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EVALUATION OF TWO-DIMENSIONAL STRESSES NEAR RIGID INCLUSIONS IN ANISOTROPIC MEDIA ACCORDING TO THE SHERMAN INTEGRAL EQUATIONS AND GREEN'S SOLUTIONS

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We construct regularized Sherman-type integral equations for the plane anisotropic problem of the theory of elasticity with displacements given on the boundaries of the holes. The integral representation of the general solution is obtained in terms of the Lekhnitskii complex potentials by using the Cauchy theorem and, in the case of a half plane and a strip, with additional application of Green's solutions. The properties of the constructed solution are established. According to the Sherman approach, we add regularizing components, which enable us to find a single-valued solution by numerical methods. The developed approach is used to determine elastic stresses in the strip under the conditions of stretching through rigid patches. We also study the distributions of stresses near rigid cylindrical inclusions in isotropic materials and a mass of aleurolite rocks and analyze the mutual influence of inclusions on the stress distribution.

Keywords: anisotropic strip, inclusions, stress state, Green's solutions, method of integral equations.

Introduction

The method of boundary integral equation (MBIE) is extensively used for the investigation of the stress-strain state (SSS) of isotropic and anisotropic multiply connected plates. In the available literature, the first basic problem in which forces are specified on the boundaries of the holes is studied most comprehensively [13–15]. The problems of determination of the SSS of anisotropic plates with given displacements on the boundaries of the holes (in particular, with rigid inclusions) are studied much less completely. In [2], for the solution of problems of this kind, the author used the method of series in combination with the method of conformal maps. The realization of this approach becomes much more complicated for domains of complex shape in the case where it is necessary to keep a large number of terms in the series representation of the mapping function. The Lekhnitskii method and, in particular, its combination with the method of integral equations was used in [3, 7, 9, 10].

As a result of direct application of the Somigliana representation to problems of elasticity with given displacements on the boundaries of the holes, we obtain integral equations with logarithmic singularity in the kernels [13]. The numerical analysis of these equations may lead to significant errors in domains of complex shapes for which it is necessary to introduce a large number of nodal points. Singular integral equations were constructed on the basis of the Lekhnitskii potentials and the Cauchy theorem in [4, 12]. These equations are especially efficient in analyzing the problems with rigid inclusions. For more difficult problems, the cor-

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responding equations contain singular integrals, which complicates the implementation of the analyzed approach.

In the investigation of stresses for the second basic plane problem of the theory of elasticity, it is customary to use simpler Sherman-type integral equations [8]. In the present work, we construct equations of this kind for anisotropic media. The procedure of regularization of the obtained equations is performed by introducing additional components in the same way as this was done in [8]. The boundary-value problems for a half plane and a strip containing inclusions are considered on the basis of Green's solution. The problem is solved numerically by the method of mechanical quadratures.

Rock masses often have anisotropic mechanical characteristics. The investigations of stresses formed in these masses were carried out only in a few works [7, 15]. In the present work, we study the stresses acting near rigidly reinforced cylindrical cavities and their systems with isotropic and transversely isotropic mechanical characteristics.

1. Statement of the Problem

Consider an infinite anisotropic elastic medium occupying a domain D with holes bounded by contours L_j, $j=1,\ldots,J$. We denote the domains occupied by the holes by D_j. Their centers of weight are denoted by C_j. We also assume that displacements (u_D, v_D) are specified on the boundaries of the holes and the medium is loaded by concentrated forces and uniformly loaded at infinity. The principal vector (X_j, Y_j) and the principal moment M_j (relative to the points C_j) of all forces applied to the boundary contours L_j, $j=1,\ldots,J$, are regarded as known.

The solution of the problem of elasticity theory is expressed via the Lekhnitskii potentials $\Phi(z_1)$ and $\Psi(z_2)$, where $z_m = x + s_m y$. Here, s_m, $m=1,2$, are the roots of the characteristic equation [3]

$$\alpha_{11}s^4 - 2\alpha_{16}s^3 + (2\alpha_{12}+\alpha_{66})s^2 - 2\alpha_{26}s + \alpha_{22} = 0,$$

with positive imaginary part and the coefficients α_{ij}, $i,j=1,2,6$, are elastic constants of the material in the plane stressed state. In analyzing the plane strained state, these coefficients are determined via the elastic constants by the relations presented in [3].

To find potentials, it is necessary to satisfy the conditions imposed on the boundaries of the holes [3, 11]

$$\begin{aligned} 2\,\mathrm{Re}\left[p_1\Phi_L(z_1)z_1' + p_2\Psi_L(z_2)z_2'\right] &= U_D, \\ 2\,\mathrm{Re}\left[q_1\Phi_L(z_1)z_1' + q_2\Psi_L(z_2)z_2'\right] &= V_D, \end{aligned} \tag{1}$$

where

$$z_i' = \frac{dz_i}{ds}, \qquad U_D = \frac{du_D}{ds}, \qquad V_D = \frac{dv_D}{ds},$$

$$p_i = \alpha_{11}s_i^2 - \alpha_{16}s_i + \alpha_{12}, \qquad q_i = \alpha_{12}s_i - \alpha_{26} + \frac{\alpha_{22}}{s_i}, \quad i=1,2.$$

Here and in what follows, the arc coordinate s increases in the course of motion along the boundary in the direction for which the domain D remains on the left side; moreover, integration is carried out in the same direction.

In the case where displacements are given on the boundaries of the holes in isotropic materials, Sherman-type integral equations proved to be quite efficient [8]. We now construct equations of this kind for anisotropic plates. To do this, we introduce bounded plates occupying the domains D_j, $j=1,\ldots,J$. Assume that the displacements $u_D(x,y)$ and $v_D(x,y)$, $(x,y)\in L_j$ are specified on their boundaries L_j. By (u^-,v^-) we denote the displacements formed in this case in the domains D_j and by (X^-,Y^-) we denote the corresponding vector of stresses on the boundary contour.

On the contours L_j, the vector of displacements is continuous, whereas the stress vectors have jumps $P=X_D-X^-$ and $Q=Y_D-Y^-$.

Consider a domain, which is either internal or external relative to an arbitrary closed contour Γ. Denote by (u,v) and (X_Γ,Y_Γ) the boundary values of the vectors of displacements and stresses on the contour Γ for this domain, respectively.

Then the following relations are true on the contour Γ [12]:

$$\Phi(z_1) = \frac{-v'+s_1u'+p_1X_\Gamma+q_1Y_\Gamma}{\Delta_1 z_1'}, \qquad \Psi(z_2) = \frac{-v'+s_2u'+p_2X_\Gamma+q_2Y_\Gamma}{\Delta_2 z_2'},$$

where $u'=du/ds$, $v'=dv/ds$,

$$\Delta_1 = \alpha_{11}(s_1-s_2)(s_1-\bar{s}_1)(s_1-\bar{s}_2), \qquad \Delta_2 = \alpha_{11}(s_2-s_1)(s_2-\bar{s}_1)(s_2-\bar{s}_2).$$

This implies that the Lekhnitskii potentials have a jump on the contours L_j, $j=1,\ldots,J$, namely,

$$\Phi^+ - \Phi^- = \frac{p_1P+q_1Q}{\Delta_1 z_1'} \quad \text{and} \quad \Psi^+ - \Psi^- = \frac{p_2P+q_2Q}{\Delta_2 z_2'},$$

where the boundary values of potentials on the contour in the domain D are marked by the superscript "+", whereas on the contour in the domain D_j, the values of potentials are marked by the superscript "–", and P and Q are unknown functions.

Thus, by virtue of the Cauchy theorem, as in [12], we obtain the following Sherman-type integral representations for the Lekhnitskii potentials:

$$\Phi(z_1) = \Phi_L(z_1) + \Phi_D(z_1),$$
$$\Psi(z_2) = \Psi_L(z_2) + \Psi_D(z_2), \tag{2}$$

where

$$\Phi_L(z_1) = \int_L \left[\Phi_3(z_1,T)P(s)+\Phi_4(z_1,T)Q(s)\right] ds,$$

$$\Psi_L(z_2) = \int_L \left[\Psi_3(z_2,T)P(s)+\Psi_4(z_2,T)Q(s)\right] ds,$$

$ds = \sqrt{(d\xi)^2+(d\eta)^2}$, T is the point with respect to which we perform integration, $T(\xi,\eta)\in L$, and $L = L_1+\ldots+L_J$,

$$\Phi_k(z_1,T) = -\frac{A_k}{z_1-t_1}, \quad \Psi_k(z_2,T) = -\frac{B_k}{z_2-t_2}, \quad k=3,4, \quad t_{1,2} = \xi+s_{1,2}\eta,$$
$$A_3 = -\frac{ip_1}{2\pi\Delta_1}, \quad A_4 = -\frac{iq_1}{2\pi\Delta_1}, \quad B_3 = -\frac{ip_2}{2\pi\Delta_2}, \quad B_4 = -\frac{iq_2}{2\pi\Delta_2}. \tag{3}$$

The potentials Φ_D and Ψ_D give the solution of the problem of elasticity theory for a solid plane loaded by concentrated forces and forces applied at infinity.

2. Properties of Solution (2)

I. We now find the resultant of forces applied to the contour L_j and corresponding to potentials (2). For this purpose, we draw a closed contour Γ in the domain D around the boundaries L_j, $j=1,\ldots,J$. The vector of stresses on this contour is determined by the following formulas [3]:

$$Y_\Gamma = -2\,\mathrm{Re}\left[\Phi_L(z_1)z_1'+\Psi_L(z_2)z_2'\right],$$
$$X_\Gamma = 2\,\mathrm{Re}\left[s_1\Phi_L(z_1)z_1'+s_2\Psi_L(z_2)z_2'\right], \quad (x,y)\in\Gamma. \tag{4}$$

The resultant of forces (P_x,P_y) corresponding to the vector of stresses (X_Γ,Y_Γ) is given by the following formulas:

$$P_x = 2\,\mathrm{Re}\left[s_1\int_\Gamma \Phi_L(z_1)z_1'\,ds_\Gamma + s_2\int_\Gamma \Psi_L(z_2)z_2'\,ds_\Gamma\right],$$

$$P_y = -2\,\mathrm{Re}\left[\int_\Gamma \Phi_L(z_1)z_1'\,ds_\Gamma + \int_\Gamma \Psi_L(z_2)z_2'\,ds_\Gamma\right].$$

According to the residue theorem, we get

$$\int_{\Gamma} \Phi_L z' \, ds_\Gamma = \int_{L_j} \left(P \int_{\Gamma} \Phi_3 z_1' \, ds_\Gamma + Q \int_{\Gamma} \Phi_4 z_1' \, ds_\Gamma \right) ds = -\frac{1}{\Delta_1} \int_{L_j} (p_1 P + q_1 Q) \, ds,$$

$$\int_{\Gamma} \Psi_{L1} z_2' \, ds_\Gamma = \int_{L_j} \left(P \int_{\Gamma} \Psi_3 z_2' \, ds_\Gamma + Q \int_{\Gamma} \Psi_2 z_2' \, ds_\Gamma \right) ds = -\frac{1}{\Delta_2} \int_{L_j} (p_2 P + q_2 Q) \, ds.$$

Hence, we find

$$P_x = -\int_{\Gamma} (s_{11} P + s_{21} Q) \, ds \quad \text{and} \quad P_y = \int_{\Gamma} (s_{10} P + s_{20} Q) \, ds,$$

where

$$s_{1k} = 2 \operatorname{Re} \left(\frac{p_1 s_1^k}{\Delta_1} + \frac{p_2 s_2^k}{\Delta_2} \right) \quad \text{and} \quad s_{2k} = 2 \operatorname{Re} \left(\frac{q_1 s_1^k}{\Delta_1} + \frac{q_2 s_2^k}{\Delta_2} \right), \quad k = 0, 1.$$

By virtue of [11, 12], we obtain

$$s_{10} = s_{21} = 0, \; s_{11} = 1, \text{ and } s_{20} = -1.$$

Then

$$P_x = -\int_{\Gamma} P \, ds \quad \text{and} \quad P_y = -\int_{\Gamma} Q \, ds.$$

Directing the contour Γ toward the contour L_j, we establish the following conditions:

$$X_j = \int_{L_j} P \, ds \quad \text{and} \quad Y_j = \int_{L_j} Q \, ds, \quad j = 1, \ldots, J. \tag{5}$$

Similarly, it can be proved that the principal moment relative to the point $C_j(x_j^C, y_j^C)$ (center-of-weight of the domain D_j) is determined as follows:

$$M_j = \int_{L_j} \Big(\big(y - y_j^C\big) P - \big(x - x_j^C\big) Q \Big) ds, \quad j = 1, \ldots, J. \tag{6}$$

Thus, the unknown functions P and Q must satisfy conditions (5) and (6).

II. We find the integrals of the derivatives of displacements on the indicated contour Γ corresponding to potentials (2). By using formula (1), we can write

$$\int_\Gamma u'\,ds = 2\,\mathrm{Re}\left[p_1\int_\Gamma \Phi(z_1)z_1'\,ds_\Gamma + p_2\int_\Gamma \Psi(z_2)z_2'\,ds_\Gamma\right],$$
$$\int_\Gamma v'\,ds = 2\,\mathrm{Re}\left[q_1\int_\Gamma \Phi(z_1)z_1'\,ds_\Gamma + q_2\int_\Gamma \Psi(z_2)z_2'\,ds_\Gamma\right]. \qquad (7)$$

Hence, for $j=1,\ldots,J$, we find

$$[u] = \int_{L_j} u'\,ds = -\int_{L_j}(r_{20}P + r_{11}Q)\,ds,$$

$$[v] = \int_{L_j} v'\,ds = -\int_{L_j}(r_{11}P + r_{02}Q)\,ds,$$

where

$$r_{20} = 2\,\mathrm{Re}\left(\frac{p_1^2}{\Delta_1}+\frac{p_2^2}{\Delta_2}\right), \qquad r_{11} = 2\,\mathrm{Re}\left(\frac{p_1q_1}{\Delta_1}+\frac{p_2q_2}{\Delta_2}\right), \qquad r_{02} = 2\,\mathrm{Re}\left(\frac{q_1^2}{\Delta_1}+\frac{q_2^2}{\Delta_2}\right).$$

According to [11, 12], we have

$$r_{20} = r_{11} = r_{02} = 0.$$

This yields the following conditions, which are identically satisfied:

$$\int_{L_j} u'\,ds = 0 \quad \text{and} \quad \int_{L_j} v'\,ds = 0, \quad j=1,\ldots,J. \qquad (8)$$

3. Integral Equations

The integral equations for finding the functions P and Q are obtained from the condition that the vector of displacements on the boundaries of the holes is equal to a given vector [11]. In this case, we assume that the boundary of each hole may rotate as a rigid body. Substituting potentials (2) in relation (1) and applying the Sochocki–Plemelj formulas, we arrive at the following system of equations:

$$2\,\mathrm{Re}\left[p_1\tilde{\Phi}_L(z_1)z_1' + p_2\tilde{\Psi}_L(z_2)z_2'\right] + H_1 = f - \omega_j\frac{dy}{ds},$$
$$2\,\mathrm{Re}\left[q_1\tilde{\Phi}_L(z_1)z_1' + q_2\tilde{\Psi}_L(z_2)z_2'\right] + H_2 = g + \omega_j\frac{dx}{ds}, \qquad (x,y)\in L, \qquad (9)$$

where $\tilde{\Phi}_L$ and $\tilde{\Psi}_L$ are potentials (2) in which integrals are understood in a sense of Cauchy principal value,

$$H_1 = (Pr_{20} + Qr_{11})/2, \qquad H_2 = (Pr_{11} + Qr_{21})/2,$$

$$f = U_D(x,y) - 2\,\mathrm{Re}\left[p_1\Phi_D(z_1)z_1' + p_2\Psi_D(z_2)z_2'\right],$$

$$g = V_D(x,y) - 2\,\mathrm{Re}\left[q_1\Phi_D(z_1)z_1' + q_2\Psi_D(z_2)z_2'\right].$$

In the system of equations (9), the rotation of the boundary of each hole as a rigid body by an unknown angle ω_j, which is determined in what follows from the condition that the moment applied to the boundary is known, is added to the displacements specified on the contours.

According to [12], we obtain $H_1 = H_2 = 0$.

Substituting potentials (2) in conditions (9), we arrive at the following equations:

$$\int_L \left[U_1^L(Z,T)P(T) + U_2^L(Z,T)Q(T)\right]ds = g(Z),$$

$$\int_L \left[V_1^L(Z,T)P(T) + V_2^L(Z,T)Q(T)\right]ds = f(Z), \tag{10}$$

$$\int_{L_j} \left((y - y_j^C)P - (x - x_j^C)Q\right)ds = M_j, \quad j = 1,\ldots,J,$$

where $U_j^L(Z,T)$ and $V_j^L(Z,T)$ are the derivatives of displacements at the point $Z(x,y) \in L$ determined by using relations (1) via the complex potentials $\Phi_j(z_1,T)$, $\Psi_j(z_2,T)$, $j = 1,\ldots,J$, and $T(\xi,\eta) \in L$.

According to the properties of the chosen general solution presented above, the integral equations have eigensolutions and, moreover, conditions (5), (6) must be satisfied. To solve equations of this kind, we use the Sherman approach [8]. For this purpose, to the general solution (2) (i.e., to the potentials Φ_L and Ψ_L) on the right side we add

$$\Phi_\Delta(z_1) = \sum_{j=1}^{J}\left[P_j^\Delta \Phi_1(z_1,C_j) + Q_j^\Delta \Phi_2(z_1,C_j)\right],$$

$$\Psi_\Delta(z_2) = \sum_{j=1}^{J}\left[P_j^\Delta \Psi_1(z_2,C_j) + Q_j^\Delta \Psi_2(z_2,C_j)\right], \tag{11}$$

where P_j^Δ and Q_j^Δ are unknown constant quantities and the functions Φ_k and Ψ_k, $k = 1,2$, are determined by using relations (3) in which

$$A_1 = -\frac{is_1}{2\pi\Delta_1}, \quad A_2 = \frac{i}{2\pi\Delta_1}, \quad B_1 = -\frac{is_2}{2\pi\Delta_2}, \quad B_2 = -\frac{i}{2\pi\Delta_2}.$$

The functions Φ_k and Ψ_k, $k = 1, 2$, are dislocation solutions [12], i.e., solutions specifying the displacements with jumps as a result of complete traversing around the points C_j. Thus, on the right-hand sides of Eqs. (10), it is necessary to replace $f(Z)$ and $g(Z)$ by $f(Z) + f_\Delta(Z)$ and $g(Z) + g_\Delta(Z)$, respectively, where

$$f_\Delta = \sum_{j=1}^{J} \left[P_j^\Delta V_{1j}(Z) + Q_j^\Delta V_{2j}(Z) \right], \quad g_\Delta = \sum_{j=1}^{J} \left[P_j^\Delta U_{1j}(Z) + Q_j^\Delta U_{2j}(Z) \right],$$

and $U_{kj}^L(Z)$ and $V_{kj}^L(Z)$ are the derivatives of displacements at the point $Z(x, y) \in L$ given by relations (1) via the corresponding complex potentials $\Phi_k(z_1, C_j)$, $\Psi_k(z_2, C_j)$, $k = 1, 2$.

We relate the constants P_j^Δ and Q_j^Δ to the functions P and Q as follows:

$$P_j^\Delta = \int_{L_j} P\, ds - X_j \quad \text{and} \quad Q_j^\Delta = \int_{L_j} Q\, ds - Y_j. \tag{12}$$

It is easy to see that if we find the solution of the integral equations (10) modified in this way, then, instead of the identities established above, we get the following equation on each contour L_j:

$$P_j^\Delta = 0, \quad Q_j^\Delta = 0, \quad j = 1, \ldots, J,$$

i.e., conditions (5) are satisfied.

The integral equations (10) are solved with the help of the quadrature formulas for regular and singular integrals on closed contours presented in [5, 11].

The solution constructed above can be generalized to the case of domains of more complex shapes with the help of Green's solutions. In particular, we write the integral equations for the half plane $y < 0$ with holes on the basis of Green's solution. These equations are obtained if, in relations (2) and (11), we set [4]

$$\Phi_k(z_1) = -\left(\frac{A_k}{z_1 - t_1} + \alpha_1 \frac{\overline{A}_k}{z_1 - \overline{t}_1} + \beta_1 \frac{\overline{B}_k}{z_1 - \overline{t}_2} \right),$$

$$\Psi_k(z_2) = -\left(\frac{B_k}{z_2 - t_2} + \alpha_2 \frac{\overline{A}_k}{z_2 - \overline{t}_1} + \beta_2 \frac{\overline{B}_k}{z_2 - \overline{t}_2} \right), \quad k = 1, \ldots, 4,$$

where

$$\alpha_1 = \frac{\overline{s}_1 - s_2}{\delta_1}, \quad \alpha_2 = \frac{s_1 - \overline{s}_1}{\delta_1},$$

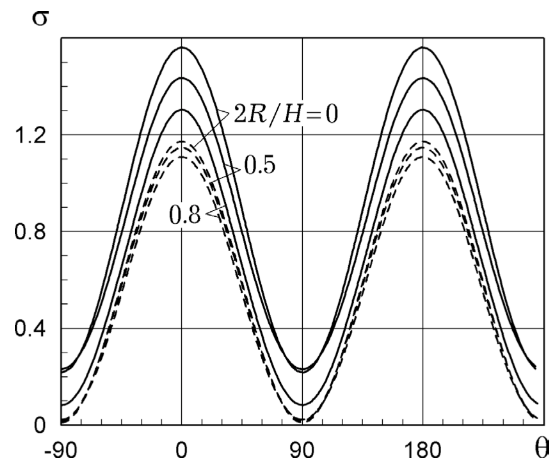

Fig. 1. Tension in a boron-epoxy strip with circular inclusion.

$$\beta_1 = \frac{\bar{s}_2 - s_2}{\delta_1}, \quad \beta_2 = \frac{s_1 - \bar{s}_2}{\delta_1}, \quad \delta_1 = s_2 - s_1.$$

Moreover, we assume that the potentials Φ_D and Ψ_D specify the solution of the problem of elasticity theory for solid half plane with load-free boundary subjected to the action of forces applied to the body.

If we use Green's solutions, then the conditions imposed on the horizontal boundary are automatically satisfied.

The functions $\Phi_{3,4}$ and $\Psi_{3,4}$ for the strip are presented in [6].

4. Numerical Results

4.1. Tension of a Strip Containing an Inclusion. Consider a boron–epoxy [6] strip $-H < y < 0$ containing a circular central inclusion of radius R. Assume that the strip is stretched by forces p and its boundaries are free of loads. We represent the solution as the sum of two components. The first component corresponds to the stresses formed in the solid strip $\sigma_x^0 = p$, $\sigma_y^0 = 0$, and $\tau_{xy}^0 = 0$. The displacements in this strip are given by the formulas

$$u_0 = \varepsilon_x^0 x + \gamma_{xy}^0 y, \quad v_0 = \varepsilon_y^0 y,$$

where

$$\varepsilon_x^0 = a_{11}\sigma_x^0 + a_{12}\sigma_y^0 + a_{16}\tau_{xy}^0,$$

$$\varepsilon_y^0 = a_{21}\sigma_x^0 + a_{22}\sigma_y^0 + a_{26}\tau_{xy}^0,$$

$$\gamma_{xy}^0 = a_{61}\sigma_x^0 + a_{62}\sigma_y^0 + a_{66}\tau_{xy}^0.$$

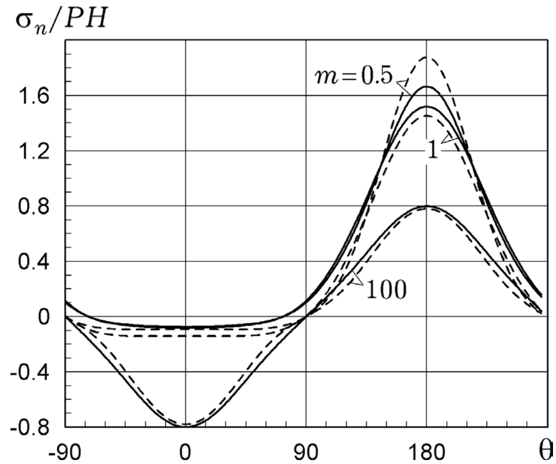

Fig. 2. Relative normal stresses on the boundary of the right inclusion in the case of tension by forces P.

The second component is a correcting solution. Its determination is reduced to the analysis of the boundary-value problem (1) in which

$$U_D = -\left(\varepsilon_x^0 \frac{dx}{ds} + \gamma_{xy}^0 \frac{dy}{ds}\right), \qquad V_D = -\varepsilon_y^0 \frac{dy}{ds},$$

and the inclusion is assumed to be unloaded.

In Fig. 1, we present the results of evaluation of the relative normal stresses $\sigma = \sigma_n / p$. The solid lines correspond to case where the stiffness of the material is maximum in the vertical direction, whereas the dashed lines correspond to case where the stiffness of the material is maximum in the horizontal direction. The curves were computed for the following ratios of the diameter of the inclusion to the width of the strip: $2R/H = 0, 0.5, 0.8$. Here and in what follows, the angular coordinate $\theta = -90°$ corresponds to the highest point of the circle and, moreover, this coordinate increases as we move in the clockwise direction.

In Fig. 1, we see that the highest stresses near the inclusion appear for small sizes of the inclusion. As the sizes of the inclusion increase, the level of normal stresses decreases. For the maximum stiffness of the material in the vertical direction, the stresses are much higher than in the case where the stiffness of the material in this direction is minimum.

The numerical analysis was carried out for an isotropic material with Poisson's ratio $\nu = 0.3$ and a small radius of the inclusion ($R/H = 0.01$). In the case of 80 chosen nodal points, the computed maximum normal and circumferential stresses referred to p are equal to 1.477 and 0.6299. These values are in good agreement with the data obtained on the basis of the analytic solution, namely, 1.4778 and 0.6333 [3].

4.2. Tension of a Strip by Forces Applied to Inclusions. We consider a boron–epoxy strip with two circular inclusions of radius $R = 0.25\,H$ centered at the points $(\pm mH, -0.5H)$. Concentrated forces $(-P, 0)$, $(P, 0)$ are applied to the centers of the inclusions. The relative normal stresses $\sigma = \sigma_n / PH$ on the boundary of the right inclusion computed for the values $m = 0.5$, 1, 100 are presented in Fig. 2. The solid curves correspond to the case where the stiffness is maximum in the vertical direction, while the dashed curves correspond to the case where the stiffness is maximum in the horizontal direction.

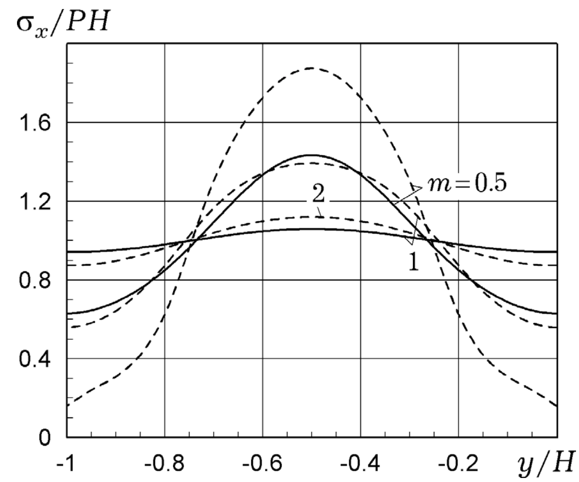

Fig. 3. Stresses between inclusions.

For small distances between the inclusions, $m<1$, the highest stresses on their boundaries are formed in the domains that are closer to the center of the strip. At the same time, at remote points, the stresses are practically absent. As the distance between inclusions increases, the maximum stresses on the boundary become lower. The lowest stresses are formed for large distances between the inclusions. Moreover, these stresses are lower than the stresses averaged over the thickness of the strip.

In Fig. 3, we show the plots of normal relative stresses σ_x/PH in the vertical section between the inclusions for $m=0.5$, 1, 2.

If the distances between the inclusions are larger than two thicknesses (for $m \geq 1$), then the stresses formed in the region between the inclusions are almost constant and equal to the stresses acting in the solid strip in the case where the stiffness is maximum in the vertical direction; the same picture is observed if the distances between the inclusions are larger than four thicknesses (for $m \geq 2$) and the stiffness is maximum in the horizontal direction. For smaller distances, $m<1$, the stresses noticeably differ from their values averaged over the thickness of the strip.

4.3. Gravitational Stresses in a homogeneous body are represented in the following form (the Oy-axis is vertical) [1]:

$$\sigma_y^0 = \gamma y, \quad \sigma_y^0 = \lambda\sigma_y^0, \quad \tau_{xy}^0 = 0, \tag{14}$$

where γ is the specific weight of the body.

The coefficient of lateral pressure λ is determined experimentally [1]. The researchers often use the Dynnyk hypothesis according to which only vertical displacements occur in a homogeneous massif and, therefore, for an isotropic material under the condition of plane deformation, we get $\lambda = \nu/(1-\nu)$.

In the case of orthotropic material, the displacements are as follows:

$$u = a_{11}\lambda\gamma yx, \quad v = \frac{a_{22}\gamma y^2 - a_{11}\lambda\gamma x^2}{2}.$$

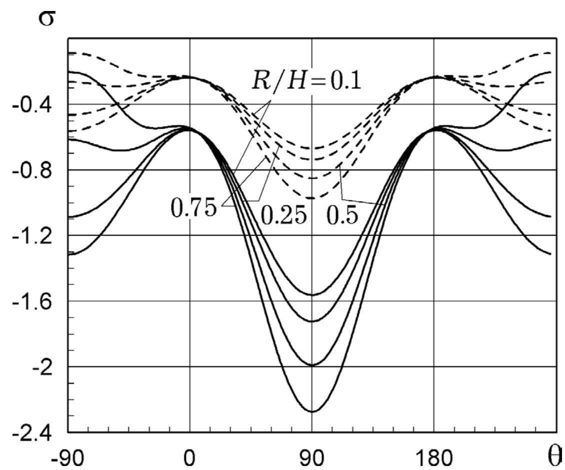

Fig. 4. Stresses formed near the inclusion in an isotropic material under the action of its own weight.

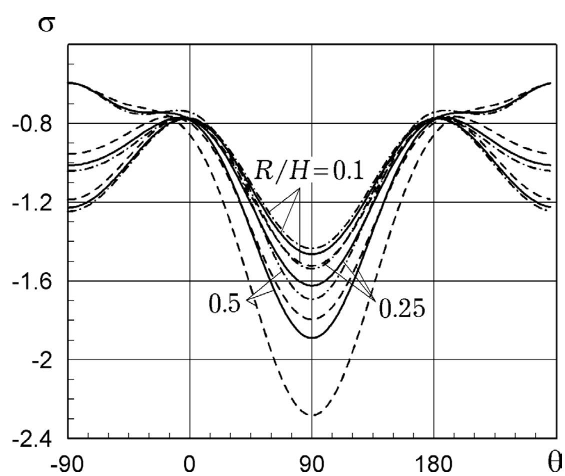

Fig. 5. Relative normal stresses on the boundary of aleurolite and the inclusion.

The stresses formed in the massif with inclusions are represented as the sum of the gravitational stresses (14) and a correcting solution. We express the correcting solution in terms of the Lekhnitskii potentials $\Phi(z_1)$ and $\Psi(z_2)$ determined from the boundary conditions (1) for

$$U_D = -a_{11}\lambda\gamma(y'x+xy') \quad \text{and} \quad V_D = -a_{22}\gamma yy' + a_{11}\lambda\gamma xx',$$

where $x' = dx/ds$ and $y' = dy/ds$.

Consider a half plane $y<0$ in the plane strain state under the action of its own weight. It is assumed that the inclusion has a circular shape of radius R and its center is located at the point $(0, -H)$. The material is supposed to be isotropic with Poisson's ratio $\nu = 0.3$. It is also assumed that the densities of the inclusion and the base material are equal. In Fig. 4, we show the plots of the computed relative normal stresses $\sigma_n/\gamma H$ (solid curves) and circumferential stresses $\sigma_\theta/\gamma H$ (dashed curves) on the boundary of the inclusion for the following ratios of the radius of the inclusion to the depth: $R/H = 0.1, 0.25, 0.5, 0.75$.

In Fig. 4, we see that the highest compressive stresses are formed near the bottom part of the inclusion. These stresses increase with the sizes of the inclusion. The magnitude of circumferential stresses is much smaller than the magnitude of normal stresses.

We consider the transport material (aleurolite) [1]. The computed relative normal stresses on the boundary of the inclusion are shown in Fig. 5. In this figure, the solid curves correspond to the case where the density of the base material and the density of the inclusion are identical; the dashed curves correspond to case where the density of the inclusion material is twice lower than the density of the base material, and the dash-dotted curves correspond to case where the density of the material of the inclusion is two times higher than the density of the base material.

It is easy to see that the magnitude of the maximum normal stresses increases with the weight of the reinforcing inclusion.

CONCLUSIONS

We propose an approach to the investigation of two-dimensional stresses in anisotropic materials with rigid inclusions based on the method of boundary integral equations. The equations are written on the basis of the Lekhnitskii potentials and the Cauchy theorem. The deduced equations are regularized by the Sherman method. The integral equations for a half plane and a strip are generalized by using Green's solutions for which the conditions imposed on the rectilinear boundaries are identically satisfied. By using the developed approach, we study the stresses formed in the strip in the course of its tension at infinity and by concentrated forces applied to the inclusions. We also performed the analysis of stresses acting on the boundaries of the inclusions and in their vicinities. In particular, it is established that the stresses formed on the boundaries of inclusions are lower than the stresses averaged over the width of the strip. In the case of tension at infinity, the stresses are concentrated in the regions located in the direction opposite to the direction of action of the forces. The stresses with the lowest magnitude are formed near inclusions for large distances between the inclusions.

We also determine elastic stresses near cavities in the massifs of rocks reinforced by rigid materials. The stresses near the cavities with circular sections are investigated for isotropic materials and a massif of aleurolite rocks with regard for their own weight. We considered inclusions with different specific weights.

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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