-1}}\alpha'(t_{n-1})\right)\ldots\left(\sigma'_{k_0}\alpha\prime(t)\right) dt, \tag{3.7}$$

where the derivatives $\sigma'_{k_j} = 1/2$ and $\alpha'(t_j) = 2/\left[1 + 3\sin^2(\pi t_j)\right]$, where $t_j := \sigma_{k_j} \circ \alpha \circ \ldots \circ \sigma_{k_0} \circ \alpha(t)$, $j = \overline{0, n-1}$, $t \in [0,1)$. In view of the fact that

$$\begin{gathered} \lambda(\sigma_0 \circ \alpha)([0,1)]) = \frac{1}{2} = \lambda(\sigma_1 \circ \alpha([0,1]), \\ \sigma_{k_j} \circ \alpha \circ \ldots \circ \sigma_{k_0} \circ \alpha([0,1)) \subset [1/2^{j+1}, 1/2^j) \end{gathered} \tag{3.8}$$

for any $j = \overline{0, n-1}$, by the classical mean-value theorem applied to (3.8), we can easily obtain that

$$\alpha'\left(\bar{t}_j\right)\right) = 2^{-j} / \left[3\sin^2\left(\pi\bar{t}_j\right) + 1\right], \tag{3.9a}$$

where the numbers $\bar{t}_j \in \left(1/2^{j+1}, 1/2^j\right) \subset [0,1)$ for all $j \in \overline{0,n}$. Further, by using the evident inequalities $2t \le \sin(\pi t) \le \pi t$ for all $t \in [0, 1/2)$, we derive the following two-sided estimations:

$$\begin{aligned}\frac{2^{-j}}{\exp 3(\pi 2^{-j})^2} &\le \frac{2^{-j}}{3\sin^2(\pi 2^{-j}) + 1} \le \frac{2^{-j}}{3\sin^2(\pi t_j) + 1} \\ &\le \frac{2^{-j}}{3\sin^2\left(\pi 2^{-(j+1)}\right) + 1} \le \frac{2^{-j}}{3(2^{-(j+1)})^2 + 1}\end{aligned} \tag{3.10}$$

for any $j = \overline{0, n}$. Thus, by using expressions (3.7) and (3.10), we immediately arrive at the needed Renyi-type [12, 28, 29, 31, 32] estimations:

$$\begin{aligned}\frac{\exp(-3\pi^2)}{2^{n(n+1)/2}} &\le \frac{\exp\left[\pi^2(-4 + 4^{-n})\right]}{2^{n(n+1)/2}} = \prod_{j=0}^{n} \frac{2^{-j}}{\exp[3(\pi 2^{-j})]^2} \le \lambda(I_n) \\ &\le \prod_{j=0}^{n} \frac{2^{-j}}{3(2^{-(j+1)})^2 + 1} \le \frac{2^{-n(n+1)/2}}{\sum_{j=0}^{n} 3(2^{-(j+1)})^2 + 1} = \frac{2^{-n(n+1)/2}}{2 - 4^{-(n+1)}} \le \frac{4/7}{2^{n(n+1)/2}}\end{aligned} \tag{3.11}$$

for all $n \in \mathbb{Z}_+$. In particular, from (3.11), we conclude that $\lim_{n\to\infty} \lambda(I_n) = 0$, which means that the family of these cylindrical sets generates [23, 30, 34] the Borel σ-algebra $\mathcal{B}$ on the interval $[0, 1)$. Thus, we arrive at the position allowing us to apply Lemmas 2.1 and 2.2. Hence, let a measurable set $B \subset [0, 1)$ be invariant: $B = \varphi^{-1}B = \varphi^{-n}B$, $n \in \mathbb{N}$. We compute the following Lebesgue measure:

$$\begin{aligned}\lambda(B \cap I_n) &= \int_{[0,1)} \chi_{B\cap I_n}(t)dt = \int_{[0,1)} \chi_B(t)\chi_{I_n}(t)dt = \int_{I_n} \chi_B(t)dt \\ &= \int_{[0,1)} \chi_B\left((\sigma_{k_n} \circ \alpha) \circ (\sigma_{k_{n-1}} \circ \alpha) \circ \ldots \circ (\sigma_{k_0} \circ \alpha)(t)\right) \\ &\qquad \times d((\sigma_{k_n} \circ \alpha) \circ (\sigma_{k_{n-1}} \circ \alpha) \circ \ldots \circ (\sigma_{k_0} \circ \alpha)(x)) \\ &= \int_{[0,1)} \chi_{\varphi^{-n}B}((\sigma_{k_n} \circ \alpha) \circ (\sigma_{k_{n-1}} \circ \alpha) \circ \ldots \circ (\sigma_{k_0} \circ \alpha)(x)) \\ &\qquad \times d((\sigma_{k_n} \circ \alpha) \circ (\sigma_{k_{n-1}} \circ \alpha) \circ \ldots \circ (\sigma_{k_0} \circ \alpha)(x)) \\ &= \int_{[0,1)} \chi_B\left(\tilde{\varphi}^n \circ (\sigma_{k_n} \circ \alpha) \circ (\sigma_{k_{n-1}} \circ \alpha) \circ \ldots \circ (\sigma_{k_0} \circ \alpha)(t)\right)\end{aligned}$$

$$\times\, d((\sigma_{k_n}\circ\alpha)\circ(\sigma_{k_{n-1}}\circ\alpha)\circ\ldots\circ(\sigma_{k_0}\circ\alpha)(t)). \tag{3.12}$$

Since the composition $\varphi\circ(\sigma_{k_j}\circ\alpha)=Id$ for any $j=\overline{0,n}$, it follows from (3.12) that

$$\lambda(B\cap I_n)=\int\limits_{[0,1)}\chi_B(x)d\left((\sigma_{k_n}\circ\alpha)\circ(\sigma_{k_{n-1}}\circ\alpha)\circ\ldots\circ(\sigma_{k_0}\circ\alpha)(x)\right)$$

$$=\int\limits_{[0,1)}\chi_B(x)\sigma'_{k_n}\alpha'\sigma'_{k_{n-1}}\alpha'\sigma'_{k_{n-2}}\alpha'\ldots\sigma'_{k_0}\alpha'(x)dx$$

$$\geq\frac{\exp(-3\pi^2)}{2^{n(n+1)/2}\lambda(I_n)}\lambda(I_n)\lambda(B)\geq\frac{7}{4\exp(3\pi^2)}\lambda(I_n)\lambda(B),$$

i.e., the Lebesgue measure $\lambda(I_n)\lambda(B)\leq C\lambda(B\cap I_n)$ for all $n\in\mathbb{Z}_+$, where the constant $C=4\exp\left(3\pi^2\right)/7$. Thus, owing to Lemma 2.2, either the Lebesgue measure $\lambda(B)=1$ or $\lambda(B)=0$, which simultaneously means that the Boole mapping (3.1) is ergodic with respect to the same invariant Lebesgue measure λ on $\mathbb{R}$, which proves the next theorem.

It is worth mentioning here that the well-known [1, 11, 28, 32, 35, 36] doubling mapping (2.5) is isomorphic to the following one-dimensional Boole-type transformation:

$$\varphi\colon\mathbb{R}\ni x\to(x-1/x)/2\in\mathbb{R}, \tag{3.13}$$

which is invariant with respect to the probability measure $d\mu(x)=dx/[\pi(1+x^2)]$, $x\in\mathbb{R}$. The Boole mapping (3.1) was generalized as follows:

$$\mathbb{R}\backslash\{b_j: j=\overline{1,N}\}\ni x\to\varphi(x):=cx+a-\sum_{j=1}^{N}\frac{\beta_j}{x-b_j}\in\mathbb{R}, \tag{3.14}$$

where a and $b_j\in\mathbb{R}$, $j=\overline{1,N}$, are some real values and $\alpha,\beta_j\in\mathbb{R}_+$, $j=\overline{1,N}$. This mapping was analyzed in [1, 2, 9, 37, 38]. In the case where $c=1$ and $a=0$, a similar ergodicity result was proved in [2, 39–41] by using a specially devised inner-function method. The related spectral aspects of mapping (3.14) were partly studied also in [1, 2]. Despite these results, the case $\alpha\neq 1$ still persists to be challenging as the only related result [1, 2] concerning the following special case of (3.14):

$$\mathbb{R}\ni x\to\varphi(x):=cx+a-\frac{\beta}{x-b}\in\mathbb{R} \tag{3.15}$$

for $0<c<1$ and arbitrary $a,b\in\mathbb{R}$ and $\beta\in\mathbb{R}_+$. The invariant measures and ergodicity related to mappings (3.15) were analyzed in [9, 35–37] due to their equivalence

$$[0,1)\ni s\colon\to T_\varphi(s)=2s \bmod 1\in[0,1) \tag{3.16}$$

that follows from the commutative diagram

$$
\begin{array}{ccc}
[0,1) & \overset{T_\varphi}{\rightarrow} & [0,1) \\
f \downarrow & & \downarrow f \\
\mathbb{R} & \overset{\varphi}{\rightarrow} & \mathbb{R}
\end{array}, \tag{3.17}
$$

for which the following condition holds: $f \circ T_\varphi = \varphi \circ f$, where $f(s) := (2\beta)^{1/2} \cot \pi s + 2a, s \in [0,1)$. It is also important to mention here that, within the framework of the theory of inner functions, in [1, 39–41], it was stated that there exists an invariant measure $d\mu(x),\ x \in \mathbb{R}$, on the axis $\mathbb{R}$ such that the generalized Boole-type transformation (3.14) is ergodic for any $N > 1$, $c = 1$, and $a = 0$.

If $\alpha = 1$ and $a \neq 0$, then transformation (3.14) appears to be not ergodic being totally dissipative, i.e., the wandering set $\mathcal{D}(\varphi) := \cup \mathcal{W}(\varphi) = \mathbb{R}$, where $\mathcal{W}(\varphi) \subset \mathbb{R}$ are subsets such that all sets $\varphi^{-n}(\mathcal{W}),\ n \in \mathbb{Z}_+$, are disjoint. A similar statement can be also formulated [1] for the generalized Boole-type transformation:

$$
\mathbb{R} \ni x \to \varphi(x) := \alpha x + a + \int_{\mathbb{R}} \frac{d\nu(s)}{s - x} \in \mathbb{R}, \tag{3.18}
$$

where $a \in \mathbb{R}$, $c \in \mathbb{R}_+$, and a measure $d\nu(s),\ s \in \mathbb{R}$, on $\mathbb{R}$ (not necessary absolutely continuous with respect to the Lebesgue measure) has a compact support $\operatorname{supp} \nu \subset \mathbb{R}$ and satisfies the following natural conditions:

$$
\int_{\mathbb{R}} \frac{d\nu(s)}{1 + s^2} = a, \qquad \int_{\mathbb{R}} d\nu(s) < \infty, \tag{3.19}
$$

ensuring the boundedness of its topological characteristics.

4. Two-Dimensional Boole-Type Transformations and Their Ergodicity

Multidimensional endomorphisms of measurable spaces are of great interest [12, 32] in mathematics from many points of view, including number-theoretical aspects, numerical theory, theory of dynamical systems, and diverse physical applications. We especially mention the works [12, 42, 43], where the author reviewed numerous very interesting measure-preserving and ergodic multidimensional mappings. Relatively recently, in [9, 35, 37, 38], the authors also proposed a set of new multidimensional Boole-type transformations $\varphi_\sigma \colon \mathbb{R}^n \to \mathbb{R}^n$, where

$$
\varphi_\sigma(x_1, x_2, \ldots, x_n) := (x_1 - 1/x_{\sigma(1)}, x_2 \pm 1/x_{\sigma(2)}, \ldots, x_n \pm 1/x_{\sigma(n)}) \tag{4.1}
$$

for any $n \in \mathbb{N}$ and arbitrary permutations $\sigma \in S_n$ (the signs "$\pm$" are chosen from the nondegeneracy condition $J_\varphi(x) \neq 0,\ x \in \mathbb{R}^n \backslash \{0\}$).

For the case $n = 2$, $(x, y) \in \mathbb{R}^2 \backslash \{0, 0\}$, we obtain the following nontrivial two-dimensional Boole-type mapping:

$$
\varphi(x, y) := (x - 1/y, y + 1/x). \tag{4.2}
$$

At the same time, in the case $n = 3$, $(x, y, z) \in \mathbb{R}^3 \backslash \{0, 0, 0\}$, we obtain the following nontrivial three-dimensional Boole-type mapping:

$$
\begin{aligned}
\varphi_+(x, y, z) &:= (x - 1/y, y + 1/z, z + 1/x), \\
\varphi_-(x, y, z) &:= (x - 1/y, y - 1/z, z - 1/x).
\end{aligned} \tag{4.3}
$$

We observe that the infinitesimal Lebesgue measure $d\lambda(x,y) := dxdy$, $(x,y) \in \mathbb{R}^2$ in the plane $\mathbb{R}^2$ is invariant under mapping (4.2), which can be easily checked by using the Perron–Frobenius condition: For the corresponding preimages

$$(u_\pm, v_\pm) := (u_\pm(x,y), v_\pm(x,y)) \in \mathbb{R}^2,$$

where $u_+u_- = xy^{-1}$, $v_+v_- = -yx^{-1}$, $u_+ + u_- = 2y^{-1} + x$, $v_+ + v_- = y - 2x^{-1}$, and $\varphi(u_\pm, v_\pm) = (x,y) \in \mathbb{R}^2$, it is possible to check that the measure

$$\begin{aligned}
\sum_\pm du_\pm dv_\pm(x,y) &= \sum_\pm \left|J_{(u_\pm,v_\pm)}(x,y)\right| dxdy \\
&= \sum_\pm \frac{dxdy}{|J_\varphi(u_\pm,v_\pm)|} = \sum_\pm \frac{dxdy}{(1+(u_\pm v_\pm)^{-2})} \\
&= \sum_\pm \frac{(u_\pm v_\pm)^2 dx}{(1+(u_\pm v_\pm)^2)} = \frac{\left[2(u_+v_+u_-v_-)^2 + (u_-v_-)^2 + (u_+v_+)^2\right] dxdy}{\left[1+(u_-v_-)^2+(u_+v_+)^2+(u_+v_+u_-v_-)^2\right]} \\
&= \frac{\left[(u_-v_-)^2+(u_+v_+)^2+2\right] dxdy}{\left[2+(u_-v_-)^2+(u_+v_+)^2\right]} = dxdy,
\end{aligned} \tag{4.4}$$

exactly coincides with the Lebesgue measure $d\lambda(x,y) := dxdy$, $(x,y) \in \mathbb{R}^2$.

As far as the ergodicity of the Lebesgue measure-preserving mapping (4.2) is concerned, the approach based on Theorem 2.1 subject to *smooth fibered* multidimensional mappings failed to be efficient. In view of the fact that the ergodicity result of [33] subject to the one-dimensional Boole mapping (3.1) is strongly based on the induced Kakutani transformation technique, one can expect that it can be also employed for the case of the two-dimensional Boole mapping (4.1).

We now proceed to the notion of induced transformation [28, 29, 32] for a measure-preserving mapping $\varphi: X \to X$, which was efficiently used by Adler and Weiss [33], when proving the ergodicity of the one-dimensional Boole transformation (3.1) being, in part, closely related to the classical Poincaré recurrence theorem [15, 28]. Namely, let $(X; \mathcal{B}, \mu, \varphi)$ be a measure-preserving discrete system and let $A \subset X$ be a measurable set with $\mu(A) > 0$ for which the following covering condition is true:

$$\cup_{n\in\mathbb{N}} \varphi^{-n} A = X \tag{4.5}$$

modulo a set of measure zero.

Remark 4.1. It is worth noting [15, 28, 29, 32] that if a measure-preserving system $(X; \mathcal{B}, \mu, \varphi)$, $\mu(X) = 1$, satisfies the covering condition (4.5) for any chosen measurable $A \subset X$, $\mu(A) > 0$, then the mapping $\varphi: X \to X$ is ergodic. Indeed, if the measure-preserving mapping $\varphi: X \to X$ is ergodic, then, for any measurable set $B \subset X$ satisfying the condition $\mu(B \Delta T^{-1} B) = 0$, we have either $\mu(B) = 1$ or $\mu(B) = 0$. Now let $A \subset X$ be a measurable set with $\mu(A) > 0$. We construct a set $B := \cup_{n\in\mathbb{N}} \varphi^{-n} A$. Since $\varphi^{-1} B \subset B$, we conclude that $\mu(\varphi^{-1} B) = \mu(B)$, which gives rise to the equality $\mu(B \Delta T^{-1} B) = 0$, i.e., either $\mu(B) = 1$ or $\mu(B) = 0$. Moreover, since $\varphi^{-1} A \subset B$, we conclude that either $\mu(B) \geq \mu(A)$ or $\mu(B) = 1$. The latter evidently means that $B = \cup_{n\in\mathbb{N}} \varphi^{-n} A = X$ modulo a set of measure zero.

Thus, owing to condition (4.5), the first return time $\tau_A \in \mathbb{N}$ can be defined by the condition that

$$\tau_A(x) := \inf_{n \in \mathbb{N}} \{n : \varphi^n(x) \in A,\ x \in A\} \tag{4.6}$$

exists almost everywhere and is finite.

Definition 4.1. *Assume that a measure-preserving system* $(X; \mathcal{B}, \mu, \varphi)$ *satisfies condition (4.5). Then a mapping* $\varphi_A: A \to A$ *defined as*

$$\varphi_A(x) := \varphi^{\tau_A(x)}(x) \tag{4.7}$$

for almost all $x \in A$ *is called the transformation induced by the measure preserving mapping* $\varphi: X \to X$ *on the set* $A \subset X$.

The induced mapping constructed above is characterized by the following important theorems [28, 29, 32]:

Theorem 4.1 (M. Kac's theorem)**.** *Let a mapping* $\varphi: X \to X$ *be ergodic and let a measurable set* $A \subset X$ *be chosen such that* $0 < \mu(A) < \infty$. *Then the average return time is proportional to the measure* $\mu(A)$, *i.e.,*

$$\int_A \tau_A(x) d\mu(x) = \mu(A). \tag{4.8}$$

Theorem 4.2. *The induced transformation (4.7) is a measure-preserving mapping in the space*

$$(A, \mathcal{B}|_A, \mu_A = \mu(A)^{-1}\mu|_A, \varphi_A),$$

where $\mathcal{B}|_A := \{B \cap A : B \in \mathcal{B}\}$ *and* $0 < \mu(A) < \infty$. *Moreover, if a mapping* $\varphi: X \to X$ *is ergodic with respect to a measure* μ, *then the induced transformation* $\varphi_A: A \to A$ *is ergodic with respect to the measure* $\mu_A := \mu/\mu(A)$ *induced on the set* A.

As already indicated, just this theorem was used in [33] to prove the ergodicity of the Boole mapping (3.1). As shown above, there also exists an efficient second essentially analytic approach that can be used to prove ergodicity. It would be also useful to present two proofs, if any, of the ergodicity property of the two-dimensional Boole type mapping (4.2).

Concerning the approach based on Theorem 4.2, its main technical ingredients are strongly related to the construction of a special generating partition of the measured space X suggested by Kakutani and Rokhlin [44, 45] for the corresponding induced mapping $\varphi_A: A \to A$ introduced above. In particular, let a mapping $\varphi: X \to X$ be ergodic. For a measurable set $A \subset X$ satisfying the condition $0 < \mu(A) < \infty$, we consider its induced mapping $\varphi_A: A \to A$. As condition (4.5) is *a priori* satisfied [15, 28, 29, 32, 33], one can construct the following disjoint measurable *first return iteration* subsets:

$$X_n := \left\{x \in X : \varphi^n(x) \in A,\ \varphi^j(x) \notin A,\ j = \overline{1, n-1}\right\}, \tag{4.9}$$

where $\sqcup_{n \in \mathbb{N}} X_n = X$, $X_n \cap X_m = \varnothing$, $m \neq n \in \mathbb{N}$, satisfying the iterative expression

$$X_{n+1} = \varphi^{-1} X_n \cap \varphi^{-1} A^c. \tag{4.10}$$

Based on the sets (4.9), we construct, for all $n \in \mathbb{N}$, the sets

$$A_n := X_n \cap A, \qquad B_n := X_n \cap A^c, \tag{4.11}$$

satisfying the following important properties of disjoint sum:

$$\varphi^{-1} B_n = B_{n+1} \sqcup A_{n+1}, \qquad \sqcup_{n\in\mathbb{N}} A_n = A, \qquad \sqcup_{n\in\mathbb{N}} B_n = A^c. \tag{4.12}$$

We now consider an arbitrary measurable subset $E \subset A$ and note that

$$\varphi_A^{-1} E = \sqcup_{n\in\mathbb{N}} \left(\varphi^{-n} E \cap A_n\right), \tag{4.13}$$

implies the equality

$$\mu\left(\varphi_A^{-1} E\right) = \mu\left(\sqcup_{n\in\mathbb{N}} \left(\varphi^{-n} E \cap A_n\right)\right) = \sum_{n\in\mathbb{N}} \mu\left(\varphi^{-n} E \cap A_n\right). \tag{4.14}$$

Further, on the basis of representation (4.12), in view of the invariance of measure, we can easily establish the equalities

$$\begin{aligned} \mu(E) &= \mu\left(\varphi^{-1} E\right) = \mu\left(\varphi^{-1} E \cap (B_1 \sqcup A_1)\right) \\ &= \mu\left(\varphi^{-1} E \cap B_1\right) + \mu\left(\varphi^{-1} E \cap A_1\right), \\ \mu(B_n) &= \mu\left(\varphi^{-1} B_n\right) = \mu\left(B_{n+1} \sqcup A_{n+2}\right) = \mu(B_{n+1}) + \mu(A_{n+2}), \\ \mu\left(\varphi^{-1} E \cap B_1\right) &= \mu\left(\varphi^{-1}\left(\varphi^{-1} E \cap B_1\right)\right) = \mu\left(\varphi^{-2} E \cap \varphi^{-1} B_1\right) \\ &= \mu\left(\varphi^{-2} E \cap (B_2 \sqcup A_2)\right) = \mu\left(\varphi^{-2} E \cap B_2\right) + \mu\left(\varphi^{-2} E \cap B_2\right), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mu\left(\varphi^{-n} E \cap B_n\right) &= \mu\left(\varphi^{-(n+1)} E \cap B_{n+1}\right) + \mu\left(\varphi^{-(n+1)} E \cap A_{n+1}\right), \end{aligned} \tag{4.15}$$

which are true for all $n \in \mathbb{N}$. As a simple consequence of equalities (4.15), we can also derive the equalities

$$\mu\left(\varphi^{-n} E \cap B_n\right) = \sum_{k=n+1}^{\infty} \mu\left(\varphi^{-n} E \cap A_n\right), \qquad \mu(B_n) = \sum_{k=n+1}^{\infty} \mu(A_n), \tag{4.16}$$

which are reduced to the following two expressions:

$$\mu\left(\varphi^{-n}E\cap B_n\right)+\sum_{k=\overline{1,n}}\mu\left(\varphi^{-n}E\cap A_n\right)=\sum_{n\in\mathbb{N}}\mu\left(\varphi^{-n}E\cap A_n\right):=\eta_A,$$
$$\mu(A)=\sum_{k=1}^{\infty}\mu(A_n),\qquad \mu(B_1)=\sum_{k=2}^{\infty}\mu(A_n), \tag{4.17}$$

which simply means the invariance of the positive quantity $\eta_A\in\mathbb{R}_+$ with respect to $n\in\mathbb{N}$ and the boundedness of the measure $\mu(B_1)\le\mu(A)$, since the measure

$$\mu(A)=\mu\big(\sqcup_{n\in\mathbb{N}}A_n\big)<\infty$$

is bounded by assumption. By using the first equality in (4.15), we immediately conclude that $\eta_A=\mu(A)>0$, i.e.,

$$\mu\left(\varphi^{-n}E\cap B_n\right)+\sum_{k=\overline{1,n}}\mu\left(\varphi^{-n}E\cap A_n\right)=\mu(E). \tag{4.18}$$

We now recall equality (4.14). Thus, by using (4.18) and (4.17), we obtain

$$\begin{aligned}\left|\mu\left(\varphi_A^{-1}E\right)-\mu(E)\right|&=\lim_{n\to\infty}\left(\sum_{k=n+1}^{\infty}\mu\left(\varphi^{-k}E\cap A_k\right)\right)+\mu\left(\varphi^{-n}E\cap B_n\right)\\&\le 2\lim_{n\to\infty}\left(\sum_{k=n+1}^{\infty}\mu(A_k)\right)=0\end{aligned} \tag{4.19}$$

due to the convergence condition (4.10) for the measure $\mu(A)<\infty$. Thus, we conclude that $\mu\big(\varphi_A^{-1}E\big)=\mu(E)$ for any measurable set $E\subset A$. This means that the measure $\mu_A=\mu/\mu(A)$ suitably induced on the set $A\subset X$ is also invariant under the induced mapping $\varphi_A\colon A\to A$.

We now assume that the induced mapping $\varphi_A\colon A\to A$ is ergodic and take a set $D\subset X$, $\mu(D\cap A)>0$, since either $\mu(D\cap A)>0$ or $\mu\big(D\cap A^c\big)>0$, which is invariant under the mapping $\varphi\colon X\to X$, that is $\varphi^{-1}D=D$. It follows from expansion (4.13) that

$$\begin{aligned}\varphi_A^{-1}(D\cap A)&=\sqcup_{n\in\mathbb{N}}\left(\varphi^{-n}(D\cap A)\cap A_n\right)=\sqcup_{n\in\mathbb{N}}\left(D\cap\varphi^{-n}A\cap A_n\right)\\&=D\cap\left(\sqcup_{n\in\mathbb{N}}\left(\varphi^{-n}A\cap A_n\right)\right)=D\cap\varphi_A^{-1}A=D\cap A,\end{aligned} \tag{4.20}$$

since the initial assumption $\cup_{n\in\mathbb{N}}\varphi^{-n}A=X$ assures that $\varphi_A^{-1}A=A$ modulo a set of measure zero. As the induced mapping is assumed to be ergodic, it follows from (4.20) and the condition $\mu(D\cap A)>0$ that $D\cap A=A$. Thus, based once again on the initial assumption $\cup_{n\in\mathbb{N}}\varphi^{-n}A=X$, we can simply obtain that

$$\begin{aligned}X&=\cup_{n\in\mathbb{N}}\varphi^{-n}(D\cap A)=\cup_{n\in\mathbb{N}}\left(\varphi^{-n}D\cap\varphi^{-n}A\right)\\&=\cup_{n\in\mathbb{N}}\left(D\cap\varphi^{-n}A\right)=D\cap\left(\cup_{n\in\mathbb{N}}\varphi^{-n}A\right)=D\cap X=D,\end{aligned} \tag{4.21}$$

which means that the mapping $\varphi\colon X \to X$ is also ergodic.

Similarly, we state that the converse statement is also true. Indeed, if a mapping $\varphi\colon X \to X$ is ergodic and a set $E \subset A$, $\mu(E) > 0$, is φ_A-invariant, then

$$\varphi_A^{-1} E = \sqcup_{n\in\mathbb{N}} \left(\varphi^{-n} E \cap A_n\right) = E. \tag{4.22}$$

By using the invariance condition (4.22), we construct a set $F := E \cup \sqcup_{n\in\mathbb{N}}\left(B_n \cap \varphi^{-n} E\right)$ and compute its φ-mapping inverse:

$$\begin{aligned}
\varphi^{-1} F &= \varphi^{-1} E \cup \sqcup_{n\in\mathbb{N}} \left(\varphi^{-1} B_n \cap \varphi^{-(n+1)} E\right) \\
&= \varphi^{-1} E \cup \sqcup_{n\in\mathbb{N}} \left((B_{n+1} \sqcup A_{n+1}) \cap \varphi^{-(n+1)} E)\right) \\
&= \varphi^{-1} E \cup \left(\sqcup_{n\in\mathbb{N}} \left(B_{n+1} \cap \varphi^{-(n+1)} E\right)\right) \\
&\qquad \bigcup \varphi^{-1} E \cup \left(\sqcup_{n\in\mathbb{N}} \left(A_{n+1} \cap \varphi^{-(n+1)} E\right)\right) \\
&= \left(A_1 \cap \varphi^{-1} E \sqcup B_1 \cap \varphi^{-1} E\right) \cup \left(\sqcup_{n\in\mathbb{N}} \left(B_{n+1} \cap \varphi^{-(n+1)} E\right)\right) \\
&\qquad \bigcup \left(A_1 \cap \varphi^{-1} E \sqcup B_1 \cap \varphi^{-1} E\right) \cup \left(\sqcup_{n\in\mathbb{N}} \left(A_{n+1} \cap \varphi^{-(n+1)} E\right)\right) \\
&= \left(\sqcup_{n\in\mathbb{N}} \left(B_n \cap \varphi^{-n} E\right)\right) \bigcup \left(\sqcup_{n\in\mathbb{N}} \left(A_n \cap \varphi^{-n} E\right)\right) \\
&= \left(\sqcup_{n\in\mathbb{N}} \left(B_n \cap \varphi^{-n} E\right)\right) \cup E = F,
\end{aligned} \tag{4.23}$$

that is $\varphi^{-1} F = F$, which means its invariance under the mapping $\varphi\colon X \to X$. Thus, in view of its ergodicity, we find that $F = X$ modulo a null subset of X. If we now take into account that, by construction, the subset $\sqcup_{n\in\mathbb{N}}\left(B_n \cap \varphi^{-n} E\right) \subset A^c$, then we conclude that the set $A \subseteq E$ modulo a null subset of X. Since, by assumption, $E \subset A$, we finally get $A = E$, which means, respectively, that the induced mapping $\varphi_A\colon A \to A$ is also ergodic.

If we now try to apply the measure-theoretic construction proposed in [33] in order to prove the ergodicity of the two-dimensional Boole mapping (4.2), then we soon arrive at very cumbersome technical complications, which are quite difficult to overcome. Thus, it is reasonable to apply to this ergodicity problem the analytic approach based on Theorem 2.1 if we take into account the fact that the two-dimensional Boole mapping (4.2) is related to the following two-dimensional transformation:

$$T_\varphi\colon [0,1)^2 \ni (s,t) \to (\{2s\}, \{2t\}) \in [0,1)^2$$

on the square $Y = [0,1)^2 \subset \mathbb{R}^2$ owing to the following commutative diagram:

$$\begin{array}{ccccc}
[0,1)^2 & \overset{\cot(\pi\circ)}{\to} & \mathbb{R}^2 & \overset{\varphi}{\to} & \mathbb{R}^2 \\
S\downarrow & & & & \downarrow \cot^{-1}\pi, \\
[0,1)^2 & \overset{T_\varphi(\circ)}{\to} & [0,1)^2 & \overset{\alpha^{-1}}{\to} & [0,1)^2
\end{array} \tag{4.24}$$

where $\alpha^{-1}\colon [0,1)^2 \to \mathbb{R}^2$ stands for the mapping

$$\alpha^{-1}\begin{pmatrix} s \\ t \end{pmatrix} := \begin{pmatrix} \alpha_1^{-1}(s,t) \\ \alpha_2^{-1}(s,t) \end{pmatrix} = \begin{pmatrix} \pi^{-1}\cot^{-1}\left(\dfrac{2\cot\{\pi(s+t)\}}{1+\sin\{\pi(s-t)\}/\sin\{\pi(s+t)\}}\right) \\ \pi^{-1}\cot^{-1}\left(\dfrac{2\cot\{\pi(s-t)\}}{-1+\sin\{\pi(s+t)\}/\sin\{\pi(s-t)\}}\right) \end{pmatrix} \tag{4.25}$$

due to the change of variables $x = \cot(\pi s)$, $y = \cot(\pi t)$, $(s,t) \in [0,1)^2$, $(x,y) \in \mathbb{R}^2$, subject to the new coordinates $(s,t) \in [0,1)^2$ and the transformation

$$S^{-1}\colon [0,1)^2 \ni (s,t) \to (\{s+t\},\{s-t\}) \in [0,1)^2.$$

This approach was proved to be successful and made it possible to obtain a proof of the ergodicity theorem for the two-dimensional Boole transformation (4.1) announced earlier in [35–37].

Theorem 4.3. *The two-dimensional Boole transformation (4.1) is ergodic with respect to the invariant Lebesgue measure λ on $\mathbb{R}^2$.*

Proof. We can now construct the proper cylindrical sets $I_n := I_n(k_0, k_{1,}, \ldots, k_{n-1}; l_0, l_{1,}, \ldots, l_{n-1}) \subset [0,1)^2$, $n \in \mathbb{Z}_+$:

$$I_n = \left\{ \overrightarrow{\prod_{j=\overline{0,n-1}}} \left(S^{-1} \circ \sigma_{k_j,l_j} \circ \alpha\right) : (u,v) \in [0,1)^2 \right\} \tag{4.26}$$

for the diffeomorphically equivalent $\tilde{\varphi}$-mapping $\tilde{\varphi} = (\tilde{\varphi}_1, \tilde{\varphi}_2)^\intercal \colon [0,1)^2 \to [0,1)^2$, where, by definition, $T_\varphi \circ \sigma_{k_j,l_j} = Id : [0,1)^2 \to [0,1)^2$, $\sigma_{k_j,l_j} := (\sigma_{k_j}, \sigma_{l_j})^{-1}$, $k_j, l_j \in \{0,1\}$, $j = \overline{0,n-1}$, $\sigma_0(s) = s/2$ if $s \in [0,1/2)$, $\sigma_1(s) = (1+s)/2$ if $s \in [1/2,1)$, and

$$(\tilde{\varphi}_1, \tilde{\varphi}_2)^\intercal = \cot^{-1}(\pi\circ)\,\alpha^{-1} \circ T_\varphi \circ S, \tag{4.27}$$

satisfy the following obvious conditions:

$$\tilde{\varphi}_1 \circ \left(S^{-1} \circ \pi^{-1}\cot^{-1} \circ \pi\sigma_{k_j} \circ \alpha_1 \circ \cot(\pi\circ)\right)(u,v) = u,$$

$$\tilde{\varphi}_2 \circ \left(S^{-1} \circ \pi^{-1}\cot^{-1} \circ \pi\sigma_{k_j} \circ \alpha_1 \circ \cot(\pi\circ)\right)(u,v) = v$$

for every $(u,v) \in [0,1)^2$, $k_j, l_j \in \overline{0,1}$, $j \in \overline{0,n-1}$. The Lebesgue measure of the cylindrical interval (4.26) can be now easily estimated as follows: By the definition of the Lebesgue measure of an interval, we get

$$\lambda(I_n) = \int_{I_n} du\,dv = \int_{[0,1)^2} \chi_{I_n}(u,v)\,du\,dv$$

$$= \int_{[0,1)^2} \left| J_{(S^{-1}\circ\sigma_{k_{n-1},l_{n-1}}\circ\alpha)\circ(S^{-1}\circ\sigma_{k_{n-2},l_{n-2}}\circ\alpha)\circ\ldots\circ(S^{-1}\circ\sigma_{k_0,l_0}\circ\alpha)}(u,v) \right| du\,dv$$

$$= \int\limits_{[0,1)^2} \prod_{j=\overline{0,n-1}} |J_{S^{-1}}|\, J_{\sigma_{k_j,l_j}} \left|J_\alpha(u_j,v_j)\right| dudv$$

$$= \frac{1}{4^n} \int\limits_{[0,1)^2} \prod_{j=\overline{0,n-1}} \left|J_\alpha(u_j,v_j)\right| dudv, \tag{4.28}$$

where, by definition, $S^{-1} \circ \sigma_{k_j,l_j} \circ \alpha(u_j,v_j) := (u_{j+1},v_{j+1}) \in [2^{-(j+1)},2^{-j})^2$, $\tilde{\varphi}(u_{j+1},v_{j+1}) = (u_j,v_j)$, $j = \overline{0,n-1}$, and $(u_0,v_0) := (u,v) \in [0,1)^2$. In view of the fact that

$$\left(S^{-1} \circ \sigma_{k_{j-1},l_{j-1}} \circ \alpha\right) \circ \left(S^{-1} \circ \sigma_{k_{n-2},l_{n-2}} \circ \alpha\right)$$
$$\ldots \circ \left(S^{-1} \circ \sigma_{k_0,l_0} \circ \alpha\right) \left([0,1)^2\right) \subset [1/2^{j+1}, 1/2^j) \tag{4.29}$$

and $2t \le \sin \pi t \le \pi t$, $2s \le \sin \pi s \le \pi s$ for all $s,t \in [0,1/2]$, the Jacobian product in the integrand of (4.28) can be represented as

$$\prod_{j=\overline{0,n-1}} J_\alpha(u_j,v_j)$$

$$= \prod_{j=\overline{0,n-1}} \frac{\left[\cos^2 \pi(u_j+v_j) + \sin^2 \pi u_j \cos^2 \pi v_j\right]\left[\cos^2 \pi(u_j-v_j) + \sin^2 \pi v_j \cos^2 \pi u_j\right]}{(1-\sin^2 \pi u_j - \sin^2 \pi v_j + 2\sin^2 \pi u_j \sin^2 \pi v_j)}$$

$$= \prod_{j=\overline{0,n-1}} \frac{\left[1-\sin^2 \pi v_j + \sin^2 \pi u_j \sin^2 \pi v_j - 1/2 \sin^2 2\pi u_j \sin 2\pi v_j\right]}{\left(1-\sin^2 \pi u_j - \sin^2 \pi v_j + 2\sin^2 \pi u_j \sin^2 \pi v_j\right)}$$

$$\times \prod_{j=\overline{0,n-1}} \left[1-\sin^2 \pi u_j + \sin^2 \pi u_j \sin^2 \pi v_j + 1/2 \sin^2 2\pi u_j \sin 2\pi v_j\right]. \tag{4.30}$$

One can easily obtain its estimation as follows:

$$\left(\frac{3\pi^2}{4} + \frac{1}{4^2} - 1\right)\left(\frac{\pi^2}{4} - \frac{3}{2} + \frac{1}{4^2}\right) \exp\left[\sum_{j\in\mathbb{Z}_+}\left(\frac{1-\pi^2}{4^j} + \frac{2(1-\pi^4)}{16^j}\right)\right]$$

$$\le \prod_{j=\overline{0,n-1}} J_\alpha(u_j,v_j) \le \left[1 - \frac{\pi^2}{2} + \left(\frac{\pi}{2}\right)^4\right]^{-1} \exp\left[\sum_{j\in\mathbb{Z}_+}\left(\frac{2\pi^2+6}{4^{j+1}} + \frac{1}{16^{j+1}}\right)\right]. \tag{4.31}$$

Thus, by using estimations (4.30), we get the following inequalities for the measure (4.28):

$$\frac{C_1}{4^n} \le \lambda(I_n) \le \frac{C_2}{4^n} \tag{4.32}$$

for any $n \in \mathbb{Z}_+$, where the bounded constants are given by the formulas

$$C_1 := \left(\frac{3\pi^2}{4} + \frac{1}{4^2} - 1\right)\left(\frac{\pi^2}{4} - \frac{3}{2} + \frac{1}{4^2}\right)\exp\left[\sum_{j\in\mathbb{Z}_+}\left(\frac{1-\pi^2}{4^j} + \frac{2(1-\pi^4)}{16^j}\right)\right],$$

$$C_2 := \left[1 - \frac{\pi^2}{2} + \left(\frac{\pi}{2}\right)^4\right]^{-1}\exp\left[\sum_{j\in\mathbb{Z}_+}\left(\frac{2\pi^2+6}{4^{j+1}} + \frac{1}{16^{j+1}}\right)\right]. \tag{4.33}$$

Estimation (4.32) means that we can apply Lemmas 2.1 and 2.2. Thus, we assume that a measurable set $B \subset [0,1)^2$ is invariant:

$$B = \varphi^{-1}B = \varphi^{-n}B, \quad n \in \mathbb{N},$$

and find the following Lebesgue measure:

$$\lambda(B \cap I_n) = \int\limits_{[0,1)^2} \chi_{B\cap I_n}(u,v)dudv$$

$$= \int\limits_{[0,1)^2} \chi_B(u,v)\chi_{I_n}(u,v)dudv = \int\limits_{I_n} \chi_B(u,v)dudv$$

$$= \int\limits_{[0,1)^2} \chi_B\Big(\big(S^{-1}\circ\sigma_{k_{n-1},l_{n-1}}\circ\alpha\big)\circ\big(S^{-1}\circ\sigma_{k_{n-2},l_{n-2}}\circ\alpha\big)$$

$$\ldots\circ\big(S^{-1}\circ\sigma_{k_0,l_0}\circ\alpha\big)(t)\Big)d\lambda\Big(\big(S^{-1}\circ\sigma_{k_{n-1},l_{n-1}}\circ\alpha\big)$$

$$\circ\big(S^{-1}\circ\sigma_{k_{n-2},l_{n-2}}\circ\alpha\big)\circ\ldots\circ\big(S^{-1}\circ\sigma_{k_0,l_0}\circ\alpha\big)(u,v)\Big)$$

$$= \int\limits_{[0,1)} \chi_{\varphi^{-n}B}\Big(\big(S^{-1}\circ\sigma_{k_{n-1},l_{n-1}}\circ\alpha\big)\circ\big(S^{-1}\circ\sigma_{k_{n-2},l_{n-2}}\circ\alpha\big)$$

$$\ldots\circ\big(S^{-1}\circ\sigma_{k_0,l_0}\circ\alpha\big)\Big)(u,v)d\lambda\Big(\big(S^{-1}\circ\sigma_{k_{n-1},l_{n-1}}\circ\alpha\big)$$

$$\circ\big(S^{-1}\circ\sigma_{k_{n-2},l_{n-2}}\circ\alpha\big)\circ\ldots\circ\big(S^{-1}\circ\sigma_{k_0,l_0}\circ\alpha\big)\big)(u,v))$$

$$= \int\limits_{[0,1)} \chi_B\Big(\tilde{\varphi}^n\circ\big(S^{-1}\circ\sigma_{k_{n-1},l_{n-1}}\circ\alpha\big)\circ\big(S^{-1}\circ\sigma_{k_{n-2},l_{n-2}}\circ\alpha\big)$$

$$\ldots\circ\big(S^{-1}\circ\sigma_{k_0,l_0}\circ\alpha\big)(u,v)\Big)d\lambda\big(\big(S^{-1}\circ\sigma_{k_{n-1},l_{n-1}}\circ\alpha\big)$$

$$\circ \left(S^{-1} \circ \sigma_{k_{n-2},l_{n-2}} \circ \alpha\right) \circ \ldots \circ \left(S^{-1} \circ \sigma_{k_0,l_0} \circ \alpha\right) \Big)(u, v))$$

$$= \int\limits_{[0,1)^2} \chi_B(u, v) \left| J_{(S^{-1}\circ\sigma_{k_{n-1},l_{n-1}}\circ\alpha)\circ(S^{-1}\circ\sigma_{k_{n-2},l_{n-2}}\circ\alpha)\circ\ldots\circ(S^{-1}\circ\sigma_{k_0,l_0}\circ\alpha)}(u, v)\right| dudv$$

$$= \int\limits_{[0,1)^2} \chi_B(u, v) \prod_{j=\overline{0,n-1}} |J_{S^{-1}}|\, J_{\sigma_{k_j,l_j}} \left|J_\alpha(u_j, v_j)\right| dudv$$

$$= \frac{1}{4^n} \int\limits_B \prod_{j=\overline{0,n-1}} J_\alpha(u_j, v_j) dudv, \tag{4.34}$$

where we have used the property that the composition $\varphi \circ \left(S^{-1} \circ \sigma_{k_j,l_j} \circ \alpha\right) = Id$ for any $j = \overline{0, n-1}$. Further, it follows from (4.32) that

$$\lambda(B \cap I_n) = \frac{1}{4^n} \int\limits_{[0,1)^2} \chi_B(u, v) \prod_{j=\overline{0,n-1}} J_\alpha(u_j, v_j)$$

$$\geq \frac{C_1}{4^n \lambda(I_n)} \lambda(I_n)\lambda(B) \geq C_1 C_2^{-1} \lambda(I_n)\lambda(B),$$

i.e., the Lebesgue measure $\lambda(I_n)\lambda(B) \leq C\lambda(B \cap I_n)$ for all $n \in \mathbb{Z}_+$, where the constant $C := C_2 C_1^{-1}$. Thus, by virtue of Lemma 2.2, either the Lebesgue measure $\lambda(B) = 1$ or $\lambda(B) = 0$, which simultaneously means the ergodicity of the two-dimensional Boole mapping (4.1) with respect to the same invariant Lebesgue measure λ on $\mathbb{R}^2$, which completes the proof.

As mentioned above, the Lebesgue measure on $\mathbb{R}^3$ is also invariant under the three-dimensional Boole-type transformations (4.3), which are also plausibly ergodic. However, the proofs of this statement are still under search.

5. Conclusions

It is shown that Schweiger's smooth fibered approach based on the Bernoulli-type shift transformations technique is an efficient tool for proving the ergodicity of discrete measure-invariant dynamical systems. In particular, we prove that the one- and two-dimensional Boole-type transformations are ergodic due to the infinite Lebesgue measures.

6. Acknowledgements

The authors are deeply grateful to Ya. V. Mykytyuk and R. A. Kycia for valuable discussions of the discrete Boole-type transformations and their ergodic measure properties. Our cordial thanks belong to O. Boichuk for his kind suggestion to prepare a commemorative research dedicated to the memory of our longtime friend and colleague, prominent Ukrainian mathematician, A. M. Samoilenko.

On behalf of all authors, the corresponding author states that there is no conflict of interest.

REFERENCES

1. J. Aaronson, *An Introduction to Infinite Ergodic Theory*, American Mathematical Society, Providence, RI (1997).
2. R. Bayless and J. Hawkings, "A special class of infinite measure-preserving quadratic rational maps," *Dynam. Syst.*, **34**, No. 2, 218–233 (2019).
3. N. N. Bogoljubov, J. A. Mitropolskii, and A. M. Samoilenko, *Method of Accelerated Convergence in Nonlinear Mechanics*, Hindustan Publ. Corp., Delhi; Springer-Verlag, Berlin, New York (1976).
4. G. Boole, "On the comparison of transcendents with certain applications to the theory of definite integrals," *Philos. Trans. Roy. Soc. London, Ser. B*, **147**, 745–803 (1857).
5. W. Bosma, K. Dajani, and C. Kraaikamp, *Entropy Quotient and Correct Digits in Number-Theoretical Expansions*, Dynamics and Stochastics, Inst. Math. Statist., Beachwood, OH, 176–188 (2006).
6. R. Bowen, "Invariant measures for Markov maps of the interval," *Comm. Math. Phys.*, **69**, 1–17 (1979).
7. B. Leal, G. Mata, and D. Ramirez, "Traslaciones de transformaciones tipo boole robustamente transitivas," *Revista Digital Novasinergia*, **1**, No. 1, 6–13 (2018).
8. T. Meyerovitch, "Ergodicity of Poisson products and applications," *Ann. Probab.*, **41**, No. 5, 3181–3200 (2013).
9. Y. A. Prykarpatsky, D. Blackmore, Y. Golenia, and A. K. Prykarpatski, "Invariant measures for discrete dynamical systems and ergodic properties of generalized Boole type transformations," *Ukr. Math. Zh.*, **65**, No. 1, 44–57 (2013); ***English translation:*** *Ukr. Math. J.*, **65**, No. 1, 47–63 (2013).
10. A. M. Samoilenko, et al., "A geometrical approach to quantum holonomic computing algorithms," *Math. Comput. Simulat.*, **66**, 1–20 (2004).
11. T. Schindler, *A Central Limit Theorem for the Birkhoff-Sum of the Riemann Zeta-Function over a Boolean Type Transformation*, arXiv:2003.02118v1 [math.NT] 4 Mar (2020).
12. F. Schweiger, *Ergodic Theory of Fibred Systems and Metric Number Theory*, Clarendon Press, Oxford Univ. Press, New York (1995).
13. T. O. Banakh, and A. K. Prykarpatsky, "Ergodic deformations of nonlinear Hamilton systems and local homeomorphism of metric spaces," *J. Math. Sci. (N.Y.)*, **241**, No. 1, 27–35 (2019).
14. P. Billingsley, *Ergodic Theory and Information*, John Wiley & Sons, New York, London, Sydney (1965).
15. P. R. Halmosh, *Lectures on Ergodic Theory*, Chelsea Publ. Co., New York (1960).
16. J. Lebowitz and O. Penrose, "Modern ergodic theory," *Phys. Today*, No. 2, 23–29 (1973).
17. P. J. Mitkowski and W. Mitkowski, "Ergodic theory approach to chaos: Remarks and computational aspects," *Int. J. Appl. Math. Comput. Sci.*, **22**, No. 2, 259–267 (2012); DOI: 10.2478/v10006-012-0019-4.
18. K. Okubo and K. Umeno, "Infinite ergodicity that preserves the Lebesgue measure," *Chaos*, **31**, Paper No. 033135 (2021); Doi: 10.1063/5.0029751.
19. P. Oprocha, "Distributional chaos revisited," *Trans. Amer. Math. Soc.*, **361**, No. 9, 4901–4925 (2009).
20. J. A. Roberts and D. T. Tran, "Algebraic entropy of integrable lattice equations and their reductions," *Nonlinearity*, **32**, 622–653 (2019).
21. D. J. Rudolph, *Fundamentals of Measurable Dynamics: Ergodic Theory on Lebesgue Spaces*, Clarendon Press, Oxford Univ. Press, New York (1990).
22. A. M. Samoilenko, *Elements of the Mathematical Theory of Multi-Frequency Oscillations*, Kluwer Acad. Publ., Dordrecht (1991).
23. F. Schweiger, "Number theoretical endomorphisms with σ-finite invariant measure," *Israel J. Math.*, **21**, No. 4, 308–318 (1975).
24. B. Weiss, "The isomorphism problem in ergodic theory," *Bull. Amer. Math. Soc.*, **78**, No. 5, 668–684 (1972).
25. A. Skorokhod, "Homogeneous Markov chains in compact spaces," *Theory Stochast. Process*, **13(29)**, No. 3, 80–95 (2007).
26. S. Ito and M. Yuri, "Number theoretical transformations with finite range structure and their ergodic properties," *Tokyo J. Math.*, **10**, No. 1, 1–32 (1987).
27. M. Yuri, "On a Bernoulli property for multi-dimensional mappings with finite range structure," *Tokyo J. Math.*, **9**, No. 2, 459–485 (1986).
28. M. Pollycott and M. Yuri, *Dynamical Systems and Ergodic Theory*, Cambridge Univ. Press, Cambridge (1998).
29. M. Einsiedler and T. Ward, *Ergodic Theory with a View Towards Number Theory*, Springer-Verlag London, Ltd., London (2011).
30. R. L. Wheedon and A. Zygmund, *Measure and Integral. An Introduction to Real Analysis*, Marcel Dekker, New York, Basel (1977).
31. A. Rényi, "Representations for real numbers and their ergodic properties," *Acta Math. Acad. Sci. Hungar.*, **8**, 477–4937 (1957).
32. A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge Univ. Press (1999).
33. R. Adler and B. Weiss, "The ergodic, infinite measure preserving transformation of Boole," *Israel J. Math.*, **16**, 263–278 (1973).
34. F. Schweiger, "Some remarks on ergodicity and invariant measures," *Michigan Math. J.*, **22**, 308–318 (1975).
35. A. K. Prykarpatsky, "On invariant measure structure of a class of ergodic discrete dynamical systems," *Nonlin. Oscillat.*, **3**, No. 1, 78–83 (2000).
36. A. K. Prykarpatski, *Ergodic Theory, Boole Type Transformations, Dynamical Systems Theory, Current Trends in Analysis and Its Applications,* Birkhäuser/Springer, Cham, 325–333 (2015).
37. A. K. Prykarpatski, "On discretizations of the generalized Boole type transformations and their ergodicity," *J. Phys. Math.*, **7**, No. 3, 1–3 (2016); DOI: 10.4172/2090-0902.1000199.

38. A. K. Prykarpatsky and J. Feldman, "On the ergodic and spectral properties of generalized Boole transformations. I," *Miskolc Math. Notes*, **7**, No. 1, 91–99 (2006).
39. J. Aaronson, "Ergodic theory for inner functions of the upper half plane," *Ann. Inst. H. Poincaré Sect. B (N.S.)*, **14**, No. 3, 233–253 (1978).
40. J. Aaronson, "A remark on the exactness of inner functions," *J. London Math. Soc.*, (2), **23**, No. 3, 469–474 (1981).
41. J. Aaronson, "The eigenvalues of nonsingular transformations," *Israel J. Math.*, **45**, 297–312 (1983).
42. F. Schweiger, *Invariant Measures and Ergodic Properties of Number Theoretical Endomorphisms, Dynamical Systems and Ergodic Theory (Warsaw, 1986)*, PWN, Warszawa (1989).
43. F. Schweiger, "Invariant measures for maps of continued fraction type," *J. Number Theory*, **39**, 162–174 (1991).
44. S. Kakutani, "Induced measure preserving transformations," *Proc. Imp. Acad. Jap.*, **12**, 82–84 (1936).
45. V. A. Rohlin, "Exact endomorphism of a Lebesgue space," *Amer. Math. Soc. Transl. Ser. 2*, **39**, No. 2, 1–36 (1964).