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AN IMPROVED BLOW-UP CRITERION FOR THE MAGNETOHYDRODYNAMICS WITH THE HALL AND ION-SLIP EFFECTS

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Abstract. In this work, we consider the magnetohydrodynamics system with the Hall and ion-slip effects in $\mathbb{R}^3$. The main result is a sufficient condition for regularity on a time interval $[0, T]$ expressed in terms of the norm of the homogeneous Besov space $\dot{B}^0_{\infty,\infty}$ with respect to the pressure and the $BMO-$norm with respect to the gradient of the magnetic field, respectively

$$\int_0^T \left(\|\nabla \pi(t)\|^{\frac{2}{3}}_{\dot{B}^0_{\infty,\infty}} + \|\nabla B(t)\|^2_{BMO} \right) dt < \infty,$$

which can be regarded as improvement of the result in [3].

Keywords: magnetohydrodynamics system, Hall effect, ion-slip effect, homogeneous Besov space, blow-up criterion.

Conflict-of-interest. The authors declare no conflicts of interest.

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CONTENTS

1. Introduction and Main Result

Magnetohydrodynamics (MHD) is concerned with interaction between fluid flow and magnetic field. The fundamental governing equations consist of the compressible Navier–Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. In this paper, we consider the Cauchy problem of the following incompressible MHD equations with the Hall and ion-slip effects in $\mathbb{R}^3$:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla \pi = (\nabla \times B) \times B + \mu \Delta u, \\ \partial_t B + \nabla \times (u \times B) + \sigma \nabla \times ((\nabla \times B) \times B) = \kappa \nabla \times [B \times (B \times (\nabla \times B))] + \eta \Delta B, \\ \nabla \cdot u = \nabla \cdot B = 0, \\ u(x,0) = u_0(x), \quad B(x,0) = B_0(x). \end{cases} \tag{1.1}$$

Here, the nonnegative parameters μ and η are associated with the properties of the materials: μ denotes the kinematic viscosity coefficient of the fluid and η denotes the reciprocal of the magnetic Reynolds number. $\kappa \geq 0$, σ are constants. Because of their mathematical and physical importance, there is a great amount of literature on the mathematical theory of MHD equations with the Hall and ion-slip effects, for instances see [3, 4] and references therein.

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The mathematical study of the above system has important applications in fluid mechanics and material sciences, and has recently attracted considerable attention in the community of mathematical fluids (see, e.g., [6–8]). Physically u denotes the velocity of the fluid, π the pressure, B denotes the magnetic field, while $u_0(x)$ and $B_0(x)$ are the given initial velocity and initial magnetic field with $\nabla \cdot u_0 = 0$ and $\nabla \cdot B_0 = 0$, respectively in the sense of distribution. Comparing with the usual viscous incompressible MHD equations, system (1.1) contains the extra term $\nabla \times ((\nabla \times B) \times B)$, which is the so-called *Hall term* and $\nabla \times [B \times (B \times (\nabla \times B))]$ the ion-slip effect. Here we have normalized the viscous coefficient and the magnetic diffusion coefficient to be 1 for convenience.

System (1.1) is important to describe some physical phenomena, e.g., in the magnetic reconnection in space plasmas, star formation, neutron stars and dynamo. In the case $\sigma = \kappa = 0$, system (1.1) reduces to the standard MHD equations; when $\kappa = 0$, system (1.1) reduces to the Hall-MHD system.

In [8], Mulone and Solonnikov proved the small data global existence of strong solutions in a bounded domain. Therefore, it is important to study a global regularity criterion and structure of possible singularities of strong solutions. In a recent paper, Fan et al. [3] proved the existence of local-in-time strong solutions. Various criteria for regularity in terms of the velocity field, the magnetic field, the pressure and their derivatives have been proposed to (1.1) in [3]. In particular, they proved that if (u, π, B) satisfies one of the following conditions

$$\left\{ \begin{array}{c} u \in L^{\frac{2q}{q-3}}(0,T; L^q(\mathbb{R}^3)), \quad 3 < q < \infty, \\ \nabla \pi \in L^{\frac{2s}{3s-3}}(0,T; L^s(\mathbb{R}^3)), \quad 3 < s \leq \infty, \\ \nabla \pi \in L^{\frac{2}{3}}(0,T; BMO(\mathbb{R}^3)), \end{array} \right. \tag{1.2}$$

and

$$B \in L^\infty(0,T; L^\infty(\mathbb{R}^3)), \ \nabla B \in L^{\frac{2s}{s-3}}(0,T; L^s(\mathbb{R}^3)) \quad \text{with} \ \ 3 < s \leq \infty \tag{1.3}$$

and $0 < T < \infty$, then the solution (u, B) can be extended beyond time T. Here BMO denotes the Bounded Mean Oscillation space [9].

Later on, Gala and Ragusa [4] extended the results of [3] to the critical Besov space $\dot{B}^{-1}_{\infty,\infty}$ and multiplier spaces. Despite a great deal of efforts by mathematicians, the question of global existence or finite time blow-up of smooth solutions for the 3D MHD equations is still one of the most outstanding open problems in applied analysis. For further progresses on this topic, the interested readers are referred to [1, 6, 7] and references therein.

Motivated by this work in [3, 4], it is reasonable to establish a blow-up criterion on the pressure and magnetic field for system (1.1). The main purpose is to improve and extend the above regularity results in [3, 4] and to consider the main mechanism for possible breakdown of strong solutions to problem (1.1) in terms of the critical Besov spaces $\dot{B}^0_{\infty,\infty}$ by removing the assumption on the magnetic field $B \in L^\infty(0,T; L^\infty(\mathbb{R}^3))$ in [3].

For the sharp blow-up criterion, we need to introduce the following functional setting. We recall the homogeneous Besov space $\dot{B}^0_{\infty,\infty}$, which is defined as follows. Let $\{\varphi_j\}_{j\in\mathbb{Z}}$ be the Littlewood–Paley dyadic decomposition of unity, where the Fourier transform is supported on the annulus $\{\xi \in \mathbb{R}^3 : 2^{j-1} \leq |\xi| < 2^j\}$ (see, e.g. [2, 9]). Then,

$$f \in \dot{B}^0_{\infty,\infty}(\mathbb{R}^3) \quad \text{if and only if} \ \sup_{j\in\mathbb{Z}} \|\varphi_j * f\|_{L^\infty} = \|f\|_{\dot{B}^0_{\infty,\infty}} < \infty.$$

The following is a well-known embedding result (cf. [9, pp. 244]):

$$L^\infty(\mathbb{R}^3) \hookrightarrow BMO(\mathbb{R}^3) \hookrightarrow \dot{B}^0_{\infty,\infty}(\mathbb{R}^3). \tag{1.4}$$

Now our result reads as:

Theorem 1.1. *Let (u,B) be the local strong solution to system* (1.1) *with initial data $(u_0,B_0) \in H^2(\mathbb{R}^3)$ and $\nabla \cdot u_0 = \nabla \cdot B_0 = 0$. Then (u,B) can be extended beyond time T provided that*

$$\nabla \pi \in L^{\frac{2}{3}}(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)) \quad \textit{and} \quad \nabla B \in L^2(0,T;BMO(\mathbb{R}^3)), \tag{1.5}$$

with $0 < T < \infty$.

Remark 1.1. We would like to compare (1.5) with the corresponding results $(1.2)_2$ and $(1.2)_3$ on pressure fields. Due to the embedding relation (1.4), our result here obviously improves the previous ones $(1.2)_2$ and $(1.2)_3$. And it should be mentioned that our regularity criterion result (1.5) covers the limiting case $s = \infty$ in $(1.2)_2$, and also extends it into the larger $\dot{B}^0_{\infty,\infty}$ spaces. Moreover, the assumption on the magnetic field $B \in L^\infty(0,T;L^\infty(\mathbb{R}^3))$ was removed.

Remark 1.2. Since the margin case $s = \infty$ in (1.3) on magnetic field appears to be more challenging, we have refined the above results in critical Lebesgue spaces to the BMO spaces in the following sense

$$\nabla B \in L^2(0,T;BMO(\mathbb{R}^3)).$$

Remark 1.3. In the absence of global well-posedness, the development of blow-up / non blow-up theory is of major importance for both theoretical and practical purposes. For incompressible Euler and Navier–Stokes equations, the well-known Beale–Kato–Majda's criterion [1] says that any solution u is smooth up to time T under the assumption that

$$\int_0^T \|\nabla \times u(\cdot,t)\|_{L^\infty} \, dt < \infty.$$

Later, Beale–Kato–Majda's criterion is slightly improved by Kozono–Taniuchi [5] under the assumption

$$\int_0^T \|\nabla \times u(\cdot,t)\|_{BMO} \, dt < \infty.$$

In this paper, we obtain a Beale–Kato–Majda type blow-up criterion of smooth solutions to Cauchy problem for the Hall-magnetohydrodynamics system in terms of the pressure and magnetic field.

Before proceeding further, we estimate the pressure in $(1.1)_1$, which will be needed later. Because of $\nabla \cdot u = \nabla \cdot B = 0$, we have

$$\nabla \times (u \times B) = (B \cdot \nabla)u - (u \cdot \nabla)B.$$

Along the arguments in [10], taking ∇div operator of both sides $(1.1)_1$, and using the identity

$$(\nabla \times B) \times B = (B \cdot \nabla)B - \nabla \left(\frac{|B|^2}{2} \right),$$

it follows that

$$\nabla \left(\pi + \frac{|B|^2}{2} \right) = (-\Delta)^{-1} \sum_{i,j=1}^{3} \frac{\partial^2}{\partial x_i \partial x_j} \left(\nabla (u_i u_j - B_i B_j) \right),$$

where we have used the facts $\nabla \cdot u = \nabla \cdot B = 0$, and then the Calderón–Zygmund inequality implies:

$$\|\nabla \pi\|_{L^q} \le C \|(u \cdot \nabla)u\|_{L^q} + C \|(B \cdot \nabla)B\|_{L^q}, \quad 1 < q < \infty. \tag{1.6}$$

Remark 1.4. Due to the velocity growth condition $(1.2)_1$ and estimate (1.6), it is reasonable to expect the regularity of the strong solutions by imposing some suitable growth conditions on the pressure.

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. The existence and uniqueness of local strong solutions was be done in [3], thus it is sufficient to establish a priori estimates for (u, B) for any $T > 0$. The key step is to establish that $\|u(\cdot,t)\|_{L^4}$ and $\|B(\cdot,t)\|_{L^4}$ are bounded due to the standard Serrin type criterion on the 3D MHD equations.

Proof. First, taking the inner product of $(1.1)_1$ with u, after integration by parts and taking the divergence-free property into account, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 = \int\limits_{\mathbb{R}^3} ((\nabla\times B)\times B)\cdot u dx$$
$$= \int\limits_{\mathbb{R}^3} ((B\cdot\nabla)B - \frac{1}{2}\nabla|B|^2)\cdot u dx = \int\limits_{\mathbb{R}^3} (B\cdot\nabla)B\cdot u dx. \quad (2.1)$$

Similarly, taking the inner product of $(1.1)_2$ with B, using the divergence free property, we obtain

$$\frac{1}{2}\frac{d}{dt}\|B\|_{L^2}^2 + \|\nabla B\|_{L^2}^2 + \|B\times(\nabla\times B)\|_{L^2}^2 = \int\limits_{\mathbb{R}^3} \nabla\times(u\times B)\cdot B dx$$
$$= \int\limits_{\mathbb{R}^3} [(B\cdot\nabla)u - (u\cdot\nabla)B]\cdot B dx = \int\limits_{\mathbb{R}^3} (B\cdot\nabla)u\cdot B dx = -\int\limits_{\mathbb{R}^3} (B\cdot\nabla)B\cdot u dx, \quad (2.2)$$

where the following cancellation property have been applied:

$$\int\limits_{\mathbb{R}^3} \nabla\times((\nabla\times B)\times B)\cdot B dx = \int\limits_{\mathbb{R}^3} ((\nabla\times B)\times B)\cdot(\nabla\times B) dx = 0.$$

Summing up (2.1) and (2.2), we easily get

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2 + \|B\|_{L^2}^2) + \|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2 + \|B\times(\nabla\times B)\|_{L^2}^2 = 0.$$

This proves

$$\|(u,B)\|_{L^\infty(0,T;L^2)} + \|(u,B)\|_{L^2(0,T;H^1)} \le C. \quad (2.3)$$

Now, we are devoted to obtaining the L^4 estimate of u and B. Multiplying $(1.1)_1$ by $|u|^2 u$, taking the divergence-free property into account and integrating the resulting equation lead to

$$\frac{1}{4}\frac{d}{dt}\|u\|_{L^4}^4 + \int\limits_{\mathbb{R}^3} |u|^2|\nabla u|^2 dx + \frac{1}{2}\int\limits_{\mathbb{R}^3} \left|\nabla|u|^2\right|^2 dx$$
$$= \int\limits_{\mathbb{R}^3} ((B\cdot\nabla)B - \frac{1}{2}\nabla|B|^2)\cdot|u|^2 u dx - \int\limits_{\mathbb{R}^3} u|u|^2\cdot\nabla\pi dx = K_1 + K_2. \quad (2.4)$$

In similar way, multipling $(1.1)_2$ by $|B|^2 B$, we find that

$$\frac{1}{4}\frac{d}{dt}\|B\|_{L^4}^4 + \int\limits_{\mathbb{R}^3} |B|^2|\nabla B|^2 dx + \frac{1}{2}\int\limits_{\mathbb{R}^3} \left|\nabla|B|^2\right|^2 dx$$
$$= \int\limits_{\mathbb{R}^3} (B\cdot\nabla)u\cdot|B|^2 B dx + \int\limits_{\mathbb{R}^3} (B\times(\nabla\times B))\cdot\nabla\times(|B|^2 B) dx$$

$$+\int\limits_{\mathbb{R}^3}[((\nabla\times B)\times B)\times B]\cdot(\nabla\times(|B|^2 B))dx = K_3+K_4+K_5. \quad (2.5)$$

Combining (2.4) and (2.5), we obtain

$$\frac{1}{4}\frac{d}{dt}\left(\|u\|_{L^4}^4+\|B\|_{L^4}^4\right)+\||u|\,|\nabla u|\|_{L^2}^2+\||B|\,|\nabla B|\|_{L^2}^2+\frac{1}{2}(\left\|\nabla|u|^2\right\|_{L^2}^2+\frac{1}{2}\left\|\nabla|B|^2\right\|_{L^2}^2)=\sum_{m=1}^{5}K_m. \quad (2.6)$$

In what follows, we will deal with each term on the right-hand side of (2.6) separately.

For the term K_1, using Hölder's and Young's inequalities, it follows that

$$\begin{aligned}K_1 &\le C\int\limits_{\mathbb{R}^3}|B|\,|\nabla B|\,|u|^3\,dx \le C\left\||u|^3\right\|_{L^{\frac{4}{3}}}\left\|\nabla|B|^2\right\|_{L^4}\le C\|u\|_{L^4}^3\|B\|_{L^4}\|\nabla B\|_{BMO}\\ &\le C\left(\|u\|_{L^4}^3+\|B\|_{L^4}^4\right)\|\nabla B\|_{BMO}\le C\left(\|u\|_{L^4}^3+\|B\|_{L^4}^4\right)(1+\|\nabla B\|_{BMO}^2), \quad (2.7)\end{aligned}$$

where we have used the following fact (see [5]):

$$\left\|\nabla|B|^2\right\|_{L^4}\le C\,\||B|\,\nabla B\|_{L^4}\le C\,\|B\|_{L^4}\,\|\nabla B\|_{BMO}\,.$$

In order to estimate $K_2=\int\limits_{\mathbb{R}^3}\nabla\pi\cdot|u|^2\,udx$, we decompose K_2 into three parts as follows:

$$\nabla\pi=\sum_{j\in\mathbb{Z}}\varphi_j*\nabla\pi=\sum_{j<-N}\varphi_j*\nabla\pi+\sum_{j=-N}^{N}\varphi_j*\nabla\pi+\sum_{j>N}\varphi_j*\nabla\pi$$

by using the Littlewood–Paley decomposition, where N is a positive integer to be determined later. Plugging this decomposition into K_2 produces that

$$\begin{aligned}K_2 &\le \left|\int\limits_{\mathbb{R}^3}\sum_{j<-N}\varphi_j*\nabla\pi\cdot|u|^2\,udx\right|+\left|\int\limits_{\mathbb{R}^3}\sum_{j=-N}^{N}\varphi_j*\nabla\pi\cdot|u|^2\,udx\right|+\left|\int\limits_{\mathbb{R}^3}\sum_{j>N}\varphi_j*\nabla\pi\cdot|u|^2\,udx\right|\\ &= K_{21}+K_{22}+K_{23}.\end{aligned}$$

Recalling the Bernstein inequality [2]:

$$\|\varphi_j*f\|_{L^q}\le C2^{3j(\frac{1}{p}-\frac{1}{q})}\|\varphi_j*f\|_{L^p}\,,\quad 1\le p\le q\le\infty, \quad (2.8)$$

with C being a positive constant independent of f and j, we apply the Hölder inequality to deduce that

$$\begin{aligned}K_{21} &\le \sum_{j<-N}\|\varphi_j*\nabla\pi\|_{L^4}\|u\|_{L^4}^3\le C\|u\|_{L^4}^3\sum_{j<-N}2^{3j(\frac{1}{2}-\frac{1}{4})}\|\varphi_j*\nabla\pi\|_{L^2}\\ &\le C\|u\|_{L^4}^3\left(\sum_{j<-N}2^{\frac{3}{2}j}\right)^{\frac{1}{2}}\left(\sum_{j<-N}\|\varphi_j*\nabla\pi\|_{L^2}^2\right)^{\frac{1}{2}}\\ &\le C2^{-\frac{3}{4}N}\|u\|_{L^4}^3\|\nabla\pi\|_{L^2}\le C2^{-\frac{3}{4}N}\|u\|_{L^4}^3(\|(u\cdot\nabla)u\|_{L^2}+\|(\nabla\times B)\times B\|_{L^2})\\ &= C\left(2^{-\frac{3}{2}N}\|u\|_{L^4}^6\right)^{\frac{1}{2}}\left(\|(u\cdot\nabla)u\|_{L^2}^2+\|(\nabla\times B)\times B\|_{L^2}^2\right)^{\frac{1}{2}}\\ &\le C\left(2^{-N}\|u\|_{L^4}^4\right)^{\frac{6}{4}}+\frac{1}{8}\|(u\cdot\nabla)u\|_{L^2}^2+\frac{1}{8}\|(\nabla\times B)\times B\|_{L^2}^2\,.\end{aligned}$$

Using the Hölder and Young inequalities, K_{22} is controlled as

$$\begin{aligned}
K_{22} &\le \int_{\mathbb{R}^3} \sum_{j=-N}^{N} |\varphi_j * \nabla\pi|\,|u|^3\,dx = \int_{\mathbb{R}^3} \sum_{j=-N}^{N} |\varphi_j * \nabla\pi|^{\frac{1}{2}}\,|\varphi_j * \nabla\pi|^{\frac{1}{2}}\,|u|^3\,dx \\
&= \sum_{j=-N}^{N} \left\| |\varphi_j * \nabla\pi|^{\frac{1}{2}} \right\|_{L^\infty} \left\| |\varphi_j * \nabla\pi|^{\frac{1}{2}} \right\|_{L^4} \|u\|_{L^4}^3 \\
&= \|u\|_{L^4}^3 \left(\sum_{j=-N}^{N} \|\varphi_j * \nabla\pi\|_{L^\infty}^{\frac{1}{2}} \|\varphi_j * \nabla\pi\|_{L^2}^{\frac{1}{2}} \right) \\
&\le C\|u\|_{L^4}^3 \left(\sup_{j\in\mathbb{Z}} \|\varphi_j * \nabla\pi\|_{L^\infty}^{\frac{1}{2}} \right) \left(\sum_{j=-N}^{N} \|\varphi_j * \nabla\pi\|_{L^2}^{\frac{1}{2}} \right) \\
&\le CN^{\frac{3}{4}} \|u\|_{L^4}^3 \|\nabla\pi\|_{\dot{B}^0_{\infty,\infty}}^{\frac{1}{2}} \|\nabla\pi\|_{L^2}^{\frac{1}{2}} \\
&\le CN^{\frac{3}{4}} \|u\|_{L^4}^3 \|\nabla\pi\|_{\dot{B}^0_{\infty,\infty}}^{\frac{1}{2}} \left(\|(u\cdot\nabla)u\|_{L^2} + \|(\nabla\times B)\times B\|_{L^2}\right)^{\frac{1}{2}} \\
&\le CN \|u\|_{L^4}^4 \|\nabla\pi\|_{\dot{B}^0_{\infty,\infty}}^{\frac{2}{3}} + \frac{1}{8}\|(u\cdot\nabla)u\|_{L^2}^2 + \frac{1}{8}\|(\nabla\times B)\times B\|_{L^2}^2 .
\end{aligned}$$

For K_{23}, with the aid of the Bernstein inequality, the Hölder inequality, and the Sobolev inequality, one shows that

$$\begin{aligned}
K_{23} &\le \sum_{j>N} \int_{\mathbb{R}^3} |\varphi_j * \nabla\pi|\,|u|^3\,dx \le \sum_{j>N} \|\varphi_j * \nabla\pi\|_{L^{\frac{12}{7}}} \|u\|_{L^4} \left\| |u|^2 \right\|_{L^6} \\
&\le C\|u\|_{L^4} \left\| \nabla |u|^2 \right\|_{L^2} \sum_{j>N} 2^{-\frac{j}{4}} \|\varphi_j * \nabla\pi\|_{L^2} \\
&\le C\|u\|_{L^4} \left\| \nabla |u|^2 \right\|_{L^2} \left(\sum_{j>N} 2^{-\frac{j}{2}} \right)^{\frac{1}{2}} \left(\sum_{j>N} \|\varphi_j * \nabla\pi\|_{L^2}^2 \right)^{\frac{1}{2}} \\
&\le C2^{-\frac{N}{4}} \|u\|_{L^4} \left\| \nabla |u|^2 \right\|_{L^2} \|\nabla\pi\|_{L^2} \\
&\le C\left(2^{-N}\|u\|_{L^4}^4\right)^{\frac{1}{4}} \left(\|(u\cdot\nabla)u\|_{L^2} + \|(\nabla\times B)\times B\|_{L^2}\right)^2 \\
&\le C\left(2^{-N}\|u\|_{L^4}^4\right)^{\frac{1}{4}} \left(\|(u\cdot\nabla)u\|_{L^2}^2 + \|(\nabla\times B)\times B\|_{L^2}^2\right).
\end{aligned}$$

Plugging the estimates of K_{21}, K_{22}, K_{23} into K_2, it follows that

$$\begin{aligned}
K_2 \le \left(C2^{-N}\|u\|_{L^4}^4\right)^{\frac{6}{4}} + \frac{1}{4}\|(u\cdot\nabla)u\|_{L^2}^2 + \frac{1}{4}\|(\nabla\times B)\times B\|_{L^2}^2 \\
+ CN\|u\|_{L^4}^4 \|\nabla\pi\|_{\dot{B}^0_{\infty,\infty}}^{\frac{2}{3}} + \left(C2^{-N}\|u\|_{L^4}^4\right)^{\frac{1}{4}} \left(\|u\cdot\nabla u\|_{L^2}^2 + \|(\nabla\times B)\times B\|_{L^2}^2\right). \quad (2.9)
\end{aligned}$$

Now let us choose a fixed positive integer N so that $C2^{-N}\|u\|_{L^4}^4 \approx \frac{1}{4}$, i.e., $N = \left[\frac{\log C + \log(\|u\|_{L^4}^4 + e)}{\log 4}\right] + 1$, where $[a]$ denotes the largest integer less than or equal to a. Therefore, inserting N into (2.9), we arrive at

$$K_2 \le C + C\left(\log C + \log(\|u\|_{L^4}^4 + e)\right)\|\nabla\pi\|_{\dot{B}^0_{\infty,\infty}}^{\frac{2}{3}}\|u\|_{L^4}^4 + \frac{1}{8}\|(u\cdot\nabla)u\|_{L^2}^2 + \frac{1}{8}\|(\nabla\times B)\times B\|_{L^2}^2. \tag{2.10}$$

For K_3, using integration by parts, Hölder's and Young's inequalities, we can derive

$$\begin{aligned} K_3 &= \sum_{i=1}^3 \int_{\mathbb{R}^3} B_i\partial_i u\cdot |B|^2 B dx = -\sum_{i=1}^3\int_{\mathbb{R}^3} B_i u\partial_i(|B|^2 B)dx \\ &= -\int_{\mathbb{R}^3} u|B|^2(B\cdot\nabla)Bdx - \int_{\mathbb{R}^3} B\cdot u\cdot\sum_{i=1}^3 B_i\partial_i B^2 dx \\ &\le C\|(B\cdot\nabla)B\|_{L^4}\|u\|_{L^4}\left\||B|^2\right\|_{L^2} \le C\|B\|_{L^4}^3\|u\|_{L^4}\|\nabla B\|_{BMO} \\ &\le C\left(\|u\|_{L^4}^3 + \|B\|_{L^4}^4\right)(1+\|\nabla B\|_{BMO}^2). \end{aligned} \tag{2.11}$$

In the same way, we get the estimates to K_4 and K_5 as follows:

$$\begin{aligned} K_4 &= \int_{\mathbb{R}^3}((B\times(\nabla\times B)))(\nabla|B|^2\times B)dx \le C\|B\times(\nabla\times B)\|_{L^4}\left\|\nabla|B|^2\right\|_{L^4}\|B\|_{L^2} \\ &\le C\|B\|_{L^4}^2\|\nabla B\|_{BMO}^2 \le C\left(1+\|B\|_{L^4}^4\right)\|\nabla B\|_{BMO}^2, \end{aligned} \tag{2.12}$$

where we have used the fact (see [5]) that

$$\|B\times(\nabla\times B)\|_{L^4} \le C\||B|\nabla B\|_{L^4} \le C\|B\|_{L^4}\|\nabla B\|_{BMO}$$

and

$$\begin{aligned} K_5 &= \int_{\mathbb{R}^3}[((\nabla\times B)\times B)\times B](\nabla|B|^2\times B)dx \\ &\le C\|B\times(\nabla\times B)\|_{L^4}\left\|\nabla|B|^2\right\|_{L^4}\left\||B|^2\right\|_{L^2} \le C\|B\|_{L^4}^4\|\nabla B\|_{BMO}^2. \end{aligned} \tag{2.13}$$

We substitute the estimate of K_m $(m = 1, 2, \ldots, 5)$ into (2.6) and obtain

$$\begin{aligned} &\frac{1}{4}\frac{d}{dt}\left(\|u\|_{L^4}^4 + \|B\|_{L^4}^4\right) + \frac{1}{2}\||u|\,|\nabla u|\|_{L^2}^2 + \frac{1}{2}\||B|\,|\nabla B|\|_{L^2}^2 + \frac{1}{2}\left\|\nabla|u|^2\right\|_{L^2}^2 + \frac{1}{2}\left\|\nabla|B|^2\right\|_{L^2}^2 \\ &\quad\le C + C\left(\log C + \log(\|u\|_{L^4}^4 + e)\right)\|\nabla\pi\|_{\dot{B}^0_{\infty,\infty}}^{\frac{2}{3}}\|u\|_{L^4}^4 + C(1+\|\nabla B\|_{BMO}^2)\left(\|B\|_{L^4}^4 + \|u\|_{L^4}^4\right), \end{aligned}$$

the above inequality reduces to

$$\begin{aligned} \frac{d}{dt}\left(\|u\|_{L^4}^4 + \|B\|_{L^4}^4\right) &\le C + C(1+\|\nabla B\|_{BMO}^2)\left(\|B\|_{L^4}^4 + \|u\|_{L^4}^4\right) \\ &\quad + C\|\nabla\pi\|_{\dot{B}^0_{\infty,\infty}}^{\frac{2}{3}}\log(\|u\|_{L^4}^4 + \|B\|_{L^4}^4 + e)\left(\|B\|_{L^4}^4 + \|u\|_{L^4}^4\right), \end{aligned}$$

for all $0 \le t < T$. For the sake of clear presentation, we set $F(t) = e + \|u(\cdot,t)\|_{L^4}^4 + \|B(\cdot,t)\|_{L^4}^4$. Thus, we have

$$\frac{dF}{dt}(t) \le C(1+\|\nabla B\|_{BMO}^2)F(t) + CF(t)\|\nabla\pi\|_{\dot{B}^0_{\infty,\infty}}^{\frac{2}{3}}\log(F(t)) + C.$$

Using Gronwall's inequality, we obtain for all $0 \le t \le T$:

$$F(t) \le (F(0)+CT)\exp\left(C\int_0^T\|\nabla\pi(\tau)\|_{\dot{B}^0_{\infty,\infty}}^{\frac{2}{3}}\log F(\tau)d\tau\right)\exp\left(\int_0^T 1+\|\nabla B(\tau)\|_{BMO}^2\,d\tau\right). \tag{2.14}$$

Taking logarithmic on both sides of (2.14), one has

$$\log F(t) \leq \log\left(F(0)+CT\right) + C\int\limits_0^T \|\nabla\pi(\tau)\|^{\frac{2}{3}}_{\dot{B}^0_{\infty,\infty}} \log F(\tau)d\tau + \int\limits_0^T \left\{1+\|\nabla B(\tau)\|^2_{BMO}\right\} d\tau.$$

Employing the Gronwall inequality again, we obtain that

$$\log F(t) \leq \log\left(F(0)+CT\right) \int\limits_0^T \|\nabla\pi(\tau)\|^{\frac{2}{3}}_{\dot{B}^0_{\infty,\infty}} d\tau < \infty$$

for any $0 \leq t \leq T$. This implies that

$$(u,B) \in L^\infty(0,T;L^4(\mathbb{R}^3)) \subset L^8(0,T;L^4(\mathbb{R}^3)). \tag{2.15}$$

From (2.15) and (1.2), it is easy to conclude that the solution (u,B) can be extended beyond $t=T$. This completes the proof of Theorem 1.1. □

REFERENCES

1. J. Beale, T. Kato, and A. Majda, "Remarks on breakdown of smooth solutions for the three-dimensional Euler equations," *Commun. Math. Phys.*, **94**, 61–66 (1984).
2. J.-Y. Chemin, *Perfect Incompressible Fluids*, Clarendon Press & Oxford University Press, New York (1998).
3. J. Fan, X. Jia, G. Nakamura, and Y. Zhou, "On well-posedness and blow-up criteria for the magnetohydrodynamics with the Hall and ion-slip effects," *Z. Angew. Math. Phys.*, **66**, 1695–1706 (2015).
4. S. Gala and M. A. Ragusa, "On the blow-up criterion of strong solutions for the MHD equations with the Hall and ion-slip effects in $\mathbb{R}^3$," *Z. Angew. Math. Phys.*, **67**, 18 (2016).
5. H. Kozono and Y. Taniuchi, "Bilinear estimates in *BMO* and the Navier—Stokes equations," *Math. Z.*, **235**, 173–194 (2000).
6. M. Maiellaro, "Uniqueness of MHD thermodiffusive mixture flows with Hall and ion-slip effects," *Meccanica*, **12**, 9–14 (1977).
7. G. Mulone and F. Salemi, "Some continuous dependence theorems in MHD with Hall and ion-slip currents in unbounded domains," *Rend. Accad. Sci. Fis. Mat. Napoli*, **55**, 139–152 (1988).
8. G. Mulone and V. A. Solonnikov, "On an initial boundary-value problem for the equation of magnetohydrodynamics with the Hall and ion-slip effects," *J. Math. Sci.* (*N.Y.*), **87**, 3381–3392 (1997).
9. H. Triebel, *Theory of Function Spaces*, Birkhäuser, Basel (1983).
10. Y. Zhou, "Regularity criteria for the 3D MHD equations in terms of the pressure," *Int. J. Nonlinear Mech.*, **41**, 1174–1180 (2006).

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