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HOLOMORPHIC CONTINUATION OF FUNCTIONS ALONG A FIXED DIRECTION (SURVEY)

A. S. Sadullaev UDC 517.55

Abstract. In this article, we give an overview of the most significant and important results on holomorphic extensions of functions along a fixed direction. We discuss the following geometric questions of multidimensional complex analysis:

- holomorphic extension along a bundle of complex straight line, the Forelly theorem;
- holomorphic continuation of functions with thin singularities along a fixed direction;
- holomorphic continuation of functions along a family of analytic curves.

Keywords: holomorphic extension, holomorphic continuation, multidimensional complex analysis, Forelly theorem.

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CONTENTS

1. Introduction

The classical Hartogs's theorem states that if the function $f\left('z, z_n\right)$ is holomorphic in the polycircle $'U \times \{|z_n| < r_n\}$, $r_n > 0$ and is such that for any fixed $'z^0 \in {}'U$ the function $f\left('z^0, z_n\right)$, which is holomorphic by z_n in the circle $\{|z_n| < r_n\}$, is holomorphically extended to the bigger circle $\{|z_n| < R_n\}$, $R_n > r_n$, then $f\left('z, z_n\right)$ is holomorpically extended to the polycircle $'U \times \{|z_n| < R_n\}$ by $('z, z_n)$.

Hartog's theorem proved in the beginning of the previous century was developed and extended in many ways. In the literature one can find a large variety of papers and books contatining such extensions and applications in various fields of mathematics. In this paper we briefly describe the next Hartog's type theorems:

- a holomorhpic extension of formal series along a bundle of complex straight lines (Forelli's theorem);
- a holomorphic extension of the functions with thin singularities along a fixed direction (Chirki–Sadullaev theorem);
- a holomorphic extension of the function along a family of analytic curves.

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The main method used in the study of the considered questions is pluripotential theory, as well as the object and methods of this theory such as polyharmonic measures, Green's functions, capacities etc. We recall the basics of pluripotential theory in Sec. 2 (more details can be found, e.g., in [9, 24, 25, 27]).

2. Elements of Pluripotential Theory

The classical potential theory is based on subharmonic functions and Laplace operator Δ. Pluripotential theory developed in 1980-1990 based on plurisubharmonic (*psh*) functions is connected to Monge–Ampère operator $(dd^c u)^n$, where, as usual, $d = \partial + \overline{\partial}$, $d^c = i\left(\bar{\partial} - \partial\right)$. The theory was built thanks to the research of many authors, is being successfully developed and applied in various branches of science (see [1–35] and the bibliography).

For the twice differentiable function $u \in C^2(G)$, $G \subset \mathbb{C}^n$, by definition

$$(dd^c u)^k = \underbrace{dd^c u \wedge dd^c u \wedge \ldots \wedge dd^c u}_{k \text{ times}}$$

represents the differential form of bidegree (k,k). It is easy to prove that

$$(dd^c u)^n = \pi^n n! \det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right)\beta_n,$$

where $\beta_n = \left(\frac{i}{2}\right)^n \prod\limits_{j=1}^{n} dz_j \wedge d\,\bar{z}_j$ is the capacity form in the space $\mathbb{C}^n$.

The operator $(dd^c u)^k$ for an arbitrary bounded plurisubharmonic function $-M \le u(z) \le M$, $M = const$, is defined in the generalized sense as the stream. The recurrent relation

$$\int (dd^c u)^k \wedge \phi = \int u\,(dd^c u)^{k-1} \wedge dd^c\phi, \quad \phi \in D^{(n-k,n-k)}(G), \quad k = 1, 2, \ldots, n, \tag{2.1}$$

where $D^{(n-k,n-k)}(G)$ is the space of infinitely smooth finitely differentiable forms of the bidegree $(n-k, n-k)$, defines $(dd^c u)^k$ as a positive stream of the bidegree (k,k). The fundamental theorem of pluripotential theory states that streams $(dd^c u_j)^k$ weakly converges for monotonically decreasing locally bounded sequence of *psh* functions: if $u_j(z) \in psh(G) \bigcap L^\infty_{loc}(G)$, $u_j(z) \downarrow u(z)$, then $(dd^c u_j)^k \mapsto (dd^c u)^k$. This circumstance allows us to apply the stream of the measure type $(dd^c u)^k$ to the class $u(z) \in psh(G) \bigcap L^\infty_{loc}(G)$ as the differential form $(dd^c u)^k$ to the class $u(z) \in psh(G) \bigcap C^2(G)$.

For a subset $E \subset G$ of the space $G \subset \mathbb{C}^n$ define a $\mathcal{P}$-measure. Put

$$\omega(z, E, G) = \sup\left\{u(z) \in psh(G): \ u|_E \le -1, \ u|_G < 0\right\}.$$

Then the regularization $\omega^*(z, E, G) = \overline{\lim\limits_{w \to z}}\, \omega(z, E, G)$ is called a *$\mathcal{P}$-measure* of the set E relative to the domain G. In substantive theory we usually suppose that the domain is G-regular and that there exists a function $\rho(z) \in psh(G)$ such that $\rho|_G < 0$, $\lim\limits_{z \to \partial G} \rho(z) = 0$. In these assumptions the $\mathcal{P}$-measure $\omega^*(z, E, G)$ either not equals zero anywhere, $-1 \le \omega^*(z, E, G) < 0$, or $\omega^*(z, E, G) \equiv 0$. The last expression holds iff E is pluripolar in G, i.e., there exists a plurisubharmonic function $\sigma(z) \not\equiv -\infty$ such that $\sigma|_E \equiv -\infty$. In applications, especially in the estimates of holomorphic functions, the following two constants theorem is used: if $u \in psh(G)$, $u|_G \le M$, $u|_E \le m$, then

$$u(z) \le M(1 + \omega * (z, E, G)) - m\omega * (z, E, G) \ \ \forall z \in G. \tag{2.2}$$

Inequality (2.2) is meaningful if E is not a pluripolar set, i.e., if $\omega^* \not\equiv 0$.

Similarly to harmonic functions in the classical potential theory, a locally bounded pluriharmonic function $u(z) \in psh(G) \bigcap L^\infty_{loc}(G)$ is called *maximal function* if its Monge–Ampère operator equals zero: $(dd^c u)^n = 0$. For $n = 1$ Monge–Amp'ere operator $dd^c u = 4\Delta u dx \wedge dy$. Hence, maximal functions in the case $n = 1$ are pluriharmonic and hence they are infinitely smooth. Unlike the case $n = 1$, for $n > 1$ maximal functions are not necessarily smooth: they can be even discontinuous. Nevertheless, the following theorem holds.

Theorem 2.1. *For any compact set $K \subset G$ the plurisubharmonic function $\omega^*(z, K, G)$ is maximal and $(dd^c\omega^*(z, K, G))^n = 0$ in the domain $G \backslash K$.*

Next, the function $\omega(z, K, G)$, which is equal to -1 at the points $z \in K$, can be discontinuous after regularization: at some points $z^0 \in K$ there may be the case $\omega\left(z^0, K, G\right) > -1$. Such points are called *(pluri)irregular points* of the compact set K. Is the set of irregular points $I_K = \left\{z^0 \in K : \omega^*\left(z^0, K, G\right) > -1\right\} = \varnothing$, i.e., all the poins of the compact set K are pluriregular, K is called a *pluriregular compact set*. Pluriregular compact sets play an important role in the pluripotential theory. For them the following theorem holds.

Theorem 2.2. *If K is a pluriregular compact set in the pluriregular domain $G \subset \mathbb{C}^n$, then $\mathcal{P}$-measure $\omega^*(z, K, G)$ is continuous in G.*

The quantity

$$C(K, G) = \int_K (dd^c\omega^*)^n = \int_G (dd^c\omega^*)^n$$

is called the *capacity* of (K, G). For an open set $U \subset G$ its capacity is defined as $C(U, G) = \sup\{C(K, G) : K \subset U,\ K \text{ is a compact set}\}$. And finally, the quantity $C^*(E, G) = \inf\{C(U, G) : U \supset E,\ U \text{ is open}\}$ is called the *outer capacity* of an arbitrary set $E \subset G$.

Outer capacity $C^*(E, G)$ of the set $E \subset G$ is nonnegative, $C^*(E, G) \geq 0$, and it equals zero, $C^*(E, G) = 0$, iff E is pluripolar. Moreover, the function of the set $C^*(E, G)$ satisfies all Choquet measurability conditions and Borel sets are measurable relative to $C(E, G)$: if $E \subset G$ is a Borel set, then its inner and outer capacities coincide,

$$C^*(E, G) = C_*(E, G) = \sup\{C(K, G) : \ K \subset E,\ K \text{ is a compact set}\}.$$

In the descriptions of holomorphic hulls Green's function plays a large role in the space $\mathbb{C}^n$. For a set $E \subset \mathbb{C}^n$ Green's function $V(z, E)$ is defined by Lelon class

$$L = \{u(z) \in psh(\mathbb{C}^n) : \quad u(z) \leq \alpha + \log(1 + |z|) \ \ \forall z \in \mathbb{C}^n, \ \ \alpha = \alpha(u) = \text{const}\}.$$

Put $V(z, E) = \sup\{u(z) \in L : \ u|_E \leq 0\}$; then the regularization $V^*(z, E)$ is called a *generalized Green's function*, or, simply, *Green's function*. The function $V^*(z, E) \geq 0$ either belongs to the class L or $V^*(z, E) \equiv +\infty$. The latter holds iff the set E is pluripolar. The function $V^*(z, E)$ is monotonical function by E : from $E_1 \subset E_2$ should $V^*(z, E_1) \geq V^*(z, E_2)$. If an open set $U \subset \mathbb{C}^n$ is represented as the union of the increasing sequence of compact sets, $U = \bigcup\limits_{j=1}^{\infty} K_j,\ K_j \subset K_{j+1}$, then $V^*(z, U) = \lim\limits_{j\to\infty} V^*(z, K_j)$. For an abitrary set $E \subset \mathbb{C}^n$ there exists a decreasing sequence of compact sets

$$U_1 \supset U_2 \supset \ldots, \quad U_j \supset E : \quad V^*(z, E) = \left[\lim_{j\to\infty} V^*(z, U_j)\right]^*.$$

For pluriregular compact sets $K \subset \mathbb{C}^n$, $V^*(z, K)|_K \equiv 0$, Green's function is continuous in the whole $\mathbb{C}^n$. In this case the open set $\{V^*(z, K) < \beta\}$ contains a compact set K, $K \subset \{V^*(z, K) < \beta\}$ $\forall \beta > 0$.

For any polynomial $P_m(z)$, $\deg P \leq m$, the function $\frac{1}{m}\log|P(z)| \in L$. Hence, the Bernstein–Walsh inequality holds:

$$\frac{1}{m}\log|P(z)| \leq \frac{1}{m}\log\|P(z)\|_K + V(z, K). \tag{2.3}$$

The set of all function of the form $\frac{1}{\deg P}\log|P(z)|$, where $P(z)$ are polynomials in $\mathbb{C}^n$, is a proper subclass in L. Therefore,

$$V(z,K) \geq \sup\left\{\frac{1}{\deg P}\log|P(z)|: \ \|P\|_K \leq 1\right\}. \tag{2.4}$$

Actually, (2.4) is an equality.

Theorem 2.3. *For any compact set $K \subset \mathbb{C}^n$ the following equality holds*:

$$V(z,K) = \sup\left\{\frac{1}{\deg P}\log|P(z)|: \ \|P\|_K \leq 1, \ P \textit{ are polynomials}\right\}. \tag{2.5}$$

Below we need Green's function for circle compact sets. Let $K \subset \mathbb{C}^n$ be a circle compact set, i.e., with each point $z^0 \in K$ the compact set K contains all point of the form $e^{i\phi}z^0$, $\phi \in \mathbb{R}$. It follows from (2.5) that Green's function $V(z,K)$ of the circle compact set K coincides with Green's function $V\left(z,\hat{K}\right)$ of the polynomial convex hull $\hat{K}$ with is the complete circle compact set.

Theorem 2.4. *Let $K \subset \mathbb{C}^n$ be a circle compact set. Then*

$$V(z,K) = \sup\left\{\frac{1}{\deg P}\log|P(z)|: \ \|P\|_K \leq 1, \ P \textit{ are polynomials}\right\}$$

$$= \sup\left\{\frac{1}{\deg Q}\log|Q(z)|: \ \|Q\|_K \leq 1, \ Q \textit{ are homogeneous polynomials}\right\}.$$

Proof. The proof is conducted in two steps.

Step 1 (see [24, 25]). For the circle compact set $K \subset \mathbb{C}^n$ its polynomial convex hull $\hat{K}$ coincides with the convex hull $\tilde{K}$ relative to homogeneous polynomials. To show this, we have to show that if $z^0 \in \tilde{K}$, i.e., if $\left|Q\left(z^0\right)\right| \leq \|Q\|_K$ for any homogeneous polynomial Q, this equality holds for any polynomial $P(z)$.

We fix a polynomial $P(z) = \sum_{j=0}^{m} Q_j(z)$, $m = \deg P$ such that $\|P\|_K \leq 1$. For any complex line l defined by $z = \lambda\xi$, where $\lambda \in \mathbb{C}^n$ is fixed, $\|\lambda\| = 1$ and $\xi \in \mathbb{C}$ is a parameter, the intersection $l \bigcap K$ is the circle $\{|\xi| \leq r(l)\}$. Applying the Cauchy inequality for the cross-section $P(z)|_l = \sum_{j=0}^{m} Q_j(\lambda)\,\xi^m$, we obtain $|Q_j(\lambda)| \leq \frac{1}{r^j(l)}$, $j = 0,1,\ldots,m$. This implies $|Q_j(\lambda)\, r(l)| \leq 1$, which is equivalent to inequality $\|Q_j\|_{K\bigcap l} \leq 1$. Therefore, $\|Q_j\|_K \leq 1$, $j = 0,1,\ldots,m$, and for a fixed $\sigma < 1$ the following holds: $\left|P\left(\sigma z^0\right)\right| \leq \sum_{j=0}^{m}\left|Q_j\left(z^0\right)\right|\sigma^j \leq \frac{1}{1-\sigma}$. This inequality holds for any polynomial R such that $\|R\|_K = 1$, in particular, for P^k, $k \in \mathbb{N}$, i.e., $\left|P^k\left(\sigma z^0\right)\right| \leq \frac{1}{1-\sigma}$, or $\left|P\left(\sigma z^0\right)\right| \leq \frac{1}{(1-\sigma)^{1/k}}$. By taking $k \to \infty$ and then $\sigma \to 1$ we obtain that $\left|P\left(z^0\right)\right| \leq 1$.

Step 2. It is sufficient to prove the theorem for a polynomially convex compact set $\hat{K} = K$. Put

$$V_O(z,K) = \sup\left\{\frac{1}{\deg Q}\log|Q(z)|: \ \|Q\|_K \leq 1, \ Q \text{ are homogeneous polynomials}\right\}.$$

Then, on the one hand, $V_O(z,K) \leq V(z,K)$. On the other hand, we fix a number $\varepsilon > 0$, a point $z^0 \in \mathbb{C}^n \backslash K$ and find a polynomial $P_m(z) = \sum_{j=0}^{m} Q_j(z)$, $m = \deg P_m$ such that

$$\|P_m\|_K \leq 1, \quad V\left(z^0,K\right) - \frac{1}{m}\log\left|P_m\left(z^0\right)\right| < \varepsilon. \tag{2.6}$$

By considering the cross-sections $P_m|_l = P_m(\lambda\xi) = \sum\limits_{j=0}^{m} Q_j(\lambda)\xi^j$ of the complex lines l of the type $z = \lambda\xi$, $\lambda \in \mathbb{C}^n$, $\xi \in \mathbb{C}$, as shown above, by the Cauchy inequalities we get that $\|Q_j\|_K \le 1$, $j = 0, 1, \ldots, m$. From $P_m(z) = \sum\limits_{j=0}^{m} Q_j(z)$, $V(z^0, K) - \frac{1}{m}\log|P_m(z^0)| < \varepsilon$ we obtain that for at least one $0 \le j_0 \le m$

$$P_m(z) = \sum_{j=0}^{m} Q_j(z), \quad |Q_{j_0}(z^0)| \ge \frac{|P_m(z^0)|}{m}$$

and

$$V(z^0, K) - \frac{1}{m}\log|mQ_{j_0}(z^0)| < \varepsilon. \tag{2.7}$$

Since $\|Q_{j_0}\|_K \le 1$ and $j_0 \le m$, $\frac{1}{m}\log|Q_{j_0}(z^0)| \le V_O(z^0, K)$. Hence, by (2.7),

$$V(z^0, K) - V_O(z^0, K) = V(z^0, K) - \frac{1}{m}\log|mQ_{j_0}(z^0)| + \frac{\log m}{m} \le \varepsilon + \frac{\log m}{m}.$$

Here we can put $\varepsilon > 0$ sufficiently small and $m = \deg P_m$ sufficiently large. From this we obtain that $V(z^0, K) \le V_O(z^0, K)$, and combining with $V_O(z, K) \le V(z, K)$ we get that $V_O(z, K) = V(z, K)$. The theorem is proved. □

Corollary 2.1. *If K is a circle compact set belonging to the closed unit ball $\|z\| \le 1$, then*

$$\hat{K} = \left\{z : \ |z| \cdot \exp V\left(\frac{z}{|z|}, \ K\right) \le 1\right\}.$$

Proof. Indeed, for any homogeneous polynomial Q_m, $\|Q_m\|_K \le 1$ we have

$$\left|Q_m\left(\frac{z}{|z|}\right)\right|^{1/m} \le \exp V\left(\frac{z}{|z|}, \ K\right).$$

Hence,

$$|Q_m(z)| = |z|^m \left|Q_m\left(\frac{z}{|z|}\right)\right| \le \left[|z| \exp V\left(\frac{z}{|z|}, \ K\right)\right]^m$$

and, consequently,

$$\tilde{K} \supset \left\{z : \ |z| \exp V\left(\frac{z}{|z|}, \ K\right) \le 1\right\}.$$

By step 1 of the proof of Theorem 2.4 we have $\tilde{K} = \hat{K}$, then

$$\hat{K} \supset \left\{z : \ |z| \exp V\left(\frac{z}{|z|}, \ K\right) \le 1\right\}.$$

Actually, we can write $=$ instead of $\supset$, since the compact set in the right-hand side is polynomially convex. □

Corollary 2.2 (see [5]). *If a circle compact set K belongs to the unit sphere $S(0,1)$, then its polynomially convex hull $\hat{K}$ contains the ball $|z| \le \exp\left[-\sup\limits_{|\xi|=1} V(\xi, K)\right]$.*

Proof. Follows from Corollary 2.1. □

Corollary 2.3. *For the circle compact set $K \subset \mathbb{C}^n$ the following identity holds*: $\exp V(Rz, K) = R\exp V(z, K) \ \ \forall z \notin \hat{K} \bigcap R\hat{K}$.

Proof. Indeed, we can assume that K is polynomially convex, $K = \hat{K}$ and $R > 1$. Then for $z \notin \hat{K}$ by Theorem 2.4

$$\exp V\left(Rz, K\right) = \sup\left\{\left|Q\left(Rz\right)\right|^{1/\deg Q}: \ \left\|Q\right\|_K \leq 1, \ Q \text{ are homogeneous polynomials}\right\}$$

$$= \sup\left\{\left|Q\left(Rz\right)\right|^{1/\deg Q}: \ \left\|Q\right\|_K \leq 1, \ Q \text{ are homogeneous polynomials deg}Q \geq 1\right\}$$

$$= \sup\left\{R\left|Q\left(z\right)\right|^{1/\deg Q}: \ \left\|Q\right\|_K \leq 1, \ Q \text{ are homogeneous polynomials deg}Q \geq 1\right\}$$

$$= R\sup\left\{\left|Q\left(z\right)\right|^{1/\deg Q}: \ \left\|Q\right\|_K \leq 1, \ Q \text{ are homogeneous polynomials}\right\} = R\exp V\left(z, K\right).$$

thus $\exp V\left(Rz, K\right) = R\exp V\left(z, K\right) \ \ \forall z \notin \hat{K}$. □

Corollary 2.4. *For the circle compact set* $K \subset \mathbb{C}^n$ *the following identity holds*: $\exp V\left(z, RK\right) = R^{-1}\exp V\left(z, K\right) \ \ \forall z \notin \hat{K} \bigcap R\hat{K}$.

Proof. Indeed, if we assume that K is polynomially convex, $K = \hat{K}$ and $R > 1$, for $z \notin \hat{K}$ by Theorem 2.4 we have

$$\exp V\left(Rz, RK\right) = R\exp V\left(z, RK\right), \ \ z \notin RK.$$

Hence, $\exp V\left(Rz, RK\right) = \exp V\left(z, K\right)$. Therefore, $\exp V\left(z, RK\right) = R^{-1}\exp V\left(z, K\right) \ \ \forall z \notin R\hat{K}$. □

3. The Convergence of Formal Hartogs Series

(1) Consider a formal Hartogs series

$$\sum_{k=0}^{\infty} c_k\left('z\right) \ z_n^k, \tag{3.1}$$

where $c_k\left('z\right)$ are holomorphic functions at some domain $'D \subset \mathbb{C}^{n-1}$. This series is formal because we do not know how it converges and where. Suppose that for any fixed $'z \in {'D}$ the series $\sum\limits_{k=0}^{\infty} c_k\left('z\right) \ z_n^k$ converge in the circle $\left|z_n\right| < R\left('z^0\right)$. We assume that $R\left('z\right)$ is the maximal convergence radius,

$$R^{-1}\left('z\right) = \overline{\lim_{k\to\infty}} \sqrt[k]{\left|c_k\left('z\right)\right|}.$$

If series (3.1) uniformly converges at some neighbourhood of the plane $\{z_n = 0\}$, i.e., the series sum $f\left('z, z_n\right)$ is holomorphic in the neighbourhood $\{z_n = 0\}$, then, as known, $f\left('z, z_n\right)$ is holomorphically extended to the domain

$$D = \left\{'z \in {'D}: \ \ |z_n| < R_*('z)\right\}, \tag{3.2}$$

where $R_*\left('z\right) = \underline{\lim}_{'w\to z} R\left(z\right)$ is the lower regularization of the function $R\left('z\right)$. We note that $-\log R_*\left('z\right) \in psh\left('D\right)$, and the set $\{'z \in {'D}: \ \ R_*\left('z\right) < R\left('z\right)\}$ is pluripolar.

Without series (3.1) convergence at some neighbourhood of the plane $\{z_n = 0\}$ by all variables the holomorphic extension of the function $f\left('z, z_n\right)$ to the domain of type (3.2) does not exist.

Example 3.1. Take the sequence of compact sets

$$K_m = K_m^1 \bigcup K_m^2 \subset \mathbb{C}, \quad K_m^1 = \left\{|z_1| \leq 1, \ \frac{1}{m} \leq \arg z_1 \leq 2\pi\right\},$$

$$K_m^2 = \left\{\frac{1}{m} \leq |z_1| \leq 1, \ \arg z_1 = \frac{1}{2m}\right\}, \quad m = 1, 2, \ldots$$

Then the continuous function on the compact set K_m

$$g_m(z) = \begin{cases} \dfrac{1}{m!}, & \text{if } z_1 \in K_m^1, \\ m!, & \text{if } z_1 \in K_m^2, \end{cases}$$

is approximated by the polynomials uniformly on K_m, i.e., there exists a polynomial $P_m(z_1)$ such that $\|g_m - P_m\|_{K_m} \le \dfrac{1}{m!}$. The series

$$f(z_1, z_2) = \sum_{m=1}^{\infty} P_m(z_1)\, z_2^m$$

has the following property: for any fixed z_1^0 such that $|z_1^0| < 1$, it converges on the whole plane $|z_2| < \infty$, but its sum $f(z_1, z_2)$ is not holomorphic on the set $S = \{|z_1| < 1,\ \arg z_1 = 0\} \subset \mathbb{C}^2$.

Nevertheless, the following theorem holds.

Theorem 3.1 (see [31, 34]). *Consider Hartogs series*

$$\sum_{k=0}^{\infty} c_k('z)\, z_n^k$$

with holomorphic coefficients $c_k('z)$, $k = 0, 1, \ldots,$ *on the domain* $'D \subset \mathbb{C}^{n-1}$. *Suppose that the convergence radius is positive for any fixed* $'z \in {'D}$, *thus* $R('z) = 1/\overline{\lim\limits_{k\to\infty}} \sqrt[k]{|c_k('z)|} > 0\ \forall' z \in {'D}$.

Then there exists a nowhere dense closed set $'S \subset {'D}$ *such that*

(a) $-\log R_*('z) \in psh('D \backslash 'S)$;

(b) *the series sum is holomorphic by all variables on*

$$\{'z \in {'D} \backslash S,\ |z_n| < R_*('z)\}.$$

Note that if we omit the condition $R('z) > 0\ \forall' z \in {'D}$, then Theorem 3.1 does not hold true.

Example 3.2. Let $K = \{|z_1| \le 1\}$ be the closed unit circle on the complex plane $\mathbb{C}_{z_1}$. Take a sequence of polynomially convex compact sets

$$F_m = \left\{z_1 \in \mathbb{C}:\ 1 + \frac{1}{m} \le |z_1| \le m,\ 0 \le \arg z_1 \le 2\pi - \frac{1}{m}\right\}.$$

Then $F_m \subset F_{m+1}$ and $\bigcup\limits_{m=1}^{\infty} F_m = \mathbb{C} \backslash K$. Moreover, the compact sets $F_m \bigcup K$ are polynomially convex. Put

$$g_m(z_1) = \begin{cases} m!, & \text{if } z_1 \in F_m \\ \dfrac{1}{m!}, & \text{if } z_1 \in K. \end{cases}$$

By the Mergelyan theorem these functions are polynomially approximated uniformy on $F_m \bigcup K$, i.e., there exist poynomials $p_m(z_1)$ such that

$$\|g_m(z_1) - p_m(z_1)\|_{K \bigcup F_m} < \frac{1}{m!}, \quad m = 1, 2, \ldots$$

Consider the formal series

$$f(z_1, z_2) = \sum_{m=1}^{\infty} p_m(z_1)\, z_2^m.$$

This series converges on the whole plane $\mathbb{C}_{z_2}$ for any fixed point $z_1 \in K$ and its sum is holomorphic on $\{|z_1| < 1\} \times \mathbb{C}_{z_2}$. But its convergence radius $R(z_1) = 0$ for all $z_1 \in \mathbb{C} \backslash K$, and this series does not define a holomorphic function on $[\mathbb{C}_{z_1} \backslash K] \times \mathbb{C}_{z_2}$.

(2) For formal series on a bundle of complex straight lines the case is more natural.

Let on the bundle of complex straight lines $\{l: \ z=\lambda\xi, \ \lambda\in\mathbb{C}^n, \ \xi\in\mathbb{C}\}\approx P^{n-1}$ the following formal series be defined:

$$\sum_{k=0}^{\infty} c_k(\lambda)\ \xi^k=\sum_{k=0}^{\infty} c_k(\lambda\xi)=\sum_{k=0}^{\infty} c_k(z), \tag{3.3}$$

where $c_k(z)$ are homogeneous polynomials on $\mathbb{C}^n$.

Theorem 3.2 (see [25]). *If for any complex line from some bundle* $\Im\subset\{l: \ z=\lambda\xi, \ \lambda\in\mathbb{C}^n, \ \xi\in\mathbb{C}\}$ *series* (3.3) *converges on the circle* $l\bigcap B(0,1)$, *this series converges on the open set* $G=\Big\{z\in\mathbb{C}^n: |z|\cdot\exp V^*\Big(\frac{z}{|z|},E\Big)<1\Big\}$. *Here* $E=\bigcup\limits_{l\in\Im} l\bigcap B(0,1)$.

Proof. Indeed, without loss of generality assume that $\Im$ coincides with the set of all complex lines l for which series (3.3) converges on the circle $l\bigcap B(0,1)$. Fix $\varepsilon>0$ and put

$$F_N=\left\{\lambda\in S(0,1): \ \left|\sum_{k=0}^{\infty} c_k(\lambda)\ \xi^k\right|\le N \ \text{ as } \ |\xi|\le 1-\varepsilon\right\}.$$

By the Cauchy inequalities $|c_k(\lambda)|\le\dfrac{N}{(1-\varepsilon)^k}$, $\lambda\in F_N$, $k=0,1,\ldots$ Hence, by the Bernstein–Walsh inequality we have

$$|c_k(\lambda)|\le\frac{N}{(1-\varepsilon)^k}\left[\exp V^*(\lambda,F_N)\right]^k, \ \ \lambda\in\mathbb{C}^n, \ k=0,1,\ldots$$

In particular, for $\lambda=\dfrac{z}{|z|}\in S(0,1)$

$$\left|c_k\left(\frac{z}{|z|}\right)\right|\le\frac{N}{(1-\varepsilon)^k}\left[\exp V^*\left(\frac{z}{|z|},F_N\right)\right]^k, \ \ z\in\mathbb{C}^n\setminus\{0\},$$

which is equivalent to

$$|c_k(z)|\le\frac{N}{(1-\varepsilon)^k}\left[|z|\exp V^*\left(\frac{z}{|z|},F_N\right)\right]^k, \ \ z\in\mathbb{C}^n\setminus\{0\}, \ k=0,1,\ldots$$

Hence, the homogeneous series $\sum\limits_{k=0}^{\infty} c_k(z)$ converges in

$$G_{N,\varepsilon}=\left\{z\in\mathbb{C}^n: \ |z|\cdot\exp V^*\left(\frac{z}{|z|},F_N\right)<1-2\varepsilon\right\}.$$

Note that the set F_N is a closed compact set on the sphere $S(0,1)$, with $F_N\subset F_{N+1}$, $N=1,2,\ldots$ and $\bigcup\limits_{N=1}^{\infty}\hat{F}_N=E$. Therefore, by taking $N\to\infty$ and then $\varepsilon\to 0$ we obtain the convergence of the series $\sum\limits_{k=0}^{\infty} c_k(z)$ inside the open set $G=\Big\{z\in\mathbb{C}^n: \ |z|\cdot\exp V^*\Big(\frac{z}{|z|},E\Big)<1\Big\}$. $\square$

Remark 3.1. If $f(z)$ is some function which is inifintely smooth on the neighbourhood of zero, $f(z)\in C^\infty\{0\}$, it correspond to the formal power series

$$f\sim\sum_{|I|+|J|=0}^{\infty} c_{IJ}z^I\bar{z}^J, \tag{3.4}$$

where $I = (i_1, i_2, \ldots, i_n)$ and $J = (j_1, j_2, \ldots, j_n)$ are multiindexes, $|I| = i_1 + i_2 + \ldots + i_n$, $|J| = j_1 + j_2 + \ldots + j_n$, $z^I = z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}$, $\bar{z}^J = \bar{z}_1^{j_1} \bar{z}_2^{j_2} \ldots \bar{z}_n^{j_n}$. Rewrite the series in (3.4) in the form

$$\sum_{|I|+|J|=0}^{\infty} c_{IJ} z^I \bar{z}^J = \sum_{|I|=0}^{\infty} c_{I0} z^I + \sum_{|I|+|J|=0, J\neq 0}^{\infty} c_{IJ} z^I \bar{z}^J. \tag{3.5}$$

If the contraction of the function $f(z)$ to each complex plane $\{l: \ z = w\xi, \ w \in \mathbb{C}^n, \ \xi \in \mathbb{C}\} \subset P^{n-1}$ is holomorphically extended to the unit circle, then the last series in (3.5) vanishes, i.e., $f(z) = \sum_{|I|=0}^{\infty} c_{I0} z^I$, and by Theorem 3.2 we obtain that the function is holomorphic on the ball $B(0,1) \subset \mathbb{C}^n$. Therefore, we obtain the Forelli theorem.

Theorem 3.3 (Forelli theorem [16])**.** *If f is an infinitely smooth function at the point* 0 *and its contraction $f|_l$ is holomorphic in the circle $l \bigcap B(0,1)$ for all complex lines $l \ni 0$, then f is holomorphically exnteded to the ball $l \bigcap B(0,1)$.*

Remark 3.2.

(1) For the function $f(z_1, z_2) = \dfrac{z_1^{k+1} \bar{z}_2}{z_1 \bar{z}_1 + z_2 \bar{z}_2} \in C^k(\mathbb{C}^2)$ Theorem 3.2 cannot be applied, although the contraction of the function is holomorphic for all comples lines $l \ni 0$ (f does not define series (3.4)).

(2) The function $f(z_1, z_2) = |z_1|^2 - |z_2|^2 \in C^\infty(\mathbb{C}^2)$, $f|_l \equiv 0$ on complex lines $\Im = \{z_2 = e^{i\theta} z_1, \theta \in [0, 2\pi]\}$. Although the set $\Im \subset P^1$ forms a nonpluripolar set on the projective plane P^1, Theorem 3.2 can be applied here. To apply Theorem 3.2 to the formal series by z and by $\bar{z}$, we must require that the set $\Im \subset P^{n-1}$ is nonpolar in the sense of real analysis.

Theorem 3.2 can be proved in the general form when on the complex line $l \in \Im$ we do not require the convergence of series (3.3) on the circle $l \bigcap B(0,1)$, but its convergence on an arbitrary circle $l \bigcap B(0, r_l)$, $0 < r_l \leq \infty$. The following main theorem holds.

Theorem 3.4. *Let the bundle of complex lines, $\Im \subset \{l : z = w\xi, \ w \in \mathbb{C}^n, \ \|w\| = 1, \ \xi \in \mathbb{C}\}$ intersecting zero, be given. If for each complex line $l \in \Im$ the contraction $\sum_{k=0}^{\infty} c_k(\lambda)\, \xi^k$ of series* (3.3) *converges on the circle $l \bigcap B(0, r_l)$, $0 < r_l \leq \infty$, this series converges on the open set $G = \Big\{z \in \mathbb{C}^n : |z| \cdot \exp V^*\Big(\frac{z}{|z|}, E\Big) < 1\Big\}$. Here $E = \bigcup_{l \in \Im} l \bigcap B(0, r_l)$.*

Proof. The proof of this theorem is conducted in several steps by supposing firstly that $\Im$ is the set of all complex lines with $r_l > 0$, thus $\Im = \{l \in P^{n-1} : \ r_l > 0\}$.

1. For $0 < r_l \leq 1 \ \forall l \in \Im$ we fix the numbers $N \in \mathbb{N}$, $r > 0$ and $0 < \varepsilon < r$. Put $\Im_r = \{l \in \Im : \ r_l \geq r\}$ and denote

$$F_{N,r,\varepsilon} = \left\{\lambda \in S(0,1),\ l = \{z = \lambda\xi\} \in \Im_r : \ \left|\sum_{k=0}^{\infty} c_k(\lambda)\, \xi^k\right| \leq N \ \text{ for } \ |\xi| \leq r_l - \varepsilon\right\}.$$

By the Cauchy inequalities $|c_k(\lambda)| \leq \dfrac{N}{(r_l - \varepsilon)^k}$, $\lambda \in F_{N,r,\varepsilon}$, $k = 0, 1, \ldots$ This is equivalent to $|c_k((r_l - \varepsilon) \ \times \lambda)| \leq N$, $\lambda \in F_{N,r,\varepsilon}$, $k = 0, 1, \ldots$ Since $F_{N,r,\varepsilon}$ is a circle compact set, $|c_k(z)| \leq N$, $z = \lambda\xi$, $\lambda \in F_{N,r,\varepsilon}$, $|\xi| \leq r_l - \varepsilon$, $k = 0, 1, \ldots$ Hence, $\|c_k(z)\|_{E_{N,r,\varepsilon}} \leq N$, where $E_{N,r,\varepsilon} = \{|z| \leq r_l - \varepsilon, z = \lambda\xi, \lambda \in F_{N,r,\varepsilon}\}$, and by the Bernstein–Walsh inequality we have $|c_k(z)| \leq N[\exp V^*(z, E_{N,r,\varepsilon})]^k$, $z \in \mathbb{C}^n$, $k = 0, 1, \ldots$

In particular, for $\lambda = \dfrac{z}{|z|} \in S(0,1)$

$$\left| c_k\left(\frac{z}{|z|}\right)\right| \le N\left[\exp V^*\left(\frac{z}{|z|}, E_{N,r,\varepsilon}\right)\right]^k, \quad z \in \mathbb{C}^n \setminus \{0\},$$

which is equivalent to the inequality

$$|c_k(z)| \le N\left[|z| \exp V^*\left(\frac{z}{|z|}, E_{N,r,\varepsilon}\right)\right]^k, \quad z \in \mathbb{C}^n \setminus \{0\}, \; k = 0, 1, \ldots$$

This implies that homogeneous series $\sum\limits_{k=0}^{\infty} c_k(z)$ converges on

$$G_{N,r,\varepsilon} = \left\{ z \in \mathbb{C}^n : \; |z| \cdot \exp V^*\left(\frac{z}{|z|}, E_{N,r,\varepsilon}\right) < 1 \right\}.$$

By taking $N \to \infty$, and next $\varepsilon \to 0$, we obtain the convergence of the series $\sum\limits_{k=0}^{\infty} c_k(z)$ inside the open set

$$G_r = \left\{ z \in \mathbb{C}^n : \; |z| \cdot \exp V^*\left(\frac{z}{|z|}, E_r\right) < 1 \right\},$$

where $E_r = \bigcup\limits_{l \in \Im_r} l \bigcap B(0, r_l)$. As $r \downarrow 0$ the set E_r increasingly converges to E. Therefore, $V^*\Big(\dfrac{z}{|z|}, E_r\Big) \downarrow V^*\Big(\dfrac{z}{|z|}, E\Big)$ and the series $\sum\limits_{k=0}^{\infty} c_k(z)$ uniformly converges inside the open set

$$G = \left\{ z \in \mathbb{C}^n : \; |z| \cdot \exp V^*\left(\frac{z}{|z|}, E\right) < 1 \right\}.$$

2. Series (3.3) converges on the circle $l \bigcap B(0, r_l)$ of variable radius r_l, $0 < r_l \le R$, $l \in \Im$. We make the transformation $z = Rw$. The corresponding series $\sum\limits_{k=0}^{\infty} c_k(Rw)$ has the following property: its contraction $l \in \Im$ converges on the circle of radius r_l/R, $0 < r_l/R \le 1$. Therefore, this series converges on the open set

$$\left\{ w \in \mathbb{C}^n : \; |w| \cdot \exp V^*\left(\frac{w}{|w|}, \frac{E}{R}\right) < 1 \right\}, \quad \frac{E}{R} = \bigcup_{l \in \Im} l \bigcap B(0, r_l/R).$$

Hence, the series $\sum\limits_{k=0}^{\infty} c_k(z)$ converges on the open set

$$\left\{ z \in \mathbb{C}^n : \; \frac{|z|}{R} \exp V^*\left(\frac{z}{|z|}, \frac{E}{R}\right) < 1 \right\}.$$

But, by Corollary 2.4 of Theorem 2.4, Green's function $V^*\Big(\xi, \dfrac{E}{R}\Big) = RV^*(\xi, E)$. Then the series $\sum\limits_{k=0}^{\infty} c_k(z)$ converges on the open set $\left\{ z \in \mathbb{C}^n : \; |z| \exp V^*\Big(\dfrac{z}{|z|}, E\Big) < 1 \right\}$, where $E = \bigcup\limits_{l \in \Im} l \bigcap B(0, r_l)$.

3. Consider the general case: $0 < r_l \le \infty$, $l \in \Im$. Fix $R > 1$ and denote $E_R = E \bigcap \{\|z\| \le R\}$. By step 2 of the proof the series $\sum\limits_{k=0}^{\infty} c_k(z)$ converges on the open set $\left\{ z \in \mathbb{C}^n : |z| \exp V^*\Big(\dfrac{z}{|z|}, E_R\Big) < 1 \right\}$. Now the proof of the theorem is easily obtained by taking R to infinity: $V^*\Big(\dfrac{z}{|z|}, E_R\Big) \downarrow V^*\Big(\dfrac{z}{|z|}, E\Big)$ as $R \uparrow \infty$. □

4. Holomorphic Extensions of the Functions with a Thin Singularity Along a Fixed Direction

We begin with the following theorem proved in [28].

Theorem 4.1. *Let the function* $f('z, z_n)$ *be holomorphic on the polycircle* $U = 'U \times U_n \subset \mathbb{C}^{n-1}_{'z} \times \mathbb{C}_{z_n}$ *and for each fixed* $'a$ *in some nonpluripolar set* $E \subset 'U$ *the function* $f('a, z_n)$ *of the variable* z_n *is extended to the function which is holomoprhic on the whole plane without some polar (discrete) set of singularities* $S_{'a}$. *Then* f *is holomorphically extended to* $('U \times \mathbb{C}) \setminus S$, *where* S *is the closed pluripolar (analytic) subset* $'U \times \mathbb{C}$.

The difficult moment of the proof in the description of the set of singularities outside U; apriori $\bigcup\limits_{'a \in 'U} S_{'a}$ can be dense in $'U \times [\mathbb{C} \setminus U_n]$. This difficulty is overcome by the expansion of the function to Jacoby–Hartogs series $f(z', z_n) = \sum\limits_{k=0}^{\infty} c_k('z, z_n)\, g^k(z_n)$ by all rational functions

$$g(z_n) = \frac{z_n^m}{p_m(z_n)},\ p_m(z_n) \text{ is a polynomial of the degree } m > 0. \tag{4.1}$$

Below we often use the pluripotential theory, the potential properties of the family of plurisubharmonic function and pseudoconcave sets.

For the formal seires on the bundle of lines, i.e., for the formal series of homogeneous polynomials $\sum\limits_{k=0}^{\infty} c_k(z)$, where $c_k(z)$ are homogeneous polynomials, the following theorem holds (see Example 3.2).

Theorem 4.2. *Let a nonpluripolar bundle of complex lines*

$$\Im \subset \{l : z = \lambda\xi,\ \lambda \in \mathbb{C}^n,\ \|\lambda\| = 1,\ \xi \in \mathbb{C}\} = P^n,$$

intersecting zero, be given. If for each comples line $l \in \Im$ *the contraction* $\sum\limits_{k=0}^{\infty} c_k(\lambda)\, \xi^k$ *of the series* $\sum\limits_{k=0}^{\infty} c_k(z)$ *to the complex line* l *converges on the circle of radius* $r_l > 0$ *and its sum is holomorphic on* $\mathbb{C}$ *without a polar (discrete) set, then the series* $\sum\limits_{k=0}^{\infty} c_k(z)$ *defines a holomorphic function on the space* $\mathbb{C}^n$ *without some pluripolar (analytic) set* $S \subset \mathbb{C}^n$.

Proof. By Theorem 3.4 the series $\sum\limits_{k=0}^{\infty} c_k(z)$ converges on the open set

$$G = \left\{ z \in \mathbb{C}^n :\ |z| \cdot \exp V^*\left(\frac{z}{|z|}, E\right) < 1 \right\},$$

where $E = \bigcup\limits_{l \in \Im} l \bigcap B(0, r_l)$. The sum of this series $f(z) = \sum\limits_{k=0}^{\infty} c_k(z)$ is a holomorphic function on G. By Theorem 4.2 the set E is not pluripolar. Hence, $V^*(\cdot, E) \neq +\infty$ and the domain G contains the point 0.

Consider the standard transformation in the space $\mathbb{C}^n$:

$$\pi : (z_1, z_2, \dots, z_{n-1}, z_n) \to (z_1 z_n, z_2 z_n, \dots, z_{n-1} z_n, z_n),$$

upon which vertical complex lines $('z^0, z_n) = (z_1^0, z_2^0, \dots, z_{n-1}^0, z_n)$, $z_n \in \mathbb{C}$, are taken into the bundle of lines $(z_1^0 z_n, z_2^0 z_n, \dots, z_{n-1}^0 z_n, z_n)$, $z_n \in \mathbb{C}$. Hence, the function $\tilde{f}('z, z_n) = \pi^{-1} \circ f = f(z_1 z_n, z_2 z_n, \dots, z_{n-1} z_n, z_n)$, $z_n \in \mathbb{C}$, is holomorphic by all variables on some neighbourhood of the plane $\{z_n = 0\}$, and for each fixed $'z^0 = (z_1^0, z_2^0, \dots, z_{n-1}^0) :\ l = \{z_1^0 z_n, z_2^0 z_n, \dots, z_{n-1}^0 z_n, z_n,\ z_n \in \mathbb{C}\} \in \Im$ the function $\tilde{f}('z^0, z_n)$, $z_n \in \mathbb{C}$ is holomorphically extended to the whole plane $\mathbb{C}_{z_n}$ without a polar (discrete) set. By Theorem 4.1 the funciton $\tilde{f}\left({}'z, z_n\right)$ is holomorphically extended to $\mathbb{C}^n \backslash S$, where $S \subset \mathbb{C}^n$ is a pluripolar (anaytic) set. Hence, the function $f(z) = \pi \circ f(z)$ is holomorphically extended to $\left[\mathbb{C}^n \backslash \{z_n = 0\}\right] \backslash \pi(S)$, where $\pi(S)$ is a pluripolar (analytic) set in $\mathbb{C}^n \backslash \{z_n = 0\}$.

Above we considered the transformation $\pi : (z_1, z_2, \ldots, z_{n-1}, z_n) \to (z_1 z_n, z_2 z_n, \ldots, z_{n-1} z_n, z_n)$ by taking the coordinate oz_n. If we commence this procedure for any index $k = n, n-1, \ldots, 1$, we obtain that the function $f(z)$ is holomorphically exnteded to $\left[\mathbb{C}^n \backslash \{z_k = 0\}\right] \backslash A_k$, where A_k is a pluripolar (analytic) set in $\mathbb{C}^n \backslash \{z_k = 0\}$. Hence we easily obtain that $f(z)$ is holomorphically extended to $\mathbb{C}^n \backslash A$, where A is a pluripolar (analytic) set in $\mathbb{C}^n$. □

5. Holomorphic Extension of the Functions Along the Family of Analytic Curves

The version of Hartogs theorem in the sense of the replacement of coordinate curves with the families of analytic curves was probably firstly considered in the work of Chirka [13]. Here we introduce three statements from this work, in which new approaches to the study of the functions holomorphic on holomorphic laminations are introduced:

(1) *Let in the domain $\Omega \subset \mathbb{C}^n$ be given n lineary independent delaminations $\left\{S_\xi^j\right\}$, $\Omega = \bigcup\limits_\xi S_\xi^j$, by holomorphic curves S_ξ^j, $j = 1, 2, \ldots, n$. If the function f is locally bounded on Ω and all contractions of $f|_{S_\xi^j}$ are holomorhpic, then f is holomorphic on Ω.*

The proof on this statement is based on the fact that in the described conditions f is Lipschitz-continuous with locally bounded differential df a.e. on Ω and $\bar{\partial} f = 0$.

(2) *Let the domain $\Omega \subset D \times \mathbb{C}_w^k$, $D \subset \mathbb{C}_z^m$ be delaminated by holomorphic graphs S_ξ such that $w = \phi_\xi(z)$. If all laminations $f|_{S_\xi}$ of the function $f(z, w)$ are holomorphic on S_ξ and $f(c, w)$ are holomorphic on $\Omega \bigcap \{z = c\}$, $c \in D$, then $f(z, w)$ is holomorphic on Ω.*

In the proof of this statement with the application of the Baire theorem one can find an open part $\Omega_1 \subset \Omega$, where $f(z, w)$ is bounded, and then with the application of the proof method used in Proposition (1) they prove that this function is holomorphic on Ω_1. Next, since $f(c, w)$ is holomorphic on $\Omega \bigcap \{z = c\}$, then by the classic Hartogs lemma we conclude that $f(z, w)$ is holomorphic on Ω.

The following proposition is a curvilinear analog of the classic Hartogs lemma.

(3) *Let the domain $\Omega \subset D \times \mathbb{C}_w^k$, $D \subset \mathbb{C}_z^m$, $0 \in D$, as in Proposition (2) be delaminated by holomorphic graphs S_ξ such that $w = \phi_\xi(z)$. If all contractions $f|_{S_\xi}$ of the function $f(z, w)$ are holomorphic on S_ξ and $f(z, w)$ is holomorphic by all variables on some neighbourhood $\Omega \bigcap \{z = 0\}$, then $f(z, w)$ is holomorphic on Ω.*

Note that the Forelli theorem (see Sec. 3) is some kind of the Hartogs theorem. In the work [13] Chirka showed that the Forelli theorem curvilinear analog holds in the case $n = 2$. The next variations of the Hartogs theorem, as well as the variations of the Forelli theorem, are obtained in [18, 19, 22].

Theorem 5.1 (see [22]). *If the function $f : B(0,1) \to \mathbb{C}$ is infinitely smooth at the point 0, $f \in C^\infty(0)$ and is holomorphic along the integral analytic curves of the vector field $X = \sum\limits_{j=1}^n \alpha_j z_j \dfrac{\partial}{\partial z}$, where α_j are constant, $\dfrac{\alpha_j}{\alpha_k} > 0\ \forall j, k$, then f is holomorphic on $B(0,1)$.*

Theorem 5.2 (see [18]). *Let the domain $\Omega \subset D \times \mathbb{C}_w^k$, $D \subset \mathbb{C}_z^m$, be delaminated by smooth family of analytic curves $\{S_\xi\}$, $\xi \in P^{n-1}$, $0 \in S_\xi$ radial at the point 0 such that $\bigcup\limits_\xi S_\xi = \Omega$. If the function $f \in \mathbb{C}^\infty\{0\}$ has the property that all its contractions $f|_{S_\xi}$ are holomorphic on S_ξ, then f is holomorphically extended to Ω.*

In this section we study the holomorphic extension of the formal series of homogeneous polynomials which is holomoprhic along the given family of analytic curves intersecting zero. With that we do not place any other conditions of the family of analytic curves. We start with the following lemma, which is quite self-sufficient.

Lemma 5.1. *The series of the form* $\sum_{k=0}^{\infty} a_k(\xi)\,\xi^k$, *where* $a_k(\xi) \in O(U)$, $k = 0, 1, \ldots,$ *uniformly converges inside the circle* $U : |\xi| < 1$ *iff for any* $\varepsilon > 0$

$$\overline{\lim_{k\to\infty}} \|a_k(\xi)\|^{1/k}_{|\xi|\le 1-\varepsilon} \le 1. \tag{5.1}$$

Proof. Indeed, if (5.1) holds, for fixed $\varepsilon > 0$ there exists k_0 such that $|a_k(\xi)| < (1+\varepsilon)^k$, $|\xi| \le 1-\varepsilon$, $k \ge k_0$. Hence, $\left|a_k(\xi)\,\xi^k\right| \le |(1+\varepsilon)\,\xi|^k \le \left(1-\varepsilon^2\right)^k$ and the series $\sum_{k=0}^{\infty} a_k(\xi)\,\xi^k$ uniformly converges inside the circle $|\xi| \le 1-\varepsilon$. And vice versa, if the series $\sum_{k=0}^{\infty} a_k(\xi)\,\xi^k$ uniformly converges inside the circle $|\xi| \le 1-\varepsilon$, then $\left\|a_k(\xi)\,\xi^k\right\|_{|\xi|\le 1-\varepsilon} \le const$, $k = 0, 1, 2, \ldots$ Hence, $\overline{\lim_{k\to\infty}} \|a_k(\xi)\|^{1/k}_{|\xi|\le 1-\varepsilon} \le 1$. □

Remark 5.1. If the inequality $\overline{\lim_{k\to\infty}} |a_k(\xi)|^{1/k} \le 1$ holds pointwise for a fixed $\xi \in U$, then there exists a dense open set $\tilde{U} \subset U$ inside which the series converges uniformly.

Indeed, put $F_m = \left\{\xi \in U : \left|a_k(\xi)\,\xi^k\right| \le m,\ k = 0, 1, 2, \ldots,\right\}$. Then F_m are closed subsets of U and $U = \bigcup_{m=1}^{\infty} F_m$. By Boier theorem we obtain that the open kernel $\tilde{U} = \bigcup_{m=1}^{\infty} F_m^0$ is dense in U and the series $\sum_{k=0}^{\infty} a_k(\xi)\,\xi^k$ converges uniformly inside $\tilde{U}$.

The following theorem plays a key role in the study of holomorphic functions of many variables along the fixed curves. Let $A = \{z = p(\xi), |\xi| < 1\}$ be an analytic curve where $p(\xi) = (p_1(\xi), \ldots, p_n(\xi))$ is a vector function holomorphic in the unit circle $U : |\xi| < 1$, $p(0) = 0$. Put $A_\varepsilon = A \bigcap \{|\xi| < 1-\varepsilon\}, 0 < \varepsilon < 1$.

Theorem 5.3. *Suppose that* $A \subset \mathbb{C}^n$ *is an analytic curve intersecting zero,* $0 \in A$, *such that the series of homogeneous polynomials* $f(z) = \sum_{k=0}^{\infty} c_k(z)$, *where* $c_k(z)$ *is a homogeneous polynomial of the degree* k, *converges on the set* A. *Then* $\overline{\lim_{k\to\infty}} \|c_k(z)\|^{1/k}_{A_\varepsilon} \le 1$.

Proof. Indeed, let $A : \{z = p(\xi)\}$ be an analytic curve intersecting 0, where $p(\xi)$ is a vector function holomorphic in the unit circle $U : |\xi| < 1$, $p(0) = 0$. Suppose that $f_A(z) = \sum_{k=0}^{\infty} c_k(p(\xi))$ converges in the circle $|\xi| < 1$. Write

$$\sum_{k=0}^{\infty} c_k(p(\xi)) = \sum_{k=0}^{\infty} \frac{c_k(p(\xi))}{\xi^k}\xi^k,$$

where $\frac{c_k(p(\xi))}{\xi^k}$ are holomorphic functions in the unit circle $|\xi| < 1$. It is clear that the series $\sum_{k=0}^{\infty} c_k(p(\xi))$ uniformly converges in the circle $|\xi| < 1-\varepsilon$, $\varepsilon > 0$. Hence, by Lemma 5.1

$$\overline{\lim_{k\to\infty}} \left\|\frac{c_k(p(\xi))}{\xi^k}\right\|^{1/k}_{|\xi|\le 1-\varepsilon} \le 1.$$

Therefore,

$$\overline{\lim_{k\to\infty}} \|c_k (p(\xi))\|^{1/k}_{|\xi|\le 1-\varepsilon} \le 1.$$

□

The main result of Sec. 4 is the following theorem.

Theorem 5.4. *Let be given an arbitrary family* $\aleph = \{A_\alpha,\ \alpha\in\Lambda\}$ *of analytic curves* A_α: $z = p_\alpha(\xi),\ \xi\in U,\ p_\alpha(0)=0$. *If the series of homogeneous polynomials* $f(z) = \sum_{k=0}^{\infty} c_k(z)$, *where* $c_k(z)$ *is a homogeneous polynomial of the degree* k, *converges on each set* $A_\alpha,\ \alpha\in\Lambda$, *then this series uniformly converges inside the ball*

$$B\left(0, \frac{1}{\exp\underline{\gamma}(E)}\right) = \left\{z\in\mathbb{C}^n:\quad \|z\| < \frac{1}{\exp\underline{\gamma}(E)}\right\}, \tag{5.2}$$

Here $E = \bigcup_{\alpha\in\Lambda} A_\alpha$ *and* $\underline{\gamma}(E) = \underline{\lim}_{z\to\infty} [V^*(z,\ E) - \log\|z\|]$ *is the lower Robin constant of the set* E.

Corollary 5.1. *In the conditions of Theorem* 5.4, *is the set* $E = \bigcup_{\alpha\in\Lambda} A_\alpha$ *is not pluripolar on* $\mathbb{C}^n$, *the formal series* $\sum_{k=0}^{\infty} c_k(z)$ *has the sum* $f(z)$ *which is holomorphic inside the nonempty ball* $\|z\| < \exp^{-1}\underline{\gamma}(E)$.

Remark 5.2 (see Sec. 3)**.** If $f(z)\in C^\infty\{0\}$, it corresponds to the formal power series

$$f \sim \sum_{|I|=0}^{\infty} c_{I0} z^I + \sum_{|I|+|J|=0, J\ne 0}^{\infty} c_{IJ} z^I \bar{z}^J, \tag{5.3}$$

where $I = (i_1, i_2, \ldots, i_n)$ and $J = (j_1, j_2, \ldots, j_n)$ are multiindexes, $|I| = i_1 + i_2 + \ldots + i_n,\ |J| = j_1 + j_2 + \ldots + j_n,\ z^I = z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n},\ \bar{z}^J = \bar{z}_1^{j_1} \bar{z}_2^{j_2} \ldots \bar{z}_n^{j_n}$. If the contractions $f|_{A_\alpha},\ \alpha\in\Lambda$ are holomorphic and the set $E = \bigcup_{\alpha\in\Lambda} A_\alpha$ is not $\mathbb{R}^{2n}$-polar, the second series in (5.3) vanishes. Therefore, applying Theorem 5.4 in this case we obtain that f is holomorphic on some heighbourhood of the point zero.

Proof of Theorem 5.2. Let $0<\varepsilon<1$ be fixed. By Theorem 5.1 the following $\overline{\lim_{k\to\infty}} \|c_k(z)\|^{1/k}_{A_{\alpha,\varepsilon}} \le 1$ holds for each $\alpha\in\Lambda$, where $A_{\alpha,\varepsilon} = \{z\in\mathbb{C}^n:\ z = p_\alpha(\xi),\ |\xi|\le 1-\varepsilon\} \subset\subset A_\alpha$. For any fixed $j\in\mathbb{N}$ put

$$\Lambda_{j,\varepsilon} = \left\{\alpha\in\Lambda:\ \|c_k(z)\|^{1/k}_{A_{\alpha,\varepsilon}} \le 1+\varepsilon,\ \ k\ge j\right\}$$

and

$$E_{j,\varepsilon} = \left\{\bigcup A_{\alpha,\varepsilon}:\ \alpha\in\Lambda_{j,\varepsilon}\right\}.$$

Then $\|c_k(z)\|^{1/k}_{E_{j,\varepsilon}} \le 1+\varepsilon,\ \ k\ge j$. By continuity this inequality holds up to the closure of $\overline{E}_{j,\varepsilon}$, i.e., $\|c_k(z)\|^{1/k}_{\overline{E}_{j,\varepsilon}} \le 1+\varepsilon,\ \ k\ge j$. By the Bernstein–Walsh inequality

$$|c_k(z)|^{1/k} \le (1+\varepsilon)\exp V^*\left(z, \overline{E}_{j,\varepsilon}\right),\ \ z\in\mathbb{C}^n,\ k\ge j.$$

Hence, for a fixed radius $R>0$

$$|c_k(z)|^{1/k} \le (1+\varepsilon)\max_{\|z\|=R} \exp V^*\left(z, \overline{E}_{j,\varepsilon}\right),\ \ z\in\partial B(0,R),\ k\ge j,$$

and arbitrary $z\in\mathbb{C}^n$ we have

$$|c_k(z)|^{1/k} = \left|c_k\left(\frac{\|z\|}{R}\frac{Rz}{\|z\|}\right)\right|^{1/k} = \frac{\|z\|}{R}\left|c_k\left(\frac{Rz}{\|z\|}\right)\right|^{1/k} \le (1+\varepsilon)\|z\| \frac{\max_{\|\xi\|=R} \exp V^*\left(\xi, \overline{E}_{j,\varepsilon}\right)}{R},\ \ k\ge j.$$

For $R \to \infty$ from this inequality we obtain that

$$|c_k(z)|^{1/k} \leq (1+\varepsilon)\|z\| \exp \underline{\gamma}\left(\overline{E}_{j,\varepsilon}\right), \quad k \geq j. \tag{5.4}$$

From (5.4) we get that the series $\sum\limits_{k=0}^{\infty} c_k(z)$ uniformly converges inside the ball

$$B\left(0, \frac{1}{(1+\varepsilon)\exp \underline{\gamma}\left(\overline{E}_{j,\varepsilon}\right)}\right) = \left\{\|z\| < \frac{1}{(1+\varepsilon)\exp \underline{\gamma}\left(\overline{E}_{j,\varepsilon}\right)}\right\}. \tag{5.5}$$

By taking $j \to \infty$ and then $\varepsilon \to 0$ from (5.5) we obtain that the series $\sum\limits_{k=0}^{\infty} c_k(z)$ uniformly converges inside the ball

$$B\left(0, \frac{1}{\exp \underline{\gamma}(E)}\right) = \left\{z \in \mathbb{C}^n : \quad \|z\| < \frac{1}{\exp \underline{\gamma}(E)}\right\}.$$

Theorem is proved. □

Remark 5.3. As we can see from the proof of Theorem 5.4, the series convergence domain $f(z) = \sum\limits_{k=0}^{\infty} c_k(z)$ may be larger than ball (5.2) if we use the estimates of homogeneous polynomials in circle domains.

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A. S. Sadullaev
National University of Uzbekistan, Tashkent, Uzbekistan
E-mail: sadullaev@mail.ru