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ON CALCULATION OF THE NORM OF A MONOTONE OPERATOR IN IDEAL SPACES

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Abstract. This paper contains the proof of general results on the calculation of the norms of monotone operators acting from one ideal space to another under matching convexity and concavity properties of the operator and the norms in ideal spaces.

Keywords: monotone operator, norm, ideal space, convexity, concavity.

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CONTENTS

1. Introduction

The work is devoted to the justification of some general results on the computation of the norm of a monotone operator acting from one normalized (quazinormalized in more general case) ideal space to another and having the convexity property which matches the convexity and concavity properties of the (quazi)norms of the ideal spaces. The term of the ideal space of measurable functions generalizes the construction of the Banach functional space introduced by Bennett and Sharpley [3]. The general questions of theory ideal structures and Banach functional spaces are considered in papers by Kantorovich and Akilov [8], Kreyn, Petunin and Semenov [9] and also Berg and Löfström [4] and Triebel [10].

In the current paper we use the definitions and the general properties of the spaces represented in [1]. Here we note that the (quasi)norm $\|\cdot\|_X$ in the ideal space X possesses the monotone property: if the function f is measurable and $|f| \le g \in X$, then $f \in X$, $\|f\|_X \le \|g\|_X$; also $\|f\|_X < \infty \Rightarrow |f| < \infty$ almost everywhere and, moreover, if $0 \le f_m \le f_{m+1}$, $f_m \to f$ $(m \to \infty)$ almost everywhere, then $\|f\|_X = \lim\limits_{m\to\infty} \|f_m\|_X$ (the Fatou property). In [1] there was proven, in particular, that the ideal space is complete, i.e., that it is (quasi)Banach, and the ideal spans for the cones of functions with the monotone property are also described there.

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Justifying the results on the norm calculations in the current paper we generalize and modify the approach developed in paper by Burenkov and Goldman [5]. We also give the example of the usage of the general formulas when computing the norm of the integral operator. The number of applications of the general results proven here in the theory of Lorentz weighted spaces and in the calculation of the associated norms on the cones of functions with the monotone property is given in [2]. Note that some other approaches to the problems of such type were developed by Gogatishvili and Stepanov in [6, 7].

The structure of the paper is as follows. In Sec. 2 there are stated the main definitions and given the results on the calculation of the norm of the operator on the cone of decreasing nonnegative functions in the ideal space both in the nondegenerate case and in the case of degeneration (Theorem 2.1). The lemmas necessary for the proof of this lemma are considered in Sec. 3. Sec. 4 contains the proof of Theorem 2.1 and some of its corollaries. In Sec. 5 there are given generalizations of the obtained results onto more general cones of functions under monotone conditions (Theorem 5.1). In Sec. 6, we consider an example of the application of the general results to the calculation of the norm of an integral operator on the cone of functions with the monotone property.

2. General Theorems on the Norms of Operators on the Cones of Functions with the Monotone Property

Let (M, Σ_M, β), (N, Σ_N, γ) be spaces with nonnegative σ-additive complete measures β, γ; $S(M, \Sigma_M, \beta)$, $S(N, \Sigma_N, \gamma)$ be spaces of real-valued measurable functions.

We say that the norm in the ideal space $X \subset S(M, \Sigma_M, \beta)$ is *order continuous* if

$$\{x_m \in X, m \in \mathbb{N};\ 0 \le x_m \downarrow 0 \quad \beta - \text{a.e.}\} \Rightarrow \|x_m\|_X \downarrow 0. \tag{2.1}$$

The ideal space $X \subset S(M, \Sigma_M, \beta)$ is called l_p-*concave* for $p \in \mathbb{R}_+$ if

$$\left(\sum_m \|x_m\|_X^p\right)^{1/p} \le \left\|\left(\sum_m |x_m|^p\right)^{1/p}\right\|_X. \tag{2.2}$$

The ideal space $Y \subset S(N, \Sigma_N, \gamma)$ is called l_q-*convex* for $q \in \mathbb{R}_+$ if

$$\left\|\left(\sum_m |y_m|^q\right)^{1/q}\right\|_Y \le \left(\sum_m \|y_m\|_Y^q\right)^{1/q}. \tag{2.3}$$

That means that the convergence of series on the right-hand side of Eq. (2.2) or Eq. (2.3) implies the convergence of series on the left-hand side and the corresponding inequalities hold.

Note that every normalized ideal space is l_1-convex and the l_q-concavity for $0 < q < 1$ leads to the triangle inequality in the following form:

$$\|f + g\|_Y \le \left(\|f\|_Y^q + \|g\|_Y^q\right)^{1/q} \le 2^{1/q-1}\left(\|f\|_Y + \|g\|_Y\right). \tag{2.4}$$

Indeed, by the Jensen inequality for $0 < q < 1$ we have

$$|f + g| \le |f| + |g| \le (|f|^q + |g|^q)^{1/q}.$$

Next, the application of Eq. (2.3) and Hölder's inequality (for two summands) implies

$$\|f + g\|_Y \le \left\|(|f|^q + |g|^q)^{1/q}\right\|_Y \le \left(\|f\|_Y^q + \|g\|_Y^q\right)^{1/q} \le 2^{1/q-1}\left(\|f\|_Y + \|g\|_Y\right).$$

Also note that $Y = L_q(N, \gamma)$, $0 < q < \infty$ is l_ρ-convex for each $\rho \in (0, q]$ (see Lemma 4.1 below) and it is l_p-concave for each $p \in [q, \infty)$.

Consider a cone $D \subset X$ of nonnegative functions with the condition

$$f, g \in D;\ \alpha, \beta \ge 0 \Rightarrow \alpha f + \beta g \in D.$$

The operator $T : D \to Y$ is called l_r*-convex* as $0 < r < \infty$ if $\forall f_m \in D$, $m \in \mathbb{Z}$ such that $\left(\sum_m f_m^r\right)^{1/r} \in D$,

$$\left| T\left[\left(\sum_m f_m^r\right)^{1/r}\right]\right| \le \left(\sum_m |Tf_m|^r\right)^{1/r} \tag{2.5}$$

almost everywhere on M; and $\forall f \in D$; $\alpha \ge 0 \Rightarrow T[\alpha f] = \alpha T[f]$.

Note that the l_1-convexity of T coincides with the sublinearity

$$\left| T\left[\left(\sum_m f_m\right)\right]\right| \le \left(\sum_m |Tf_m|\right).$$

The operator T is called *monotone* if

$$\{f, g \in D;\ 0 \le f \le g \quad \beta - \text{a.e.}\} \Rightarrow \{0 \le Tf \le Tg \quad \gamma - \text{a.e.}\}. \tag{2.6}$$

An example of an l_r-convex monotone operator is the operator

$$T\,[f] = (L[f^r])^{1/r},$$

where L is a sublinear monotone operator. More, this formula illustrates the correspondence between the l_r-convex and sublinear operators.

We consider the case $M = J := (a, b)$, $-\infty \le a, b \le \infty$, with noninear Borel measure β and the restrictions of operator onto the following cones of nonnegative decreasing left-continuous functions on $J := (a, b)$:

$$\Omega = \{g \in X :\ 0 \le g \downarrow;\ g(t) = g(t-0),\ t \in (a, b)\},$$
$$\dot\Omega = \left\{g \in \Omega :\ \lim_{t\to b-0} g(t) = 0\right\}. \tag{2.7}$$

Let us define the norms of the restrictions of operator:

$$\|T\|_\Omega = \sup\left\{\|T[g]\|_Y : g \in \Omega,\ \|g\|_X \le 1\right\}, \tag{2.8}$$

$$\|T\|_{\dot\Omega} = \sup\left\{\|T[g]\|_Y : g \in \dot\Omega,\ \|g\|_X \le 1\right\}. \tag{2.9}$$

We denote by

$$\dot\Omega_0 := \left\{\chi_{(a,t]} : a < t < b\right\},\ \Omega_0 := \dot\Omega_0 \bigcup \chi_{(a,b)}; \tag{2.10}$$

$$F(x, t) = T\left[\chi_{(a,t]}\right](x),\ a < t < b; \quad F(x, b) = T\left[\chi_{(a,b)}\right](x). \tag{2.11}$$

Theorem 2.1. *Let* $0 < p \le q \le r < \infty$; $X \subset S(J, \beta)$ *be an ideal* l_p*-concave space with order continuous (quasi)norm;* $Y \subset S(N, \gamma)$ *be an ideal* l_q*-convex space, and* $T : \Omega \to Y$ *be an* l_r*-convex monotone operator.*

(1) *Then there hold the relations*

$$\|T\|_{\dot\Omega} = \|T\|_{\dot\Omega_0} := \sup_{a<t<b}\left[\|F(\cdot, t)\|_Y \|\chi_{(a,t]}(\cdot)\|_X^{-1}\right]. \tag{2.12}$$

(2) *Under the additional nondegeneracy condition*

$$\|\chi_{(a,b)}\|_X = \infty \tag{2.13}$$

there hold the relations

$$\|T\|_\Omega = \|T\|_{\dot\Omega} = \|T\|_{\dot\Omega_0} := \sup_{a<t<b}\left[\|F(\cdot, t)\|_Y \|\chi_{(a,t]}(\cdot)\|_X^{-1}\right]. \tag{2.14}$$

(3) *Under the degeneracy*

$$\|\chi_{(a,b)}\|_X < \infty \tag{2.15}$$

there hold the relations

$$\|T\|_\Omega = \|T\|_{\Omega_0} := \max\left\{\|T\|_{\dot{\Omega}_0}, \|F(\cdot,b)\|_Y \|\chi_{(a,b)}(\cdot)\|_X^{-1}\right\}, \tag{2.16}$$

see Eq. (2.11).

Remark 2.1. Note that in nondegenerate case Eq. (2.13)

$$\|\chi_{(a,b)}\|_X = \infty \Rightarrow \Omega = \dot{\Omega},$$

since

$$\left\{0 \le g \downarrow, \lim_{t\to b-0} g(t) > 0\right\} \Rightarrow g \notin X. \tag{2.17}$$

This means that in the case of Eq. (2.13) there hold the equality $\|T\|_\Omega = \|T\|_{\dot{\Omega}}$. Thus for the calculation of $\|T\|_\Omega$ one can apply Eq. (2.12). Thus, there holds the relation (2.14) and the item (2) of Theorem 2.1 follows from its item (1).

Remark 2.2. When proving Theorem 2.1 we firstly prove the general statement, which comprises item (3) of this theorem. Next we note the simplifications (quite significant), which arise in this reasoning while proving item (1) of theorem, no matter if the nondegeneracy condition (2.13) holds or fails. These simplifications are connected to the fact that in item (1) of theorem we consider the cone $\dot{\Omega}$ instead of the general cone Ω, see Eq. (2.7).

3. Lemmas

Lemma 3.1. *Let* $0 < q < s < \infty$; ω, ψ *be nonegative functions on* (a,b), $-\infty \le a < b \le \infty$, $\omega \uparrow$, $\psi \downarrow$; $\omega, \psi, \psi' \in C(a,b)$, $\psi(a) > \psi(b)$. *We define*

$$\omega(a) := \lim_{t\to a+0} \omega(t),\ \omega(b) := \lim_{t\to b-0} \omega(t),\ \psi(a) := \lim_{t\to a+0} \psi(t),\ \psi(b) := \lim_{t\to b-0} \psi(t).$$

If $\psi(b) = 0$, *then we set* $C := 0$. *If* $\psi(b) > 0$, *then we set that the condition* $C := \omega(b)\psi(b) < \infty$ *is valid. We introduce for* $r \in [q,s]$ *the operator*

$$A_r(a,b) = \left\{ \int_{(a,b)} \omega(t)^r \left(-d\left[\psi(t)^r\right]\right) \right\}^{1/r}. \tag{3.1}$$

Then

$$\{C^s + A_s(a,b)^s\}^{1/s} \le \{C^q + A_q(a,b)^q\}^{1/q}. \tag{3.2}$$

Proof. We set $A_q(a,b) < \infty$.

1. Firstly we consider the case when $b \in (a,\infty]$, $\psi(b) = 0$.

$$A_s(a,b)^s = \int_{(a,b)} \omega^s \left(-d\left[(\psi^q)^{s/q}\right]\right) = \frac{s}{q}\int_{(a,b)} \omega^s \psi^{s-q}\left(-d\left[\psi^q\right]\right) = \frac{s}{q}\int_{(a,b)} [\omega\psi]^{s-q}\omega^q\left(-d\left[\psi^q\right]\right).$$

Note that the increase of ω^q together with the condition $\psi(b) = 0$ implies that

$$[\omega(t)\psi(t)]^{s-q} \le \left[\int_{[t,b)} \omega(\tau)^q \left(-d\left[\psi(\tau)^q\right]\right)\right]^{s/q-1},\ t \in (a,b).$$

Thus,

$$A_s(a,b)^s \le \frac{s}{q} \int\limits_{(a,b)} \left[\int\limits_{[t,b)} \omega(\tau)^q \left(-d\left[\psi(\tau)^q\right]\right) \right]^{s/q-1} \omega(t)^q \left(-d\left[\psi(t)^q\right]\right)$$

$$= \int\limits_{(a,b)} \left(-d \left[\left(\int\limits_{[t,b)} \omega(\tau)^q \left(-d\left[\psi(\tau)^q\right]\right) \right]^{s/q} \right] \right) = \left[\int\limits_{(a,b)} \omega^q \left(-d\left[\psi^q\right]\right) \right]^{s/q} = A_q(a,b)^s.$$

2. Now let $b \in \mathbb{R}_+$, $\psi(b) > 0$, $C = \omega(b)\psi(b) < \infty$. We extend the functions ω, ψ onto $[b, b+1)$ in such way that $\omega(\tau) = \omega(b)$, $\tau \in [b, b+1)$; $\psi(\tau) \downarrow$, $\psi(b+1) = 0$. Next we apply the result obtained on the first step on $(a, b+1)$ and obtain

$$A_s(a, b+1) \le A_q(a, b+1). \tag{3.3}$$

Next,

$$A_s(a,b+1)^s = A_s(a,b)^s + \int\limits_{[b,b+1)} \omega(t)^s \left(-d\left[\psi(t)^s\right]\right.$$

$$= A_s(a,b)^s + \omega(b)^s \int\limits_{[b,b+1)} \left(-d\left[\psi(t)^s\right]\right) = A_s(a,b)^s + \omega(b)^s \psi(b)^s.$$

Analogically, $A_q(a,b+1)^q = A_q(a,b)^q + \omega(b)^q \psi(b)^q$, and Eq. (3.3) leads to Eq. (3.2).

3. Consider the last case $b = \infty$, $\psi(\infty) > 0$. We apply the inequality obtained above in the case when $b < \infty$, $\psi(b) > 0$:

$$\left\{\omega(b)^s \psi(b)^s + A_s(a,b)^s\right\}^{1/s} \le \left\{\omega(b)^q \psi(b)^q + A_q(a,b)^q\right\}^{1/q}.$$

Then we pass to the limit as $b \to +\infty$. Then

$$\left\{\omega(\infty)^s \psi(\infty)^s + A_s(a,\infty)^s\right\}^{1/s} \le \left\{\omega(\infty)^q \psi(\infty)^q + A_q(a,\infty)^q\right\}^{1/q}.$$

□

Remark 3.1. Estimate (3.2) remain valid in the case of substitution of C with any constant $D \in (C, \infty)$.

Proof. As $A, B > 0$, $0 < q < s < \infty$ consider the function $\varphi(x) = (x^s + A^s)^{1/s}(x^q + B^q)^{-1/q}$, $x \in \mathbb{R}_+$.

1. If $A \le B$, then $\varphi(x) \le (x^s + B^s)^{1/s}(x^q + B^q)^{-1/q} \le 1$, $x \in \mathbb{R}_+$.
2. Let $A > B$. Then $\varphi(0) = AB^{-1} > 1$. Moreover,

$$\varphi'(x) = (x^s + A^s)^{1/s-1}(x^q + B^q)^{-1/q-1} x^{q-1}(x^{s-q}B^q - A^s).$$

Thus, the function φ decreases from $\varphi(0) > 1$ to $\varphi(x_1) < 1$, $x_1 = (A^s B^{-q})^{1/(s-q)}$, then increases to $\varphi(+\infty) = 1$. Thus, if $\varphi(x_0) \le 1$ for $x_0 > 0$, then $\varphi(x) \le 1$ for all $x \ge x_0$. Then we denote $A = A_s(a,b)$, $B = A_q(a,b)$ and note that by Eq. (3.2)

$$\varphi(C) = \left(C^s + A_s(a,b)^s\right)^{1/s} \left(C^q + A_q(a,b)^q\right)^{-1/q} \le 1.$$

Thus for any $D \ge C$ we have $\varphi(D) \le 1$. □

Lemma 3.2. *Let*

$$\omega_m,\ \psi_m \ge 0,\ m \in \mathbb{Z};\ \omega_m \uparrow,\ \psi_m \downarrow.$$

We define

$$\omega_\infty = \lim_{m\to+\infty} \omega_m,\ \psi_\infty = \lim_{m\to+\infty} \psi_m,\ C := \omega_\infty \cdot \psi_\infty. \tag{3.4}$$

Here we set that

$$\psi_\infty = 0 \Rightarrow C = 0;\ \psi_\infty > 0 \Rightarrow C < \infty.$$

Then as $0 < q < s < \infty$

$$\left\{ C^s + \sum_{m\in\mathbb{Z}} \omega_m^s \left(\psi_m^s - \psi_{m+1}^s\right) \right\}^{1/s} \le \left\{ C^q + \sum_{m\in\mathbb{Z}} \omega_m^q \left(\psi_m^q - \psi_{m+1}^q\right) \right\}^{1/q}. \tag{3.5}$$

Estimate (3.5) *remains valid under the substitution of* C *with any constant* $D \in (C, \infty)$.

Proof. We introduce the function $\psi \in C^1(\mathbb{R}_+)$, $0 \le \psi \downarrow$; $\psi(2^m) = \psi_m, m \in \mathbb{Z}$.

Next, we introduce for $n \in \mathbb{N}$ the function $\omega(n,\cdot) \in C(\mathbb{R}_+)$ on $[2^m, 2^{m+1}]$, $m \in \mathbb{Z}$, in the following way:

$$\omega(n,t) = \begin{cases} \omega_m,\ 2^m \le t \le 2^{m+1} - 2^{m-1}/n; \\ \text{linear as } 2^{m+1} - 2^{m-1}/n \le t \le 2^{m+1}. \end{cases}$$

Note that

$$\psi(\infty) := \lim_{t\to+\infty} \psi(t) = \lim_{m\to+\infty} \psi_m = \psi_\infty.$$

If $\psi_\infty = 0$, then we set $C = 0$. Next, if $\omega_\infty = \lim_{m\to+\infty} \omega_m < \infty$, then we have $\omega(n,\infty) = \lim_{t\to+\infty} \omega(n,t) = \omega_\infty < \infty$ (ω_∞ does not depend on $n \in \mathbb{N}$), and we set $C := \omega(n,\infty)\psi(\infty) = \omega_\infty\psi_\infty < \infty$. By Eq. (3.2) there holds the inequality

$$I_n := \left\{ C^s + \int_{\mathbb{R}_+} \omega(n,t)^s \left(-d\left[\psi(t)^s\right]\right) \right\}^{1/s} \le J_n := \left\{ C^q + \int_{\mathbb{R}_+} \omega(n,t)^q \left(-d\left[\psi(t)^q\right]\right) \right\}^{1/q}.$$

Note that $\{\omega(n,t)\}_{n\in\mathbb{N}}$ is the decreasing sequence and

$$\lim_{n\to+\infty} \omega(n,t) = \omega(\infty,t) = \omega_m,\ t \in [2^m, 2^{m+1}), \quad m \in \mathbb{Z}.$$

Then by Levi theorem on the monotone convergence we can pass to the limit as $n \to +\infty$ in the latter inequality, which implies

$$I_\infty := \left\{ C^s + \int_{\mathbb{R}_+} \omega(\infty,t)^s \left(-d\left[\psi(t)^s\right]\right) \right\}^{1/s} \le J_\infty := \left\{ C^q + \int_{\mathbb{R}_+} \omega(\infty,t)^q \left(-d\left[\psi(t)^q\right]\right) \right\}^{1/q}.$$

However,

$$I_\infty := \left\{ C^s + \sum_{m\in\mathbb{Z}} \int_{[2^m,2^{m+1})} \omega(\infty,t)^s \left(-d\left[\psi(t)^s\right]\right) \right\}^{1/s} = \left\{ C^s + \sum_{m\in\mathbb{Z}} \omega_m^s \int_{[2^m,2^{m+1})} \left(-d\left[\psi(t)^s\right]\right) \right\}^{1/s},$$

which coincides with the left-hand side of Eq. (3.5). Analogically,

$$J_\infty = \left\{ C^q + \sum_{m\in\mathbb{Z}} \omega_m^q \int_{[2^m,2^{m+1})} \left(-d\left[\psi(t)^q\right]\right) \right\}^{1/q} = \left\{ C^q + \sum_{m\in\mathbb{Z}} \omega_m^q \left(\psi_m^q - \psi_{m+1}^q\right) \right\}^{1/q}.$$

These reasonings lead to Eq. (3.5). □

Corollary 3.1. *Let*

$$\omega_m,\, \psi_m \geq 0,\; m \in \{0, 1, \ldots, m_0+1\},\; \omega_m \uparrow,\; \psi_m \downarrow,\; C = \omega_{m_0+1} \cdot \psi_{m_0+1}. \tag{3.6}$$

Then as $0 < q \leq s < \infty$

$$\left\{ C^s + \sum_{m=0}^{m_0} \omega_m^s \left(\psi_m^s - \psi_{m+1}^s \right) \right\}^{1/s} \leq \left\{ C^q + \sum_{m=0}^{m_0} \omega_m^q \left(\psi_m^q - \psi_{m+1}^q \right) \right\}^{1/q}. \tag{3.7}$$

Estimate (3.7) *holds as* $m_0 = \infty$ *with constant*

$$C = \omega_\infty \cdot \psi_\infty, \quad \omega_\infty = \lim_{m\to+\infty} \omega_m, \quad \psi_\infty = \lim_{m\to+\infty} \psi_m; \quad \infty \cdot 0 := 0.$$

This estimate remain valid in the case of substitution of C *with any constant* $D \in (C, \infty)$.

Indeed, in Eq. (3.5) we set $\omega_m = 0$, $\psi_m = \psi_0$, $m \leq -1$; if $m_0 < \infty$,

$$\omega_m = \omega_{m_0+1}, \quad \psi_m = \psi_{m_0+1}, \quad m \geq m_0 + 2,$$

then Eq. (3.5) implies Eq. (3.7).

Remark 3.2. We obtained the discrete estimates (3.5) and (3.7) as the corollaries of the integral estimate (3.2). For the applications if suffices to show that Eq. (3.2), in its turn, can be obtained from the discrete estimates. In order to do this it suffices for ω to be left-continuous, and for ψ to be right-continuous on (a, b). For completeness of the material we give the following statement.

Lemma 3.3. *Let* $0 < q < s < \infty$; ω, ψ *be the nonegative functions on* (a, b); $-\infty \leq a < b \leq \infty$, $\omega \uparrow$, $\psi \downarrow$; ω *be left-continuous on* (a, b), ψ *be right-continuous on* (a, b); $\psi(b) < \psi(a) < \infty$. *We define*

$$\omega(a) := \lim_{t\to a+0} \omega(t), \quad \omega(b) := \lim_{t\to b-0} \omega(t), \quad \psi(a) := \lim_{t\to a+0} \psi(t), \quad \psi(b) := \lim_{t\to b-0} \psi(t).$$

If $\psi(b) = 0$, *we set* $C := 0$. *If* $\psi(b) > 0$, *we claim that* $C := \omega(b)\psi(b) < \infty$. *As* $r \in [q, s]$ *we introduce* $A_r(a, b)$ *by Eq.* (3.1). *Then estimate* (3.2) *holds.*
Proof.

1. Firstly we obtain some useful inequalities for Lebesgue–Stieltjes integral (in the case considered here it coincides with the Riemann–Stieltjes integral). For any $r \in \mathbb{R}_+$, $a \leq t < \tau < b$ and decreasing on (a, b) right-continuous at t (note that it happens automatically at $t = a$ since $\psi(a) = \psi(a + 0)$) function ψ there holds the quality

$$\int_{(t,\tau]} (-d\,[\psi^r]) = \psi(t)^r - \psi(\tau)^r. \tag{3.8}$$

Indeed,

$$\int_{(t,\tau]} (-d\,[\psi^r]) = \lim_{\rho\to t+0} \int_{[\rho,\tau]} (-d\,[\psi^r]) = \lim_{\rho\to t+0} [\psi(\rho)^r - \psi(\tau)^r] = \psi(t)^r - \psi(\tau)^r.$$

Here we use the equality

$$\int_{[\rho,\tau]} (-d\,[\psi^r]) = \psi(\rho)^r - \psi(\tau)^r. \tag{3.9}$$

It follows from the definition of the Riemann–Stieltjes integral, since all integral sums of the integral on the segment $[\rho, \tau]$ coincide with the right-hand side of this formula. Thus, there holds Eq. (3.8).

Passing to the limit in this formula as $t = a$, $\tau \to b - 0$, we obtain

$$\int_{(a,b)} (-d\,[\psi(t)^r]) = \psi(a)^r - \psi(b)^r. \tag{3.10}$$

For any $r \in \mathbb{R}_+$, $a < t < \tau \le b$ and function ψ decreasing on (a,b) left-continuous at τ (note that at point $\tau = b$ it happens automatically, since $\psi(b) = \psi(b-0)$) there holds the equality

$$\int\limits_{[t,\tau)} (-d\,[\psi(t)^r]) = \psi(t)^r - \psi(\tau)^r. \tag{3.11}$$

Indeed,

$$\int\limits_{[t,\tau)} (-d\,[\psi(t)^r]) = \lim_{\rho\to\tau-0} \int\limits_{[t,\rho]} (-d\,[\psi(t)^r]) = \lim_{\rho\to\tau-0} [\psi(t)^r - \psi(\rho)^r] = \psi(t)^r - \psi(\tau)^r.$$

Moreover, if ψ is left-continuous at τ and right-continuous at t, then

$$\int\limits_{(t,\tau)} (-d\,[\psi(t)^r]) = \psi(t)^r - \psi(\tau)^r. \tag{3.12}$$

Applying Eq. (3.11), we obtain

$$\int\limits_{(t,\tau)} (-d\,[\psi(t)^r]) = \lim_{\rho\to t+0} \int\limits_{[\rho,\tau)} (-d\,[\psi(t)^r]) = \lim_{\rho\to t+0} [\psi(\rho)^r - \psi(\tau)^r] = \psi(t)^r - \psi(\tau)^r.$$

Set $A_q(a,b) < \infty$ (otherwise there is nothing to prove in Eq. (3.2)). If $\omega(a) > 0$, then this supposition implies $\psi(a) < \infty$. Indeed, ω increases on (a,b), thus, $\omega(t) \ge \omega(a)$, $t \in (a,b)$, and

$$A_q(a,b)^q = \int\limits_{(a,b)} \omega^q\,(-d\,[\psi^q]) \ge \omega(a)^q \int\limits_{(a,b)} (-d\,[\psi^q]) = \omega(a)^q\,[\psi(a)^q - \psi(b)^q]\,.$$

The case $\omega(a) = 0$ will be considered in the fifth step below.

2. At first we consider the case $0 < \omega(a) = \omega(b)$. Then $\omega(t) = \omega(b)$, $t \in [a,b)$, thus Eq. (3.10) as $r = s$ and $r = q$ leads to the equality

$$A_s^s = \omega(b)^s \int\limits_{(a,b)} (-d\,[\psi(t)^s]) = \omega(b)^s\,[\psi(a)^s - \psi(b)^s]\,,$$

$$\{\omega(b)^s\psi(b)^s + A_s^s\}^{1/s} = \omega(b)\psi(a) = \left\{\omega(b)^q\psi(b)^q + A_q^q\right\}^{1/q}.$$

3. Let $0 < \omega(a) < \omega(b) < \infty$. As $1 < d < \omega(b)/\omega(a)$ we choose $t_m \in [a,b)$ in the following way: $t_0 = a$, $t_1 = \sup\{t > a : \omega(t) \le d\omega(a)\}$,

$$\begin{aligned} &t_{m+1} = \sup\{t > a : \omega(t) \le d\omega(t_m + 0)\}\,,\ m = 1,2,\ldots,m_0 - 1;\\ &\omega(t_{m_0}+0) < \omega(b) \le d\omega(t_{m_0}+0). \end{aligned} \tag{3.13}$$

Note that

$$\begin{aligned} &(1)\quad t_{m+1} > t_m,\ m = 0,1,\ldots,m_0;\\ &(2)\quad \omega(t_m+0) \le \omega(t) \le d\omega(t_m+0), t \in \delta_m := (t_m, t_{m+1}],\\ &m = 0,1,\ldots,m_0-1;\\ &(3)\quad \omega(t) > d\omega(t_m+0),\ \forall t \in (t_{m+1}, b);\ m = 0,1,\ldots,m_0-1. \end{aligned} \tag{3.14}$$

Thus, $a = t_0 < t_1 < \ldots < t_{m_0} < t_{m_0+1} := b$.

Now we obtain the estimates for A_s^s, A_q^q. We denote by $\delta_m := (t_m, t_{m+1}]$, $m = 0,1,\ldots,m_0-1$, $\delta_{m_0} = (t_{m_0}, t_{m_0+1})$.

$$A_s^s = \int\limits_{(a,b)} \omega^s\,(-d\,[\psi^s]) = \sum_{m=0}^{m_0-1} \int\limits_{\delta_m} \omega^s\,(-d\,[\psi^s]) + \int\limits_{(t_{m_0},b)} \omega^s\,(-d\,[\psi^s])\,.$$

The function ω^s increases in such way that

$$A_s^s \le \sum_{m=0}^{m_0-1} \omega(t_{m+1})^s \int_{\delta_m} (-d\,[\psi^s]) + \omega(b)^s \int_{(t_{m_0},b)} (-d\,[\psi^s]) = \sum_{m=0}^{m_0} \omega(t_{m+1})^s \,[\psi(t_m)^s - \psi(t_{m+1})^s]$$

$$\le d^s \sum_{m=0}^{m_0} \omega(t_m+0)^s \,[\psi(t_m)^s - \psi(t_{m+1})^s]\,.$$

On the second step we noted that ψ is right-continuous on (a,b) and $\psi(b-0)=\psi(b)$, such that Eq. (3.8) as $0 \le m \le m_0-1$ and Eq. (3.12) as $m=m_0$ are applicable. Then, by Eq. (3.12) as $t=t_{m_0}$, $\tau=b$, $r=s$ we obtain

$$\int_{(t_{m_0},b)} (-d\,[\psi^s]) = \psi(t_{m_0})^s - \psi(b)^s = \psi(t_{m_0})^s - \psi(t_{m_0+1})^s.$$

On the last step we apply Eq. (3.13). Denote

$$\omega_m := \omega(t_m+0), \quad \psi_m := \psi(t_m), \quad m=0,\ldots,m_0; \quad \omega_{m_0+1} := \omega(b), \quad \psi_{m_0+1} := \psi(b).$$

Then

$$A_s^s \le d^s \sum_{m=0}^{m_0} \omega_m^s \left[\psi_m^s - \psi_{m+1}^s\right]. \tag{3.15}$$

Analogically,

$$A_q^q = \int_{(a,b)} \omega^q \,(-d\,[\psi^q]) = \sum_{m=0}^{m_0-1} \int_{\delta_m} \omega^q \,(-d\,[\psi^q]) + \int_{(t_{m_0},b)} \omega^q \,(-d\,[\psi^q])\,,$$

$$A_q^q = \sum_{m=0}^{m_0} \int_{\delta_m} \omega^q \,(-d\,[\psi^q]) \ge \sum_{m=0}^{m_0} \omega(t_m+0)^q \,[\psi(t_m)^q - \psi(t_{m+1})^q]\,,$$

such that

$$A_q^q \ge \sum_{m=0}^{m_0} \omega_m^q \left[\psi_m^q - \psi_{m+1}^q\right]. \tag{3.16}$$

Now we apply Eq. (3.7) as $C = \omega_{m_0+1}\cdot\psi_{m_0+1} = \omega(b)\psi(b)$ and obtain

$$\{\omega(b)^s\psi(b)^s + A_s^s\}^{1/s} \le \left\{ d^s\omega(b)^s\psi(b)^s + d^s \sum_{m=0}^{m_0} \omega_m^s \left[\psi_m^s - \psi_{m+1}^s\right] \right\}^{1/s}$$

$$\le d\left\{ \omega(b)^q\psi(b)^q + \sum_{m=0}^{m_0} \omega_m^q \left[\psi_m^q - \psi_{m+1}^q\right] \right\}^{1/q} \le d\left\{\omega(b)^q\psi(b)^q + A_q^q\right\}^{1/q}.$$

Thus,

$$\{\omega(b)^s\psi(b)^s + A_s^s\}^{1/s} \le d\left\{\omega(b)^q\psi(b)^q + A_q^q\right\}^{1/q}.$$

In the latter inequality all summands in {} do not depend on $d>1$. Then the passage to the limit as $d \to 1+0$ leads to Eq. (3.2).

4. Let $0 < \omega(a) < \omega(b) = \infty$. In this case we set that $\psi(b)=0$, $C=0$. For any $d>1$ we define t_m, $m \in \mathbb{N}_0 = \{0,1,\ldots\}$ by formulas (3.13) with properties (3.14) for $m \in \mathbb{N}_0$ (in this case $m_0=\infty$). Thus, estimates (3.15) and (3.16) are valid. More than that, here $\psi_\infty = \psi(b) = 0$. The application of Eq. (3.7) as $C=0$, $m_0=\infty$ implies

$$A_s \le dA_q,\ \forall d>1 \Rightarrow A_s \le A_q.$$

5. It remains to consider the case $\omega(a) = 0$, $\psi(a) \le \infty$. Without loss of generality we consider that $\omega(\delta) > 0$, $\psi(\delta) < \infty$, $\delta \in (a, b)$. For each $\delta \in (a, b)$ there holds the estimate

$$\{\omega(b)^s \psi(b)^s + A_s^s(\delta, b)\}^{1/s} \le \left\{\omega(b)^q \psi(b)^q + A_q^q(\delta, b)\right\}^{1/q}, \tag{3.17}$$

which was proved above (with δ instead of a). Note that $A_r(\delta, b) \to A_r(a, b)$ $(\delta \to a + 0)$ as $r = q$, $r = s$.

Thus, passing to the limit as $\delta \to a + 0$ in Eq. (3.17), we obtain Eq. (3.2). □

4. The Proof of Theorem 2.1. Corollaries

4.1. Proof of Theorem 2.1.

1. Let us now prove item (3) of this theorem, i.e., we obtain Eq. (2.16). Let

$$a < t_m < t_{m+1} < b, \quad m \in \mathbb{Z}; \quad \lim_{m \to -\infty} t_m = a, \quad \lim_{m \to +\infty} t_m = b.$$

For function $g \in \Omega$ consider its "step majorant" $\tilde{g}$:

$$\tilde{g}(u) = \sum_m g(t_m) \chi_{\Delta_m}(u); \quad \Delta_m = (t_m, t_{m+1}], \quad m \in \mathbb{Z}. \tag{4.1}$$

Note that for any $u \in (a, b)$, $s > 0$, there holds $\chi_{\Delta_m}(u)^s = \chi_{\Delta_m}(u)$, and

$$\tilde{g}(u) = \left(\sum_m g(t_m)^s \chi_{\Delta_m}(u)^s\right)^{1/s}, \tag{4.2}$$

since the summands in Eq. (4.1) do not intersect and for each $u \in (a, b)$ only one summand does not vanish. Denote

$$B := \lim_{u \to b-0} g(u); \quad 0 \le c_m(s) = \left(g(t_{m-1})^s - g(t_m)^s\right)^{1/s}. \tag{4.3}$$

Then for any $u \in (a, b)$, $s > 0$,

$$\tilde{g}(u) = \left(\sum_m c_m(s)^s \chi_{(a, t_m]}(u)^s + B^s \chi_{(a,b)}(u)^s\right)^{1/s}, \tag{4.4}$$

Equality (4.4) follows from Eq. (4.2) after applying the Abel transformation of the form

$$\sum_{m=l}^{n} e_m \left(d_{m+1} - d_m\right) = \sum_{m=l}^{n} \left(e_m - e_{m+1}\right) d_{m+1} + e_{n+1} d_{n+1} - e_l d_l, \tag{4.5}$$

if we set

$$e_m = g(t_m)^s, \quad d_m = \chi_{(a, t_m]}(u)^s = \chi_{(a, t_m]}(u)$$

and take into account that

$$\lim_{n \to +\infty} \left(e_{n+1} d_{n+1}\right) = \lim_{n \to +\infty} \left(g(t_{n+1})^s \chi_{(a, t_{n+1}]}(u)^s\right) = B^s \chi_{(a,b)}(u)^s;$$

$$\lim_{l \to -\infty} \left(e_l d_l\right) = \lim_{l \to -\infty} \left(g(t_l)^s \chi_{(a, t_l]}(u)^s\right) = 0.$$

Next, passing to the limit in Eq. (4.5) as $n \to +\infty$, $l \to -\infty$ and using the equality

$$\sum_m \left(e_m - e_{m+1}\right) d_{m+1} = \sum_m \left(e_{m-1} - e_m\right) d_m,$$

we obtain Eq. (4.4).

By Eq. (4.4) as $s = r$ we have $\tilde{g}(u) = \left(f(u)^r + h(u)^r\right)^{1/r}$;

$$f(u) = \left(\sum_m c_m(r)^r \chi_{(a, t_m]}(u)^r\right)^{1/r}; \quad h(u) = B \chi_{(a,b)}(u).$$

Here $0 \le g \le \tilde{g}$, and for the monotone l_r-convex operator T there holds

$$0 \le T[g] \le T[\tilde{g}] \le \left(T[f]^r + T[h]^r\right)^{1/r}.$$

Next, by Eq. (2.5) as $f_m = c_m(r)\chi_{(a,t_m]}(u)$, $m \in \mathbb{Z}$,

$$T[f]^r \le \sum_m c_m(r)^r T\left[\chi_{(a,t_m]}\right]^r; \quad T[h]^r = B^r T\left[\chi_{(a,b)}\right]^r.$$

Thus,

$$0 \le T[g](x) \le T[\tilde{g}](x) \le \left\{\sum_m c_m(r)^r F(x,t_m)^r + B^r F(x,b)^r\right\}^{1/r}. \tag{4.6}$$

As $0 < q < r$ we use Corollary 3.1 of Lemma 3.2 as $s = r$, $\omega_m = F(x,t_m)\uparrow$, $\psi_m = g(t_{m-1})\downarrow$. Thus,

$$c_m(r)^r = g(t_{m-1})^r - g(t_m)^r = \psi_m^r - \psi_{m+1}^r;$$
$$c_m(q)^q = g(t_{m-1})^q - g(t_m)^q = \psi_m^q - \psi_{m+1}^q.$$

Here $A = \lim\limits_{m\to+\infty} \omega_m = \lim\limits_{m\to+\infty} F(x,t_m) := F_0(x,b)$; $B = \lim\limits_{m\to-\infty} \psi_m = g(b-0)$, see (4.3). Note that $\chi_{(a,t]} \le \chi_{(a,b)}$, $t \in (a,b)$. Thus,

$$F(x,t) \le F(x,b),\ t \in (a,b) \quad \Rightarrow \quad F_0(x,b) \le F(x,b). \tag{4.7}$$

Finally,

$$C \equiv AB = F_0(x,b)B \le F(x,b)B \equiv D,$$

and we come to Eq. (3.5) as $s = r$ substituting C with D in our notation

$$\left\{\sum_m c_m(r)^r F(x,t_m)^r + B^r F(x,b)^r\right\}^{1/r} \le \left\{\sum_m c_m(q)^q F(x,t_m)^q + B^q F(x,b)^q\right\}^{1/q}.$$

By Eq. (4.6) we have

$$0 \le T[g](x) \le \left\{\sum_m \left[g(t_{m-1})^q - g(t_m)^q\right] F(x,t_m)^q + B^q F(x,b)^q\right\}^{1/q}. \tag{4.8}$$

Note that in the case $q = r$ Eq. (4.8) coincides with Eq. (4.6).

Inequality (4.8) and l_q-convexity of Y imply

$$\|T[g]\|_Y \le \left\{\sum_m \left[g(t_{m-1})^q - g(t_m)^q\right] \|F(\cdot,t_m)\|_Y^q + B^q \|F(\cdot,b)\|_Y^q\right\}^{1/q}$$

for any $g \in \Omega$. We apply the estimates

$$\|F(\cdot,t_m)\|_Y = \|T[\chi_{(a,t_m]}]\|_Y \le \|T\|_{\dot{\Omega}_0} \|\chi_{(a,t_m]}\|_X \le \|T\|_{\Omega_0} \|\chi_{(a,t_m]}\|_X,$$
$$\|F(\cdot,b)\|_Y \le \|T\|_{\Omega_0} \|\chi_{(a,b)}\|_X$$

and obtain

$$\|T[g]\|_Y \le \|T\|_{\Omega_0} \left\{\sum_m \left[g(t_{m-1})^q - g(t_m)^q\right] \|\chi_{(a,t_m]}\|_X^q + B^q \|\chi_{(a,b)}\|_X^q\right\}^{1/q}.$$

From here as $p \le q$ by Corollary 3.1 of Lemma 3.2 with q instead of s and p instead of q we obtain

$$\|T[g]\|_Y \le \|T\|_{\Omega_0} \left\{\sum_m \left[g(t_{m-1})^p - g(t_m)^p\right] \|\chi_{(a,t_m]}\|_X^p + B^p \|\chi_{(a,b)}\|_X^p\right\}^{1/p}.$$

Thus,

$$\|T[g]\|_Y \le \|T\|_{\Omega_0}\left\{\sum_m c_m(p)^p\|\chi_{(a,t_m]}\|_X^p + B^p\|\chi_{(a,b)}\|_X^p\right\}^{1/p}.$$

Consider nonnegative functions

$$\varphi_m = c_m(p)\chi_{(a,t_m]};\quad \varphi = \left(\sum_m \varphi_m^p\right)^{1/p};\quad \zeta = B\chi_{(a,b)}.$$

Then

$$\|T[g]\|_Y \le \|T\|_{\Omega_0}\left\{\sum_m \|\varphi_m\|_X^p + \|\zeta\|_X^p\right\}^{1/p}.$$

The l_p-concavity of X implies

$$\left\{\sum_m \|\varphi_m\|_X^p + \|\zeta\|_X^p\right\}^{1/p} \le \left\|\left(\sum_m \varphi_m^p + \zeta^p\right)^{1/p}\right\|_X,$$

such that for $g \in \Omega$ there holds the inequality

$$\|T[g]\|_Y \le \|T\|_{\Omega_0}\left\|\left(\sum_m \varphi_m^p + \zeta^p\right)^{1/p}\right\|_X$$

$$= \|T\|_{\Omega_0}\left\|\left(\sum_m c_m(p)^p\chi_{(a,t_m]}^p + B^p\chi_{(a,b)}^p\right)^{1/p}\right\|_X = \|T\|_{\Omega_0}\|\tilde{g}\|_X.$$

On the last step we apply inequality (4.4) as $s = p$. Thus, for any $g \in \Omega$ we obtain the inequality

$$\|T[g]\|_Y \le \|T\|_{\Omega_0}\|\tilde{g}\|_X, \tag{4.9}$$

where $\tilde{g}$ is a "step majorant" of Eq. (4.1).

Next, we construct for $n = 1, 2, 3, \ldots$ the sequences $\{t_m(n)\}_{m\in\mathbb{Z}}$ in such way that the corresponding step functions $\tilde{g}_n$ of form of Eq. (4.1) form a nonincreasing sequence everywhere tending to $g \in \Omega$. By the order continuity of (quasi)norm in X we obtain

$$\|\tilde{g}_n\|_X \to \|g\|_X \ (n \to \infty).$$

Now let us use inequality (4.9) with $\tilde{g}_n$ instead of $\tilde{g}$ and pass to the limit as $n \to \infty$. As the result, we obtain the inequality

$$\|T[g]\|_Y \le \|T\|_{\Omega_0}\|g\|_X, \quad g \in \Omega, \tag{4.10}$$

such that $\|T\|_\Omega \le \|T\|_{\Omega_0}$. The opposite is obvious, since under Eq. (2.15) there holds the embedding $\Omega_0 \subset \Omega$. Thus, we obtain Eq. (2.16).

2. In the conditions of item (1) of Theorem 2.1 we obtain that

$$g \in \dot{\Omega} \Rightarrow B := \lim_{t\to b-0} g(t) = 0.$$

Thus it suffices to set $B = 0$ ($\psi = 0$, respectively) in the reasoning above. With that all summands containing $\chi_{(a,b)}$ have the multiplier $B = 0$ and, thus, disappear independently on whether the nondegeneracy condition (2.13) holds or fails. As a result, we obtain $\|T\|_{\dot{\Omega}_0}$ instead of $\|T\|_{\Omega_0}$ in Eqs. (4.9), (4.10), such that $\|T\|_{\dot{\Omega}} \le \|T\|_{\dot{\Omega}_0}$. The opposite is obvious since there holds the embedding $\dot{\Omega}_0 \subset \dot{\Omega}$. Thus, we come to Eq. (2.12).

Remark 4.1. Thus, we have proven items (1) and (3) of Theorem 2.1. With Remark 2.1 we see that the proof of Theorem 2.1 is now complete.

Remark 4.2. In many cases there holds Eq. (4.7):

$$F_0(x,b) = F(x,b) \quad \gamma - \text{a.e.}, \tag{4.11}$$

which leads to $\|T\|_{\Omega_0} = \|T\|_{\dot{\Omega}_0}$ by Eq. (2.12).

For instance, Eq. (4.11) holds for any bounded linear operator $T : X \to Y$. Indeed, for any $\{t_m\}_{m\in\mathbb{Z}}$; $t_m \uparrow b$ $(m \uparrow +\infty)$

$$\|F(\cdot,b) - F(\cdot,t_m)\|_Y = \left\|T\left[\chi_{(a,b)} - \chi_{(a,t_m]}\right]\right\|_Y \le \|T\| \left\|\chi_{(t_m,b)}\right\|_X \to 0\,(m\uparrow +\infty).$$

Here we take into account that $\chi_{(t_m,b)} \downarrow 0\,(m \uparrow +\infty)$, and the ideal space X has the order continuous (quasi)norm. Next,

$$0 \le F(x,b) - F_0(x,b) \le F(x,b) - F(x,t_m), \quad m \in \mathbb{Z}.$$

Thus, for the ideal space Y we have

$$\|F(\cdot,b) - F_0(\cdot,b)\|_Y \le \|F(\cdot,b) - F(\cdot,t_m)\|_Y.$$

This implies $\|F(\cdot,b) - F_0(\cdot,b)\|_Y = 0 \Rightarrow$ Eq. (4.11).

Remark 4.3. There are cases where Eq. (4.11) does not hold. For instance, consider the case $N = J = (a,b)$, $T_0[g](x) = g(b-0)\chi_J(x)$, $g \in G$. Then

$$T_0[g](x) = 0, \quad g \in \dot{\Omega}_0 \Rightarrow \|T_0\|_{\dot{\Omega}_0} = 0; \quad T_0[\chi_J] = \chi_J \Rightarrow \|T_0\|_{\Omega_0} = \frac{\|\chi_J\|_Y}{\|\chi_J\|_X} > 0.$$

4.2. Corollaries.

Corollary 4.1. *Let* $0 < p \le \min\{q,r\} < \infty$; $X \subset S(J,\beta)$ *be the ideal* l_p*-concave space with order continuous* (*quasi*)*norm, and let condition* (2.13) *hold. Let* $Y = L_q(N,\gamma)$, *and let* T *be a* l_r*-convex monotone operator. Then Eq.* (2.14) *holds.*

Corollary 4.2. *Let* $0 < p \le \min\{q,r\} < \infty$; $X \subset S(J,\beta)$ *be the ideal* l_p*-concave space with order continuous* (*quasi*)*norm, and let condition* (2.15) *hold. Let* $Y = L_q(N,\gamma)$ *and let* T *be the* l_r*-convex monotone operator. Then Eq.* (2.16) *holds.*

In order to prove these corollaries we need a lemma on the convexity properties of the Lebesgue spaces.

Lemma 4.1. *Let* $0 < q < \infty$. *Then* $Y = L_q(N,\gamma)$ *is the ideal* l_ρ*-convex space for any* $\rho \in (0,q]$.

Proof. Denote

$$A = \left\| \left(\sum_m |y_m|^\rho \right)^{1/\rho} \right\|_Y. \tag{4.12}$$

Let us show that

$$A \le \left(\sum_m \|y_m\|_Y^\rho \right)^{1/\rho}. \tag{4.13}$$

As $\rho = q$ we obtain

$$A^q = \int_N \sum_m |y_m|^q d\gamma = \sum_m \int_N |y_m|^q d\gamma = \sum_m \|y_m\|_Y^q,$$

such that Eq. (4.13) is the equality. Let now $0 < \rho < q$. Then

$$A^q = \int_N \left(\sum_m |y_m|^\rho\right) \left(\sum_l |y_l|^\rho\right)^{q/\rho - 1} d\gamma = \sum_m \int_N |y_m|^\rho \left(\sum_l |y_l|^\rho\right)^{\frac{q-\rho}{\rho}} d\gamma.$$

Then we apply the Hölder's inequality to every term with powers $p = q/\rho > 1$ and $p' = q/(q-\rho)$, $1/p + 1/p' = 1$, and obtain

$$A^q \le \sum_m \left(\int_N |y_m|^q d\gamma \right)^{\rho/q} \left(\int_N \left(\sum_l |y_l|^\rho \right)^{q/\rho} d\gamma \right)^{\frac{q-\rho}{q}} = A^{q-\rho} \sum_m \left(\int_N |y_m|^q d\gamma \right)^{\rho/q}.$$

Hence,

$$A^\rho \le \sum_m \left(\int_N |y_m|^q d\gamma \right)^{\rho/q} = \sum_m \|y_m\|_Y^\rho.$$

Thus we obtain the inequality (4.13), which means the l_ρ-convexity of the ideal space $Y = L_q(N,\gamma)$. □

Proof of Corollaries 4.1 *and* 4.2. Let us denote $\rho = min\{q, r\}$. Then $p \le \rho \le r$. By Lemma 4.1 the space $Y = L_q(N,\gamma)$ is l_ρ-convex and we can apply Theorem 2.1 with ρ instead of q. Thus, Eq. (4.13) holds for $Y = L_q(N,\gamma)$. The proof of Corollary 4.2 is similar. □

5. The Generalization of Monotone Conditions

The analogical results are valid for the cones of functions with the monotone property related to the given positive function $k \in C(J)$. We define

$$\Omega_k \equiv \Omega(X,k) = \{g \in X : g \ge 0,\ g(t)/k(t) \downarrow;\ g(t) = g(t-0),\ t \in (a,b)\}, \tag{5.1}$$

$$\dot\Omega_k \equiv \dot\Omega(X,k) = \{g \in \Omega_k :\ g(t)/k(t) \to 0, t \to b-0\} \tag{5.2}$$

(in this notation as $k(t) \equiv 1$ we have: $\Omega_1 = \Omega$, $\dot\Omega_1 = \dot\Omega$, see Eq. (2.7)). We denote

$$\begin{aligned} \dot\Omega_{k,0} &\equiv \dot\Omega_0(X,k) = \left\{k\chi_{(a,t]} : a < t < b\right\}, \\ \Omega_{k,0} &\equiv \Omega_0(X,k) = \dot\Omega_{k,0} \cup \left\{k\chi_{(a,b)}\right\}. \end{aligned} \tag{5.3}$$

Theorem 5.1. *Let* $0 < p \le q \le r < \infty$; $X \subset S(J,\beta)$ *be the ideal* l_p*-concave space with order continuous* (*quasi*)*norm;* $Y \subset S(N,\gamma)$ *be the ideal* l_q*-convex space, and let* $T : \Omega_k \to Y$ *be the* l_r*-convex monotone operator.*

(1) *Then there hold the relations*

$$\|T\|_{\dot\Omega_k} = \|T\|_{\dot\Omega_{k,0}} := \sup_{a<t<b} \left[\|F_k(\cdot,t)\|_Y \|k(\cdot)\chi_{(a,t]}(\cdot)\|_X^{-1} \right], \tag{5.4}$$

where

$$F_k(x,t) = T[k\chi_{(a,t]}](x), \quad a < t < b. \tag{5.5}$$

(2) *Under the additional nondegeneracy condition*

$$\|k\chi_{(a,b)}\|_X = \infty \tag{5.6}$$

there hold the relations

$$\|T\|_{\Omega_k} = \|T\|_{\dot\Omega_k} = \|T\|_{\dot\Omega_{k,0}} = \sup_{a<t<b} \left[\|F_k(\cdot,t)\|_Y \|k(\cdot)\chi_{(a,t]}(\cdot)\|_X^{-1} \right]. \tag{5.7}$$

(3) *Under the degeneracy*

$$\|k\chi_{(a,b)}\|_X < \infty \tag{5.8}$$

there hold the relations

$$\|T\|_{\Omega_k} = \|T\|_{\Omega_{k,0}} := \max\left\{ \|T\|_{\dot\Omega_{k,0}}, \|F_k(\cdot,b)\|_Y \|k\chi_{(a,b)}(\cdot)\|_X^{-1} \right\}, \tag{5.9}$$

where

$$F_k(x,b) = T[k\chi_{(a,b)}](x). \tag{5.10}$$

Remark 5.1. Note that in the nondegenerate case

$$\|k\chi_{(a,b)}\|_X = \infty \quad \Rightarrow \quad \Omega_k = \dot{\Omega}_k, \tag{5.11}$$

since

$$\left\{0 \le g/k \downarrow, \lim_{t\to b-0}[g(t)/k(t)] > 0\right\} \quad \Rightarrow \quad g \notin X. \tag{5.12}$$

This means that in the case of Eq. (5.6) there holds the equality $\|T\|_{\Omega_k} = \|T\|_{\dot{\Omega}_k}$. Thus, in order to calculate $\|T\|_{\Omega_k}$ one can apply Eq. (5.4). Thus, there holds Eq. (5.7), and item (2) of Theorem 5.1 follows from its item (1).

Proof. Formally this theorem is more general than Theorem 2.1, however we can easily reduce it to Theorem 2.1.

Consider the l_r-convex monotone operator $T : \Omega(X,k) \to Y$. Define

$$X_k := \{f \in S(J,\beta) : kf \in X\} = \{f = g/k : g \in X\}, \quad \|f\|_{X_k} = \|g\|_X. \tag{5.13}$$

Then we have the equivalence

$$g \in \Omega(X,k) \quad \Leftrightarrow \quad f = g/k \in \Omega(X_k, 1); \quad \|f\|_{X_k} = \|kf\|_X. \tag{5.14}$$

see Eqs. (5.1), (5.2), (5.13). Thus

$$\|T\|_{\Omega(X,k)} = \sup\left\{\frac{\|T[g]\|_Y}{\|g\|_X} : 0 \ne g \in \Omega(X,k)\right\} = \sup\left\{\frac{\|T[kf]\|_Y}{\|f\|_{X_k}} : 0 \ne f \in \Omega(X_k, 1)\right\}.$$

Note that X_k, as well as X, is the ideal l_p-concave space with order continuous (quasi)norm, and the operator

$$T_k : \Omega(X_k, 1) \to Y; \quad T_k[f] := T[kf], \quad f \in \Omega(X_k, 1),$$

is l_r-convex along with the operator T. Remark 5.1 here takes the form of Remark 2.1. Thus, Theorem 2.1 is applicable, and we obtain all statements of Theorem 5.1. □

6. Applications. The Computation of the Norm of Integral Operator on the Cone of Functions with the Monotone Property

Here we consider one application of the general results given in Secs. 2–5, namely, the computation of the norm of the integral operator on the cone of functions with the monotone property. In our paper [2] we gave a number of other applications of these general results in the theory of weighted Lorentz spaces, in computation of associated norms on the cones of the monotone functions, etc.

Let $K = K(x,\tau)$ be a nonnegative measurable function of $(x,\tau) \in N \otimes J$, where $(N;\gamma)$ and $(J;\mu)$ are the spaces with nonnegative σ-finite σ-additive measure γ and nonnegative Borel measure μ on $J = (a,b)$.

$$T_{r\mu}[f](x) = \left(\int_{(a,b)} K(x,\tau)|f(\tau)|^r d\mu(\tau)\right)^{1/r}, \quad r \in (0,\infty). \tag{6.1}$$

This is the l_r-convex monotone operator. As $r = 1$ its restriction onto the set of nonnegative μ-measurable functions coincides with the restriction of the linear integral operator

$$T[f](x) = \int_{(a,b)} K(x,\tau)f(\tau)d\mu(\tau). \tag{6.2}$$

Here we can apply the results of Secs. 2–5. In particular, for the restriction of operator $T_{r\mu}$ onto the cone Ω_k the application of Theorem 5.1 gives the following results.

Theorem 6.1. *Let $0 < p \le q \le r < \infty$; $X \subset S(J,\beta)$ be the ideal l_p-concave space with order continuous (quasi)norm; $Y \subset S(N,\gamma)$ be the l_q-convex ideal space, and*

$$\|k\chi_{(a,b)}\|_X = \infty. \tag{6.3}$$

Then

$$\|T_{r\mu}\|_{\Omega_k} = \|T_{r\mu}\|_{\dot{\Omega}_{k,0}} := \sup_{a<t<b} \left\{ \|T_{r\mu}[k\chi_{(a,t]}]\|_Y \|k\chi_{(a,t]}\|_X^{-1} \right\}. \tag{6.4}$$

Here

$$T_{r\mu}[k\chi_{(a,t]}](x) = \int\limits_{(a,t]} K(x,\tau)k(\tau)^r d\mu(\tau), \quad x \in N. \tag{6.5}$$

In the case

$$\|k\chi_{(a,b)}\|_X < \infty \tag{6.6}$$

we have $\|T_{r\mu}\|_{\dot{\Omega}_k} = \|T_{r\mu}\|_{\dot{\Omega}_{k,0}}$ *(see Eq.* (6.4)*),*

$$\|T_{r\mu}\|_{\Omega_k} = \max\left\{ \|T_{r\mu}\|_{\dot{\Omega}_{k,0}}; \quad \|T_{r\mu}[k\chi_{(a,b)}]\|_Y \|k\chi_{(a,b)}\|_X^{-1} \right\}. \tag{6.7}$$

Remark 6.1. For the restriction onto the cone $\Omega = \Omega_1$ of nonnegative decreasing left-continuous functions we need to set $k(\tau) = 1$ in Eqs. (6.3)–(6.7).

Remark 6.2. In the case $Y = L_q(N,\gamma)$ the results of Theorem 6.1 remain valid if $0 < p \le \min\{q,r\} < \infty$, see Corollaries 4.1 and 4.2.

Remark 6.3. As a concretization of operator (6.1) we consider the case where $(N,\gamma) = (J,\gamma)$ with nonnegative Borel measure γ on $J = (a,b)$, and $T_{r\mu}$ coincides with the generalized operator of Hardy type

$$A_{r\mu}[f](x) = \left(\int\limits_{(a,x]} |f(\tau)|^r d\mu(\tau) \right)^{1/r}, \quad x \in (a,b). \tag{6.8}$$

Then

$$A_{r\mu}[k\chi_{(a,t]}](x) = \left(\int\limits_{(a,x]} k(\tau)^r d\mu(\tau) \right)^{1/r}, \quad x \le t,$$

$$A_{r\mu}[k\chi_{(a,t]}](x) = \left(\int\limits_{(a,t]} k(\tau)^r d\mu(\tau) \right)^{1/r}, \quad x > t, \tag{6.9}$$

and we have the equalities in the case of Eq. (6.3):

$$\|A_{r\mu}\|_{\Omega_k} = \|A_{r\mu}\|_{\dot{\Omega}_{k,0}} := \sup_{a<t<b} \left\{ \left\|A_{r\mu}[k\chi_{(a,t]}]\right\|_Y \left\|k\chi_{(a,t]}\right\|_X^{-1} \right\}. \tag{6.10}$$

In the case of Eq. (6.6) formula (6.10) remains valid for $\|A_{r\mu}\|_{\dot{\Omega}_k}$, but

$$\|A_{r\mu}\|_{\Omega_k} = \max\left\{ \|A_{r\mu}\|_{\dot{\Omega}_{k,0}}; \quad \left\|A_{r\mu}[k\chi_{(a,b)}]\right\|_Y \left\|k\chi_{(a,b)}\right\|_X^{-1} \right\}. \tag{6.11}$$

REFERENCES

1. E. G. Bakhtigareeva and M. L. Goldman, "Construction of an optimal envelope for a cone of non-negative functions with monotonicity properties," *Proc. Steklov Inst. Math.*, **293**, 37–55 (2016).
2. E. G. Bakhtigareeva and M. L. Goldman, "Calculation of the norms for monotone operators on the cones of functions with monotonicity properties," *Lobachevskii J. Math.*, **42**, No. 5, 857–874 (2021).
3. C. Bennett and R. Sharpley, *Interpolation of Operators*, Acad. Press, Boston (1988).
4. J. Berg and J. Löfström, *Interpolation Spaces. An Introduction*, Springer, Berlin (1976).
5. V. I. Burenkov and M. L. Goldman, "Calculation of the norm of a positive operator on the cone of monotone functions," *Proc. Steklov Inst. Math.*, **210**, 47–65 (1995).
6. A. Gogatishvili and V. D. Stepanov, "Reduction theorems for weighted integral inequalities on the cone of monotone functions," *J. Math. Anal. Appl.*, **68**, No. 4, 597–664 (2013).
7. A. Gogatishvili and V. D. Stepanov, "Reduction theorems for operators on the cone of monotone functions," *Russ. Math. Surv.*, **405**, No. 1, 156–172 (2013).
8. L. V. Kantorovich and G. P. Akilov, *Functional Analysis* [in Russian], Nauka, Moscow (1984).
9. S. G. Kreyn, Yu. I. Petunin, and E. M. Semenov, *Interpolation of Linear Operators* [in Russian], Nauka, Moscow (1978).
10. H. Triebel, *Interpolation Theory. Function Spaces. Differential Operators* [Russian translation], Mir, Moscow (1980).

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