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ASYMPTOTICS OF THE SPECTRUM OF VARIATIONAL PROBLEMS ARISING IN THE THEORY OF FLUID OSCILLATIONS

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Abstract. This work is a survey of results on the spectral asymptotics of variational problems arising in the theory of small oscillations of a fluid in a vessel near the equilibrium position. These problems were posed by Kopachevsky in the late 1970s and cover various fluid models. The statements of the problems are given both in the form of boundary-value problems in the domain $\Omega \subset \mathbb{R}^3$ occupied by the fluid in the equilibrium state and in the form of variational problems on the spectrum of the ratio of quadratic forms. The common features of all the problems under consideration are the presence of an “elliptic” constraint (the Laplace equation for an ideal fluid or a homogeneous Stokes system for a viscous fluid), as well as the occurrence of the spectral parameter in the boundary condition on the free (equilibrium) surface Γ. The spectrum in the considered problems is discrete; the spectral counting functions have power-law asymptotics.

Keywords: variational problem, spectral asymptotics, small oscillation, fluid oscillation, boundary-value problem, variational problem.

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CONTENTS

To the bright memory of Nikolai Dmitrievich Kopachevsky

1. Introduction

1.1. Background of the topic. The present paper gives a survey of the results on the spectral asymptotics of variational problems arising in the theory of fluid oscillations. We study small (linear) oscillations of the fluid in a vessel near the equilibrium position. In [23, Appendix 1] (see also [3, 21, 22]), Nikolai Dmitrievich Kopachevsky posed a number of problems on the spectrum of normal fluid oscillations for various physical models: heavy ideal, capillary ideal, heavy viscous, and capillary viscous fluids. (These problems were also discussed in the later monograph [26] by Kopachevsky, Krein, and Ngo Zui Kan and in books [24, 25] by Kopachevsky and Krein.) Recall that for the heavy fluid the main role is played by the mass forces (force of gravity, centrifugal force during the vessel

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rotation). For the capillary fluid the main role is played by the surface forces (surface tension force). We consider the case of one fluid partially filling a vessel and the case of a system of immiscible fluids. By the heuristic arguments, Kopachevsky has found the formulas for the principal terms of the spectral asymptotics in these problems. In the end of 1970s, he raised the question about the rigorous proof of these formulas and turned for advice to Leningrad mathematicians Birman and Solomyak, well known specialists in the spectral operator theory. It turned out that in one auxiliary problem for the heavy viscous fluid the spectral asymptotics followed from the general results of Metivier [31]. In other problems, the question about the rigorous proof of the asymptotic formulas was open at that time.

Soon, in the case of the heavy ideal fluid the spectral asymptotics was justified by Karazeeva and Solomyak [19]. In all other problems stated in [23] the spectral asymptotics was justified by the author (at that time, the PhD student of Birman). On this subject, the author has published the short note [40] and has deposited manuscript [38]. A detailed exposition of the results has not been previously published. The present survey fills this gap.

Solving these problems marked the beginning of many years of close scientific cooperation and strong friendship between the author and Nikolai Dmitrievich Kopachevsky, a wonderful mathematician and person.

1.2. On the statements of the problems and the results. The statements of the problems are given both in the form of boundary-value problems in the domain $\Omega \subset \mathbb{R}^3$ occupied by a fluid in the equilibrium state and in the form of variational problems on the spectrum of the ratio of quadratic forms. The common features of all the problems under consideration are the presence of an "elliptic" constraint (the Laplace equation for the ideal fluid or the homogeneous Stokes system for the viscous fluid), as well as the occurrence of the spectral parameter in the boundary condition on the free (equilibrium) surface Γ. Problems with elliptic constraints in smooth domains are amenable to the technique of pseudodifferential operators; see [11, 12]. However, now the boundary $\partial\Omega$ is not smooth, since the free surface Γ (or the interface of two fluids) and the solid wall S of the vessel form an edge at the intersection. This is the main difficulty. In addition to the nonsmoothness of $\partial\Omega$, other complications arise in the problems under consideration. They are related to the presence of the nonlocal operator $\mathbf{B}_\Gamma^{-1}$ in the variational formulation for the capillary fluid (this is the resolving operator for some elliptic boundary-value problem on Γ; see Secs. 4, 6), to the vector nature of the problems and to the additional constraints ($\operatorname{div} \mathbf{u} = 0$) for a viscous fluid.

The spectrum of the problems under consideration is discrete; the counting functions of the spectrum have power-like asymptotics. The main terms of the spectral asymptotic formulas are obtained.

1.3. Method. We start from the variational statements of the problems, i.e., study the spectrum of the ratio of quadratic forms

$$\frac{\mathcal{B}_\Gamma[u]}{\mathcal{A}_\Omega[u]}, \tag{1.1}$$

where $\mathcal{A}_\Omega[u]$ is a positive definite differential quadratic form in the domain Ω, and $\mathcal{B}_\Gamma[u]$ is a form on Γ, which is compact with respect to $\mathcal{A}_\Omega[u]$. The proof of the spectral asymptotics is based on the general approach developed by Birman and Solomyak in [6, 7]. The scheme of the proof is as follows. First, we obtain estimates of the spectrum in terms of appropriate $L_r(\Gamma)$-norms of the coefficients of the form $\mathcal{B}_\Gamma$. These estimates allow us to consider only coefficients from some dense set in $L_r(\Gamma)$, when calculating the principal term of the spectral asymptotics. It is convenient to take $C_0^\infty(\Gamma)$ as such set. For the case of smooth coefficients compactly supported on Γ, it is possible to compare ratio (1.1) with a similar ratio defined on a domain $\widetilde{\Omega}$ with smooth boundary (the domain $\widetilde{\Omega}$ "smooths out" Ω). Next, the problem in the domain $\widetilde{\Omega}$ is reduced to the boundary, by parametrizing the solutions of a homogeneous elliptic equation in terms of their Dirichlet data. From the properties of the Boutet de

Monvel algebra of pseudodifferential operators [14, 17] it follows that the initial forms are represented[1] as the quadratic forms of some pseudodifferential operators on $\partial\widetilde{\Omega}$. After reducing to the boundary, we arrive at the problem on the spectrum of the ratio of pseudodifferential forms (probably, with additional constraints on a part of the boundary). Then the spectral asymptotics follows from the results of [8, 10, 37].

A similar method was applied in the author's works [39, 42], where, in a rather general statement, the spectral asymptotics of variational problems on the solutions of an elliptic equation in a domain with piecewise smooth boundary was obtained. Two more model problems of the theory of fluid oscillations were studied in [41].

1.4. A short survey of the results on the spectral asymptotics for problems with the spectral parameter in the boundary condition. Boundary-value problems considered in the present paper involve the spectral parameter in the boundary condition. Spectral properties of such problems have been actively studied by many authors during a long time. Asymptotics of the spectrum for such problems is discussed in survey [9, Sec. 7] by Birman and Solomyak. We will mention only a few most important papers, sending the reader to more complete bibliography in [9]. The simplest problem with the spectral parameter in the boundary condition is the Steklov problem:

$$-\Delta u = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = \lambda u \quad \text{on } \partial\Omega, \qquad \int_{\partial\Omega} u\, dS = 0.$$

Here $\Omega \subset \mathbb{R}^{m+1}$ is a bounded domain. The eigenvalues of this problem coincide with the consecutive maxima of the ratio of forms

$$\frac{\int_{\partial\Omega} |u|^2\, dS}{\int_\Omega |\nabla u|^2\, dx}, \qquad u \in H^1(\Omega), \qquad \int_{\partial\Omega} u\, dS = 0.$$

The counting function $N(\lambda)$ for the eigenvalues of the Steklov problem has the following power-like asymptotics: $N(\lambda) \sim \lambda^{-m}\omega_m \operatorname{meas}\partial\Omega$, as $\lambda \to +0$. Here ω_m is the volume of the unit m-dimensional ball. A more general "Steklov-type problem" with variable coefficients is equivalent to the variational problem on the spectrum of the ratio

$$\frac{\int_{\partial\Omega} b(\mathbf{y})|u(\mathbf{y})|^2\, dS(\mathbf{y})}{\int_\Omega \left(\sum_{i,j=1}^n a_{ij}\partial_i u \overline{\partial_j u} + |u|^2 \right) d\mathbf{x}}, \qquad u \in H^1(\Omega),$$

where the coefficients are real-valued, and the matrix $\{a_{ij}(\mathbf{x})\}$ is symmetric and positive definite. First, the case where the boundary of the domain and the coefficients are smooth was studied. Sandgren [35] was the first who obtained the spectral asymptotics, by using the variational method. Rather general results for the problems with the spectral parameter in the boundary condition were obtained by Kozhevnikov [20, 27] with the help of the pseudodifferential technique. Vulis and Solomyak proved the spectral asympotics for the degenerating Steklov-type problem [45]. In the two-dimensional case, some delicate results on the behavior of the eigenvalues of the Steklov-type problem were obtained by Rozenblum [33].

The spectral asymptotics in the problems with the spectral parameter in the boundary condition in a domain with piecewise smooth boundary (a domain with edges) was studied in the above mentioned paper [19] by Karazeeva and Solomyak and in the author's papers [38, 40] (the results from [19, 38, 40] are described below in details). Agranovich [1] proved the spectral asymptotics for the Steklov-type problem in a Lipschitz domain with the "almost smooth" boundary (it was assumed that $\partial\Omega$ is infinitely smooth outside some closed set of zero measure). Rozenblum [34] managed to obtain the same result in the Lipschitz domain replacing the "almost smoothness condition" by the following assumption: for any $\varepsilon > 0$ there exists a closed set $U_\varepsilon \subset \partial\Omega$ with measure less than ε such that on $\partial\Omega \setminus U_\varepsilon$ the vector field $\mathbf{n}(\mathbf{x})$ belongs to the VMO class (here $\mathbf{n}(\mathbf{x})$ is the unit outer normal vector

[1] In [11], where the Boutet de Monvel algebra was not used, such representation was obtained only up to the lower order terms, which does not influence the principal term of the spectral asymptotics, but explicit formulas for the principal terms of the corresponding pseudodifferential operators on $\partial\widetilde{\Omega}$ were obtained.

to $\partial\Omega$ at the point $\mathbf{x}$). For the Steklov-type problem, in the case where it is known only that the boundary is Lipschitz, Rozenblum [34] obtained the two-sided estimates of right order for the counting function, however, the problem of proving the spectral asymptotics remains open.

Note that recently the properties of eigenfunctions of the Steklov-type problem were actively studied: they decrease rapidly, as the distance from a point to the boundary grows (see, e.g., [15, 16, 46] and references therein).

1.5. Plan of the paper. The paper consists of Introduction (Sec. 1) and five more sections. In Sec. 2, the necessary information about compact operators in a Hilbert space with the power-like spectral asymptotics is provided. In Sec. 3, the Steklov-type problem (the case of a heavy ideal fluid) is considered; the results of paper [19] are presented. In Sec. 4, the spectral asymptotics of the nonclassical Steklov-type problem (the problem on the spectrum of ratio (4.1)) is studied; the result is applied to several problems of the theory of fluid oscillations, including the problem for the capillary ideal fluid. Secs. 5 and 6 are devoted to problems for the viscous fluid. In Sec. 5, the spectral asymptotics for one auxiliary problem for the heavy viscous fluid is studied. In Sec. 6, the spectrum of small oscillations of the capillary viscous fluid is studied.

1.6. Notation and preliminaries. Let $\mathfrak{H}$ be a complex separable Hilbert space. The inner product and the norm in $\mathfrak{H}$ are denoted by $(\cdot,\cdot)_{\mathfrak{H}}$ and $\|\cdot\|_{\mathfrak{H}}$, respectively. Sometimes we omit the indices.

The standard inner product and the norm in $\mathbb{C}^k$ are denoted by $\langle\cdot,\cdot\rangle$ and $|\cdot|$, respectively.

Next, $L_p(\Omega;\mathbb{C}^k)$, $1 \le p \le \infty$, and $H^s(\Omega;\mathbb{C}^k)$, $s \ge 0$, stand for the standard L_p-spaces and Sobolev spaces of $\mathbb{C}^k$-valued functions in a domain $\Omega \subset \mathbb{R}^n$. Let $H_0^s(\Omega;\mathbb{C}^k)$ be the closure of $C_0^\infty(\Omega;\mathbb{C}^k)$ in $H^s(\Omega;\mathbb{C}^k)$. For $k=1$ we write simply $H^s(\Omega)$, $H_0^s(\Omega)$; but sometimes we use such simple notation also for spaces of vector-valued functions.

If $\Omega \subset \mathbb{R}^n$ is a bounded domain with piecewise smooth boundary and $\boldsymbol{\nu}(\mathbf{x})$ is the unit (inner or outer) normal vector to $\partial\Omega$ at the point $\mathbf{x} \in \partial\Omega$ (defined on smooth parts of the boundary), then the symbol $\partial/\partial\nu$ stands for the normal derivative.

Let $\mathcal{K}$ be the *class of bounded domains* $\Omega \subset \mathbb{R}^n$ *satisfying the assumptions of usual embedding and extension theorems*. A sufficient condition for $\Omega \in \mathcal{K}$ is that Ω is Lipschitz (i.e., locally, in appropriate coordinates, the boundary $\partial\Omega$ is the graph of a Lipschitz function).

If $f(\mathbf{x})$ is a real-valued function in a domain Ω, its positive and negative parts are denoted by $f_\pm(\mathbf{x}) := \frac{1}{2}(|f(\mathbf{x})| \pm f(\mathbf{x}))$.

If $\mathcal{D}$ is a smooth compact m-dimensional manifold with or without boundary, then $T^*\mathcal{D}$ stands for the cotangent bundle, and $T^*_{\mathbf{x}}\mathcal{D}$ stands for the cotangent space at the point $\mathbf{x} \in \mathcal{D}$, $T^*\mathcal{D} \setminus \{0\}$ is the cotangent bundle without zero section. We need the notion of semidensity (see [44]). In any local coordinate system on $\mathcal{D}$, a semidensity $\mathfrak{u}$ is given by a function. If the functions $u(\mathbf{y})$ and $u'(\mathbf{y}')$ correspond to the same semidensity $\mathfrak{u}$ in coordinates related by the transformation $h : \mathbf{y}' \mapsto \mathbf{y}$, then $u' = (u \circ h) j_h$, where j_h^2 is the modulus of the Jacobian of the transformation h. The classes C^l, H^s, etc., for semidensities are introduced via local coordinates. The notion of classical pseudodifferential operator on semidensities makes sense; the definition can be given in local coordinates. The algebra of the principal symbols for pseudodifferential operators on semidensities is the same as the similar algebra for pseudodifferential operators on functions. In particular, the principal symbol of a pseudodifferential operator is a function on the cotangent bundle. The product of two semidensities is a density. For densities on $\mathcal{D}$, the integral is defined invariantly. This allows one to introduce a complex separable Hilbert space $L_2(\mathcal{D})$ of semidensities.

2. Preliminaries on Operator Theory

Now, we give the necessary information about compact operators in a Hilbert space; see [6, § 1], [7, Appendix 1], [13].

2.1. Spectrum counting functions for compact operators. Let $\mathfrak{H}$ be a complex separable Hilbert space, and let $\mathfrak{S}_\infty = \mathfrak{S}_\infty(\mathfrak{H})$ be the set of all compact operators in $\mathfrak{H}$. If $T = T^* \in \mathfrak{S}_\infty(\mathfrak{H})$, then $\lambda_n^+(T)$, $-\lambda_n^-(T)$ denote the positive and negative eigenvalues of T; the values $\lambda_n^\pm(T)$ are numbered in the nonincreasing order with multiplicities taken into account. By $N_\pm(\lambda, T)$ we denote the counting functions for the positive and negative eigenvalues: $N_\pm(\lambda, T) := \#\{n : \lambda_n^\pm(T) > \lambda\}$, $\lambda > 0$. If $T \geq 0$, we write simply $N(\lambda, T) = N_+(\lambda, T)$.

The minimaximal principle for the eigenvalues of a self-adjoint operator $T \in \mathfrak{S}_\infty(\mathfrak{H})$ claims that

$$\lambda_{n+1}^\pm(T) = \min_{\operatorname{codim} \mathfrak{G} \leq n} \max_{0 \neq u \in \mathfrak{G}} \pm \frac{(Tu, u)_\mathfrak{H}}{(u, u)_\mathfrak{H}}, \tag{2.1}$$

where $\mathfrak{G}$ is a subspace of $\mathfrak{H}$.

We will systematically use the following statement which follows from (2.1).

Lemma 2.1. *Let $\mathfrak{H}_1$ be a subspace of $\mathfrak{H}_2$. Let $T_i = T_i^* \in \mathfrak{S}_\infty(\mathfrak{H}_i)$, $i = 1, 2$. Suppose that any function $u \in \mathfrak{H}_1$ such that $\pm(T_1 u, u) > 0$ satisfies $\pm(T_1 u, u) \leq \pm(T_2 u, u)$. Then $N_\pm(\lambda, T_1) \leq N_\pm(\lambda, T_2)$ for $\lambda > 0$.*

The following lemma generalizes Lemma 2.1 and allows us to compare the spectra of operators acting in different Hilbert spaces.

Lemma 2.2 (see [7, Lemma 1.15])**.** *Let $T_i = T_i^* \in \mathfrak{S}_\infty(\mathfrak{H}_i)$, $i = 1, 2$. Let $\mathcal{S} : \mathfrak{H}_1 \to \mathfrak{H}_2$ be a continuous operator such that $(T_1 u, u)_{\mathfrak{H}_1} = 0$ for $u \in \operatorname{Ker} \mathcal{S}$. If, for some $t > 0$ and any $u \in \mathfrak{H}_1$ satisfying $\pm(T_1 u, u)_{\mathfrak{H}_1} > 0$, we have*

$$\pm \frac{(T_1 u, u)_{\mathfrak{H}_1}}{(u, u)_{\mathfrak{H}_1}} \leq \pm \frac{t\,(T_2 \mathcal{S} u, \mathcal{S} u)_{\mathfrak{H}_2}}{(\mathcal{S} u, \mathcal{S} u)_{\mathfrak{H}_2}},$$

then $N_\pm(\lambda, T_1) \leq N_\pm(t^{-1}\lambda, T_2)$ for $\lambda > 0$.

We will consider operators $T = T^* \in \mathfrak{S}_\infty(\mathfrak{H})$ with power-like asymptotics of the functions $N_\pm(\lambda, T)$. It is convenient to introduce the functionals

$$\Delta_\theta^\pm(T) := \limsup_{\lambda \to +0} \lambda^\theta N_\pm(\lambda, T), \qquad \delta_\theta^\pm(T) := \liminf_{\lambda \to +0} \lambda^\theta N_\pm(\lambda, T). \tag{2.2}$$

Here $\theta > 0$. Note that

$$\limsup_{n \to \infty} n^{1/\theta} \lambda_n^\pm(T) = \left(\Delta_\theta^\pm(T)\right)^{1/\theta}, \qquad \liminf_{n \to \infty} n^{1/\theta} \lambda_n^\pm(T) = \left(\delta_\theta^\pm(T)\right)^{1/\theta}.$$

Functionals (2.2) do not change under compact perturbations of the metric of the initial Hilbert space $\mathfrak{H}$. Let $Q = Q^* \in \mathfrak{S}_\infty(\mathfrak{H})$ and let $\lambda_1^+(Q) < 1$. We put

$$(u, v)_{\mathfrak{H}_1} := (u, v)_\mathfrak{H} - (Qu, v)_\mathfrak{H}. \tag{2.3}$$

The inner product (2.3) transforms $\mathfrak{H}$ into a new Hilbert space $\mathfrak{H}_1$. The metrics of $\mathfrak{H}$ and $\mathfrak{H}_1$ are equivalent. Let $T = T^* \in \mathfrak{S}_\infty(\mathfrak{H})$, and let T_1 be the self-adjoint operator in $\mathfrak{H}_1$ generated by the sesquilinear form of T, i.e., $(T_1 u, v)_{\mathfrak{H}_1} = (Tu, v)_\mathfrak{H}$, $u, v \in \mathfrak{H}$.

Lemma 2.3 (see [7, Lemma 1.16])**.** *Under the above assumptions, we have $\Delta_\theta^\pm(T_1) = \Delta_\theta^\pm(T)$ and $\delta_\theta^\pm(T_1) = \delta_\theta^\pm(T)$ for any $\theta > 0$.*

Now, let us discuss the behavior of the functionals $\Delta_\theta^\pm(T)$, $\delta_\theta^\pm(T)$ under additive perturbations of the operator T. First of all, note that if $T_i = T_i^* \in \mathfrak{S}_\infty(\mathfrak{H})$, $i = 1, 2$, then

$$N_\pm(\lambda + \mu, T_1 + T_2) \leq N_\pm(\lambda, T_1) + N_\pm(\mu, T_2), \quad \lambda, \mu > 0. \tag{2.4}$$

Inequalities (2.4) are equivalent to well-known H. Weyl's inequalities for eigenvalues of the sum of self-adjoint operators. The following statement also belongs to H. Weyl.

Lemma 2.4 (see [7, Lemma 1.17])**.** *Let $T_i = T_i^* \in \mathfrak{S}_\infty(\mathfrak{H})$, $i = 1, 2$. Suppose that $\Delta_\theta^+(T_2) = \Delta_\theta^-(T_2) = 0$ for some $\theta > 0$. Then $\Delta_\theta^\pm(T_1 + T_2) = \Delta_\theta^\pm(T_1)$ and $\delta_\theta^\pm(T_1 + T_2) = \delta_\theta^\pm(T_1)$.*

Lemma 2.4 is contained in the following statement, which is important for us.

Lemma 2.5 (see [13, (6)]). *Let $T_i = T_i^* \in \mathfrak{S}_\infty(\mathfrak{H})$, $i = 1, 2$. Then*

$$\left|\left(\Delta_\theta^\pm(T_1)\right)^{\frac{1}{1+\theta}} - \left(\Delta_\theta^\pm(T_2)\right)^{\frac{1}{1+\theta}}\right| \le \left(\Delta_\theta^+(T_1 - T_2) + \Delta_\theta^-(T_1 - T_2)\right)^{\frac{1}{1+\theta}},$$

$$\left|\left(\delta_\theta^\pm(T_1)\right)^{\frac{1}{1+\theta}} - \left(\delta_\theta^\pm(T_2)\right)^{\frac{1}{1+\theta}}\right| \le \left(\Delta_\theta^+(T_1 - T_2) + \Delta_\theta^-(T_1 - T_2)\right)^{\frac{1}{1+\theta}}.$$

Functionals (2.2) do not change if $\mathfrak{H}$ is replaced by a subspace of finite codimension.

Lemma 2.6. *Let $\mathfrak{H}_2$ be a subspace of $\mathfrak{H}_1$ such that $\dim \mathfrak{H}_1 \ominus \mathfrak{H}_2 < \infty$. Suppose that $J : \mathfrak{H}_2 \to \mathfrak{H}_1$ is the embedding operator and P is the orthogonal projection of $\mathfrak{H}_1$ onto $\mathfrak{H}_2$. Let $T_1 = T_1^* \in \mathfrak{S}_\infty(\mathfrak{H}_1)$ and let T_2 be an operator in $\mathfrak{H}_2$ given by $T_2 := PT_1J$. Then $\Delta_\theta^\pm(T_2) = \Delta_\theta^\pm(T_1)$ and $\delta_\theta^\pm(T_2) = \delta_\theta^\pm(T_1)$.*

When calculating the spectral asymptotics for the orthogonal sum of operators, we will use the following obvious statement. Let $T_i = T_i^* \in \mathfrak{S}_\infty(\mathfrak{H}_i)$, $i = 1, 2$. Then $T = T_1 \oplus T_2 \in \mathfrak{S}_\infty(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$ and

$$N_\pm(\lambda, T_1 \oplus T_2) = N_\pm(\lambda, T_1) + N_\pm(\lambda, T_2), \quad \lambda > 0. \tag{2.5}$$

2.2. The spectrum of the ratio of quadratic forms. We will consider compact operators generated by the ratio of quadratic forms. If $\mathcal{F}[u, v]$ is a sesquilinear form in $\mathfrak{H}$, we put $\mathcal{F}[u] := \mathcal{F}[u, u]$. Suppose that a continuous sesquilinear form $\mathcal{A}[u, v]$ generates an inner product in the Hilbert space $\mathfrak{H}$, transforming $\mathfrak{H}$ into the new Hilbert space $\mathfrak{H}_\mathcal{A}$. Suppose that $\mathcal{B}[u, v]$ is a continuous sesquilinear form in $\mathfrak{H}$. This form generates an operator $\mathbf{B}$ in the space $\mathfrak{H}_\mathcal{A}$, i.e., $\mathcal{B}[u, v] = \mathcal{A}[\mathbf{B}u, v]$, $u, v \in \mathfrak{H}$. Suppose that $\mathbf{B} = \mathbf{B}^* \in \mathfrak{S}_\infty(\mathfrak{H}_\mathcal{A})$. Then, by the minimaximal principle (2.1), the numbers $\lambda_n^\pm(\mathbf{B})$ coincide with the consecutive maxima of the ratio of quadratic forms

$$\pm\frac{\mathcal{B}[u]}{\mathcal{A}[u]}, \quad u \in \mathfrak{H}. \tag{2.6}$$

Therefore, we can speak simply about the spectrum of the form ratio (2.6) and use the notation like $N_\pm(\lambda,\ (2.6))$, $\Delta_\theta^\pm$ (2.6), $\delta_\theta^\pm$ (2.6) instead of $N_\pm(\lambda, \mathbf{B})$, $\Delta_\theta^\pm(\mathbf{B})$, $\delta_\theta^\pm(\mathbf{B})$. If $\mathcal{B}[u] \ge 0$, then the indices $\pm$ are omitted.

We will often deal with the finite-dimensional spectral problems depending on the additional parameter $\mathbf{w}$ (usually, $\mathbf{w} = (\mathbf{x}, \boldsymbol{\xi}) \in T^*\mathcal{D} \setminus \{0\}$, where $\mathcal{D}$ is a smooth compact manifold). Suppose that $a_\mathbf{w}$ and $b_\mathbf{w}$ are Hermitian sesquilinear forms on a finite-dimensional space $H_\mathbf{w}$, such that $a_\mathbf{w}[f] > 0$, $0 \ne f \in H_\mathbf{w}$. Then the spectrum counting functions for the ratio

$$\pm\frac{b_\mathbf{w}[f]}{a_\mathbf{w}[f]}, \quad f \in H_\mathbf{w}, \tag{2.7}$$

will be denoted by $n_\pm(\lambda, \mathbf{w};\ (2.7))$.

Sometimes, a finite-dimensional spectral problem will be represented in a different form. If $q(\mathbf{w})$, $p(\mathbf{w})$ are Hermitian $(l \times l)$-matrices depending on the parameter $\mathbf{w}$, such that $p(\mathbf{w}) > 0$, then the counting functions for eigenvalues of the problem

$$q(\mathbf{w})\mathbf{z} = \lambda p(\mathbf{w})\mathbf{z}, \quad \mathbf{z} \in \mathbb{C}^l, \tag{2.8}$$

are denoted by $n_\pm(\lambda, \mathbf{w};\ (2.8))$.

2.3. Estimates for the spectrum of the ratio of differential or pseudodifferential forms. Let $\Omega \subset \mathbb{R}^m$ be a bounded domain such that $\Omega \in \mathcal{K}$. Consider the quotient of quadratic forms

$$\frac{\|u\|^2_{H^{s_1}(\Omega)}}{\|u\|^2_{H^{s_2}(\Omega)}}, \quad u \in H^{s_2}(\Omega), \tag{2.9}$$

where $s_2 > s_1 \ge 0$. The following statement is well known.

Lemma 2.7. *The spectrum counting function for ratio (2.9) satisfy the estimate*

$$N(\lambda,\ (2.9)) \le C\lambda^{-\theta}, \quad \lambda > 0; \quad \theta = \frac{m}{2(s_2 - s_1)}.$$

The constant C depends on m, s_1, s_2, and the domain Ω.

Now, we consider the ratio

$$\pm\frac{\int_\Omega b(\mathbf{y})|u(\mathbf{y})|^2\,d\mathbf{y}}{\|u\|^2_{H^s(\Omega)}},\quad u\in H^s(\Omega),\tag{2.10}$$

where $s>0$ and $b(\mathbf{y})$ is a real-valued measurable function on Ω.

Lemma 2.8. *Let $b(\mathbf{y})$ be a real-valued function such that $b\in L_r(\Omega)$, where $r=1$ for $2s>m$, $r>1$ for $2s=m$, and $r=\frac{m}{2s}$ for $2s<m$. Then the spectrum counting functions for ratio* (2.10) *satisfy the estimates*

$$N_\pm(\lambda,\ (2.10))\le C\lambda^{-\theta}\|b\|^\theta_{L_r(\Omega)},\quad \lambda>0;\quad \theta=\frac{m}{2s}.$$

The constant $C=C(m,r,s,\Omega)$ does not depend on the function b.

For integer s, the statement of Lemma 2.8 coincides with the statement of [7, Theorem 4.1]. The corresponding proof can be automatically carried over to the case of noninteger s: this proof is based on theorems about approximations of functions from $H^s(\Omega)$ by piecewise polynomial functions, and these theorems were proved in [6, 7] for the case of arbitrary $s>0$.

We also need estimates for the spectrum of the ratio

$$\pm\frac{2\,\mathrm{Re}\int_\Omega b(\mathbf{y})u_1(\mathbf{y})\overline{u_2(\mathbf{y})}\,d\mathbf{y}}{\|u_1\|^2_{H^{s_1}(\Omega)}+\|u_2\|^2_{H^{s_2}(\Omega)}},\quad \{u_1,u_2\}\in H^{s_1}(\Omega)\oplus H^{s_2}(\Omega),\tag{2.11}$$

where $s_1,s_2>0$.

Lemma 2.9. *Let $b(\mathbf{y})$ be a real-valued function such that $b\in L_r(\Omega)$, where $r=1$ for $2s_1>m$, $2s_2>m$; $1<r<2$ for $2s_1=m$, $2s_2>m$ or $2s_1>m$, $2s_2=m$ or $2s_1=2s_2=m$; $\frac{1}{r}=\frac{1}{2}+\frac{s_2}{m}$ for $2s_1>m$, $2s_2<m$; $\frac{1}{r}=\frac{1}{2}+\frac{s_1}{m}$ for $2s_1<m$, $2s_2>m$; $\frac{1}{r}<\frac{1}{2}+\frac{s_2}{m}$ for $2s_1=m$, $2s_2<m$; $\frac{1}{r}<\frac{1}{2}+\frac{s_1}{m}$ for $2s_1<m$, $2s_2=m$; $r=\frac{m}{s_1+s_2}$ for $2s_1<m$, $2s_2<m$. Then the spectrum counting functions for ratio* (2.11) *satisfy the estimates*

$$N_\pm(\lambda,\ (2.11))\le C\lambda^{-\theta}\|b\|^\theta_{L_r(\Omega)},\quad \lambda>0;\quad \theta=\frac{m}{s_1+s_2}.\tag{2.12}$$

The constant $C=C(m,r,s_1,s_2,\Omega)$ does not depend on the function b.

Proof. It suffices to check inequality (2.12) in the case where $\|b\|_{L_r(\Omega)}=1$.

For any $\varepsilon>0$ and $0<\alpha<1$ the absolute value of ratio (2.11) is estimated by

$$\frac{\varepsilon\int_\Omega|b(\mathbf{y})|^{2\alpha}|u_1(\mathbf{y})|^2\,d\mathbf{y}+\varepsilon^{-1}\int_\Omega|b(\mathbf{y})|^{2(1-\alpha)}|u_2(\mathbf{y})|^2\,d\mathbf{y}}{\|u_1\|^2_{H^{s_1}(\Omega)}+\|u_2\|^2_{H^{s_2}(\Omega)}},\quad \{u_1,u_2\}\in H^{s_1}(\Omega)\oplus H^{s_2}(\Omega).$$

Consider the ratios

$$\frac{\int_\Omega|b(\mathbf{y})|^{2\alpha}|u_1(\mathbf{y})|^2\,d\mathbf{y}}{\|u_1\|^2_{H^{s_1}(\Omega)}},\quad u_1\in H^{s_1}(\Omega),\tag{2.13}$$

$$\frac{\int_\Omega|b(\mathbf{y})|^{2(1-\alpha)}|u_2(\mathbf{y})|^2\,d\mathbf{y}}{\|u_2\|^2_{H^{s_2}(\Omega)}},\quad u_2\in H^{s_2}(\Omega).\tag{2.14}$$

According to Lemma 2.1 and (2.5),

$$N_\pm(\lambda,\ (2.11))\le N(\lambda\varepsilon^{-1},\ (2.13))+N(\lambda\varepsilon,\ (2.14)),\quad \lambda>0.\tag{2.15}$$

From the conditions on r it follows that there exist numbers r_1, r_2 such that $r^{-1}=\frac{1}{2}(r_1^{-1}+r_2^{-1})$ and $r_i=1$ for $2s_i>m$; $r_i>1$ for $2s_i=m$; $r_i=\frac{m}{2s_i}$ for $2s_i<m$, $i=1,2$. Applying Lemma 2.8 for ratios (2.13), (2.14) and taking (2.15) into account, we obtain

$$N_\pm(\lambda,\ (2.11))\le C\left(\varepsilon^{\theta_1}\lambda^{-\theta_1}\left\||b|^{2\alpha}\right\|^{\theta_1}_{L_{r_1}(\Omega)}+\varepsilon^{-\theta_2}\lambda^{-\theta_2}\left\||b|^{2(1-\alpha)}\right\|^{\theta_2}_{L_{r_2}(\Omega)}\right),$$

where $\theta_i=\frac{m}{2s_i}$, $i=1,2$. We take $\varepsilon=\lambda^{\frac{s_2-s_1}{s_1+s_2}}$, $\alpha=\frac{r_2}{r_1+r_2}$. Then $N_\pm(\lambda,\ (2.11))\le C\lambda^{-\theta}$ for $\|b\|_{L_r(\Omega)}=1$. □

We also need the spectral asymptotics for the ratio of two pseudodifferential forms defined on a smooth compact orientable m-dimensional manifold $\mathcal{D}$ without boundary. (In applications, the role of the manifold $\mathcal{D}$ will be played by the boundary of the smooth domain $\widetilde{\Omega} \subset \mathbb{R}^{m+1}$.)

Let $\rho > 0$ and let $H^\rho(\mathcal{D}; \mathbb{C}^l)$ be the Sobolev space of $\mathbb{C}^l$-valued semidensities on $\mathcal{D}$. In $H^\rho(\mathcal{D}; \mathbb{C}^l)$, we consider a pseudodifferential form $(\mathcal{P}\boldsymbol{\varphi}, \boldsymbol{\psi}) = \sum_{i,j=1}^l (\mathcal{P}_{ij}\varphi_j, \psi_i)$. It is assumed that the operators $\mathcal{P}_{ij} = \mathcal{P}_{ji}^*$ are classical pseudodifferential operators of order not exceeding 2ρ acting on semidensities on $\mathcal{D}$. Suppose that the pseudodifferential form $(\mathcal{P}\boldsymbol{\varphi}, \boldsymbol{\varphi})$ defines an equivalent norm on $H^\rho(\mathcal{D}; \mathbb{C}^l)$:

$$(\mathcal{P}\boldsymbol{\varphi}, \boldsymbol{\varphi}) \asymp \|\boldsymbol{\varphi}\|^2_{H^\rho(\mathcal{D})}, \quad \boldsymbol{\varphi} \in H^\rho(\mathcal{D}; \mathbb{C}^l). \tag{2.16}$$

The principal symbol of the pseudodifferential operator $\mathcal{P}$ (of order 2ρ) is denoted by $p^\circ(\mathbf{w})$, $\mathbf{w} \in T^*\mathcal{D} \setminus \{0\}$.

Let $\varkappa > 0$ and let $\mathcal{Q}_{ij} = \mathcal{Q}_{ji}^*$ be classical pseudodifferential operators of order not exceeding $2(\rho - \varkappa)$ acting on semidensities on $\mathcal{D}$. Then the pseudodifferential form $(\mathcal{Q}\boldsymbol{\varphi}, \boldsymbol{\psi}) = \sum_{i,j=1}^l (\mathcal{Q}_{ij}\varphi_j, \psi_i)$ is continuous in $H^{\rho-\varkappa}(\mathcal{D}; \mathbb{C}^l)$. The principal symbol of the pseudodifferential operator $\mathcal{Q}$ (of order $2(\rho - \varkappa)$) is denoted by $q^\circ(\mathbf{w})$, $\mathbf{w} \in T^*\mathcal{D} \setminus \{0\}$.

We consider the ratio of the forms

$$\pm \frac{(\mathcal{Q}\boldsymbol{\varphi}, \boldsymbol{\varphi})}{(\mathcal{P}\boldsymbol{\varphi}, \boldsymbol{\varphi})}, \quad \boldsymbol{\varphi} \in H^\rho(\mathcal{D}; \mathbb{C}^l). \tag{2.17}$$

From condition (2.16) it follows that the matrix $p^\circ(\mathbf{w})$ is positive. Therefore, a finite-dimensional problem on the spectrum of the ratio

$$\pm \frac{\langle q^\circ(\mathbf{w})\mathbf{z}, \mathbf{z}\rangle}{\langle p^\circ(\mathbf{w})\mathbf{z}, \mathbf{z}\rangle}, \quad \mathbf{z} \in \mathbb{C}^l, \tag{2.18}$$

makes sense. The counting functions $n_\pm(\lambda, \mathbf{w};\ (2.18))$ are homogeneous in the following sense: $n_\pm(\lambda, \mathbf{x}, t\boldsymbol{\xi};\ (2.18)) = n_\pm(t^{2\varkappa}\lambda, \mathbf{x}, \boldsymbol{\xi};\ (2.18))$, $\mathbf{x} \in \mathcal{D}$, $\boldsymbol{\xi} \in T^*_{\mathbf{x}}\mathcal{D} \setminus \{0\}$, $t > 0$.

The next statement follows from [11, Lemma 1]; see also [8, 10].

Lemma 2.10. *Under the above assumptions, the spectrum counting functions for ratio* (2.17) *satisfy the following asymptotics for* $\lambda \to +0$:

$$N_\pm(\lambda,\ (2.17)) \sim (2\pi)^{-m} \int_{T^*\mathcal{D}} n_\pm(\lambda, \mathbf{w};\ (2.18))\, d\mathbf{w} = \lambda^{-\theta}(2\pi)^{-m} \int_{T^*\mathcal{D}} n_\pm(1, \mathbf{w};\ (2.18))\, d\mathbf{w}, \quad \theta = \frac{m}{2\varkappa}.$$

Here $d\mathbf{w}$ *is the invariant measure on* $T^*\mathcal{D}$.

Finally, we need the spectral asymptotics of the ratio of pseudodifferential forms under additional constraints on a part of the manifold $\mathcal{D}$. Let $\mathcal{D}_0$ be an open subset of the manifold $\mathcal{D}$ such that $\operatorname{meas}_m \partial\mathcal{D}_0 = 0$. Consider the ratio of the forms

$$\pm \frac{(\mathcal{Q}\boldsymbol{\varphi}, \boldsymbol{\varphi})}{(\mathcal{P}\boldsymbol{\varphi}, \boldsymbol{\varphi})}, \quad \boldsymbol{\varphi} \in H^\rho(\mathcal{D}; \mathbb{C}^l), \quad \boldsymbol{\varphi}|_{\mathcal{D}_0} = 0. \tag{2.19}$$

The following statement is a particular case of the result of [37].

Lemma 2.11. *Under the above assumptions, the spectrum counting functions for ratio* (2.19) *satisfy the following asymptotics for* $\lambda \to +0$:

$$N_\pm(\lambda,\ (2.19)) \sim \lambda^{-\theta}(2\pi)^{-m} \int_{T^*(\mathcal{D}\setminus\overline{\mathcal{D}_0})} n_\pm(1, \mathbf{w};\ (2.18))\, d\mathbf{w}, \quad \theta = \frac{m}{2\varkappa}.$$

3. The Spectral Asymptotics of Small Oscillations of the Heavy Ideal Fluid

In paper [19] by Karazeeva and Solomyak, the Steklov-type problem in composite domains was considered. As the main example, the authors obtained the spectral asymptotics for the problem of small oscillations of a system of immiscible heavy ideal fluids completely filling a vessel. The method was based on the general approach to the study of nonsmooth variational problems developed by Birman and Solomyak. The same method can be used to consider the problem in the case of one fluid partially filling a vessel. In this section, we will briefly describe the results for the heavy ideal fluid.

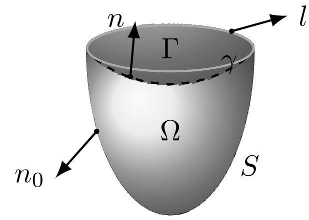

Fig. 1. One fluid partially filling a vessel

3.1. Small oscillations of the heavy ideal fluid. Let $\Omega \subset \mathbb{R}^3$ be a domain occupied by the fluid in the vessel in the equilibrium position (see Fig. 1); Γ is the equilibrium free surface of the fluid; $S = \partial\Omega \setminus \overline{\Gamma}$ is the solid wall of the vessel. We start with the classical formulation of the problem.

Let $\mathbf{n}_0(\mathbf{x})$ be the unit outer normal vector to S at the point $\mathbf{x} \in S$, and let $\mathbf{n}(\mathbf{x})$ be the unit outer normal vector to Γ at the point $\mathbf{x} \in \Gamma$. Suppose that $a(\mathbf{x})$ is a real-valued function on Γ such that $\int_\Gamma a(\mathbf{x})\,d\mathbf{x} \neq 0$. For physical reasons, the function $a(\mathbf{x})$ is positive, but the mathematical problem can be considered without this restriction.

The problem of normal oscillations of the heavy ideal fluid is reduced to the following spectral boundary-value problem[2]:

$$\begin{gathered} \Delta\Phi = 0 \ \text{ in } \Omega, \quad \frac{\partial\Phi}{\partial n_0} = 0 \ \text{ on } S, \\ \frac{\partial\Phi}{\partial n}(\mathbf{x}) = \lambda^{-1} a(\mathbf{x})\Phi(\mathbf{x}) \ \text{ on } \Gamma, \quad \int_\Gamma a(\mathbf{x})\Phi(\mathbf{x})\,dS = 0. \end{gathered} \tag{3.1}$$

Here $\Phi(\mathbf{x})$ is the amplitude of fluctuations of the potential $\Phi(\mathbf{x}, t)$ of the fluid particles velocity field: $\Phi(\mathbf{x}, t) = \Phi(\mathbf{x})e^{i\omega t}$, where $\omega^{-1} = \sqrt{\lambda}$ and t is the time. The Laplace equation is the continuity equation, the condition on S is the nonflow condition.

The boundary-value problem (3.1) is equivalent to the problem of finding successive maxima of the form ratio

$$\pm\frac{\int_\Gamma a(\mathbf{x})|\Phi(\mathbf{x})|^2\,dS}{\int_\Omega |\nabla\Phi|^2\,d\mathbf{x}}, \quad \Phi \in H^1(\Omega), \quad \int_\Gamma a(\mathbf{x})\Phi(\mathbf{x})\,dS = 0. \tag{3.2}$$

By the theorem on equivalent norms in Sobolev spaces (see, e.g., [36]), the functional $\int_\Omega |\nabla\Phi|^2\,d\mathbf{x} + |\int_\Gamma a(\mathbf{x})\Phi(\mathbf{x})\,dS|^2$ determines a norm in $H^1(\Omega)$ equivalent to the standard one. Therefore, on the subspace $\{\Phi \in H^1(\Omega) : \int_\Gamma a(\mathbf{x})\Phi(\mathbf{x})\,dS = 0\}$ the form $\int_\Omega |\nabla\Phi|^2\,d\mathbf{x}$ is equivalent to $\|\Phi\|^2_{H^1(\Omega)}$. The Laplace equation in Ω and the condition on S are natural conditions in the variational problem on the spectrum of ratio (3.2).

Theorem 3.1. *Let Ω be a bounded domain in $\mathbb{R}^3$ satisfying the assumptions of usual embedding and extension theorems*: $\Omega \in \mathcal{K}$; *let Γ be a smooth two-dimensional surface with Lipschitz boundary such that $\overline{\Gamma} \subset \partial\Omega$. Let $a \in L_2(\Gamma)$ be a real-valued function. Then the spectrum counting functions for ratio* (3.2) *satisfy the following asymptotics for $\lambda \to +0$*:

$$N_\pm(\lambda,\ (3.2)) \sim \frac{\lambda^{-2}}{4\pi}\int_\Gamma a^2_\pm(\mathbf{x})\,dS(\mathbf{x}). \tag{3.3}$$

We give the scheme of the proof of Theorem 3.1 by the method of [19]. Applying Lemmas 2.3 and 2.6, we see that the values $\Delta_2^\pm$ (3.2) and $\delta_2^\pm$ (3.2) coincide with the similar values for the ratio

$$\pm\frac{\int_\Gamma a(\mathbf{x})|\Phi(\mathbf{x})|^2\,dS}{\int_\Omega (|\nabla\Phi|^2 + |\Phi|^2)\,d\mathbf{x}}, \quad \Phi \in H^1(\Omega). \tag{3.4}$$

We start with the estimate for the spectrum.

[2] In [26, Chap. 3, § 3], this problem is discussed in the case where $a(\mathbf{x}) = 1$.

Lemma 3.1. *Under the assumptions of Theorem* 3.1, *we have* $\Delta_2^{\pm}\ (3.4) \le C\|a\|^2_{L_2(\Gamma)}$.

Proof. By the trace theorem, $\int_\Omega \left(|\nabla\Phi|^2 + |\Phi|^2\right)\,d\mathbf{x} \ge C\|\Phi\|^2_{H^{1/2}(\Gamma)}$, $\Phi \in H^1(\Omega)$, $C > 0$. Applying Lemma 2.2, we see that the functions $N_\pm(\lambda,\ (3.4))$ are estimated by the spectrum counting functions for the ratio

$$\pm C\frac{\int_\Gamma a(\mathbf{x})|\Phi(\mathbf{x})|^2\,dS}{C\|\Phi\|^2_{H^{1/2}(\Gamma)}}, \quad \Phi \in H^{1/2}(\Gamma), \quad C > 0. \tag{3.5}$$

Using Lemma 2.8, we obtain $\Delta_2^{\pm}\ (3.4) \le \Delta_2^{\pm}\ (3.5) \le C\|a\|^2_{L_2(\Gamma)}$. □

Lemmas 3.1 and 2.5 show that the values $\Delta_2^{\pm}\ (3.4)$ and $\delta_2^{\pm}\ (3.4)$ are continuous functionals of $a \in L_2(\Gamma)$. Therefore, it suffices to calculate these values for a set of coefficients a dense in $L_2(\Gamma)$. As such a set, we take $C_0^\infty(\Gamma)$.

Now, assuming that $a \in C_0^\infty(\Gamma)$, we will compare ratio (3.4) and a similar ratio in the domain $\widetilde{\Omega}$ with smooth boundary. Let $\widetilde{\Omega}$ be a bounded domain with smooth boundary such that $\widetilde{\Omega} \subset \Omega$ and $\operatorname{supp} a$ lies strictly inside the set $\partial\widetilde{\Omega} \cap \Gamma$. Let $\widetilde{a} \in C^\infty(\partial\widetilde{\Omega})$ be the function that is equal to $a(\mathbf{x})$ for $\mathbf{x} \in \partial\widetilde{\Omega} \cap \Gamma$ and equal to zero for $\mathbf{x} \in \partial\widetilde{\Omega} \setminus \Gamma$.

Our goal is to compare ratio (3.4) and the ratio

$$\pm\frac{\int_{\partial\widetilde{\Omega}} \widetilde{a}(\mathbf{x})|\Phi(\mathbf{x})|^2\,dS}{\int_{\widetilde{\Omega}} \left(|\nabla\Phi|^2 + |\Phi|^2\right)\,d\mathbf{x}}, \quad \Phi \in H^1(\widetilde{\Omega}). \tag{3.6}$$

Lemma 3.2. *Suppose that the assumptions of Theorem* 3.1 *are satisfied, and let* $a \in C_0^\infty(\Gamma)$. *Then*

$$\delta_2^{\pm}\ (3.6) \le \delta_2^{\pm}\ (3.4) \le \Delta_2^{\pm}\ (3.4) \le \Delta_2^{\pm}\ (3.6). \tag{3.7}$$

Proof. Obviously, for $\Phi \in H^1(\Omega)$ the numerators of ratios (3.4) and (3.6) are the same, and the denominator of (3.6) does not exceed the denominator of (3.4). Applying Lemma 2.2, in which the role of $\mathcal{S}$ is played by the restriction operator $\mathcal{S} : H^1(\Omega) \to H^1(\widetilde{\Omega})$, we obtain:

$$N_\pm(\lambda,\ (3.4)) \le N_\pm(\lambda,\ (3.6)), \quad \lambda > 0. \tag{3.8}$$

This implies the right inequality in (3.7).

Let us fix the cut-off function $\vartheta \in C^\infty(\overline{\Omega})$; $0 \le \vartheta(\mathbf{x}) \le 1$; $\vartheta(\mathbf{x}) = 1$ for $\mathbf{x} \in \operatorname{supp} a$; $\vartheta(\mathbf{x}) = 0$ in some neighborhood of $\Omega \setminus \widetilde{\Omega}$. We have

$$\begin{aligned}
\int_{\widetilde{\Omega}} \left(|\nabla\Phi|^2 + |\Phi|^2\right)\,d\mathbf{x} &\ge \varepsilon \int_{\widetilde{\Omega}} \left(|\nabla\Phi|^2 + |\Phi|^2\right)\,d\mathbf{x} + (1-\varepsilon)\int_{\widetilde{\Omega}} \vartheta^2\left(|\nabla\Phi|^2 + |\Phi|^2\right)\,d\mathbf{x} \\
&= \varepsilon \int_{\widetilde{\Omega}} \left(|\nabla\Phi|^2 + |\Phi|^2\right)\,d\mathbf{x} + (1-\varepsilon)\int_{\widetilde{\Omega}} \left(|\nabla(\vartheta\Phi)|^2 + |\vartheta\Phi|^2\right)\,d\mathbf{x} \\
&\quad + (1-\varepsilon)\int_{\widetilde{\Omega}} \left(\vartheta^2|\nabla\Phi|^2 - |\nabla(\vartheta\Phi)|^2\right)\,d\mathbf{x}, \quad \Phi \in H^1(\widetilde{\Omega}).
\end{aligned} \tag{3.9}$$

The sum of the first two terms in the right-hand side of (3.9) determines an equivalent metric in $H^1(\widetilde{\Omega})$, and the last term is a compact form in $H^1(\widetilde{\Omega})$.

Next, using the relation $\widetilde{a}(\mathbf{x}) = \vartheta^2(\mathbf{x})\widetilde{a}(\mathbf{x})$, $\mathbf{x} \in \partial\widetilde{\Omega}$, we obtain

$$\int_{\partial\widetilde{\Omega}} \widetilde{a}(\mathbf{x})|\Phi(\mathbf{x})|^2\,dS = \int_{\partial\widetilde{\Omega}} \widetilde{a}(\mathbf{x})|\vartheta(\mathbf{x})\Phi(\mathbf{x})|^2\,dS, \quad \Phi \in H^1(\widetilde{\Omega}).$$

According to Lemma 2.3, the values $\delta_\theta^{\pm}\ (3.6)$ do not exceed the similar values for the ratio

$$\pm\frac{\int_{\partial\widetilde{\Omega}} \widetilde{a}(\mathbf{x})|\vartheta(\mathbf{x})\Phi(\mathbf{x})|^2\,dS}{\varepsilon\int_{\widetilde{\Omega}} \left(|\nabla\Phi|^2 + |\Phi|^2\right)\,d\mathbf{x} + (1-\varepsilon)\int_{\widetilde{\Omega}} \left(|\nabla(\vartheta\Phi)|^2 + |\vartheta\Phi|^2\right)\,d\mathbf{x}}, \quad \Phi \in H^1(\widetilde{\Omega}). \tag{3.10}$$

For $\Phi \in H^1(\widetilde{\Omega})$, let $\widehat{\mathcal{S}}\Phi$ be the function equal to $\vartheta\Phi$ on $\widetilde{\Omega}$ and equal to zero on $\Omega \setminus \widetilde{\Omega}$. Then the operator $\widehat{\mathcal{S}} : H^1(\widetilde{\Omega}) \to H^1(\Omega)$ is bounded. For any $\Phi \in H^1(\widetilde{\Omega})$ such that $\pm\int_{\partial\widetilde{\Omega}} \widetilde{a}(\mathbf{x})|\vartheta(\mathbf{x})\Phi(\mathbf{x})|^2\,dS > 0$, ratio (3.10) does not exceed

$$\pm\frac{\int_\Gamma a(\mathbf{x})|(\widehat{\mathcal{S}}\Phi)(\mathbf{x})|^2\,dS}{(1-\varepsilon)\int_\Omega\left(|\nabla(\widehat{\mathcal{S}}\Phi)|^2+|\widehat{\mathcal{S}}\Phi|^2\right)\,d\mathbf{x}}.$$

By Lemma 2.2, this implies that $\delta_2^\pm$ (3.6) $\le \delta_2^\pm$ (3.10) $\le (1-\varepsilon)^{-2}\delta_2^\pm$ (3.4). Tending ε to zero, we arrive at the left inequality in (3.7). □

It remains to establish the asymptotics of the spectrum for the problem in a smooth domain.

Lemma 3.3. *Let $\widetilde{\Omega}$ be a bounded domain in $\mathbb{R}^3$ with smooth boundary. Suppose that $\widetilde{a}$ is a smooth real-valued function on $\partial\widetilde{\Omega}$. Then the spectrum counting functions for ratio* (3.6) *satisfy the following asymptotics for $\lambda\to+0$:*

$$N_\pm(\lambda,\ (3.6))\sim\frac{\lambda^{-2}}{4\pi}\int_{\partial\widetilde{\Omega}}\widetilde{a}_\pm^2(\mathbf{x})\,dS(\mathbf{x}). \tag{3.11}$$

Proof. Let $L_0=-\Delta+I$. By $H^1(\widetilde{\Omega},L_0)$ we denote the subspace in $H^1(\widetilde{\Omega})$ formed by solutions of the equation $L_0u=0$ in $\widetilde{\Omega}$. We have $H^1(\widetilde{\Omega})=H^1(\widetilde{\Omega},L_0)\oplus H_0^1(\widetilde{\Omega})$. Since the form in the numerator of (3.6) vanishes for $u\in H_0^1(\widetilde{\Omega})$, then the nonzero spectrum of ratio (3.6) will not change if we consider this ratio on $H^1(\widetilde{\Omega},L_0)$.

Let G_0 be the "Poisson operator" taking a function $\varphi\in H^{1/2}(\partial\widetilde{\Omega})$ to the solution of the corresponding Dirichlet problem for the equation $L_0u=0$: the relation $u=G_0\varphi$ means that $u\in H^1(\widetilde{\Omega},L_0)$, $u|_{\partial\widetilde{\Omega}}=\varphi$. The operator G_0 is a homeomorphism of the spaces $H^{1/2}(\partial\widetilde{\Omega})$ and $H^1(\widetilde{\Omega},L_0)$. The problem on the spectrum of ratio (3.6) is equivalent to the problem on the spectrum of the ratio

$$\pm\frac{\int_{\partial\widetilde{\Omega}}\widetilde{a}(\mathbf{x})|\varphi(\mathbf{x})|^2\,dS}{\int_{\widetilde{\Omega}}\left(|\nabla G_0\varphi|^2+|G_0\varphi|^2\right)\,d\mathbf{x}},\quad \varphi\in H^{1/2}(\partial\widetilde{\Omega}). \tag{3.12}$$

From the properties of the Boutet de Monvel algebra [14, 17] it follows that

$$\int_{\widetilde{\Omega}}\left(|\nabla G_0\varphi|^2+|G_0\varphi|^2\right)\,d\mathbf{x}=(\mathcal{P}_0\varphi,\varphi),\quad \varphi\in H^{1/2}(\partial\widetilde{\Omega}),$$

where $\mathcal{P}_0$ is a classical pseudodifferential operator on $\partial\widetilde{\Omega}$ of order 1. Calculating the principal symbol $p_0^\circ(\mathbf{x},\boldsymbol{\xi})$ of the operator $\mathcal{P}_0$ according to the known computational rules for the Boutet de Monvel algebra, we obtain: $p_0^\circ(\mathbf{x},\boldsymbol{\xi})=|\boldsymbol{\xi}|$, $\mathbf{x}\in\partial\widetilde{\Omega}$, $0\ne\boldsymbol{\xi}\perp\boldsymbol{\nu}(\mathbf{x})$. Here $\boldsymbol{\nu}(\mathbf{x})$ is the normal to $\partial\widetilde{\Omega}$. (Compare with the calculations in Sec. 4.4 below.)

Obviously, the form in the numerator of ratio (3.12) can be interpreted as the form $(\mathcal{Q}_0\varphi,\varphi)$ of a zeroth-order pseudodifferential operator $\mathcal{Q}_0$ on $\partial\widetilde{\Omega}$ with the symbol $q_0^\circ(\mathbf{x},\boldsymbol{\xi})=\widetilde{a}(\mathbf{x})$. Thus, ratio (3.12) coincides with the ratio of the pseudodifferential forms

$$\pm\frac{(\mathcal{Q}_0\varphi,\varphi)}{(\mathcal{P}_0\varphi,\varphi)},\quad \varphi\in H^{1/2}(\partial\widetilde{\Omega}). \tag{3.13}$$

We have proved that

$$N_\pm(\lambda,\ (3.6))=N_\pm(\lambda,\ (3.12))=N_\pm(\lambda,\ (3.13)),\quad \lambda>0. \tag{3.14}$$

By Lemma 2.10, the spectrum counting functions for ratio (3.13) satisfy the following asymptotics for $\lambda\to+0$:

$$N_\pm(\lambda,\ (3.13))\sim\frac{1}{4\pi^2}\int_{\partial\widetilde{\Omega}}dS(\mathbf{x})\int_{\boldsymbol{\xi}\perp\boldsymbol{\nu}(\mathbf{x})}d\boldsymbol{\xi}\,n_\pm(\lambda,\mathbf{x},\boldsymbol{\xi};\ (3.16)), \tag{3.15}$$

where $n_\pm(\lambda,\mathbf{x},\boldsymbol{\xi};\ (3.16))$ are the spectrum counting functions for the ratio of the (one-dimensional) forms

$$\pm\frac{q_0^\circ(\mathbf{x},\boldsymbol{\xi})|z|^2}{p_0^\circ(\mathbf{x},\boldsymbol{\xi})|z|^2},\quad z\in\mathbb{C}. \tag{3.16}$$

We have

$$n_\pm(\lambda,\mathbf{x},\boldsymbol{\xi};\ (3.16))=\begin{cases}1, & \lambda<\widetilde{a}_\pm(\mathbf{x})|\boldsymbol{\xi}|^{-1},\\ 0, & \lambda\ge\widetilde{a}_\pm(\mathbf{x})|\boldsymbol{\xi}|^{-1}.\end{cases} \tag{3.17}$$

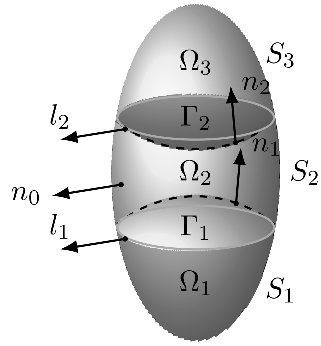

Fig. 2. System of fluids in a closed vessel

Now, calculating asymptotics of the functions $N_\pm(\lambda,\ (3.13))$ according to (3.15), (3.17) and using (3.14), we obtain the required asymptotics (3.11). $\square$

Completion of the proof of Theorem 3.1. Combining Lemma 3.2, asymptotics (3.11), and the relation $\widetilde{a}(\mathbf{x}) = a(\mathbf{x})$ for $\mathbf{x} \in \operatorname{supp} a = \operatorname{supp} \widetilde{a} \subset \Gamma$, we obtain asymptotics of form (3.3) for $N_\pm(\lambda,\ (3.4))$ in the case where $a \in C_0^\infty(\Gamma)$. By closure, this formula is true for $a \in L_2(\Gamma)$; see Lemma 3.1. It remains to recall that $\Delta_2^\pm\ (3.2) = \Delta_2^\pm\ (3.4)$ and $\delta_2^\pm\ (3.2) = \delta_2^\pm\ (3.4)$. This completes the proof of Theorem 3.1. $\square$

3.2. Small oscillations of a system of the heavy ideal fluids. By the previous way, we can obtain asymptotics of the spectrum for the problem on oscillations of a system of immiscible fluids completely or partially filling a vessel. We restrict ourselves to the statement of the problem and the formulation of the result for a system of the heavy ideal fluids completely filling a vessel (see Fig. 2); the proof can be found in [19].

Suppose that a bounded domain $\Omega \subset \mathbb{R}^3$ is divided into $(k+1)$ parts Ω_j, $j = 1, \dots, k+1$. Here $k+1$ is the number of fluids, and Ω_j is the domain occupied by the j-th fluid in the equilibrium position. Suppose that

$$\overline{\Omega} = \bigcup_{j=1}^{k+1} \overline{\Omega_j}, \quad \Omega = \operatorname{int} \overline{\Omega}, \quad \Omega_i \cap \Omega_j = \varnothing \text{ for } i \neq j, \quad \overline{\Omega_i} \cap \overline{\Omega_j} = \varnothing \text{ for } j \notin \{i-1, i, i+1\}. \tag{3.18}$$

Denote by

$$\overline{\Gamma_i} = \overline{\Omega_i} \cap \overline{\Omega_{i+1}}, \quad i = 1, \dots, k, \tag{3.19}$$

the interfaces; let $S = \partial\Omega$ be the solid wall of the vessel; let $S_j = \partial\Omega_j \cap S$, $j = 1, \dots, k+1$. We assume that $\Omega_i \in \mathcal{K}$, $i = 1, \dots, k+1$, and Γ_i, $i = 1, \dots, k$, are smooth two-dimensional surfaces with Lipschitz boundaries. Let $\mathbf{n}_0(\mathbf{x})$ be the unit outer normal vector to S at the point $\mathbf{x} \in S$; and let $\mathbf{n}_j(\mathbf{x})$ be the outer (with respect to Ω_j) unit normal vector to Γ_j at the point $\mathbf{x} \in \Gamma_j$.

The problem on normal oscillations of a system of the heavy ideal fluids is formulated for a system of functions $\{\Phi_j(\mathbf{x})\}$, $j = 1, \dots, k+1$, where Φ_j is a function in Ω_j:

$$\begin{gathered}
\Delta\Phi_j = 0 \ \text{ in } \Omega_j, \quad \frac{\partial\Phi_j}{\partial n_0} = 0 \ \text{ on } S_j, \quad j = 1, \dots, k+1, \\
\frac{\partial\Phi_j}{\partial n_j} = \frac{\partial\Phi_{j+1}}{\partial n_j} = \lambda^{-1} a_j(\mathbf{x}) \left(\rho_j \Phi_j - \rho_{j+1}\Phi_{j+1}\right) \ \text{ on } \Gamma_j, \quad j = 1, \dots, k, \\
\int_{\Gamma_j} a_j(\mathbf{x}) \left(\rho_j \Phi_j - \rho_{j+1}\Phi_{j+1}\right) dS = 0, \ \ j = 1, \dots, k; \quad \sum\nolimits_{j=1}^{k+1} \int_{\Omega_j} \rho_j^{-1} \Phi_j(\mathbf{x})\, d\mathbf{x} = 0.
\end{gathered} \tag{3.20}$$

Here $a_j \in L_2(\Gamma_j)$ are real-valued functions such that $\int_{\Gamma_j} a_j(\mathbf{x})\, d\mathbf{x} \neq 0$. The constants $\rho_j > 0$ stand for the densities of the fluids. (According to the physical meaning, the functions $a_j(\mathbf{x})$ must be positive, and the densities ρ_j must satisfy the inequalities $\rho_1 > \rho_2 > \dots > \rho_{k+1}$, but the mathematical problem can be considered without these restrictions.)

Problem (3.20) is equivalent to the variational problem on the spectrum of the form ratio

$$\frac{\sum_{j=1}^{k}\int_{\Gamma_j} a_j(\mathbf{x})|\rho_j\Phi_j-\rho_{j+1}\Phi_{j+1}|^2\,dS}{\sum_{j=1}^{k+1}\rho_j\int_{\Omega_j}|\nabla\Phi_j|^2\,d\mathbf{x}},\quad \Phi_j\in H^1(\Omega_j),\quad j=1,\dots,k+1;$$
$$\int_{\Gamma_j} a_j(\mathbf{x})\,(\rho_j\Phi_j-\rho_{j+1}\Phi_{j+1})\,dS=0,\quad j=1,\dots,k;\quad \sum_{j=1}^{k+1}\int_{\Omega_j}\rho_j^{-1}\Phi_j(\mathbf{x})\,d\mathbf{x}=0. \tag{3.21}$$

Proposition 3.1 (see [19])**.** *Under the above assumptions, the spectrum counting functions for ratio* (3.21) *satisfy the following asymptotics for* $\lambda\to+0$:

$$N_\pm(\lambda,\ (3.21))\sim\frac{\lambda^{-2}}{4\pi}\sum_{j=1}^{k}(\rho_j+\rho_{j+1})^2\int_{\Gamma_j}(a_j)_\pm^2\,d\mathbf{x}.$$

4. The Spectral Asymptotics of a Nonclassical Steklov-Type Problem. Application to the Theory of Small Fluid Oscillations

A number of problems in the theory of small fluid oscillations (see [3, 21–23]) leads to the question of the spectrum of the variational ratio

$$\frac{\operatorname{Re}\int_\Gamma b(\mathbf{y})(\mathbf{B}_\Gamma^{-1}u)(\mathbf{y})\overline{u(\mathbf{y})}\,dS(\mathbf{y})}{\mathcal{A}_\Omega[u]},\quad u\in H^1(\Omega). \tag{4.1}$$

Here $\Omega\subset\mathbb{R}^3$ (see Fig. 1) is the domain occupied by the fluid partially filling a vessel in the equilibrium position; a smooth two-dimensional surface $\Gamma\subset\partial\Omega$ has the meaning of the free surface of the fluid or the elastic bottom of the vessel. The quadratic form $\mathcal{A}_\Omega[u]$ determines an equivalent metric in $H^1(\Omega)$. The nonlocal operator $\mathbf{B}_\Gamma^{-1}$ is the resolving operator of some elliptic boundary-value problem on Γ. We call the problem of the spectrum of ratio (4.1) a *Steklov-type problem*, because: (a) the extremals automatically satisfy the homogeneous elliptic equation $Lu=0$ in the domain Ω, where the operator L corresponds to the form $\mathcal{A}_\Omega$; (b) the extremals satisfy the equation on Γ, which includes the spectral parameter. However, this problem differs from the classical Steklov-type problem in that the form ratio involves a nonlocal operator $\mathbf{B}_\Gamma^{-1}$.

In this section, we obtain the asymptotics of the spectrum of ratio (4.1). The result is applied to the question on the spectrum of small oscillations of the capillary ideal fluid, the capillary stratified fluid, as well as to one auxiliary problem of hydroelasticity. The main difficulty is related to the nonsmoothness of the boundary $\partial\Omega$ (the vessel wall and the free surface of the fluid form an edge at their intersection). Another difficulty arises from the nonlocal nature of the operator $\mathbf{B}_\Gamma^{-1}$.

4.1. Statement of the problem. Formulation of the result. Let $\Omega\subset\mathbb{R}^{m+1}$ be a bounded domain such that $\Omega\in\mathcal{K}$. Suppose that the boundary $\partial\Omega$ contains an infinitely smooth m-dimensional surface Γ with a smooth $(m-1)$-dimensional edge γ (here Γ and γ are not necessarily connected). Let $\mathbf{n}(\mathbf{x})$ be the interior unit normal vector to $\partial\Omega$ at the point $\mathbf{x}\in\Gamma$.

In the domain Ω, we consider a Hermitian quadratic form given by

$$\mathcal{A}_\Omega[u]:=\int_\Omega\Big(\sum_{i,j=1}^{m+1}a_{ij}(\mathbf{x})\partial_i u(\mathbf{x})\overline{\partial_j u(\mathbf{x})}+V(\mathbf{x})|u(\mathbf{x})|^2\Big)\,d\mathbf{x},\quad u\in H^1(\Omega). \tag{4.2}$$

Suppose that the coefficients $a_{ij}(\mathbf{x})=a_{ji}(\mathbf{x})$ and $V(\mathbf{x})$ are infinitely smooth real-valued functions in $\overline{\Omega}$. Assume also that the matrix $a(\mathbf{x})=\{a_{ij}(\mathbf{x})\}$ is positive definite:

$$\langle a(\mathbf{x})\boldsymbol{\eta},\boldsymbol{\eta}\rangle=\sum_{i,j=1}^{m+1}a_{ij}(\mathbf{x})\eta_i\overline{\eta_j}\ge c_a|\boldsymbol{\eta}|^2,\quad \mathbf{x}\in\Omega,\ \boldsymbol{\eta}\in\mathbb{C}^{m+1},\quad c_a>0, \tag{4.3}$$

and the function $V(\mathbf{x})$ is positive definite:

$$V(\mathbf{x})\ge c_a>0,\quad \mathbf{x}\in\Omega. \tag{4.4}$$

Under the above assumptions, the form $\mathcal{A}_\Omega[u]$ determines in $H^1(\Omega)$ the norm equivalent to the standard one:

$$c_a\|u\|^2_{H^1(\Omega)}\le\mathcal{A}_\Omega[u]\le C_a\|u\|^2_{H^1(\Omega)},\quad u\in H^1(\Omega). \tag{4.5}$$

By L we denote the differential expression corresponding to form (4.2):

$$L = -\sum_{i,j=1}^{m+1} \partial_j a_{ij}(\mathbf{x})\partial_i + V(\mathbf{x}).$$

Next, let B_Γ be a scalar strongly elliptic differential expression on Γ of order $2q$; let $T_1, \dots, T_q$ be the differential trace operators acting "from Γ to γ," $\operatorname{ord} T_j = \beta_j \le 2q-1$. It is assumed that the coefficients of the operators B_Γ and T_j, $j = 1, \dots, q$, are infinitely smooth, in general, complex-valued functions. Suppose that the problem $B_\Gamma u = f$ on Γ, $T_j u = \varphi_j$ on γ, $j = 1, \dots, q$, is a regular elliptic problem, i.e., the Shapiro–Lopatinsky condition is satisfied; see, e.g., [2, 4, 28]. By $\mathbf{B}_\Gamma$ we denote the operator on $L_2(\Gamma)$ given by the expression B_Γ on the domain $\operatorname{Dom} \mathbf{B}_\Gamma = \{u \in H^{2q}(\Gamma) : T_j u|_\gamma = 0,\ j = 1, \dots, q\}$. Suppose that $\mathbf{B}_\Gamma$ is self-adjoint and positive definite. Then $\mathbf{B}_\Gamma^{-1}$ is a compact operator on $L_2(\Gamma)$. Let $B(\mathbf{x}, \boldsymbol{\xi})$, $\mathbf{x} \in \Gamma$, $\boldsymbol{\xi} \perp \mathbf{n}(\mathbf{x})$, be the principal symbol of the differential expression B_Γ. From the strong ellipticity and the self-adjointness it follows that $B(\mathbf{x}, \boldsymbol{\xi}) \ge c_0 |\boldsymbol{\xi}|^{2q}$, $\mathbf{x} \in \Gamma$, $\boldsymbol{\xi} \perp \mathbf{n}(\mathbf{x})$, $c_0 > 0$.

Consider the quadratic form $\mathcal{B}_\Gamma[u] := \operatorname{Re} \int_\Gamma b(\mathbf{x}) \left(\mathbf{B}_\Gamma^{-1} u\right)(\mathbf{x}) \overline{u(\mathbf{x})}\, dS(\mathbf{x})$, where $b(\mathbf{x})$ is a real-valued function on Γ such that

$$\begin{gathered} b \in L_r(\Gamma), \quad r > 1 \text{ for } m = 1; \quad \frac{1}{r} = \frac{1}{2} + \frac{1}{2m} \text{ for } 1 < m < 4q+1; \\ \frac{1}{r} < \frac{1}{2} + \frac{1}{2m} \text{ for } m = 4q+1; \quad r = \frac{m}{2q+1} \text{ for } m > 4q+1. \end{gathered} \tag{4.6}$$

Consider the ratio of quadratic forms

$$\pm \frac{\mathcal{B}_\Gamma[u]}{\mathcal{A}_\Omega[u]}, \quad u \in H^1(\Omega). \tag{4.7}$$

Let $\mathbf{x} \in \Gamma$, $\xi \perp \mathbf{n}(x)$. Denote $M(\mathbf{x}, \boldsymbol{\xi}) := \left(\langle a(\mathbf{x})\boldsymbol{\xi}, \boldsymbol{\xi}\rangle \langle a(\mathbf{x})\mathbf{n}(\mathbf{x}), \mathbf{n}(\mathbf{x})\rangle - \langle a(\mathbf{x})\boldsymbol{\xi}, \mathbf{n}(\mathbf{x})\rangle^2\right)^{1/2}$.

The main result of this section is the following theorem.

Theorem 4.1. *Under the above assumptions, the spectrum counting functions for ratio* (4.7) *satisfy the following asymptotics for* $\lambda \to +0$:

$$N_\pm(\lambda,\ (4.7)) \sim \frac{\lambda^{-\theta}}{m(2\pi)^m} \int_\Gamma dS(\mathbf{x}) \int_{\boldsymbol{\xi} \perp \mathbf{n}(x) :\, |\boldsymbol{\xi}| = 1} dS(\boldsymbol{\xi}) \left(b_\pm(\mathbf{x}) B^{-1}(\mathbf{x}, \boldsymbol{\xi}) M^{-1}(\mathbf{x}, \boldsymbol{\xi})\right)^\theta, \quad \theta = \frac{m}{2q+1}.$$

4.2. Estimates of the spectrum.

Lemma 4.1. *Suppose that* $b(\mathbf{x})$ *is a real-valued function satisfying conditions* (4.6). *Then we have* $\Delta_\theta^\pm\ (4.7) \le C \|b\|_{L_r(\Gamma)}^\theta$, $\theta = \frac{m}{2q+1}$, *where the constant* C *does not depend on* b.

Proof. By the trace theorem (see, e.g., [2, 28]), the lower estimate (4.5) implies that $\mathcal{A}_\Omega[u] \ge c_a \|u\|_{H^1(\Omega)}^2 \ge C_1 \|u\|_{H^{1/2}(\Gamma)}^2$, $u \in H^1(\Omega)$, with some constant $C_1 > 0$. In (4.7), we denote $g = \mathbf{B}_\Gamma^{-1} u$. By the theorem on homeomorphisms (see, e.g., [2, 4, 28]), we have $\|u\|_{H^{1/2}(\Gamma)}^2 \ge C_2 \|g\|_{H^{2q+1/2}(\Gamma)}^2$, $u \in H^{1/2}(\Gamma)$, $C_2 > 0$. Applying Lemma 2.2, we obtain that the functions $N_\pm(\lambda,\ (4.7))$ are estimated from above in terms of the spectrum counting functions for the ratio

$$\pm C \frac{\operatorname{Re} \int_\Gamma b(\mathbf{x}) g(\mathbf{x}) \overline{u(\mathbf{x})}\, dS(\mathbf{x})}{\|u\|_{H^{1/2}(\Gamma)}^2 + \|g\|_{H^{2q+1/2}(\Gamma)}^2}, \quad \{u, g\} \in H^{1/2}(\Gamma) \oplus H^{2q+1/2}(\Gamma). \tag{4.8}$$

By Lemma 2.9, we have $\Delta_\theta^\pm\ (4.7) \le \Delta_\theta^\pm\ (4.8) \le C \|b\|_{L_r(\Gamma)}^\theta$. □

Lemmas 4.1 and 2.5 show that the values $\Delta_\theta^\pm\ (4.7)$ and $\delta_\theta^\pm\ (4.7)$ are continuous functionals of $b \in L_r(\Gamma)$. Therefore, it *suffices to calculate these values for a set of coefficients* $b(\mathbf{x})$ *dense in* $L_r(\Gamma)$. As such a set, it is convenient to take $C_0^\infty(\Gamma)$.

4.3. Comparison with the problem in a smooth domain. *Below it is assumed that* $b \in C_0^\infty(\Gamma)$. We proceed similarly to Sec. 3.1. Let $\widetilde{\Omega}$ be a bounded domain with smooth boundary such that $\widetilde{\Omega} \subset \Omega$, and let $\operatorname{supp} b$ lie strictly inside the set $\partial\widetilde{\Omega} \cap \Gamma$. Then there exists an open subset $\widetilde{\Gamma}$ of the boundary $\partial\widetilde{\Omega}$ lying strictly inside $\partial\widetilde{\Omega} \cap \Gamma$ and such that $\operatorname{supp} b \subset \widetilde{\Gamma}$. We can assume that $\widetilde{\Gamma}$ is an m-dimensional surface with sufficiently smooth boundary. Let $\boldsymbol{\nu}(\mathbf{x})$ be the inner unit normal vector to $\partial\widetilde{\Omega}$ at the point $\mathbf{x} \in \partial\widetilde{\Omega}$. Let $\widetilde{b} \in C^\infty(\partial\widetilde{\Omega})$ be the function equal to $b(\mathbf{x})$ for $\mathbf{x} \in \widetilde{\Gamma}$ and equal to zero outside $\widetilde{\Gamma}$.

Let $B_{\partial\widetilde{\Omega}}(\mathbf{x}, \boldsymbol{\xi})$, $\mathbf{x} \in \partial\widetilde{\Omega}$, $\boldsymbol{\xi} \perp \boldsymbol{\nu}(\mathbf{x})$, be a homogeneous polynomial of $\boldsymbol{\xi}$ of order $2q$, whose coefficients are smooth functions of $\mathbf{x}$, and $B_{\partial\widetilde{\Omega}}(\mathbf{x}, \boldsymbol{\xi}) = B(\mathbf{x}, \boldsymbol{\xi})$ for $\mathbf{x} \in \partial\widetilde{\Omega} \cap \Gamma$, $\boldsymbol{\xi} \perp \boldsymbol{\nu}(\mathbf{x})$, and also the following strong ellipticity condition is satisfied: $B_{\partial\widetilde{\Omega}}(\mathbf{x}, \boldsymbol{\xi}) \geq c|\boldsymbol{\xi}|^{2q}$, $\mathbf{x} \in \partial\widetilde{\Omega}$, $\boldsymbol{\xi} \perp \boldsymbol{\nu}(\mathbf{x})$, $c > 0$. It is easily seen that such a "strong elliptic" continuation of the symbol $B(\mathbf{x}, \boldsymbol{\xi})$ is always possible. By $B_{\partial\widetilde{\Omega}}$ we denote some differential expression of order $2q$ on $\partial\widetilde{\Omega}$ with smooth coefficients and the principal symbol $B_{\partial\widetilde{\Omega}}(\mathbf{x}, \boldsymbol{\xi})$.

Let $\mathbf{B}_{\partial\widetilde{\Omega}}$ be the operator in $L_2(\partial\widetilde{\Omega})$ given by the expression $B_{\partial\widetilde{\Omega}}$ on the domain $H^{2q}(\partial\widetilde{\Omega})$. By choosing lower order terms, the expression $B_{\partial\widetilde{\Omega}}$ can be chosen so that the operator $\mathbf{B}_{\partial\widetilde{\Omega}}$ is self-adjoint and positive definite. The inverse operator $\mathbf{B}_{\partial\widetilde{\Omega}}^{-1}$ is a pseudodifferential operator on $\partial\widetilde{\Omega}$ of order $(-2q)$. Denote $\mathcal{B}_{\partial\widetilde{\Omega}}[u] := \operatorname{Re} \int_{\partial\widetilde{\Omega}} \widetilde{b}(\mathbf{x}) \big(\mathbf{B}_{\partial\widetilde{\Omega}}^{-1} u\big)(\mathbf{x}) \overline{u(\mathbf{x})}\, dS(\mathbf{x})$.

By $\mathcal{A}_{\widetilde{\Omega}}$ we denote the form $\mathcal{A}_{\widetilde{\Omega}}[u] := \int_{\widetilde{\Omega}} \Big(\sum_{i,j=1}^{m+1} a_{ij}(\mathbf{x}) \partial_i u(\mathbf{x}) \overline{\partial_j u(\mathbf{x})} + V(\mathbf{x}) |u(\mathbf{x})|^2 \Big)\, d\mathbf{x}$, $u \in H^1(\widetilde{\Omega})$. The coefficients of this form are the same as in (4.2) (restricted to $\widetilde{\Omega}$). Our goal is to compare ratio (4.7) and the ratio

$$\pm \frac{\mathcal{B}_{\partial\widetilde{\Omega}}[u]}{\mathcal{A}_{\widetilde{\Omega}}[u]}, \quad u \in H^1(\widetilde{\Omega}). \tag{4.9}$$

This comparison is carried out in two steps; they correspond to Lemma 4.2 and Lemma 4.3. Consider the ratio

$$\pm \frac{\mathcal{B}_{\partial\widetilde{\Omega}}[u]}{\mathcal{A}_{\Omega}[u]}, \quad u \in H^1(\Omega). \tag{4.10}$$

Lemma 4.2. *We have*

$$\Delta_\theta^\pm\ (4.7) = \Delta_\theta^\pm\ (4.10), \quad \delta_\theta^\pm\ (4.7) = \delta_\theta^\pm\ (4.10), \quad \theta = \frac{m}{2q+1}. \tag{4.11}$$

Proof. Consider the ratio

$$\pm \frac{\mathcal{F}[u]}{\mathcal{A}_{\Omega}[u]}, \quad u \in H^1(\Omega), \tag{4.12}$$

where $\mathcal{F}[u] := \mathcal{B}_{\partial\widetilde{\Omega}}[u] - \mathcal{B}_\Gamma[u]$. By Lemma 2.4, relations (4.11) will be proved as soon as we show that $\Delta_\theta^+\ (4.12) = \Delta_\theta^-\ (4.12) = 0$. Let $u \in H^1(\Omega)$. Denote $g = \mathbf{B}_\Gamma^{-1} u$, $f = \mathbf{B}_{\partial\widetilde{\Omega}}^{-1} u$. Then

$$\begin{aligned} \mathcal{F}[u] &= \operatorname{Re} \int_{\widetilde{\Gamma}} b(f-g) \overline{B_{\partial\widetilde{\Omega}} f}\, dS = \operatorname{Re} \int_{\widetilde{\Gamma}} B_{\partial\widetilde{\Omega}}(b(f-g)) \overline{f}\, dS \\ &= \operatorname{Re} \int_{\widetilde{\Gamma}} b\left(B_{\partial\widetilde{\Omega}} f - B_\Gamma g + (B_\Gamma - B_{\partial\widetilde{\Omega}}) g\right) \overline{f}\, dS + \operatorname{Re} \int_{\widetilde{\Gamma}} \left(B_{\partial\widetilde{\Omega}}(b(f-g)) - b B_{\partial\widetilde{\Omega}}(f-g)\right) \overline{f}\, dS. \end{aligned} \tag{4.13}$$

We take into account the following: (a) $(B_{\partial\widetilde{\Omega}} f)(\mathbf{x}) = (B_\Gamma g)(\mathbf{x}) = u(\mathbf{x})$ for $\mathbf{x} \in \widetilde{\Gamma}$; (b) $(B_\Gamma - B_{\partial\widetilde{\Omega}})$ on $\widetilde{\Gamma}$ is a differential expression of order $2q-1$, since the principal symbols of B_Γ and $B_{\partial\widetilde{\Omega}}$ coincide for $\mathbf{x} \in \widetilde{\Gamma}$ (i.e., $B(\mathbf{x}, \boldsymbol{\xi}) = B_{\partial\widetilde{\Omega}}(\mathbf{x}, \boldsymbol{\xi})$ for $\mathbf{x} \in \widetilde{\Gamma}$, $\boldsymbol{\xi} \perp \boldsymbol{\nu}(\mathbf{x})$); (c) $(B_{\partial\widetilde{\Omega}} \widetilde{b} - \widetilde{b} B_{\partial\widetilde{\Omega}})$ is a differential expression of order $2q-1$. Then from (4.13) it follows that the form $\mathcal{F}[u]$ is represented as

$$\mathcal{F}[u] = \operatorname{Re} \int_{\widetilde{\Gamma}} (B' g + B'' f) \overline{f}\, dS, \tag{4.14}$$

where B' and B'' are some differential expressions on $\widetilde{\Gamma}$ of order $2q-1$ with smooth coefficients.

Using (4.5), the trace theorem, the homeomorphism theorem for $\mathbf{B}_\Gamma$, and the continuity properties of the pseudodifferential operators on Sobolev spaces on $\partial\widetilde{\Omega}$, we conclude that

$$\mathcal{A}_\Omega[u] \geq c_a \|u\|^2_{H^1(\Omega)} \geq C_3 \left(\|u\|^2_{H^{1/2}(\Gamma)} + \|u\|^2_{H^{1/2}(\partial\widetilde{\Omega})} \right) \geq C_4 \left(\|g\|^2_{H^{2q+1/2}(\Gamma)} + \|f\|^2_{H^{2q+1/2}(\partial\widetilde{\Omega})} \right)$$
$$\geq C_5 \left(\|B'g + B''f\|^2_{H^{3/2}(\widetilde{\Gamma})} + \|f\|^2_{H^{2q+1/2}(\partial\widetilde{\Omega})} \right), \quad u \in H^1(\Omega). \quad (4.15)$$

Denote $B'g + B''f = \psi$. Applying Lemma 2.2 and taking (4.14) and (4.15) into account, we obtain that the values $\Delta^\pm_\theta$ (4.12) do not exceed the similar values for the ratio

$$\pm C \frac{\operatorname{Re} \int_{\widetilde{\Gamma}} \psi(\mathbf{x}) \overline{f(\mathbf{x})}\, dS(\mathbf{x})}{\|\psi\|^2_{H^{3/2}(\widetilde{\Gamma})} + \|f\|^2_{H^{2q+1/2}(\widetilde{\Gamma})}}, \quad \{\psi, f\} \in H^{3/2}(\widetilde{\Gamma}) \oplus H^{2q+1/2}(\widetilde{\Gamma}). \quad (4.16)$$

By Lemma 2.9, $N_\pm(\lambda,\ (4.16)) = O(\lambda^{-\frac{m}{2q+2}})$. Hence, $\Delta^\pm_\theta$ (4.16) $= 0$ for $\theta = \frac{m}{2q+1}$. Then also Δ^+_θ (4.12) $=$ Δ^-_θ (4.12) $= 0$. □

Now, we compare ratios (4.10) and (4.9).

Lemma 4.3. *We have*

$$\delta^\pm_\theta\ (4.9) \leq \delta^\pm_\theta\ (4.10) \leq \Delta^\pm_\theta\ (4.10) \leq \Delta^\pm_\theta\ (4.9), \quad \theta = \frac{m}{2q+1}. \quad (4.17)$$

Proof. By (4.4) and (4.3), we have $\mathcal{A}_\Omega[u] \geq \mathcal{A}_{\widetilde{\Omega}}[u]$, $u \in H^1(\Omega)$. Applying Lemma 2.2, in which $\mathcal{S} : H^1(\Omega) \to H^1(\widetilde{\Omega})$ is the restriction operator, we obtain

$$N_\pm(\lambda,\ (4.10)) \leq N_\pm(\lambda,\ (4.9)), \quad \lambda > 0. \quad (4.18)$$

This implies the right inequality in (4.17).

We fix a cut-off function $\vartheta \in C^\infty(\overline{\Omega})$ such that $0 \leq \vartheta(\mathbf{x}) \leq 1$; $\vartheta(\mathbf{x}) = 1$ for $\mathbf{x} \in \operatorname{supp} b$; and $\vartheta(\mathbf{x}) = 0$ in some neighborhood of $\Omega \setminus \widetilde{\Omega}$. Then

$$\begin{aligned}\mathcal{A}_{\widetilde{\Omega}}[u] &\geq \varepsilon \mathcal{A}_{\widetilde{\Omega}}[u] + (1-\varepsilon) \int_{\widetilde{\Omega}} \vartheta^2(\mathbf{x}) \left(\langle a(\mathbf{x}) \nabla u(\mathbf{x}), \nabla u(\mathbf{x}) \rangle + V(\mathbf{x}) |u(\mathbf{x})|^2 \right) d\mathbf{x} \\ &= \varepsilon \mathcal{A}_{\widetilde{\Omega}}[u] + (1-\varepsilon) \mathcal{A}_{\widetilde{\Omega}}[\vartheta u] + (1-\varepsilon) \int_{\widetilde{\Omega}} \left(\vartheta^2 \langle a \nabla u, \nabla u \rangle - \langle a \nabla(\vartheta u), \nabla(\vartheta u) \rangle \right) d\mathbf{x}, \quad u \in H^1(\widetilde{\Omega}).\end{aligned} \quad (4.19)$$

The sum of the first two terms in the right-hand side of (4.19) determines an equivalent metric in $H^1(\widetilde{\Omega})$, and the last term is a compact form in $H^1(\widetilde{\Omega})$. Next, using that $\widetilde{b}(\mathbf{x}) = \vartheta^2(\mathbf{x}) \widetilde{b}(\mathbf{x})$, $\mathbf{x} \in \partial\widetilde{\Omega}$, we obtain

$$\mathcal{B}_{\partial\widetilde{\Omega}}[u] = \mathcal{B}_{\partial\widetilde{\Omega}}[\vartheta u] + \int_{\partial\widetilde{\Omega}} \widetilde{b}(\mathbf{x}) \left((\vartheta \mathbf{B}^{-1}_{\partial\widetilde{\Omega}} - \mathbf{B}^{-1}_{\partial\widetilde{\Omega}} \vartheta) u \right)(\mathbf{x}) \vartheta(\mathbf{x}) \overline{u(\mathbf{x})}\, dS(\mathbf{x}). \quad (4.20)$$

The second term in the right-hand side of (4.20) is the form of a pseudodifferential operator of order $-(2q+1)$, i.e., it is a lower order form. According to Lemmas 2.3 and 2.4, the values $\delta^\pm_\theta$ (4.9) do not exceed the similar values for the ratio

$$\pm \frac{\mathcal{B}_{\partial\widetilde{\Omega}}[\vartheta u]}{\varepsilon \mathcal{A}_{\widetilde{\Omega}}[u] + (1-\varepsilon) \mathcal{A}_{\widetilde{\Omega}}[\vartheta u]}, \quad u \in H^1(\widetilde{\Omega}). \quad (4.21)$$

For $u \in H^1(\widetilde{\Omega})$, by $\widehat{\mathcal{S}} u$ we denote a function coinciding with ϑu on $\widetilde{\Omega}$ and equal to zero on $\Omega \setminus \widetilde{\Omega}$. Then the operator $\widehat{\mathcal{S}} : H^1(\widetilde{\Omega}) \to H^1(\Omega)$ is bounded. For all $u \in H^1(\widetilde{\Omega})$, for which $\pm \mathcal{B}_{\partial\widetilde{\Omega}}[\vartheta u] > 0$, ratio (4.21) does not exceed $\pm \frac{\mathcal{B}_{\partial\widetilde{\Omega}}[\widehat{\mathcal{S}} u]}{(1-\varepsilon) \mathcal{A}_\Omega[\widehat{\mathcal{S}} u]}$. Then we are under the assumptions of Lemma 2.2, which implies $\delta^\pm_\theta$ (4.9) $\leq (1-\varepsilon)^{-\theta} \delta^\pm_\theta$ (4.10). Letting ε tend to zero, we arrive at the left inequality in (4.17). □

4.4. Asymptotics in the smooth case. According to Lemmas 4.2 and 4.3, we have

$$\delta_\theta^\pm\ (4.9) \le \delta_\theta^\pm\ (4.7) \le \Delta_\theta^\pm\ (4.7) \le \Delta_\theta^\pm\ (4.9), \quad \theta = \frac{m}{2q+1}. \tag{4.22}$$

Now, we establish an asymptotic formula for $N_\pm(\lambda,\ (4.9))$.

Lemma 4.4. *For $\lambda \to +0$, we have*

$$N_\pm(\lambda,\ (4.9)) \sim \frac{\lambda^{-\theta}}{m(2\pi)^m} \int_{\partial\widetilde{\Omega}} dS(\mathbf{x}) \int_{\boldsymbol{\xi}\perp\boldsymbol{\nu}(\mathbf{x}):\, |\boldsymbol{\xi}|=1} dS(\boldsymbol{\xi}) \left(\widetilde{b}_\pm(\mathbf{x}) B_{\partial\widetilde{\Omega}}^{-1}(\mathbf{x},\boldsymbol{\xi}) \widetilde{M}^{-1}(\mathbf{x},\boldsymbol{\xi})\right)^\theta, \quad \theta = \frac{m}{2q+1}. \tag{4.23}$$

Here $\widetilde{M}(\mathbf{x},\boldsymbol{\xi}) := \left(\langle a(\mathbf{x})\boldsymbol{\xi},\boldsymbol{\xi}\rangle\langle a(\mathbf{x})\boldsymbol{\nu}(\mathbf{x}),\boldsymbol{\nu}(\mathbf{x})\rangle - \langle a(\mathbf{x})\boldsymbol{\xi},\boldsymbol{\nu}(\mathbf{x})\rangle^2\right)^{1/2}$.

Proof. By $H^1(\widetilde{\Omega}, L)$ we denote the subspace in $H^1(\widetilde{\Omega})$ formed by solutions of the equation $Lu = 0$ in $\widetilde{\Omega}$. We have $H^1(\widetilde{\Omega}) = H^1(\widetilde{\Omega}, L) \oplus^{\mathcal{A}} H_0^1(\widetilde{\Omega})$. Here the orthogonal sum is understood in the sense of the inner product $\mathcal{A}_{\widetilde{\Omega}}[u,v]$. Since the form in the numerator of ratio (4.9) vanishes for $u \in H_0^1(\widetilde{\Omega})$, then the nonzero spectrum of ratio (4.9) will not change if we consider this ratio on $H^1(\widetilde{\Omega}, L)$.

Let G be the "Poisson operator" that takes a function $\varphi \in H^{1/2}(\partial\widetilde{\Omega})$ into the solution of the corresponding Dirichlet problem for the equation $Lu = 0$: the equality $u = G\varphi$ means that $u \in H^1(\widetilde{\Omega}, L)$, $u|_{\partial\widetilde{\Omega}} = \varphi$. The operator G is a homeomorphism between the spaces $H^{1/2}(\partial\widetilde{\Omega})$ and $H^1(\widetilde{\Omega}, L)$. The problem on the spectrum of ratio (4.9) is equivalent to the problem on the spectrum of the ratio

$$\pm\frac{\mathcal{B}_{\partial\widetilde{\Omega}}[\varphi]}{\mathcal{A}_{\widetilde{\Omega}}[G\varphi]}, \quad \varphi \in H^{1/2}(\partial\widetilde{\Omega}). \tag{4.24}$$

From the properties of the Boutet de Monvel algebra [14, 17] it follows that

$$\mathcal{A}_{\widetilde{\Omega}}[G\varphi] = (\mathcal{P}\varphi, \varphi), \quad \varphi \in H^{1/2}(\partial\widetilde{\Omega}), \tag{4.25}$$

where $\mathcal{P}$ is a classical pseudodifferential operator on $\partial\widetilde{\Omega}$ of order 1. Let us calculate the principal symbol of the operator $\mathcal{P}$, using the recipe from [11]. The principal symbol of the operator L is given by $L^\circ(\mathbf{x},\boldsymbol{\eta}) = a(\mathbf{x})\boldsymbol{\eta}\cdot\boldsymbol{\eta} = \sum_{j,l=1}^{m+1} a_{jl}(\mathbf{x})\eta_j\eta_l$. For each pair $(\mathbf{x},\boldsymbol{\xi})$, where $\mathbf{x} \in \partial\widetilde{\Omega}$, $0 \ne \boldsymbol{\xi} \perp \boldsymbol{\nu}(\mathbf{x})$, we should consider the ordinary differential equation $L^\circ(\mathbf{x}, \boldsymbol{\xi} + \boldsymbol{\nu}(\mathbf{x})D_t)f(t) = 0$, $t \in \mathbb{R}_+$. This equation takes the form

$$-\langle a(\mathbf{x})\boldsymbol{\nu}(\mathbf{x}),\boldsymbol{\nu}(\mathbf{x})\rangle\frac{d^2f(t)}{dt^2} - 2i\langle a(\mathbf{x})\boldsymbol{\xi},\boldsymbol{\nu}(\mathbf{x})\rangle\frac{df(t)}{dt} + \langle a(\mathbf{x})\boldsymbol{\xi},\boldsymbol{\xi}\rangle f(t) = 0.$$

Let $F(\mathbf{x},\boldsymbol{\xi})$ be the space of solutions of this equation, vanishing as $t \to +\infty$. This space is one-dimensional. We choose the basis function $Y(\mathbf{x},\boldsymbol{\xi};t)$ in it satisfying the condition $Y(\mathbf{x},\boldsymbol{\xi};0) = 1$. Then $Y(\mathbf{x},\boldsymbol{\xi};t) = e^{\kappa(\mathbf{x},\boldsymbol{\xi})t}$, where $\kappa(\mathbf{x},\boldsymbol{\xi}) = -\frac{\widetilde{M}(\mathbf{x},\boldsymbol{\xi}) + i\langle a(\mathbf{x})\boldsymbol{\xi},\boldsymbol{\nu}(\mathbf{x})\rangle}{\langle a(\mathbf{x})\boldsymbol{\nu}(\mathbf{x}),\boldsymbol{\nu}(\mathbf{x})\rangle}$. The principal symbol $p^\circ(\mathbf{x},\boldsymbol{\xi})$ of the pseudodifferential operator $\mathcal{P}$ is calculated by the rule

$$p^\circ(\mathbf{x},\boldsymbol{\xi}) = \sum_{i,j=1}^{m+1}\int_0^\infty a_{ij}(\mathbf{x})(\xi_i + \nu_i(\mathbf{x})D_t)Y(\mathbf{x},\boldsymbol{\xi};t)(\xi_j - \nu_j(\mathbf{x})D_t)\overline{Y(\mathbf{x},\boldsymbol{\xi};t)}\,dt.$$

A calculation shows that

$$p^\circ(\mathbf{x},\boldsymbol{\xi}) = \widetilde{M}(\mathbf{x},\boldsymbol{\xi}), \quad \mathbf{x} \in \partial\widetilde{\Omega}, \quad \boldsymbol{\xi} \perp \boldsymbol{\nu}(\mathbf{x}). \tag{4.26}$$

Obviously, $\mathcal{B}_{\partial\widetilde{\Omega}}[\varphi] = (\mathcal{Q}\varphi, \varphi)$, where $\mathcal{Q}$ is the pseudodifferential operator on $\partial\widetilde{\Omega}$ of order $(-2q)$ with the principal symbol

$$q^\circ(\mathbf{x},\boldsymbol{\xi}) = \widetilde{b}(\mathbf{x})B_{\partial\widetilde{\Omega}}^{-1}(\mathbf{x},\boldsymbol{\xi}), \quad \mathbf{x} \in \partial\widetilde{\Omega}, \quad \boldsymbol{\xi} \perp \boldsymbol{\nu}(\mathbf{x}). \tag{4.27}$$

Thus, ratio (4.24) coincides with the ratio of pseudodifferential forms

$$\pm\frac{(\mathcal{Q}\varphi,\varphi)}{(\mathcal{P}\varphi,\varphi)}, \quad \varphi \in H^{1/2}(\partial\widetilde{\Omega}). \tag{4.28}$$

We have proved that

$$N_\pm(\lambda,\ (4.9)) = N_\pm(\lambda,\ (4.24)) = N_\pm(\lambda,\ (4.28)), \quad \lambda > 0. \tag{4.29}$$

By Lemma 2.10, the spectrum counting functions for ratio (4.28) satisfy the following asymptotics for $\lambda \to +0$:

$$N_{\pm}(\lambda,\ (4.28)) \sim (2\pi)^{-m} \int_{\partial\widetilde{\Omega}} dS(\mathbf{x}) \int_{\boldsymbol{\xi}\perp\boldsymbol{\nu}(\mathbf{x})} d\boldsymbol{\xi}\, n_{\pm}(\lambda, \mathbf{x}, \boldsymbol{\xi};\ (4.31)), \tag{4.30}$$

where $n_{\pm}(\lambda, \mathbf{x}, \boldsymbol{\xi};\ (4.31))$ are the spectrum counting functions for the ratio of the (one-dimensional) forms

$$\pm\frac{q^{\circ}(\mathbf{x}, \boldsymbol{\xi})|z|^2}{p^{\circ}(\mathbf{x}, \boldsymbol{\xi})|z|^2}, \quad z \in \mathbb{C}. \tag{4.31}$$

Taking (4.26) and (4.27) into account, we obtain

$$n_{\pm}(\lambda, \mathbf{x}, \boldsymbol{\xi};\ (4.31)) = \begin{cases} 1, & \lambda < \widetilde{b}_{\pm}(\mathbf{x}) B^{-1}_{\partial\widetilde{\Omega}}(\mathbf{x}, \boldsymbol{\xi}) \widetilde{M}^{-1}(\mathbf{x}, \boldsymbol{\xi}), \\ 0, & \lambda \geq \widetilde{b}_{\pm}(\mathbf{x}) B^{-1}_{\partial\widetilde{\Omega}}(\mathbf{x}, \boldsymbol{\xi}) \widetilde{M}^{-1}(\mathbf{x}, \boldsymbol{\xi}). \end{cases} \tag{4.32}$$

Now, calculating asymptotics of the functions $N_{\pm}(\lambda,\ (4.28))$ by (4.30), (4.32) and taking (4.29) into account, we arrive at the required result (4.23). □

Completion of the proof of Theorem 4.1. By (4.22) and (4.23), taking into account that $\widetilde{b}(\mathbf{x}) = b(\mathbf{x})$ for $\mathbf{x} \in \operatorname{supp} b = \operatorname{supp} \widetilde{b} \subset \Gamma$ and $B_{\partial\widetilde{\Omega}}(\mathbf{x}, \boldsymbol{\xi}) = B(\mathbf{x}, \boldsymbol{\xi})$, $\widetilde{M}(\mathbf{x}, \boldsymbol{\xi}) = M(\mathbf{x}, \boldsymbol{\xi})$ for $\mathbf{x} \in \operatorname{supp} b$, $\boldsymbol{\xi} \perp \boldsymbol{\nu}(\mathbf{x})$, we obtain:

$$\Delta^{\pm}_{\theta}\ (4.7) = \delta^{\pm}_{\theta}\ (4.7) = \frac{1}{m(2\pi)^m} \int\limits_{\Gamma} dS(\mathbf{x}) \int\limits_{\boldsymbol{\xi}\perp\mathbf{n}(\mathbf{x}) \colon |\boldsymbol{\xi}|=1} dS(\boldsymbol{\xi})\ \left(b_{\pm}(\mathbf{x}) B^{-1}(\mathbf{x}, \boldsymbol{\xi}) M^{-1}(\mathbf{x}, \boldsymbol{\xi})\right)^{\theta}, \quad \theta = \frac{m}{2q+1},$$

for any $b \in C_0^{\infty}(\Gamma)$. By closure, this formula is valid for $b \in L_r(\Gamma)$; see Sec. 4.2. This completes the proof of Theorem 4.1. □

4.5. Application to the study of the spectrum of small oscillations of the capillary ideal fluid. Application of Theorem 4.1 allows us to solve the problem on the spectral asymptotics of small oscillations of the capillary ideal fluid (see [3, 23], and also [26, Chap. 4, § 1]). In this case $\Omega \subset \mathbb{R}^3$ (see Fig. 1) is the domain occupied by the fluid in a vessel in the equilibrium position; Γ is the equilibrium free surface of the fluid; $S = \partial\Omega \setminus \overline{\Gamma}$ is the solid wall of the vessel; $\gamma = \partial\Gamma$ is the wetting line. Let us first give a classical formulation of the problem assuming that the following condition is satisfied.

Condition 4.1. *Ω is a bounded domain in $\mathbb{R}^3$ with piecewise smooth boundary $\partial\Omega = \overline{\Gamma} \cup \overline{S}$, where Γ and S are smooth two-dimensional surfaces, the intersection of which is a smooth one-dimensional edge γ, and the inner angle at the edge is greater than zero and less than 2π (Γ, S and γ are not necessarily connected).*

Let $\mathbf{n}_0(\mathbf{x})$ be the unit outer normal vector to S at the point $\mathbf{x} \in S$, and let $\mathbf{n}(\mathbf{x})$ be the unit outer normal vector to Γ at the point $\mathbf{x} \in \Gamma$. Finally, let $\mathbf{l}(\mathbf{x})$ be the unit outer (with respect to Γ) normal vector to γ lying in the plane tangent to Γ at the point $\mathbf{x} \in \gamma$. Note that, after passing to the variational formulation of the problem, the smoothness requirement on S can be lifted (see Condition 4.2 below).

Let $\sigma > 0$ and let $h \in C^{\infty}(\overline{\Gamma})$, $\chi \in C^{\infty}(\gamma)$ be real-valued functions. Here σ is the surface tension coefficient (see [3]), the function h is related to the normal derivative of the potential of mass forces and to the principal curvatures of the surface Γ. It is assumed that

$$\begin{gathered} \int_{\Gamma} \left(\sigma|\nabla_{\Gamma} u(\mathbf{x})|^2 + h(\mathbf{x})|u(\mathbf{x})|^2\right) dS(\mathbf{x}) + \int_{\gamma} \sigma\chi(\mathbf{x})|u(\mathbf{x})|^2\, d\gamma \geq c \int_{\Gamma} |u(\mathbf{x})|^2\, dS(\mathbf{x}), \\ u \in H^1(\Gamma), \quad \int_{\Gamma} u(\mathbf{x})\, dS(\mathbf{x}) = 0; \quad c > 0. \end{gathered} \tag{4.33}$$

Condition (4.33) imposes a restriction on the problem data. Physically, it means that the equilibrium position of the fluid is stable. Denote $L_2(\Gamma) \ominus \{1\} := \left\{u \in L_2(\Gamma) : \int_{\Gamma} u\, dS = 0\right\}$. Let P be the orthogonal projection of the space $L_2(\Gamma)$ onto $L_2(\Gamma) \ominus \{1\}$. The quadratic form on the left-hand side of inequality (4.33) corresponds to the self-adjoint positive definite operator $\mathfrak{B}_{\Gamma}$ in the space $L_2(\Gamma) \ominus \{1\}$.

The operator $\mathfrak{B}_\Gamma$ is called the *operator of potential energy*. Let Δ_Γ be the Laplace–Beltrami operator on Γ. Then $\mathfrak{B}_\Gamma$ is given by the expression $P(-\sigma\Delta_\Gamma + h)$ on the domain

$$\operatorname{Dom}\mathfrak{B}_\Gamma = \Big\{u \in H^2(\Gamma) :\ \frac{\partial u}{\partial l} + \chi u = 0 \text{ on } \gamma,\ \int_\Gamma u\, dS = 0\Big\}.$$

The resolving operator $\mathfrak{B}_\Gamma^{-1}$ of the corresponding boundary-value problem on Γ is a compact operator on $L_2(\Gamma) \ominus \{1\}$.

The problem of normal oscillations of the capillary ideal fluid is reduced (see [3, Chap. 4, § 2]) to the spectral boundary-value problem:

$$\begin{gathered} \Delta\Phi = 0 \ \text{ in } \Omega, \quad \frac{\partial\Phi}{\partial n_0} = 0 \ \text{ on } S, \\ P(-\sigma\Delta_\Gamma + h)\frac{\partial\Phi}{\partial n} = \lambda^{-1}\Phi \ \text{ on } \Gamma, \quad \int_\Gamma \Phi\, dS = 0, \\ \frac{\partial}{\partial l}\left(\frac{\partial\Phi}{\partial n}\right) + \chi\frac{\partial\Phi}{\partial n} = 0 \ \text{ on } \gamma. \end{gathered} \tag{4.34}$$

Here $\Phi(\mathbf{x})$ is the amplitude of oscillations of the potential $\Phi(\mathbf{x}, t)$ of the fluid particles velocity field: $\Phi(\mathbf{x}, t) = \Phi(\mathbf{x})e^{i\omega t}$, where $\omega^{-1} = \sqrt{\lambda}$ and t is the time. The Laplace equation is the continuity equation, the condition on S is the impermeability condition, the third-type boundary condition on γ is the linearized condition for maintaining the contact angle during movement.

The boundary-value problem (4.34) is equivalent (see [3, Chap. 4, § 5]) to the problem of finding the successive maxima of the ratio of quadratic forms

$$\frac{\int_\Gamma (\mathfrak{B}_\Gamma^{-1}\Phi)\overline{\Phi}\, dS}{\int_\Omega |\nabla\Phi|^2\, d\mathbf{x}}, \quad \Phi \in H^1(\Omega), \quad \int_\Gamma \Phi\, dS = 0. \tag{4.35}$$

The Laplace equation in Ω and the condition on S are natural conditions in the variational problem on the spectrum of ratio (4.35). We consider ratio (4.35) under the following condition.

Condition 4.2. *Ω is a bounded domain in $\mathbb{R}^3$ such that $\Omega \in \mathcal{K}$; Γ is a smooth two-dimensional surface with smooth one-dimensional boundary γ and $\overline{\Gamma} \subset \partial\Omega$.*

Proposition 4.1. *Suppose that Condition* 4.2 *is satisfied. Let $\sigma > 0$. Suppose that $h \in C^\infty(\overline{\Gamma})$ and $\chi \in C^\infty(\gamma)$ are real-valued functions satisfying inequality* (4.33)*. Then the spectrum counting function for ratio* (4.35) *satisfies the following asymptotics for $\lambda \to +0$:*

$$N(\lambda,\ (4.35)) \sim \lambda^{-2/3}\frac{\operatorname{meas}\Gamma}{4\pi\sigma^{2/3}}. \tag{4.36}$$

Proof. Let $\widetilde{\mathfrak{B}}_\Gamma = \mathfrak{B}_\Gamma + CI$ be the operator in $L_2(\Gamma) \ominus \{1\}$ given by the expression $P(-\sigma\Delta_\Gamma + h + C)$ on the domain $\operatorname{Dom}\mathfrak{B}_\Gamma$. Let $\mathbf{B}_\Gamma$ be the operator on $L_2(\Gamma)$ given by the expression $-\sigma\Delta_\Gamma + h + C$ on the domain $\operatorname{Dom}\mathbf{B}_\Gamma = \Big\{u \in H^2(\Gamma) : \frac{\partial u}{\partial l} + \chi u = 0 \text{ on } \gamma\Big\}$. We assume that the constant C is so large that the operator $\mathbf{B}_\Gamma$ is positive definite. By the Hilbert identity, $\widetilde{\mathfrak{B}}_\Gamma^{-1} = (I + K)\mathfrak{B}_\Gamma^{-1}$, where $K = -C\widetilde{\mathfrak{B}}_\Gamma^{-1}$ is a compact self-adjoint operator in $L_2(\Gamma) \ominus \{1\}$, and K commutes with $\mathfrak{B}_\Gamma^{-1}$. Together with Lemma 2.4, this implies that the values $\Delta_{2/3}$ (4.35) and $\delta_{2/3}$ (4.35) coincide with the similar values for the ratio

$$\frac{\int_\Gamma (\widetilde{\mathfrak{B}}_\Gamma^{-1}\Phi)\overline{\Phi}\, dS}{\int_\Omega |\nabla\Phi|^2\, d\mathbf{x}}, \quad \Phi \in H^1(\Omega), \quad \int_\Gamma \Phi\, dS = 0. \tag{4.37}$$

By Lemma 2.6, the principal term of the spectral asymptotics of ratio (4.37) does not change if we consider this ratio on a subspace of finite codimension in $\mathfrak{H} = \{\Phi \in H^1(\Omega) :\ \int_\Gamma \Phi\, dS = 0\}$, namely, on the subspace $\mathfrak{G} := \Big\{\Phi \in H^1(\Omega) : \int_\Gamma \Phi\, dS = 0,\ \int_\Gamma \mathbf{B}_\Gamma^{-1}\Phi\, dS = 0\Big\}$. Note that $\widetilde{\mathfrak{B}}_\Gamma^{-1}\Phi = \mathbf{B}_\Gamma^{-1}\Phi$ for $\Phi \in \mathfrak{G}$.

Further, applying Lemma 2.3, we obtain that the values $\Delta_{2/3}$ (4.37) and $\delta_{2/3}$ (4.37) are the same as for the ratio

$$\frac{\int_\Gamma (\mathbf{B}_\Gamma^{-1}\Phi)\overline{\Phi}\,dS}{\int_\Omega \left(|\nabla\Phi|^2+|\Phi|^2\right)\,d\mathbf{x}}, \quad \Phi\in\mathfrak{G}. \tag{4.38}$$

Finally, by Lemma 2.6, the principal term of the spectral asymptotics does not change if (4.38) is replaced by the ratio

$$\frac{\int_\Gamma (\mathbf{B}_\Gamma^{-1}\Phi)\overline{\Phi}\,dS}{\int_\Omega \left(|\nabla\Phi|^2+|\Phi|^2\right)\,d\mathbf{x}}, \quad \Phi\in H^1(\Omega). \tag{4.39}$$

Applying Theorem 4.1 to ratio (4.39) and taking into account that $b(\mathbf{x})=1$, $B(\mathbf{x},\boldsymbol{\xi})=\sigma|\boldsymbol{\xi}|^2$ and $M(\mathbf{x},\boldsymbol{\xi})=|\boldsymbol{\xi}|$, we arrive at asymptotics (4.36). □

4.6. Small oscillations of the capillary stratified fluid. Theorem 4.1 also allows us to find the spectral asymptotics of small oscillations of the capillary stratified fluid. In the classical statement, we assume that Condition 4.1 is satisfied. The constant σ and the functions h and χ satisfy the same conditions as above. In addition, let $\rho\in C^\infty(\overline{\Omega})$ be a positive function, which has the meaning of the fluid density.

The problem of small oscillations of the capillary stratified fluid is reduced to the following spectral boundary-value problem:

$$\begin{gathered}
\operatorname{div}\left(\rho^{-1}\nabla\Phi\right)=0 \ \text{ in } \Omega, \quad \rho^{-1}\frac{\partial\Phi}{\partial n_0}=0 \ \text{ on } S,\\
P(-\sigma\Delta_\Gamma+h)\Big(\rho^{-1}\frac{\partial\Phi}{\partial n}\Big)=\lambda^{-1}\Phi \ \text{ on } \Gamma, \quad \int_\Gamma \Phi\,dS=0,\\
\frac{\partial}{\partial l}\Big(\rho^{-1}\frac{\partial\Phi}{\partial n}\Big)+\chi\rho^{-1}\frac{\partial\Phi}{\partial n}=0 \ \text{ on } \gamma.
\end{gathered} \tag{4.40}$$

The boundary-value problem (4.40) is equivalent to the variational problem on the spectrum of the form ratio

$$\frac{\int_\Gamma (\mathfrak{B}_\Gamma^{-1}\Phi)\overline{\Phi}\,dS}{\int_\Omega \rho^{-1}|\nabla\Phi|^2\,d\mathbf{x}}, \quad \Phi\in H^1(\Omega), \quad \int_\Gamma \Phi\,dS=0, \tag{4.41}$$

where the operator $\mathfrak{B}_\Gamma$ is the same as in Sec. 4.5. We consider the variational problem already under Condition 4.2.

By analogy with the proof of Proposition 4.1, it is easy to deduce the following statement from Theorem 4.1.

Proposition 4.2. *Suppose that Condition* 4.2 *is satisfied. Let $\sigma>0$. Suppose that $h\in C^\infty(\overline{\Gamma})$ and $\chi\in C^\infty(\gamma)$ are real-valued functions satisfying* (4.33)*. Let $\rho\in C^\infty(\overline{\Omega})$, $\rho(\mathbf{x})>0$. Then the spectrum counting function for ratio* (4.41) *satisfies the following asymptotics for $\lambda\to+0$:*

$$N(\lambda,\ (4.41))\sim\lambda^{-2/3}\frac{1}{4\pi\sigma^{2/3}}\int_\Gamma \rho^{2/3}(\mathbf{x})\,dS(\mathbf{x}).$$

4.7. An auxiliary problem of the theory of hydroelasticity. Theorem 4.1 finds application also in the theory of hydroelasticity. In this case, Γ has the meaning of the elastic bottom of the vessel. The following auxiliary problem of hydroelasticity corresponds to oscillations of the system in the case where the elastic bottom has zero mass:

$$\begin{gathered}
\Delta\Phi=0 \ \text{ in } \Omega, \quad \frac{\partial\Phi}{\partial n_0}=0 \ \text{ on } S,\\
P\Big(D\rho^{-1}\Delta_\Gamma^2\frac{\partial\Phi}{\partial n}\Big)=\lambda^{-1}\Phi \ \text{ on } \Gamma, \quad \int_\Gamma \Phi\,dS=0,\\
\frac{\partial\Phi}{\partial n}=0, \quad \frac{\partial}{\partial l}\Big(\frac{\partial\Phi}{\partial n}\Big)=0 \ \text{ on } \gamma.
\end{gathered} \tag{4.42}$$

Here Δ_Γ^2 is the biharmonic operator on Γ, conditions on γ are the rigid fixation conditions. The constant $D > 0$ is the coefficient of elasticity, the constant $\rho > 0$ is the fluid density[3].

By $\widehat{\mathfrak{B}}_\Gamma$ we denote the operator in $L_2(\Gamma) \ominus \{1\}$ given by the expression $PD\rho^{-1}\Delta_\Gamma^2$ on the domain $\operatorname{Dom} \widehat{\mathfrak{B}}_\Gamma = H^4(\Gamma) \cap H_0^2(\Gamma) \cap (L_2(\Gamma) \ominus \{1\})$. The operator $\widehat{\mathfrak{B}}_\Gamma$ is self-adjoint and positive definite. The boundary-value problem (4.42) corresponds to the variational problem on the spectrum of the ratio of quadratic forms

$$\frac{\int_\Gamma (\widehat{\mathfrak{B}}_\Gamma^{-1}\Phi)\overline{\Phi}\, dS}{\int_\Omega |\nabla \Phi|^2\, d\mathbf{x}}, \quad \Phi \in H^1(\Omega), \quad \int_\Gamma \Phi\, dS = 0. \tag{4.43}$$

Proposition 4.3. *Suppose that Condition* 4.2 *is satisfied. Let $D > 0$ and $\rho > 0$. Then the spectrum counting function for ratio* (4.43) *satifies the following asymptotics for $\lambda \to +0$:*

$$N(\lambda,\ (4.43)) \sim \lambda^{-2/5}\Big(\frac{\rho}{D}\Big)^{2/5}\frac{\operatorname{meas}\Gamma}{4\pi}. \tag{4.44}$$

Proof. Let $\widehat{\mathbf{B}}_\Gamma$ be the operator in $L_2(\Gamma)$ given by the expression $D\rho^{-1}\Delta_\Gamma^2$ on the domain $\operatorname{Dom}\widehat{\mathbf{B}}_\Gamma = H^4(\Gamma) \cap H_0^2(\Gamma)$. The operator $\widehat{\mathbf{B}}_\Gamma$ is selfa-djoint and positive definite. It is easily seen that the principal term of the spectral asymptotics will not change if we replace ratio (4.43) by the ratio

$$\frac{\int_\Gamma (\widehat{\mathbf{B}}_\Gamma^{-1}\Phi)\overline{\Phi}\, dS}{\int_\Omega (|\nabla \Phi|^2 + |\Phi|^2)\, d\mathbf{x}}, \quad \Phi \in H^1(\Omega). \tag{4.45}$$

Applying Theorem 4.1 to ratio (4.45), we arrive at asymptotics (4.44). □

4.8. Oscillations of a system of the capillary ideal fluids. The asymptotic formulas of the spectrum obtained above for the problems of small oscillations of a single fluid partially filling a vessel can be generalized to the case of oscillations of a system of immiscible fluids completely or partially filling a vessel. Let us consider as an example the problem of oscillations of a system of capillary ideal fluids completely filling a vessel (see Fig. 2).

Condition 4.3. *A bounded domain $\Omega \subset \mathbb{R}^3$ is divided into $(k+1)$ parts Ω_j, $j = 1, \dots, k+1$, and* (3.18) *is satisfied. The surfaces $\overline{\Gamma_j}$, $j = 1, \dots, k$, are defined in* (3.19). *We assume that Γ_i, $i = 1, \dots, k$, are smooth two-dimensional surfaces with smooth one-dimensional boundaries $\gamma_i = \partial\Gamma_i$, S is a smooth two-dimensional surface, and the angles between Γ_i and S are greater than zero and less than 2π. Moreover, S, Γ_i, and γ_i are not necessarily connected.*

As in the case of a single fluid, the smoothness requirement on S can be relaxed after the transition to variational formulation of the problem. Let $\mathbf{n}_0(\mathbf{x})$ be the unit outer normal vector to S at the point $\mathbf{x} \in S$; let $\mathbf{n}_j(\mathbf{x})$ be the unit outer (with respect to Ω_j) normal vector to Γ_j at the point $\mathbf{x} \in \Gamma_j$; and let $\mathbf{l}_j(\mathbf{x})$ be the unit outer (with respect to Γ_j) normal vector to γ_j at the point $\mathbf{x} \in \gamma_j$ lying in the plane tangent to Γ_j at the point $\mathbf{x}$.

The problem of normal oscillations of a system of capillary ideal fluids (see [3, Chap. 4, § 6]) is formulated for a system of functions $\{\Phi_j(\mathbf{x})\}$, $j = 1, \dots, k+1$, where Φ_j is a function in Ω_j:

$$\begin{gathered}
\Delta\Phi_j = 0 \text{ in } \Omega_j, \quad \frac{\partial \Phi_j}{\partial n_0} = 0 \text{ on } S_j, \quad j = 1, \dots, k+1, \\
\frac{\partial \Phi_j}{\partial n_j} = \frac{\partial \Phi_{j+1}}{\partial n_j} \text{ on } \Gamma_j, \quad j = 1, \dots, k, \\
P_j\left(-\sigma_j \Delta_{\Gamma_j} + h_j\right)\frac{\partial \Phi_j}{\partial n_j} = \lambda^{-1}\left(\rho_j \Phi_j - \rho_{j+1}\Phi_{j+1}\right) \text{ on } \Gamma_j, \quad j = 1, \dots, k, \\
\frac{\partial}{\partial l_j}\frac{\partial \Phi_j}{\partial n_j} + \chi_j \frac{\partial \Phi_j}{\partial n_j} = 0 \text{ on } \gamma_j, \quad j = 1, \dots, k, \\
\int_{\Gamma_j} (\rho_j \Phi_j - \rho_{j+1}\Phi_{j+1})\, dS = 0, \quad j = 1, \dots, k; \quad \sum\nolimits_{j=1}^{k+1} \int_{\Omega_j} \rho_j^{-1}\Phi_j\, d\mathbf{x} = 0.
\end{gathered} \tag{4.46}$$

[3]Boundary-value problems, in which the order of the operator in the boundary condition is higher than the order of the equation in the domain were discussed, e.g., in [29].

Here $\sigma_j > 0$ are constants, $h_j \in C^\infty(\overline{\Gamma_j})$ and $\chi_j \in C^\infty(\gamma_j)$ are real-valued functions. The constants $\rho_j > 0$ are the densities of the fluids. The operator P_j is the orthogonal projection of the space $L_2(\Gamma_j)$ onto $L_2(\Gamma_j) \ominus \{1\}$. By $\mathfrak{B}_j$ we denote the self-adjoint operator in the space $L_2(\Gamma_j) \ominus \{1\}$ given by the expression $P_j(-\sigma_j \Delta_{\Gamma_j} + h_j)$ on the domain $\operatorname{Dom} \mathfrak{B}_j = \big\{u \in H^2(\Gamma_j) : \frac{\partial u}{\partial l_j} + \chi_j u = 0$ on γ_j, $\int_{\Gamma_j} u\, dS = 0\big\}$. It is assumed that the operators $\mathfrak{B}_j$ are positive definite for all $j = 1, \dots, k$ (cf. (4.33)). Then the operators $\mathfrak{B}_j^{-1}$ are compact on $L_2(\Gamma_j) \ominus \{1\}$.

Problem (4.46) is equivalent to the variational problem on the spectrum of the form ratio

$$\begin{gathered} \frac{\sum_{j=1}^k \int_{\Gamma_j} (\mathfrak{B}_j^{-1}\Psi_j)\overline{\Psi_j}\, dS}{\sum_{j=1}^{k+1} \rho_j \int_{\Omega_j} |\nabla \Phi_j|^2\, d\mathbf{x}}, \quad \Phi_j \in H^1(\Omega_j), \quad j = 1, \dots, k+1; \\ \Psi_j := \rho_j \Phi_j - \rho_{j+1}\Phi_{j+1}, \quad \int_{\Gamma_j} \Psi_j\, dS = 0, \quad j = 1, \dots k; \quad \sum_{j=1}^{k+1} \int_{\Omega_j} \rho_j^{-1}\Phi_j\, d\mathbf{x} = 0. \end{gathered} \tag{4.47}$$

We consider the variational problem under the following condition.

Condition 4.4. *A bounded domain $\Omega \subset \mathbb{R}^3$ is divided by the smooth two-dimensional surfaces $\Gamma_1, \dots,$ Γ_k into $(k+1)$ disjoint domains $\Omega_1, \dots, \Omega_{k+1}$. Suppose that relations* (3.18) *and* (3.19) *are satisfied. Assume that $\Omega_j \in \mathcal{K}$, $j = 1, \dots, k+1$, and Γ_j, $j = 1, \dots, k$, are smooth two-dimensional surfaces with smooth boundaries $\gamma_j = \partial \Gamma_j$.*

Proposition 4.4. *Under the above assumptions, the spectrum counting function for ratio* (4.47) *satisfies the following asymptotics for $\lambda \to +0$:*

$$N(\lambda,\ (4.47)) \sim \lambda^{-2/3} \sum_{j=1}^k \Big(\frac{\rho_j + \rho_{j+1}}{\sigma_j}\Big)^{2/3} \frac{\operatorname{meas} \Gamma_j}{4\pi}.$$

By $\mathbf{B}_{\Gamma_j}$ we denote the self-adjoint operator in $L_2(\Gamma_j)$ given by $B_{\Gamma_j} = -\sigma_j \Delta_{\Gamma_j} + h_j(\mathbf{x}) + C_j$ on the domain $\operatorname{Dom} \mathbf{B}_{\Gamma_j} = \big\{u \in H^2(\Gamma_j) : \frac{\partial u}{\partial l_j} + \chi_j u = 0 \text{ on } \gamma_j\big\}$. The constants $C_j > 0$ are so large that the operators $\mathbf{B}_{\Gamma_j}$ are positive definite for all $j = 1, \dots, k$. For $\Phi = \{\Phi_j\}_{1 \le j \le k+1} \in \sum_{j=1}^{k+1} \oplus H^1(\Omega_j)$, we put

$$\begin{gathered} \mathcal{B}[\Phi] := \sum_{j=1}^k \operatorname{Re} \int_{\Gamma_j} b_j(\mathbf{x}) \big(\mathbf{B}_{\Gamma_j}^{-1}\Psi_j\big)\overline{\Psi_j} dS, \quad \Psi_j := \rho_j\Phi_j - \rho_{j+1}\Phi_{j+1}, \\ \mathcal{A}[\Phi] := \sum_{j=1}^{k+1} \rho_j \int_{\Omega_j} \big(|\nabla \Phi_j|^2 + |\Phi_j|^2\big) d\mathbf{x}. \end{gathered}$$

Here $b_j(\mathbf{x})$ are real-valued functions on Γ_j such that $b_j \in L_{4/3}(\Gamma_j)$, $j = 1, \dots, k$.

By analogy with the reasoning from the proof of Proposition 4.1, it is easy to show that the values $\Delta_{2/3}$ (4.47) and $\delta_{2/3}$ (4.47) for $b_j = 1$ $(j = 1, \dots, k)$ coincide with the similar values for the ratio

$$\pm \frac{\mathcal{B}[\Phi]}{\mathcal{A}[\Phi]}, \quad \Phi \in \sum_{j=1}^{k+1} \oplus H^1(\Omega_j). \tag{4.48}$$

Proposition 4.4 now follows from the following statement.

Proposition 4.5. *Suppose that the assumptions of Proposition* 4.4 *are satisfied. Let $b_j(\mathbf{x})$ be real-valued functions on Γ_j such that $b_j \in L_{4/3}(\Gamma_j)$. Then the spectrum counting functions for ratio* (4.48) *satisfy the following asymptotics for $\lambda \to +0$:*

$$N_\pm(\lambda,\ (4.48)) \sim \frac{\lambda^{-2/3}}{4\pi} \sum_{j=1}^k \Big(\frac{\rho_j + \rho_{j+1}}{\sigma_j}\Big)^{2/3} \int_{\Gamma_j} (b_j)_\pm^{2/3} dS.$$

Remark 4.1. It would be possible to obtain a generalization of Theorem 4.1 for the case of composite domains, and then Proposition 4.5 would be a special case. We will confine ourselves to discussing Proposition 4.5, so as not to go into details of the general formulation of the problem.

Proposition 4.5 is proved by the same way as Theorem 4.1. We outline the main steps omitting details (details of the proof can be found in [38]). By analogy with the proof of Lemma 4.1 it is not difficult to obtain estimates of the spectrum: $\Delta^{\pm}_{2/3}$ (4.48) $\le C\sum_{j=1}^k \|b_j\|^{2/3}_{L_{4/3}(\Gamma_j)}$. Together with Lemma 2.5, this allows us, when calculating the spectral asymptotics of ratio (4.48), to consider only the case where $b_j \in C_0^\infty(\Gamma_j)$, $j = 1, \dots, k$.

Further, in the case of smooth compactly supported coefficients b_j, ratio (4.48) is compared with similar ratios of forms defined on smooth domains. Let $\Omega_j^{\pm}$, $j = 1, \dots, k$, be disjoint domains in $\mathbb{R}^3$ with smooth boundaries such that $\Omega_j^- \subset \Omega_j$, $\Omega_j^+ \subset \Omega_{j+1}$, and $\operatorname{supp} b_j$ lies strictly inside the set $\partial\Omega_j^- \cap \partial\Omega_j^+$, $j = 1, \dots, k$. Then there are open sets $\widetilde{\Gamma}_j \subset \partial\Omega_j^- \cap \partial\Omega_j^+$ such that $\operatorname{supp} b_j \subset \widetilde{\Gamma}_j$. It can be assumed that $\widetilde{\Gamma}_j$ are two-dimensional surfaces with smooth boundaries.

Let $h_j^{\pm}(\mathbf{x})$ be smooth positive functions on $\partial\Omega_j^{\pm}$. Consider the operators $\mathbf{B}_{\partial\Omega_j^{\pm}}$ given by the expressions $B_{\partial\Omega_j^{\pm}} = -\sigma_j \Delta_{\partial\Omega_j^{\pm}} + h_j^{\pm}$ on the domains $\operatorname{Dom} \mathbf{B}_{\partial\Omega_j^{\pm}} = H^2(\partial\Omega_j^{\pm})$, $j = 1, \dots, k$. The inverse operators $\mathbf{B}^{-1}_{\partial\Omega_j^{\pm}}$ are pseudodifferential operators on $\partial\Omega_j^{\pm}$ of order (-2). For $\widehat{\Phi} = \{\Phi_j^-, \Phi_j^+\}_{1\le j\le k} \in \sum_{j=1}^k \oplus \Big(H^1(\Omega_j^-) \oplus H^1(\Omega_j^+)\Big) =: \widehat{H}$, we put

$$\widehat{\mathcal{B}}[\widehat{\Phi}] := \sum\nolimits_{j=1}^k \operatorname{Re} \int_{\widetilde{\Gamma}_j} b_j(\mathbf{x}) \big(\rho_j \mathbf{B}^{-1}_{\partial\Omega_j^-} \Phi_j^- - \rho_{j+1} \mathbf{B}^{-1}_{\partial\Omega_j^+} \Phi_j^+\big)\big(\rho_j \overline{\Phi_j^-} - \rho_{j+1} \overline{\Phi_j^+}\big)\, dS,$$

$$\widehat{\mathcal{A}}[\widehat{\Phi}] := \sum\nolimits_{j=1}^k \left(\rho_j \int_{\Omega_j^-} \big(|\nabla \Phi_j^-|^2 + |\Phi_j^-|^2\big)\, d\mathbf{x} + \rho_{j+1} \int_{\Omega_j^+} \big(|\nabla \Phi_j^+|^2 + |\Phi_j^+|^2\big)\, d\mathbf{x}\right).$$

We consider the form ratio

$$\pm \frac{\widehat{\mathcal{B}}[\widehat{\Phi}]}{\widehat{\mathcal{A}}[\widehat{\Phi}]}, \quad \widehat{\Phi} \in \widehat{H}. \tag{4.49}$$

By analogy with the proofs of Lemma 4.2 and Lemma 4.3, it can be verified that $\delta^{\pm}_{2/3}$ (4.49) $\le \delta^{\pm}_{2/3}$ (4.48) $\le \Delta^{\pm}_{2/3}$ (4.48) $\le \Delta^{\pm}_{2/3}$ (4.49).

It remains to establish the spectral asymptotics of ratio (4.49). Clearly, the problem decomposes into the orthogonal sum of k independent problems on the spectra of the ratios

$$\pm \frac{\widehat{\mathcal{B}}_j[\widehat{\Phi}_j]}{\widehat{\mathcal{A}}_j[\widehat{\Phi}_j]}, \quad \widehat{\Phi}_j = \{\Phi_j^-, \Phi_j^+\} \in \widehat{H}_j = H^1(\Omega_j^-) \oplus H^1(\Omega_j^+), \tag{4.50}$$

where

$$\widehat{\mathcal{B}}_j[\widehat{\Phi}_j] := \operatorname{Re} \int_{\widetilde{\Gamma}_j} b_j(\mathbf{x}) \big(\rho_j \mathbf{B}^{-1}_{\partial\Omega_j^-} \Phi_j^- - \rho_{j+1} \mathbf{B}^{-1}_{\partial\Omega_j^+} \Phi_j^+\big)\big(\rho_j \overline{\Phi_j^-} - \rho_{j+1} \overline{\Phi_j^+}\big)\, dS,$$

$$\widehat{\mathcal{A}}_j[\widehat{\Phi}_j] := \rho_j \int_{\Omega_j^-} \big(|\nabla \Phi_j^-|^2 + |\Phi_j^-|^2\big)\, d\mathbf{x} + \rho_{j+1} \int_{\Omega_j^+} \big(|\nabla \Phi_j^+|^2 + |\Phi_j^+|^2\big)\, d\mathbf{x}.$$

It is clear that the nonzero spectrum of ratio (4.50) will not change if we consider this ratio on the subspace $\widehat{H}_j(L_0) := H^1(\Omega_j^-, L_0) \oplus H^1(\Omega_j^+, L_0)$, $H^1(\Omega_j^{\pm}, L_0) := \{u \in H^1(\Omega_j^{\pm}) : L_0 u = -\Delta u + u = 0\}$. By $G_j^{\pm} : H^{1/2}(\partial\Omega_j^{\pm}) \to H^1(\Omega_j^{\pm}, L_0)$ we denote the Poisson operator solving the corresponding Dirichlet problem in $\Omega_j^{\pm}$ (cf. the definition of the operator G in Sec. 4.4). Putting $\Phi_j^{\pm} = G_j^{\pm} \varphi_j^{\pm}$ and using the properties of the Boutet de Monvel algebra, we obtain the representation

$$\widehat{\mathcal{A}}_j[\widehat{\Phi}_j] = \widehat{\mathcal{A}}_j[G_j^- \varphi_j^- \oplus G_j^+ \varphi_j^+] = \rho_j (\mathcal{P}_j^- \varphi_j^-, \varphi_j^-)_{L_2(\partial\Omega_j^-)} + \rho_{j+1} (\mathcal{P}_j^+ \varphi_j^+, \varphi_j^+)_{L_2(\partial\Omega_j^+)}, \quad \varphi_j^{\pm} \in H^{1/2}(\partial\Omega_j^{\pm}).$$

Here $\mathcal{P}_j^{\pm}$ are first-order positive definite pseudodifferential operators on $\partial\Omega_j^{\pm}$.

Substituting $\psi_j^- = \rho_j^{1/2} (\mathcal{P}_j^-)^{1/2} \varphi_j^-$, $\psi_j^+ = \rho_{j+1}^{1/2} (\mathcal{P}_j^+)^{1/2} \varphi_j^+$, we see that the problem on the spectrum of ratio (4.50) is equivalent to the problem on the spectrum of the ratio

$$\pm \frac{\mathcal{T}_j[\psi_j]}{\|\psi_j^-\|^2_{L_2(\partial\Omega_j^-)} + \|\psi_j^+\|^2_{L_2(\partial\Omega_j^+)}}, \quad \psi_j = \{\psi_j^-, \psi_j^+\} \in L_2(\partial\Omega_j^-) \oplus L_2(\partial\Omega_j^+), \tag{4.51}$$

where

$$\mathcal{T}_j[\psi_j] := \operatorname{Re}\int_{\widetilde{\Gamma}_j} b_j \left(\rho_j^{1/2}\mathbf{B}^{-1}_{\partial\Omega_j^-}(\mathcal{P}_j^-)^{-1/2}\psi_j^- - \rho_{j+1}^{1/2}\mathbf{B}^{-1}_{\partial\Omega_j^+}(\mathcal{P}_j^+)^{-1/2}\psi_j^+\right) \times \left(\rho_j^{1/2}\overline{(\mathcal{P}_j^-)^{-1/2}\psi_j^-} - \rho_{j+1}^{1/2}\overline{(\mathcal{P}_j^+)^{-1/2}\psi_j^+}\right) dS.$$

The problem on the spectrum of ratio (4.51) is reduced[4] to the problem on the spectrum of a matrix pseudodifferential operator of order $(-3/2)$ on $\partial\Omega_j^- \cap \partial\Omega_j^+$. Therefore, the spectral asymptotics of ratio (4.51) follows from the results of [8, 10]. This completes the proof of Proposition 4.5.

Remark 4.2. In the same way, one can obtain a result on the spectral asymptotics of small oscillations of a system of capillary ideal fluids in the case where the vessel is partially filled, as well as in the case where ρ_j are positive smooth functions in Ω_j (not constants).

5. The Spectral Asymptotics of Small Oscillations of the Heavy Viscous Fluid

In the present and the following sections, we study problems related to viscous fluid oscillations; for problem statements, see [3, 21–23].

Let $\Omega \subset \mathbb{R}^3$ be the domain occupied by the fluid in a vessel in the equilibrium position. The problems are posed for the vector-valued function $\mathbf{u}(\mathbf{x})$ having the meaning of the fluid particles velocity field and the scalar function $p(\mathbf{x})$ having the meaning of pressure. Statements are given in the form of spectral boundary-value problems in the domain Ω, as well as in the form of variational problems on the spectra of the ratios of quadratic forms. The functions $\mathbf{u}$ and p satisfy a homogeneous system, which is elliptic in the generalized sense, namely, the Stokes system; the spectral parameter is included into the boundary condition on the free surface Γ.

5.1. Statement of the problem and formulation of the result for the heavy viscous fluid. Suppose that Ω satisfies Condition 4.1. (After passing to the variational statement of the problem, the smoothness conditions on S and γ will be weakened.) Let $\mu > 0$ be a constant having the meaning of the viscosity coefficient. Consider a sesquilinear form

$$E_\Omega[\mathbf{u},\mathbf{v}] := \frac{1}{2}\sum\nolimits_{i,j=1}^{3}\int_\Omega \Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)\Big(\frac{\partial \overline{v_i}}{\partial x_j} + \frac{\partial \overline{v_j}}{\partial x_i}\Big)\, d\mathbf{x}, \quad \mathbf{u},\mathbf{v} \in H^1(\Omega;\mathbb{C}^3).$$

Note that $\rho\mu E_\Omega[\mathbf{u}]$ is the rate of energy dissipation in the entire volume of the fluid.

Let $\mathbf{n}(\mathbf{x})$ be the unit outer normal vector to $\partial\Omega$ at the point $\mathbf{x}$. By $u_n(\mathbf{x})$ we denote the normal component of the vector-valued function $\mathbf{u}(\mathbf{x})$ on the boundary: $u_n(\mathbf{x}) = \langle \mathbf{u}(\mathbf{x}), \mathbf{n}(\mathbf{x})\rangle$, $\mathbf{x} \in \partial\Omega$. By $\tau(\mathbf{x}) = \tau(\mathbf{u}(\mathbf{x}), p(\mathbf{x}))$ we denote the stress tensor in the fluid:

$$\tau_{ik}(\mathbf{x}) = \tau_{ik}(\mathbf{u}(\mathbf{x}), p(\mathbf{x})) := -p(\mathbf{x})\delta_{ik} + \mu\Big(\frac{\partial u_i(\mathbf{x})}{\partial x_k} + \frac{\partial u_k(\mathbf{x})}{\partial x_i}\Big), \quad i,k = 1,2,3.$$

Next, on Γ we define a vector field $\boldsymbol{\tau}_n(\mathbf{x})$ with the coordinates $\tau_{in}(\mathbf{x}) = \sum_{k=1}^{3} \tau_{ik}(\mathbf{x}) n_k(\mathbf{x})$, $i = 1,2,3$. Let $\boldsymbol{\tau}_{tn}(\mathbf{x})$ be the vector field tangent to Γ that is the tangent component of the field $\boldsymbol{\tau}_n(\mathbf{x})$. Let $\tau_{nn}(\mathbf{x}) = \langle \tau(\mathbf{x})\mathbf{n}(\mathbf{x}), \mathbf{n}(\mathbf{x})\rangle$ be the normal component of the field $\boldsymbol{\tau}_n(\mathbf{x})$.

For any sufficiently smooth functions $\mathbf{u}, p$ and $\mathbf{v}$, the Green formula is valid:

$$\mu E_\Omega[\mathbf{u},\mathbf{v}] = \int_\Omega \langle -\mu\Delta\mathbf{u} + \nabla p, \mathbf{v}\rangle\, d\mathbf{x} + \int_\Omega (-\mu\langle \nabla \operatorname{div}\mathbf{u}, \mathbf{v}\rangle + p \operatorname{div}\overline{\mathbf{v}})\, d\mathbf{x} + \int_{\partial\Omega} \langle \boldsymbol{\tau}_n(\mathbf{u},p), \mathbf{v}\rangle\, dS(\mathbf{x}). \quad (5.1)$$

Below we use the following notation

$$J^1(\Omega) := \{\mathbf{u} \in H^1(\Omega;\mathbb{C}^3) : \operatorname{div}\mathbf{u} = 0\}, \qquad J^1_S(\Omega) := \{\mathbf{u} \in J^1(\Omega) : \mathbf{u}|_S = 0\}. \quad (5.2)$$

[4]Difficulties related to the fact that $\partial\Omega_j^- \cap \partial\Omega_j^+$ is a manifold with boundary do not arise, since we are dealing with a negative order pseudodifferential operator.

The following spectral boundary-value problem is related to small oscillations of the heavy viscous fluid (see [23]):

$$\begin{gathered} -\mu\Delta\mathbf{u} + \nabla p = 0, \quad \operatorname{div}\mathbf{u} = 0 \ \text{ in } \Omega, \\ \mathbf{u} = 0 \ \text{ on } S, \quad \boldsymbol{\tau}_{tn} = 0 \ \text{ on } \Gamma, \\ a(\mathbf{x})u_n(\mathbf{x}) = \lambda\tau_{nn}(\mathbf{x}) \ \text{ on } \Gamma. \end{gathered} \tag{5.3}$$

The real-valued function $a \in C^\infty(\overline{\Gamma})$ is related to the normal derivative of the potential of mass forces.

To proceed to the variational statement, one should multiply the last equation in (5.3) by $\overline{u_n}$ and integrate over Γ. By the Green formula (5.1), taking conditions on $\mathbf{u}$ and p into account, from (5.3) we obtain: $\int_\Gamma \tau_{nn}\overline{u_n}\,dS = \mu E_\Omega[\mathbf{u}]$. Problem (5.3) is equivalent to the variational problem on the spectrum of the ratio of quadratic forms

$$\frac{\int_\Gamma a(\mathbf{x})|u_n(\mathbf{x})|^2\,dS(\mathbf{x})}{\mu E_\Omega[\mathbf{u}]}, \quad \mathbf{u} \in J^1_S(\Omega). \tag{5.4}$$

The first equation in (5.3) and the condition of zero tangential stresses on Γ are natural conditions in the variational problem on the spectrum of ratio (5.4).

We will study the spectrum of the form ratio (5.4) under the following condition on the domain Ω and the function $a(\mathbf{x})$.

Condition 5.1. *Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded domain such that $\Omega \in \mathcal{K}$. Suppose that Γ is a smooth two-dimensional surface with Lipschitz boundary $\gamma = \partial\Gamma$, and $\overline{\Gamma} \subset \partial\Omega$. Let $a \in L_2(\Gamma)$ be a real-valued function.*

Note that, in contrast to the case of the capillary ideal fluid (see Sec. 4), now the smoothness of γ is not needed. (In Sec. 3 we did not require the smoothness of γ either.)

General theorems on the conditions of coercivity of differential operators (see, e.g., [5, § 11]) imply the inequality

$$E_\Omega[\mathbf{u}] \geq c\|\mathbf{u}\|^2_{H^1(\Omega)} - C\|\mathbf{u}\|^2_{L_2(\Omega)}, \quad \mathbf{u} \in H^1(\Omega;\mathbb{C}^3), \quad c > 0. \tag{5.5}$$

Moreover, the requirements on the domain ensuring (5.5) are rather weak. If the sticking condition $\mathbf{u}|_S = 0$ is satisfied, then the Korn inequality holds (see [32]):

$$E_\Omega[\mathbf{u}] \geq c\|\mathbf{u}\|^2_{H^1(\Omega)}, \quad \mathbf{u} \in H^1(\Omega;\mathbb{C}^3), \ \ \mathbf{u}|_S = 0, \quad c > 0. \tag{5.6}$$

Now, we formulate the result on the spectral asymptotics of ratio (5.4).

Theorem 5.1. *Suppose that Condition* 5.1 *is satisfied. Then the spectrum counting functions for ratio* (5.4) *satisfy the following asymptotics*:

$$N_\pm(\lambda,\ (5.4)) \sim \lambda^{-2}\frac{1}{16\pi\mu^2}\int_\Gamma a^2_\pm(\mathbf{x})\,dS(\mathbf{x}), \quad \lambda \to +0.$$

5.2. Estimates of the spectrum.

Lemma 5.1. *Suppose that Condition* 5.1 *is satisfied. Then*

$$\Delta_2^\pm\ (5.4) \leq C\|a\|^2_{L_2(\Gamma)}, \tag{5.7}$$

where the constant C does not depend on the function a.

Proof. From (5.6) and the trace theorem it follows that $E_\Omega[\mathbf{u}] \geq \check{C}\|\mathbf{u}\|^2_{H^{1/2}(\Gamma)} \geq \check{C}\|u_n\|^2_{H^{1/2}(\Gamma)}$, $\mathbf{u} \in H^1(\Omega;\mathbb{C}^3)$, $\mathbf{u}|_S = 0$. By Lemma 2.2, the functions $N_\pm(\lambda,\ (5.4))$ do not exceed the spectrum counting functions for the ratio

$$\pm\frac{\int_\Gamma a(\mathbf{x})|v(\mathbf{x})|^2\,dS(\mathbf{x})}{\check{C}\mu\|v\|^2_{H^{1/2}(\Gamma)}}, \quad v \in H^{1/2}(\Gamma). \tag{5.8}$$

For $\Delta_2^\pm$ (5.8) the required estimate follows from Lemma 2.8. □

From Lemma 2.5 and inequality (5.7) it follows that the functionals $\Delta_2^\pm$ (5.4), $\delta_2^\pm$ (5.4) continuously depend on the coefficient a in $L_2(\Gamma)$. Therefore, it suffices to calculate the principal term of the spectral asymptotics of ratio (5.4) in the case where $a \in C_0^\infty(\Gamma)$.

5.3. Comparison with a problem in a smooth domain. *Below it is assumed that* $a \in C_0^\infty(\Gamma)$. Let $\widetilde{\Omega}$ be a bounded domain with smooth boundary such that $\widetilde{\Omega} \subset \Omega$, and let $\operatorname{supp} a$ lie strictly inside the set $\partial\widetilde{\Omega} \cap \Gamma$. Then there exists an open subset $\widetilde{\Gamma}$ of the boundary $\partial\widetilde{\Omega}$ lying strictly inside $\partial\widetilde{\Omega} \cap \Gamma$ and such that $\operatorname{supp} a \subset \widetilde{\Gamma}$. We can assume that the boundary of $\widetilde{\Gamma}$ is sufficiently smooth. Let $\widetilde{a} \in C^\infty(\partial\widetilde{\Omega})$ be the function equal to $a(\mathbf{x})$ for $\mathbf{x} \in \partial\widetilde{\Omega} \cap \Gamma$ and equal to zero for $\mathbf{x} \in \partial\widetilde{\Omega} \setminus \Gamma$.

To estimate $\Delta_2^\pm$ (5.4) from above, we consider the ratio

$$\pm\frac{\int_{\partial\widetilde{\Omega}} \widetilde{a}(\mathbf{x})|u_\nu(\mathbf{x})|^2\, dS(\mathbf{x})}{\mu E_{\widetilde{\Omega}}[\mathbf{u}] + C\|\mathbf{u}\|^2_{L_2(\widetilde{\Omega})}}, \quad \mathbf{u} \in J^1(\widetilde{\Omega}). \tag{5.9}$$

Here $u_\nu(\mathbf{x})$ is the normal component of the function $\mathbf{u}$ on $\partial\widetilde{\Omega}$, and the constant C is so large that the form in the denominator determines an equivalent metric in $H^1(\Omega;\mathbb{C}^3)$; see (5.5). Let $\mathcal{S}$ be the operator of restriction of functions $\mathbf{u} \in H^1(\Omega;\mathbb{C}^3)$ onto $\widetilde{\Omega}$. Then $\mathcal{S}$ takes $J^1_S(\Omega)$ to $J^1(\widetilde{\Omega})$. For any $\mathbf{u} \in J^1_S(\Omega)$ such that $\pm\int_\Gamma a(\mathbf{x})|u_n|^2\, dS > 0$, we have

$$\pm\frac{\int_\Gamma a(\mathbf{x})|u_n|^2\, dS}{\mu E_\Omega[\mathbf{u}] + C\|\mathbf{u}\|^2_{L_2(\Omega)}} \le \pm\frac{\int_{\partial\widetilde{\Omega}} \widetilde{a}(\mathbf{x})|(\mathcal{S}\mathbf{u})_\nu|^2\, dS}{\mu E_{\widetilde{\Omega}}[\mathcal{S}\mathbf{u}] + C\|\mathcal{S}\mathbf{u}\|^2_{L_2(\widetilde{\Omega})}}.$$

By Lemmas 2.3 and 2.2,

$$\Delta_2^\pm\ (5.4) \le \Delta_2^\pm\ (5.9). \tag{5.10}$$

To estimate $N_\pm(\lambda,\ (5.4))$ from below, we consider the ratio

$$\pm\frac{\int_{\partial\widetilde{\Omega}} \widetilde{a}(\mathbf{x})|u_\nu(\mathbf{x})|^2\, dS(\mathbf{x})}{\mu E_{\widetilde{\Omega}}[\mathbf{u}]}, \quad \mathbf{u} \in J^1_{\widetilde{S}}(\widetilde{\Omega}). \tag{5.11}$$

Here $\widetilde{S} := \partial\widetilde{\Omega} \setminus \widetilde{\Gamma}$ and $J^1_{\widetilde{S}}(\widetilde{\Omega}) := \{\mathbf{u} \in J^1(\widetilde{\Omega}) :\ \mathbf{u}|_{\widetilde{S}} = 0\}$. By the Korn inequality, $E_{\widetilde{\Omega}}[\mathbf{u}] \ge c\|\mathbf{u}\|^2_{H^1(\widetilde{\Omega})}$, $\mathbf{u} \in J^1_{\widetilde{S}}(\widetilde{\Omega})$, $c > 0$.

Let Π be the operator of extension of functions in $\widetilde{\Omega}$ by zero to $\Omega \setminus \widetilde{\Omega}$. Let us check that Π is a linear continuous operator from $J^1_{\widetilde{S}}(\widetilde{\Omega})$ to $J^1_S(\Omega)$. Fix a function $\zeta \in C^\infty(\overline{\widetilde{\Omega}})$ such that $\zeta(\mathbf{x}) = 1$ for $\mathbf{x} \in \widetilde{\Gamma}$ and $\zeta(\mathbf{x}) = 0$ in some neighborhood of $\partial\widetilde{\Omega} \setminus \Gamma$. Let $\mathbf{u} \in J^1_{\widetilde{S}}(\widetilde{\Omega})$. Obviously, $\Pi(\zeta\mathbf{u}) \in H^1(\Omega;\mathbb{C}^3)$. Next, $(1-\zeta)\mathbf{u} \in H^1(\widetilde{\Omega};\mathbb{C}^3)$ and $(1-\zeta)\mathbf{u} = 0$ on $\partial\widetilde{\Omega}$. Then we have $\Pi\left((1-\zeta)\mathbf{u}\right) \in H^1(\Omega;\mathbb{C}^3)$; see [28] or [36]. Thus, $\Pi\mathbf{u} = \Pi\left(\zeta\mathbf{u}\right) + \Pi\left((1-\zeta)\mathbf{u}\right) \in H^1(\Omega;\mathbb{C}^3)$. Obviously, $\Pi\mathbf{u} = 0$ on S. The condition $\operatorname{div}\Pi\mathbf{u} = 0$ in Ω is satisfied, because $\operatorname{div}\mathbf{u} = 0$ in $\widetilde{\Omega}$, $\Pi\mathbf{u} = 0$ in $\Omega \setminus \widetilde{\Omega}$, and $\Pi\mathbf{u} \in H^1(\Omega;\mathbb{C}^3)$. Thus, $\Pi\mathbf{u} \in J^1_S(\Omega)$.

We have

$$\pm\frac{\int_{\partial\widetilde{\Omega}} \widetilde{a}(\mathbf{x})|u_\nu(\mathbf{x})|^2\, dS(\mathbf{x})}{\mu E_{\widetilde{\Omega}}[\mathbf{u}]} = \pm\frac{\int_\Gamma a(\mathbf{x})|(\Pi\mathbf{u})_n(\mathbf{x})|^2\, dS(\mathbf{x})}{\mu E_\Omega[\Pi\mathbf{u}]}, \quad \mathbf{u} \in J^1_{\widetilde{S}}(\widetilde{\Omega}).$$

Applying Lemma 2.2, we obtain that

$$N_\pm(\lambda,\ (5.11)) \le N_\pm(\lambda,\ (5.4)), \quad \lambda > 0. \tag{5.12}$$

5.4. Asymptotics of the spectrum of the problem for the heavy viscous fluid in the smooth case. Inequlities (5.10) and (5.12) show that Theorem 5.1 will be proved if we establish the following lemma.

Lemma 5.2. *For* $\lambda \to +0$ *the following asymptotic formulas hold*:

$$N_\pm(\lambda,\ (5.9)) \sim N_\pm(\lambda,\ (5.11)) \sim \lambda^{-2}\frac{1}{16\pi\mu^2}\int_{\partial\widetilde{\Omega}} \widetilde{a}^2_\pm(\mathbf{x})\, dS(\mathbf{x}). \tag{5.13}$$

Proof. By inequality (5.5) for the domain $\widetilde{\Omega}$, the space

$$Z_0 := \left\{\mathbf{u} \in H^1(\widetilde{\Omega};\mathbb{C}^3) : \operatorname{div}\mathbf{u} = 0,\ E_{\widetilde{\Omega}}[\mathbf{u}] = 0\right\}$$

is finite-dimensional. By Z we denote the set of the traces of functions from Z_0 on $\partial\widetilde{\Omega}$. Then Z is a finite-dimensional subspace of $L_2(\partial\widetilde{\Omega};\mathbb{C}^3)$. We put $W(\widetilde{\Omega}) := \left\{\mathbf{u} \in J^1(\widetilde{\Omega}) : (\mathbf{u},\boldsymbol{\varphi})_{L_2(\partial\widetilde{\Omega})} = 0,\ \forall\boldsymbol{\varphi} \in Z\right\}$. Note that if $\mathbf{u} \in H^1_0(\widetilde{\Omega};\mathbb{C}^3)$ and $E_{\widetilde{\Omega}}[\mathbf{u}] = 0$, then $\mathbf{u} = 0$. Hence, if $\mathbf{u} \in W(\widetilde{\Omega})$ and $E_{\widetilde{\Omega}}[\mathbf{u}] = 0$, then $\mathbf{u} = 0$.

In a standard way, this implies that $E_{\widetilde{\Omega}}[\mathbf{u}] \asymp \|\mathbf{u}\|^2_{H^1(\widetilde{\Omega})}$ for $\mathbf{u} \in W(\widetilde{\Omega})$ (the symbol $\asymp$ is understood as two-sided estimates with some constants).

Consider the form ratio

$$\pm \frac{\int_{\partial\widetilde{\Omega}} \widetilde{a}(\mathbf{x})|u_\nu(\mathbf{x})|^2\, dS(\mathbf{x})}{\mu E_{\widetilde{\Omega}}[\mathbf{u}]}, \quad \mathbf{u} \in W(\widetilde{\Omega}). \tag{5.14}$$

Applying Lemmas 2.3 and 2.6, we obtain

$$\Delta_2^\pm\ (5.9) = \Delta_2^\pm\ (5.14), \quad \delta_2^\pm\ (5.9) = \delta_2^\pm\ (5.14). \tag{5.15}$$

We put $W_0(\widetilde{\Omega}) := \{\mathbf{u} \in H_0^1(\widetilde{\Omega}) : \operatorname{div} \mathbf{u} = 0\}$. Let $W(\widetilde{\Omega}, \mathcal{L})$ be the subspace of $W(\widetilde{\Omega})$ formed by the solutions of the Stokes system, i.e.,

$$W(\widetilde{\Omega}, \mathcal{L}) := \{\mathbf{u} \in W(\widetilde{\Omega}) :\ \text{there exists } p \in L_2(\widetilde{\Omega}) \text{ such that }\ \mathcal{L}\{\mathbf{u}, p\} = 0 \text{ in } \widetilde{\Omega}\}.$$

Here $\mathcal{L}\{\mathbf{u}, p\} := \{-\mu\Delta\mathbf{u} + \nabla p, \operatorname{div}\mathbf{u}\}$. The relation $\mathcal{L}\{\mathbf{u}, p\} = 0$ is understood in the sense of distributions.

We have the following orthogonal decomposition: $W(\widetilde{\Omega}) = W_0(\widetilde{\Omega}) \oplus^E W(\widetilde{\Omega}, \mathcal{L})$, where orthogonality is understood in the sense of the inner product $E_{\widetilde{\Omega}}[\mathbf{u}, \mathbf{v}]$. Since ratio (5.14) vanishes on $W_0(\widetilde{\Omega})$, then the nonzero spectrum of ratio (5.14) will not change if we consider this ratio on $W(\widetilde{\Omega}, \mathcal{L})$.

Similarly, the nonzero spectrum of ratio (5.11) will not change if we consider this ratio on the subspace $H^1_{\widetilde{S}}(\widetilde{\Omega}, \mathcal{L}) := \{\mathbf{u} \in H^1(\widetilde{\Omega}; \mathbb{C}^3) : \mathcal{L}\{\mathbf{u}, p\} = 0 \text{ for some } p \in L_2(\widetilde{\Omega}),\ \mathbf{u} = 0 \text{ on } \widetilde{S}\}$.

Let $\mathcal{G}$ be the operator taking a vector-valued function $\boldsymbol{\varphi} \in H^{1/2}(\partial\widetilde{\Omega}; \mathbb{C}^3)$, for which $\int_{\partial\widetilde{\Omega}} \varphi_\nu\, dS = 0$, to the solution of the first boundary-value problem for the Stokes system (see [43]): the relation $\{\mathbf{u}, p\} = \mathcal{G}\boldsymbol{\varphi}$ means that the pair of functions $\mathbf{u} \in H^1(\widetilde{\Omega}; \mathbb{C}^3)$, $p \in L_2(\widetilde{\Omega})$ is a weak solution of the boundary-value problem $\mathcal{L}\{\mathbf{u}, p\} = 0$ in $\widetilde{\Omega}$, $\mathbf{u} = \boldsymbol{\varphi}$ on $\partial\widetilde{\Omega}$.

The operator $\mathcal{G}$ is a homeomorphism of the space $\{\boldsymbol{\varphi} \in H^{1/2}(\partial\widetilde{\Omega}; \mathbb{C}^3) : \int_{\partial\widetilde{\Omega}} \varphi_\nu\, dS = 0\}$ and the space $\{\{\mathbf{u}, p\} \in H^1(\widetilde{\Omega}; \mathbb{C}^3) \times (L_2(\widetilde{\Omega})/\{1\}) : \mathcal{L}\{\mathbf{u}, p\} = 0\}$. Here $L_2(\widetilde{\Omega})/\{1\}$ is the quotient space of $L_2(\widetilde{\Omega})$ by the one-dimensional subspace of constants. As noted in [18], $\mathcal{G}$ is a Poisson operator from the Boutet de Monvel algebra. From the properties of this algebra it follows that

$$E_{\widetilde{\Omega}}[\mathbf{u}] = (\mathcal{E}\boldsymbol{\varphi}, \boldsymbol{\varphi})_{L_2(\partial\widetilde{\Omega})}, \quad \{\mathbf{u}, p\} = \mathcal{G}\boldsymbol{\varphi}, \tag{5.16}$$

where $\mathcal{E}$ is a matrix first-order pseudodifferential operator. Up to lower order terms, representation (5.16) can also be obtained from the considerations generalizing the arguments from [11] to the case of systems elliptic in the sense of Douglis–Nirenberg. Similarly to (5.16), we have

$$\|\mathbf{u}\|^2_{L_2(\widetilde{\Omega})} = (\mathcal{Q}_0\boldsymbol{\varphi}, \boldsymbol{\varphi})_{L_2(\partial\widetilde{\Omega})}, \quad \{\mathbf{u}, p\} = \mathcal{G}\boldsymbol{\varphi}, \tag{5.17}$$

where $\mathcal{Q}_0$ is a matrix pseudodifferential operator of order (-1). From (5.16), (5.17) and inequality (5.5) for $\widetilde{\Omega}$ it follows that

$$((\mathcal{E} + C\mathcal{Q}_0)\boldsymbol{\varphi}, \boldsymbol{\varphi})_{L_2(\partial\widetilde{\Omega})} \geq c\|\boldsymbol{\varphi}\|^2_{H^{1/2}(\partial\widetilde{\Omega})}, \quad \boldsymbol{\varphi} \in H^{1/2}(\partial\widetilde{\Omega}; \mathbb{C}^3), \quad \int_{\partial\widetilde{\Omega}} \varphi_\nu\, dS = 0, \quad c > 0. \tag{5.18}$$

Inequality (5.18) shows that, changing the lower order terms in the pseudodifferential operator $\mathcal{E}$ if necessary, we can assume that

$$(\mathcal{E}\boldsymbol{\varphi}, \boldsymbol{\varphi})_{L_2(\partial\widetilde{\Omega})} \geq c\|\boldsymbol{\varphi}\|^2_{H^{1/2}(\partial\widetilde{\Omega})}, \quad \boldsymbol{\varphi} \in H^{1/2}(\partial\widetilde{\Omega}; \mathbb{C}^3), \quad c > 0. \tag{5.19}$$

Assuming that (5.19) is satisfied, consider the form ratios

$$\pm \frac{\int_{\partial\widetilde{\Omega}} \widetilde{a}(\mathbf{x})|\varphi_\nu(\mathbf{x})|^2\, dS(\mathbf{x})}{\mu(\mathcal{E}\boldsymbol{\varphi}, \boldsymbol{\varphi})_{L_2(\partial\widetilde{\Omega})}}, \quad \boldsymbol{\varphi} \in H^{1/2}(\partial\widetilde{\Omega}; \mathbb{C}^3), \tag{5.20}$$

$$\pm \frac{\int_{\partial\widetilde{\Omega}} \widetilde{a}(\mathbf{x})|\varphi_\nu(\mathbf{x})|^2\, dS(\mathbf{x})}{\mu(\mathcal{E}\boldsymbol{\varphi}, \boldsymbol{\varphi})_{L_2(\partial\widetilde{\Omega})}}, \quad \boldsymbol{\varphi} \in H^{1/2}(\partial\widetilde{\Omega}; \mathbb{C}^3), \quad \boldsymbol{\varphi}|_{\widetilde{S}} = 0. \tag{5.21}$$

Summarizing all that has been said and applying Lemmas 2.3 and 2.6, we obtain that

$$\Delta_2^\pm\ (5.14) = \Delta_2^\pm\ (5.20), \quad \delta_2^\pm\ (5.14) = \delta_2^\pm\ (5.20), \tag{5.22}$$

$$\Delta_2^\pm\ (5.11) = \Delta_2^\pm\ (5.21), \quad \delta_2^\pm\ (5.11) = \delta_2^\pm\ (5.21). \tag{5.23}$$

Ratios (5.20) and (5.21) are ratios of pseudodifferential forms like (2.17) and (2.19), respectively. The spectral asymptotic formulas for them follow from Lemma 2.10 and Lemma 2.11. We need to calculate the principal symbols of the corresponding pseudodifferential operators. Denote by $\mathcal{Q}$ the matrix pseudodifferential operator on $\partial\widetilde{\Omega}$, corresponding to the form in the numerator of (5.20) and (5.21): $\int_{\partial\widetilde{\Omega}} \widetilde{a}(\mathbf{x})|\varphi_\nu(\mathbf{x})|^2\, dS(\mathbf{x}) = (\mathcal{Q}\boldsymbol{\varphi},\boldsymbol{\varphi})_{L_2(\partial\widetilde{\Omega})}$. Locally, in a neighborhood $\mathcal{U}$ of some point $\mathbf{x}_0 \in \partial\widetilde{\Omega}$, we choose a curvilinear orthogonal coordinate system so that the coordinate lines for the third coordinate on $\partial\widetilde{\Omega}$ are directed along the inner normal $\boldsymbol{\nu} = \boldsymbol{\nu}(\mathbf{x})$, and the corresponding Lamé coefficient on $\partial\widetilde{\Omega}$ is equal to 1. With this choice of the coordinate system, the symbol of the pseudodifferential operator $\mathcal{Q}$ is given by

$$q^\circ(\mathbf{x},\boldsymbol{\xi}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \widetilde{a}(\mathbf{x}) \end{pmatrix}, \quad \mathbf{x} \in \mathcal{U}, \quad \boldsymbol{\xi} \perp \boldsymbol{\nu}(\mathbf{x}). \tag{5.24}$$

Let $e^\circ(\mathbf{x},\boldsymbol{\xi})$ be the principal symbol of the pseudodifferential operator $\mathcal{E}$ calculated in the same local coordinates. Consider the algebraic problem

$$q^\circ(\mathbf{x},\boldsymbol{\xi})\mathbf{z} = \lambda\mu e^\circ(\mathbf{x},\boldsymbol{\xi})\mathbf{z}, \quad \mathbf{z} \in \mathbb{C}^3. \tag{5.25}$$

By Lemma 2.10, for $\lambda \to +0$ we have

$$N_\pm(\lambda,\ (5.20)) \sim \frac{1}{(2\pi)^2} \int_{\partial\widetilde{\Omega}} dS(\mathbf{x}) \int_{\boldsymbol{\xi}\perp\boldsymbol{\nu}(\mathbf{x})} d\boldsymbol{\xi}\, n_\pm(\lambda,\mathbf{x},\boldsymbol{\xi};\ (5.25)). \tag{5.26}$$

Lemma 2.11 implies the following asymptotics for $\lambda \to +0$:

$$N_\pm(\lambda,\ (5.21)) \sim \frac{1}{(2\pi)^2} \int_{\widetilde{\Gamma}} dS(\mathbf{x}) \int_{\boldsymbol{\xi}\perp\boldsymbol{\nu}(\mathbf{x})} d\boldsymbol{\xi}\, n_\pm(\lambda,\mathbf{x},\boldsymbol{\xi};\ (5.25)). \tag{5.27}$$

We calculate the symbol $e^\circ(\mathbf{x},\boldsymbol{\xi})$ according to the rules from [18]. We write down the principal symbol $L^\circ(\mathbf{x},\boldsymbol{\eta}) = \{L^\circ_{sj}(\mathbf{x},\boldsymbol{\eta})\}_{1\le s,j\le 4}$ of the operator $\mathcal{L}$: $L^\circ_{sj}(\mathbf{x},\boldsymbol{\eta}) = \mu|\boldsymbol{\eta}|^2\delta_{sj}$, $s,j = 1,2,3$; $L^\circ_{s4}(\mathbf{x},\boldsymbol{\eta}) = \overline{L^\circ_{4s}(\mathbf{x},\boldsymbol{\eta})} = i\eta_s$, $s = 1,2,3$; $L^\circ_{44}(\mathbf{x},\boldsymbol{\eta}) = 0$.

Further, for each point $(\mathbf{x},\boldsymbol{\xi})$, $\mathbf{x} \in \partial\widetilde{\Omega}$, $\boldsymbol{\xi} \perp \boldsymbol{\nu}(\mathbf{x})$, we consider the following system of ordinary differential equations on the semiaxis:

$$\sum_{j=1}^4 L^\circ_{sj}(\mathbf{x},\boldsymbol{\xi} + \boldsymbol{\nu}(\mathbf{x})D_t)f_j(t) = 0, \quad s = 1,2,3,4; \quad t \in \mathbb{R}_+. \tag{5.28}$$

With our choice of the coordinate system, we have $\xi_3 = 0$, $\nu_1 = \nu_2 = 0$, $\nu_3 = 1$, and system (5.28) takes the form

$$\begin{gathered} \mu\Big(|\boldsymbol{\xi}|^2 - \frac{d^2}{dt^2}\Big)f_j(t) + i\xi_j f_4(t) = 0, \quad j = 1,2, \\ \mu\Big(|\boldsymbol{\xi}|^2 - \frac{d^2}{dt^2}\Big)f_3(t) + \frac{d}{dt}f_4(t) = 0, \quad i\xi_1 f_1(t) + i\xi_2 f_2(t) + \frac{d}{dt}f_3(t) = 0. \end{gathered} \tag{5.29}$$

By $F(\mathbf{x},\boldsymbol{\xi})$ we denote the linear space of the solutions of system (5.29) vanishing as $t \to +\infty$. The characteristic determinant of system (5.29) is equal to $D(k) = \mu^2(|\boldsymbol{\xi}|^2 - k^2)^3$. The solution vanishing as $t \to +\infty$ corresponds to the threefold root $k = -|\boldsymbol{\xi}|$. Hence, the space $F(\mathbf{x},\boldsymbol{\xi})$ is three-dimensional. We consider a basis in $F(\mathbf{x},\boldsymbol{\xi})$ consisting of vector-valued functions $Y^{(j)}(\mathbf{x},\boldsymbol{\xi},t) = \{Y_i^{(j)}(\mathbf{x},\boldsymbol{\xi},t)\}_{1\le i\le 4}$, $j = 1,2,3$, which are the solutions of system (5.29) vanishing as $t \to +\infty$ and satisfying the initial conditions $Y_i^{(j)}(\mathbf{x},\boldsymbol{\xi},0) = \delta_{ij}$, $i,j = 1,2,3$. Calculations show that $Y^{(j)}$ do not depend on $\mathbf{x}$ and are given by

$$Y^{(1)}(\boldsymbol{\xi},t) = \begin{pmatrix} -\frac{\xi_1^2 t}{|\boldsymbol{\xi}|} + 1 \\ -\frac{\xi_1\xi_2 t}{|\boldsymbol{\xi}|} \\ -i\xi_1 t \\ -2\mu i\xi_1 \end{pmatrix} e^{-|\boldsymbol{\xi}|t}, \quad Y^{(2)}(\boldsymbol{\xi},t) = \begin{pmatrix} -\frac{\xi_1\xi_2 t}{|\boldsymbol{\xi}|} \\ -\frac{\xi_2^2 t}{|\boldsymbol{\xi}|} + 1 \\ -i\xi_2 t \\ -2\mu i\xi_2 \end{pmatrix} e^{-|\boldsymbol{\xi}|t}, \quad Y^{(3)}(\boldsymbol{\xi},t) = \begin{pmatrix} -i\xi_1 t \\ -i\xi_2 t \\ |\boldsymbol{\xi}|t + 1 \\ 2\mu|\boldsymbol{\xi}| \end{pmatrix} e^{-|\boldsymbol{\xi}|t}. \tag{5.30}$$

The following finite-dimensional sesquilinear form on $F(\mathbf{x},\boldsymbol{\xi})$ corresponds to the form $E_{\widetilde{\Omega}}[\mathbf{u},\mathbf{v}]$:

$$\mathbf{e}_{\mathbf{x},\boldsymbol{\xi}}[f,g] := \frac{1}{2}\sum\nolimits_{k,j=1}^{3}\int_0^\infty \big((\xi_j+\nu_j(\mathbf{x})D_t)f_k(t)+(\xi_k+\nu_k(\mathbf{x})D_t)f_j(t)\big) \times \big((\xi_j-\nu_j(\mathbf{x})D_t)\overline{g_k(t)}+(\xi_k-\nu_k(\mathbf{x})D_t)\overline{g_j(t)}\big)\,dt, \quad f,g\in F(\mathbf{x},\boldsymbol{\xi}).$$

The principal symbol $e^\circ(\mathbf{x},\boldsymbol{\xi}) = \left\{e^\circ_{jl}(\mathbf{x},\boldsymbol{\xi})\right\}_{1\le j,l\le 3}$ of the pseudodifferential operator $\mathcal{E}$ can be calculated by the formula $e^\circ_{jl}(\mathbf{x},\boldsymbol{\xi}) = \mathbf{e}_{\mathbf{x},\boldsymbol{\xi}}[Y^{(j)}(\boldsymbol{\xi},\cdot),Y^{(l)}(\boldsymbol{\xi},\cdot)]$, $j,l=1,2,3$. Calculations show that

$$e^\circ_{33}(\mathbf{x},\boldsymbol{\xi}) = 2|\boldsymbol{\xi}|, \quad e^\circ_{j3}(\mathbf{x},\boldsymbol{\xi}) = e^\circ_{3j}(\mathbf{x},\boldsymbol{\xi}) = 0, \quad j=1,2. \tag{5.31}$$

(There is no need to calculate the rest of e°_{jl}.)

From (5.24) and (5.31) it follows that the algebraic problem (5.25) has the eigenvalues $0,0$ and $\widetilde{a}(\mathbf{x})/2\mu|\boldsymbol{\xi}|$. Therefore,

$$n_\pm(\lambda,\mathbf{x},\boldsymbol{\xi};\ (5.25)) = \begin{cases}1, & \lambda<\widetilde{a}_\pm(\mathbf{x})/2\mu|\boldsymbol{\xi}|\\ 0, & \lambda\ge\widetilde{a}_\pm(\mathbf{x})/2\mu|\boldsymbol{\xi}|\end{cases}.$$

By (5.26) and (5.27), for $\lambda\to+0$ we have

$$N_\pm(\lambda,\ (5.20)) \sim N_\pm(\lambda,\ (5.21)) \sim \lambda^{-2}\frac{1}{16\pi\mu^2}\int_{\partial\widetilde{\Omega}}\widetilde{a}^2_\pm(\mathbf{x})\,dS(\mathbf{x}).$$

Combining this with (5.15), (5.22), and (5.23), we obtain the required asymptotics (5.13). □

Lemma 5.2, and Theorem 5.1 with it, are proved.

5.5. Oscillations of a system of the heavy viscous fluids. Theorem 5.1 can be generalized to the case of oscillations of a system of the heavy viscous fluids partially or completely filling a vessel. For definiteness, consider the case of complete filling (see Fig. 2). Here we restrict ourselves to the statement of the problem and formulation of the result; the proof can be found in [38].

Suppose that $\Omega\subset\mathbb{R}^3$ is a domain satisfying the assumptions of Sec. 3.2. We consider the following boundary-value problem for a system of vector-valued functions $\{\mathbf{u}_j(\mathbf{x})\}$, $j=1,\dots,k+1$, and a system of scalar functions $\{p_j(\mathbf{x})\}$, $j=1,\dots,k+1$:

$$\begin{gathered}-\mu_j\Delta\mathbf{u}_j+\nabla p_j=0, \quad \operatorname{div}\mathbf{u}_j=0 \ \text{ in } \Omega_j, \quad j=1,\dots,k+1,\\ \mathbf{u}_j=0 \ \text{ on } S_j, \quad j=1,\dots,k+1,\\ \mathbf{u}_j=\mathbf{u}_{j+1}, \quad \boldsymbol{\tau}_{tn}(\mathbf{u}_j)=\boldsymbol{\tau}_{tn}(\mathbf{u}_{j+1}) \ \text{ on } \Gamma_j, \quad j=1,\dots,k,\\ a_j(\mathbf{x})u_{jn}=\lambda\left(\tau_{nn}(\mathbf{u}_j,p_j)-\tau_{nn}(\mathbf{u}_{j+1},p_{j+1})\right) \ \text{ on } \Gamma_j, \quad j=1,\dots,k.\end{gathered} \tag{5.32}$$

Here we use the notation $u_{jn}(\mathbf{x}) := \langle\mathbf{u}_j(\mathbf{x}),\mathbf{n}_j(\mathbf{x})\rangle$. The constants $\mu_j>0$, $j=1,\dots,k+1$, are the viscosity coefficients of the fluids; $a_j(\mathbf{x})$ are smooth real-valued functions on Γ_j, $j=1,\dots,k$. Problem (5.32) is equivalent to the problem on the spectrum of the form ratio

$$\begin{gathered}\pm\frac{\sum_{j=1}^{k}\int_{\Gamma_j}a_j(\mathbf{x})|u_{jn}|^2\,d\mathbf{x}}{\sum_{j=1}^{k+1}\mu_jE_{\Omega_j}[\mathbf{u}_j]}, \quad \mathbf{u}_j\in H^1(\Omega_j;\mathbb{C}^3), \quad j=1,\dots,k+1,\\ \operatorname{div}\mathbf{u}_j=0 \ \text{ in } \Omega_j, \quad \mathbf{u}_j=0 \ \text{ on } S_j, \quad j=1,\dots,k+1,\\ \mathbf{u}_j=\mathbf{u}_{j+1} \ \text{ on } \Gamma_j, \quad j=1,\dots,k.\end{gathered} \tag{5.33}$$

We consider the form ratio (5.33) under the following condition.

Condition 5.2. *A bounded domain $\Omega\subset\mathbb{R}^3$ is divided by smooth two-dimensional surfaces $\Gamma_1,\dots,\Gamma_k$ into $(k+1)$ disjoint domains $\Omega_1,\dots,\Omega_{k+1}$. Suppose also that relations* (3.18), (3.19) *are satisfied. Let $\Omega_j\in\mathcal{K}$, $j=1,\dots,k+1$. Suppose that the curves $\gamma_j=\partial\Gamma_j$, $j=1,\dots,k$, are Lipschitz. Let $a_j\in L_2(\Gamma_j)$, $j=1,\dots,k$, be real-valued functions.*

Proposition 5.1 (see [38]). *Suppose that Condition* 5.2 *is satisfied. Then the spectrum counting functions for ratio* (5.33) *satisfy the following asymptotics for* $\lambda \to +0$:

$$N_{\pm}(\lambda,\ (5.33)) \sim \frac{\lambda^{-2}}{16\pi} \sum\nolimits_{j=1}^{k} (\mu_j + \mu_{j+1})^{-2} \int_{\Gamma_j} (a_j)_{\pm}^2 \, dS. \tag{5.34}$$

6. The Spectral Asymptotics of Small Oscillations of the Capillary Viscous Fluid

6.1. Properties of solutions of the Stokes system in a domain with edges. In the present subsection, we establish a number of technical statements needed to solve the problem associated with oscillations of the capillary viscous fluid. We rely essentially on the results from [30] on the solvability in weighted Sobolev classes of the first boundary-value problem for the Stokes system in domains with edges.

Suppose that a domain $\Omega \subset \mathbb{R}^3$ satisfies Condition 4.1. We need to introduce some notation. Let $l \geq 0$ be an integer number, $s \in \mathbb{R}$, and let $V_s^l(\Omega)$ be the Kondratyev space (the weighted Sobolev space) with the norm

$$\|u\|_{V_s^l(\Omega)} := \left(\int_{\Omega} \left(|\nabla_l u|^2 r^{2s} + |\nabla_{l-1} u|^2 r^{2(s-1)} + \dots + |u|^2 r^{2(s-l)} \right) d\mathbf{x} \right)^{1/2},$$

where $r = r(\mathbf{x})$ is the distance from a point $\mathbf{x} \in \Omega$ to the edge γ. The symbol $\nabla_j u$ means the "gradient of u of order j," i.e., the set of all derivatives $\partial^\alpha u$ of order $|\alpha| = j$. By $V_s^{l-1/2}(\partial\Omega)$ we denote the space of traces on $\partial\Omega$ of the functions from $V_s^l(\Omega)$ with the induced norm:

$$\|\varphi\|_{V_s^{l-1/2}(\partial\Omega)} := \inf_{v \in V_s^l(\Omega): v|_{\partial\Omega} = \varphi} \|v\|_{V_s^l(\Omega)}.$$

For $s = 0$ we write simply $V^l(\Omega) = V_0^l(\Omega)$ and $V^{l-1/2}(\partial\Omega) = V_0^{l-1/2}(\partial\Omega)$. Let $\rho(\mathbf{x})$ be the regularized distance on $\partial\Omega$ from a point $\mathbf{x}$ to γ. In the space $V^{l-1/2}(\partial\Omega)$, we can define an equivalent norm by

$$\|\varphi\|^2_{V^{l-1/2}(\partial\Omega)} := \int_{\partial\Omega} \int_{\partial\Omega} dS(\mathbf{x}) \, dS(\mathbf{y}) \, \frac{|\nabla_{l-1}\varphi(\mathbf{x}) - \nabla_{l-1}\varphi(\mathbf{y})|^2}{|\mathbf{x} - \mathbf{y}|^3} + \int_{\partial\Omega} dS(\mathbf{x}) \, |\varphi(\mathbf{x})|^2 \rho(\mathbf{x})^{1-2l},$$

and in $V_s^{l-1/2}(\partial\Omega)$ we consider an equivalent norm given by $\|\varphi\|_{V_s^{l-1/2}(\partial\Omega)} = \|\rho^s \varphi\|_{V^{l-1/2}(\partial\Omega)}$. By $V_s^l(\Omega; \mathbb{C}^3)$ and $V_s^{l-1/2}(\partial\Omega; \mathbb{C}^3)$ we denote the corresponding spaces of vector-valued functions.

Let $H_{00}^{1/2}(\Gamma)$ (see [28]) be the closure of $C_0^\infty(\Gamma)$ in the norm

$$\|\varphi\|_{H_{00}^{1/2}(\Gamma)} := \left(\int_{\Gamma} \int_{\Gamma} dS(\mathbf{x}) \, dS(\mathbf{y}) \, \frac{|\varphi(\mathbf{x}) - \varphi(\mathbf{y})|^2}{|\mathbf{x} - \mathbf{y}|^3} + \int_{\Gamma} dS(\mathbf{x}) \, \frac{|\varphi(\mathbf{x})|^2}{\rho(\mathbf{x})} \right)^{1/2}.$$

Note that $\{\varphi : \varphi \in V^{1/2}(\partial\Omega),\ \varphi|_S = 0\} = \{\varphi : \varphi|_\Gamma \in H_{00}^{1/2}(\Gamma),\ \varphi|_S = 0\}$. By $(H_{00}^{1/2}(\Gamma))'$ we denote the dual space to $H_{00}^{1/2}(\Gamma)$ with respect to the duality in $L_2(\Gamma)$ (see [4]).

An important role in the study of the problem for the capillary viscous fluid will be played by the space

$$\mathcal{H}_1 := \left\{ \{\mathbf{u}, p\} : \mathbf{u} \in H^1(\Omega; \mathbb{C}^3),\ p \in L_2(\Omega)/\{1\}, -\mu\Delta\mathbf{u} + \nabla p = 0,\ \operatorname{div} \mathbf{u} = 0,\ \mathbf{u}|_S = 0 \right\}.$$

Here $L_2(\Omega)/\{1\}$ is the quotient space of $L_2(\Omega)$ by the one-dimensional subspace of constants; the equation $-\mu\Delta\mathbf{u} + \nabla p = 0$ is understood in the sense of distributions.

We will need the following statements.

Proposition 6.1. *Let* $v \in H^1(\Omega)$. *Suppose that* $v = 0$ *on* S. *Then* $v \in V^1(\Omega)$ *and*

$$\|v\|_{V^1(\Omega)} \leq C \|v\|_{H^1(\Omega)}. \tag{6.1}$$

Proof. To prove (6.1), it suffices to check the inequality

$$\int_{\Omega} \frac{|v(\mathbf{x})|^2}{r^2(\mathbf{x})} \, d\mathbf{x} \leq C \|v\|^2_{H^1(\Omega)}, \quad v \in H^1(\Omega),\ v|_S = 0.$$

Let $\varepsilon > 0$ be sufficiently small. We put $\Omega_\varepsilon := \{\mathbf{x} \in \Omega : \ r(\mathbf{x}) > \varepsilon\}$. Obviously,

$$\int_{\Omega_\varepsilon} \frac{|v(\mathbf{x})|^2}{r^2(\mathbf{x})}\, d\mathbf{x} \le \frac{1}{\varepsilon^2} \int_\Omega |v(\mathbf{x})|^2 \, d\mathbf{x}.$$

For sufficiently small ε the set $\Omega \setminus \overline{\Omega_\varepsilon}$ can be divided into a finite number of parts U_j, $1 \le j \le N$, such that there exist C^1-diffeomorphisms f_j mapping $\overline{U_j}$ onto $\overline{\mathcal{U}_j}$, where $\mathcal{U}_j$ is a subset of the dihedral angle $\mathcal{U}_j := \{(r, \omega, x_3) : \ 0 < r < 1, \ 0 < \omega < \omega_j, \ 0 < x_3 < 1\}$, $0 < \omega_j < 2\pi$, and $S \cap \partial U_j$ transforms to the set $\{(r, \omega, x_3) : \ 0 \le r \le 1, \ \omega = 0, \ 0 \le x_3 \le 1\}$. Here (r, ω, x_3) are cylindrical coordinates in $\mathbb{R}^3$. Let $v \in H^1(\mathcal{U}_j)$ and $v(r, \omega, x_3) = 0$ for $\omega = 0$. Then

$$\int_{\mathcal{U}_j} |v|^2 r^{-2}\, d\mathbf{x} = \int_{\mathcal{U}_j} |v|^2 r^{-2}\, r dr\, d\omega\, dx_3 = \int_{\mathcal{U}_j} \left| \int_0^\omega \frac{1}{r} \frac{\partial v(r, \omega', x_3)}{\partial \omega'}\, d\omega' \right|^2 r dr\, d\omega\, dx_3$$
$$\le \omega_j \int_{\mathcal{U}_j} \left| \frac{1}{r} \frac{\partial v(r, \omega', x_3)}{\partial \omega'} \right|^2 r dr\, d\omega'\, dx_3 \le C \int_{\mathcal{U}_j} |\nabla v|^2 d\mathbf{x}.$$

□

Proposition 6.2. *The form $E_\Omega[\mathbf{u}]$ determines an equivalent norm in the space $\mathcal{H}_1$, i.e.,*

$$E_\Omega[\mathbf{u}] \asymp \left(\|\mathbf{u}\|^2_{H^1(\Omega)} + \|p\|^2_{L_2(\Omega)/\{1\}} \right), \quad \{\mathbf{u}, p\} \in \mathcal{H}_1.$$

Proof. By Proposition 6.1, for $\{\mathbf{u}, p\} \in \mathcal{H}_1$ we have $\mathbf{u} \in V^1(\Omega; \mathbb{C}^3)$ and $\|\mathbf{u}\|_{V^1(\Omega)} \asymp \|\mathbf{u}\|_{H^1(\Omega)}$. Let T be the trace operator taking a pair $\{\mathbf{u}, p\} \in \mathcal{H}_1$ to the trace of $\mathbf{u}$ on $\partial\Omega$: $\boldsymbol{\varphi} = T\{\mathbf{u}, p\} = \mathbf{u}|_{\partial\Omega}$. By the theorem on solvability of the first boundary-value problem for the Stokes system (see [30]), we have $\|\mathbf{u}\|_{V^1(\Omega)} + \|p\|_{L_2(\Omega)/\{1\}} \le C\|\boldsymbol{\varphi}\|_{V^{1/2}(\partial\Omega)}$. Obviously, $\|\boldsymbol{\varphi}\|_{V^{1/2}(\partial\Omega)} \le \|\mathbf{u}\|_{V^1(\Omega)}$. Combining what has been said with the Korn inequality (5.6), we obtain the required statement. □

Proposition 6.3. *The space $\mathcal{H}_2 := \big\{\{\mathbf{u}, p\} : \mathbf{u} \in V^2_1(\Omega; \mathbb{C}^3),\ p \in V^1_1(\Omega)/\{1\},\ -\mu\Delta\mathbf{u} + \nabla p = 0,\ \operatorname{div}\mathbf{u} = 0,\ \mathbf{u}|_S = 0\big\}$ is dense in $\mathcal{H}_1$.*

Proof. From the theorem on solvability in the weighted spaces of the first boundary-value problem for the Stokes system (see [30]) it follows that T is a homeomorphism of the following pairs of spaces:

$$\mathcal{H}_1 \to \mathcal{H}_{1/2} := \Big\{ \boldsymbol{\varphi} \in V^{1/2}(\partial\Omega; \mathbb{C}^3) : \ \boldsymbol{\varphi}|_S = 0, \ \int_\Gamma \varphi_n\, dS = 0 \Big\},$$
$$\mathcal{H}_2 \to \mathcal{H}_{3/2} := \Big\{ \boldsymbol{\varphi} \in V^{3/2}_1(\partial\Omega; \mathbb{C}^3) : \boldsymbol{\varphi}|_S = 0, \ \int_\Gamma \varphi_n\, dS = 0 \Big\}.$$

Note that for $\boldsymbol{\varphi} \in \mathcal{H}_{1/2}$ we have $\boldsymbol{\varphi}|_\Gamma \in H^{1/2}_{00}(\Gamma; \mathbb{C}^3)$. Since $C_0^\infty(\Gamma; \mathbb{C}^3)$ is dense in $H^{1/2}_{00}(\Gamma; \mathbb{C}^3)$, then the set $\big\{ \boldsymbol{\varphi} : \boldsymbol{\varphi}|_\Gamma \in C_0^\infty(\Gamma; \mathbb{C}^3), \ \boldsymbol{\varphi}|_S = 0, \ \int_\Gamma \varphi_n\, dS = 0 \big\}$ is dense in $\mathcal{H}_{1/2}$. Hence, a wider set $\mathcal{H}_{3/2}$ is also dense in $\mathcal{H}_{1/2}$. It follows from what has been said that $\mathcal{H}_2$ is dense in $\mathcal{H}_1$. □

Let $z \in L_2(\Gamma)$ and $\int_\Gamma z\, dS \ne 0$. By P_z we denote the (nonorthogonal) projection in $L_2(\Gamma)$ acting as follows:

$$(P_z f)(\mathbf{x}) = f(\mathbf{x}) - \frac{\int_\Gamma f(\mathbf{y}) \overline{z(\mathbf{y})}\, dS(\mathbf{y})}{\int_\Gamma \overline{z(\mathbf{y})}\, dS(\mathbf{y})}. \tag{6.2}$$

The operator P_z projects onto the subspace $\big\{ v \in L_2(\Gamma) : \ \int_\Gamma v\bar{z}\, dS = 0 \big\}$. The adjoint projection P_z^* acts by the formula $(P_z^* f)(\mathbf{x}) = f(\mathbf{x}) - z(\mathbf{x}) \frac{\int_\Gamma f(\mathbf{y})\, dS(\mathbf{y})}{\int_\Gamma z(\mathbf{y})\, dS(\mathbf{y})}$. Note that $\int_\Gamma (P_z^* f)(\mathbf{x})\, dS(\mathbf{x}) = 0$. The following statement plays an important role below.

Proposition 6.4. *Suppose that $\{\mathbf{u}, p\} \in \mathcal{H}_1$, $z \in H^{1/2}_{00}(\Gamma)$, $\int_\Gamma z\, dS \ne 0$. Let $\tau_{nn}(\mathbf{u}, p)$ be defined in Sec.* 5.1. *Then*

$$\|P_z \tau_{nn}(\mathbf{u}, p)\|_{(H^{1/2}_{00}(\Gamma))'} \le C\|\mathbf{u}\|_{H^1(\Omega)}. \tag{6.3}$$

Proof. For a function $w \in H_{00}^{1/2}(\Gamma)$ satisfying $\int_\Gamma w\,dS = 0$, let $\mathbf{f}_w(\mathbf{x})$ denote the vector-valued function on $\partial\Omega$, equal to $w(\mathbf{x})\mathbf{n}(\mathbf{x})$ for $\mathbf{x} \in \Gamma$ and equal to zero for $\mathbf{x} \in S$. Then $\mathbf{f}_w \in V^{1/2}(\partial\Omega; \mathbb{C}^3)$ and $\int_{\partial\Omega} (\mathbf{f}_w)_n\,dS = 0$. Let $\mathcal{M}$ be the operator resolving the first boundary-value problem for the Stokes system. Then

$$\|\mathcal{M}\mathbf{f}_w\|_{V^1(\Omega)} \leq C\|w\|_{H_{00}^{1/2}(\Gamma)}. \tag{6.4}$$

It suffices to prove estimate (6.3) on the set $\mathcal{H}_2$, which is dense in $\mathcal{H}_1$. Let us apply the Green formula (5.1) to functions $\{\mathbf{u}, p\} \in \mathcal{H}_2$ and $\mathbf{v}_w = \mathcal{M}\mathbf{f}_w$ (it is easy to check that all expressions in (5.1) make sense and are finite on these functions). We obtain: $\mu E_\Omega[\mathbf{u}, \mathcal{M}\mathbf{f}_w] = \int_\Gamma \tau_{nn}\overline{w}\,dS$. Together with (6.4), this implies that

$$\left|\int_\Gamma \tau_{nn}\overline{w}\,dS\right| \leq C'\|\mathbf{u}\|_{H^1(\Omega)}\|w\|_{H_{00}^{1/2}(\Gamma)}, \quad w \in H_{00}^{1/2}(\Gamma),\ \int_\Gamma w\,dS = 0. \tag{6.5}$$

Next, let $f \in H_{00}^{1/2}(\Gamma)$. Then $P_z^* f \in H_{00}^{1/2}(\Gamma)$, $\int_\Gamma P_z^* f\,dS = 0$, and $\|P_z^* f\|_{H_{00}^{1/2}(\Gamma)} \leq C_z\,\|f\|_{H_{00}^{1/2}(\Gamma)}$. By (6.5),

$$\begin{aligned} \left|\int_\Gamma P_z\tau_{nn}\overline{f}\,dS\right| &= \left|\int_\Gamma \tau_{nn}\overline{P_z^* f}\,dS\right| \leq C'\|\mathbf{u}\|_{H^1(\Omega)}\|P_z^* f\|_{H_{00}^{1/2}(\Gamma)} \\ &\leq C'C_z\|\mathbf{u}\|_{H^1(\Omega)}\|f\|_{H_{00}^{1/2}(\Gamma)}, \quad f \in H_{00}^{1/2}(\Gamma). \end{aligned} \tag{6.6}$$

By definition of the negative norm, from (6.6) we obtain:

$$\|P_z\tau_{nn}\|_{(H_{00}^{1/2}(\Gamma))'} = \sup_{0\neq f\in H_{00}^{1/2}(\Gamma)} \frac{\left|\int_\Gamma P_z\tau_{nn}\overline{f}\,dS\right|}{\|f\|_{H_{00}^{1/2}(\Gamma)}} \leq C'C_z\|\mathbf{u}\|_{H^1(\Omega)}.$$

□

Corollary 6.1. *Let* $z_i \in H_{00}^{1/2}(\Gamma)$, $i=1,2$, $\int_\Gamma z_1 dS \neq 0$. *Then the functional* $I(\{\mathbf{u},p\}) = \int_\Gamma (P_{z_1}\tau_{nn})\overline{z_2}dS$ *is a linear continuous functional over* $\mathcal{H}_1$.

Proposition 6.5. *Let* $\{\mathbf{u}, p\} \in \mathcal{H}_1$. *Suppose that* $\tau_{tn}(\mathbf{u})$ *is defined in Sec.* 5.1. *Then*

$$\|\boldsymbol{\tau}_{tn}(\mathbf{u})\|_{(H_{00}^{1/2}(\Gamma))'} \leq C\|\mathbf{u}\|_{H^1(\Omega)}. \tag{6.7}$$

Proof. Let us carry out the estimates locally. We choose some finite atlas $\{U_j, \alpha_j\}_{1\leq j\leq N}$ on the manifold Γ. Let $\{\omega_j\}_{1\leq j\leq N}$ be a partition of unity subordinate to the covering of Γ by the sets $\{U_j\}$. Let $\mathbf{e}_1^{(j)}(\mathbf{x})$, $\mathbf{e}_2^{(j)}(\mathbf{x})$ be smooth tangent vector fields in U_j forming a basis in the tangent plane to Γ for each $\mathbf{x} \in U_j$.

Let $w \in H_{00}^{1/2}(\Gamma)$. Then $\omega_j w \in H_{00}^{1/2}(\Gamma)$ and $\|\omega_j w\|_{H_{00}^{1/2}(\Gamma)} \leq C_j\|w\|_{H_{00}^{1/2}(\Gamma)}$. By $\mathbf{g}_w^{ij}(\mathbf{x})$ we denote the vector-valued function on $\partial\Omega$ equal to $\omega_j(\mathbf{x})w(\mathbf{x})\mathbf{e}_i^{(j)}(\mathbf{x})$ for $\mathbf{x} \in U_j$ and equal to zero for $\mathbf{x} \in \partial\Omega \setminus \overline{U_j}$. Then $\mathbf{g}_w^{ij} \in V^{1/2}(\partial\Omega; \mathbb{C}^3)$ and $\int_{\partial\Omega}(\mathbf{g}_w^{ij})_n\,dS = 0$. Let $\mathcal{M}$ be the same operator as in the proof of Proposition 6.4. Then

$$\left\|\mathcal{M}\mathbf{g}_w^{ij}\right\|_{V^1(\Omega)} \leq C_{ij}\,\|w\|_{H_{00}^{1/2}(\Gamma)}, \quad 1 \leq j \leq N, \quad i = 1,2. \tag{6.8}$$

It suffices to prove inequality (6.7) for $\{\mathbf{u}, p\} \in \mathcal{H}_2$. Applying the Green formula (5.1) to the functions $\{\mathbf{u}, p\} \in \mathcal{H}_2$ and $\mathcal{M}\mathbf{g}_w^{ij}$, we obtain $\mu E_\Omega[\mathbf{u}, \mathcal{M}\mathbf{g}_w^{ij}] = \int_\Gamma \langle\boldsymbol{\tau}_{tn}(\mathbf{u}), \mathbf{e}_i^{(j)}\rangle\omega_j\overline{w}\,dS$. Together with (6.8), this implies that

$$\left|\int_\Gamma \langle\boldsymbol{\tau}_{tn}(\mathbf{u}), \mathbf{e}_i^{(j)}\rangle\omega_j\overline{w}\,dS\right| \leq \check{C}_{ij}\|\mathbf{u}\|_{H^1(\Omega)}\,\|w\|_{H_{00}^{1/2}(\Gamma)}, \quad w \in H_{00}^{1/2}(\Gamma), \quad 1 \leq j \leq N, \quad i = 1,2. \tag{6.9}$$

Since $\|\boldsymbol{\tau}_{tn}\|_{(H_{00}^{1/2}(\Gamma))'} \leq \sum_{j=1}^N\sum_{i=1}^2 \left\|\omega_j\langle\boldsymbol{\tau}_{tn}, \mathbf{e}_i^{(j)}\rangle\right\|_{(H_{00}^{1/2}(\Gamma))'}$, then (6.9) and the definition of the negative norm imply (6.7). □

We put $\mathcal{H}_1^\tau := \{\{\mathbf{u}, p\} \in \mathcal{H}_1 : \boldsymbol{\tau}_{tn}(\mathbf{u})|_\Gamma = 0\}$. It is meant that $\boldsymbol{\tau}_{tn}(\mathbf{u})$ is the zero element of the space $(H_{00}^{1/2}(\Gamma))'$. The set $\mathcal{H}_1^\tau$ is a closed subspace of the space $\mathcal{H}_1$. Let $\widetilde{\mathcal{H}}_1 := \{\{\mathbf{u}, p\} \in \mathcal{H}_1 : u_n|_\Gamma = 0\}$.

Proposition 6.6. *The following orthogonal decomposition is true*: $\mathcal{H}_1 = \mathcal{H}_1^\tau \oplus^E \widetilde{\mathcal{H}}_1$, *where the orthogonality is understood in the sense of the inner product* $E_\Omega[\mathbf{u}, \mathbf{v}]$.

The proof of Proposition 6.6 is preceded by the following considerations. Consider the boundary-value problem

$$\begin{gathered} -\mu\Delta\mathbf{u} + \nabla p = 0, \quad \operatorname{div}\mathbf{u} = 0 \ \text{ in } \Omega, \\ \mathbf{u} = 0 \ \text{ on } S, \quad \boldsymbol{\tau}_{tn} = 0, \ \tau_{nn} = \psi \ \text{ on } \Gamma, \end{gathered} \tag{6.10}$$

with $\psi \in (H_{00}^{1/2}(\Gamma))'$. Suppose that the space $J_S^1(\Omega)$ defined in (5.2) is endowed with the inner product $E_\Omega[\mathbf{u}, \mathbf{v}]$. A function $\mathbf{u} \in J_S^1(\Omega)$ satisfying the integral identity

$$\mu E_\Omega[\mathbf{u}, \mathbf{v}] = \int_\Gamma \psi \overline{v_n}\, dS, \quad \mathbf{v} \in J_S^1(\Omega). \tag{6.11}$$

is called a *generalized solution* of problem (6.10). If $\mathbf{u}$, p, and ψ are smooth functions satisfying (6.10) in the classical sense, then it is easily seen that $\mathbf{u}$ is also a generalized solution of problem (6.10).

Proposition 6.7. *For any* $\psi \in (H_{00}^{1/2}(\Gamma))'$ *there exists a unique generalized solution of problem* (6.10).

Proof. If $\mathbf{v} \in J_S^1(\Omega)$, then from Proposition 6.1 it follows that $\mathbf{v} \in V^1(\Omega; \mathbb{C}^3)$ and $\|\mathbf{v}\|_{V^1(\Omega)} \le C\|\mathbf{v}\|_{H^1(\Omega)}$. Then $\mathbf{v}|_{\partial\Omega} \in V^{1/2}(\partial\Omega; \mathbb{C}^3)$ and $\mathbf{v}|_S = 0$. Consequently, $\mathbf{v}|_\Gamma \in H_{00}^{1/2}(\Gamma; \mathbb{C}^3)$ and $\|\mathbf{v}\|_{H_{00}^{1/2}(\Gamma)} \le C\|\mathbf{v}\|_{H^1(\Omega)}$.

Let $\psi \in (H_{00}^{1/2}(\Gamma))'$. Then $\int_\Gamma \psi\overline{v_n}\, dS$ is an antilinear continuous functional over $\mathbf{v} \in J_S^1(\Omega)$. By the Riesz theorem, there exists a unique function $\mathbf{u} = \mathbf{u}(\psi) \in J_S^1(\Omega)$ such that $\int_\Gamma \psi\overline{v_n}\, dS = \mu E_\Omega[\mathbf{u}, \mathbf{v}]$, $\mathbf{v} \in J_S^1(\Omega)$. □

As usual, it can be shown that if $\mathbf{u}$ is a generalized solution of problem (6.10), then there exists a function $p \in L_2(\Omega)$ such that $\{\mathbf{u}, p\}$ is a solution of problem (6.10) in the following weak sense: (a) $\mathbf{u} \in J_S^1(\Omega)$; (b) $-\mu\Delta\mathbf{u} + \nabla p = 0$ in the sense of distributions; (c) $\boldsymbol{\tau}_{tn} = 0$, $\tau_{nn} = \psi$ as elements of $(H_{00}^{1/2}(\Gamma))'$. Note that then $\{\mathbf{u}, p\} \in \mathcal{H}_1^\tau$.

Proof of Proposition 6.6. Let $\{\mathbf{v}, q\} \in \mathcal{H}_1$ and $\{\mathbf{v}, q\} \perp \mathcal{H}_1^\tau$, i.e., $E_\Omega[\mathbf{u}, \mathbf{v}] = 0$ for any $\{\mathbf{u}, p\} \in \mathcal{H}_1^\tau$. For any function $\psi \in (H_{00}^{1/2}(\Gamma))'$, we denote by $\mathbf{u}_\psi$ the generalized solution of problem (6.10). Then $E_\Omega[\mathbf{u}_\psi, \mathbf{v}] = 0$ for any $\psi \in (H_{00}^{1/2}(\Gamma))'$. According to (6.11), this means that $\int_\Gamma \psi\overline{v_n}\, dS = 0$ for any $\psi \in (H_{00}^{1/2}(\Gamma))'$. Therefore, $v_n = 0$ on Γ, i.e., $\{\mathbf{v}, q\} \in \widetilde{\mathcal{H}}_1$. We have proved that $(\mathcal{H}_1^\tau)^\perp \subset \widetilde{\mathcal{H}}_1$.

Now we prove the reverse inclusion $\widetilde{\mathcal{H}}_1 \subset (\mathcal{H}_1^\tau)^\perp$. Let $\{\mathbf{v}, q\} \in \widetilde{\mathcal{H}}_1$. Then, by the Green formula (5.1), we have $E_\Omega[\mathbf{u}, \mathbf{v}] = 0$, $\{\mathbf{u}, p\} \in \mathcal{H}_2$, $\boldsymbol{\tau}_{tn}(\mathbf{u}) = 0$. By closure, $E_\Omega[\mathbf{u}, \mathbf{v}] = 0$ for any $\{\mathbf{u}, p\} \in \mathcal{H}_1^\tau$, i.e., $\{\mathbf{v}, q\} \in (\mathcal{H}_1^\tau)^\perp$. □

6.2. Statement of the problem associated with oscillations of the capillary viscous fluid. Formulation of the result. The problem was formulated in [26, Chap. 8, § 2]. Suppose that a domain $\Omega \subset \mathbb{R}^3$ satisfies Condition 4.1. By $\mathbf{B}_\Gamma$ we denote the differential operator given by the expression $B_\Gamma = -\sigma\Delta_\Gamma + h(\mathbf{x})$ on the domain $\operatorname{Dom}\mathbf{B}_\Gamma = H^2(\Gamma) \cap H_0^1(\Gamma)$. Here $\sigma > 0$, $h(\mathbf{x})$ is a smooth real-valued function on Γ. The coefficients σ and h have the same meaning as in Sec. 4.5. The operator $\mathbf{B}_\Gamma$ is self-adjoint in $L_2(\Gamma)$, its kernel $Z_B := \{v \in \operatorname{Dom}\mathbf{B}_\Gamma : (-\sigma\Delta_\Gamma + h)v = 0\}$ coincides with the cokernel, is finite-dimensional and consists of infinitely smooth functions. The inverse operator $\mathbf{B}_\Gamma^{-1}$ is defined on $L_2(\Gamma) \ominus Z_B$; it is the resolving operator for the first boundary-value problem on Γ: $\varphi = \mathbf{B}_\Gamma^{-1} f$ with $f \in L_2(\Gamma) \ominus Z_B$ means that $(-\sigma\Delta_\Gamma + h)\varphi = f$ on Γ, $\varphi = 0$ on γ, and $\varphi \perp Z_B$.

We assume that the operator $\mathbf{B}_\Gamma$ is positive definite on $\operatorname{Dom}\mathbf{B}_\Gamma \cap (L_2(\Gamma) \ominus \{1\})$. This condition is equivalent to the inequality

$$\int_\Gamma \left(\sigma|\nabla_\Gamma u|^2 + h(\mathbf{x})|u|^2\right) dS \ge c\int_\Gamma |u|^2\, dS, \quad u \in H_0^1(\Gamma), \quad \int_\Gamma u\, dS = 0; \quad c > 0. \tag{6.12}$$

Condition (6.12) imposes a restriction on the problem data. Physically, it means that the equilibrium position of the fluid is stable in the linear approximation. From (6.12) it follows that $\dim Z_B \le 1$. Denote by z_1 the basis vector in Z_B; in the case where $Z_B = \{0\}$ we put $z_1 = 0$.

Consider the following boundary-value problem in Ω:

$$\begin{gathered} -\mu\Delta\mathbf{u} + \nabla p = 0, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in} \quad \Omega, \\ \mathbf{u} = 0 \quad \text{on } S, \quad \boldsymbol{\tau}_{tn} = 0 \quad \text{on } \Gamma, \\ B_\Gamma u_n = \lambda^{-1}(\tau_{nn} + c_\tau) \quad \text{on } \Gamma. \end{gathered} \tag{6.13}$$

Here the constant c_τ is not specified, but is searched for together with the solution. The boundary condition $u_n|_\gamma = 0$ is fulfilled automatically due to $\mathbf{u}|_S = 0$.

As will be shown below, the boundary-value problem (6.13) corresponds to the variational form ratio, which defines a nonnegative compact operator in the space $\mathcal{H}_1^\tau$. The main result of this section is the following theorem.

Theorem 6.1. *Under the above assumptions, the spectrum counting function for problem* (6.13) *satisfies the following asymptotics for* $\lambda \to +0$:

$$N(\lambda,\ (6.13)) \sim \lambda^{-2}\frac{\mu^2}{\pi\sigma^2}\operatorname{meas}\Gamma.$$

6.3. Variational formulation of the problem for the capillary viscous fluid. On the solutions of problem (6.13), the function $\tau_{nn} + c_\tau$ belongs to the range of the operator $\mathbf{B}_\Gamma$, and therefore

$$\int_\Gamma (\tau_{nn} + c_\tau)\overline{z_1}\, dS = 0. \tag{6.14}$$

In addition, from the continuity condition ($\operatorname{div}\mathbf{u} = 0$) and the adhesion condition $\mathbf{u}|_S = 0$ it follows that

$$\int_\Gamma u_n\, dS = 0. \tag{6.15}$$

We start with the case where $Z_B \ne \{0\}$. Then, by (6.12), $\int_\Gamma \overline{z_1}\, dS \ne 0$, and the constant c_τ is given by $c_\tau = -\frac{\int_\Gamma \tau_{nn}\overline{z_1}\, dS}{\int_\Gamma \overline{z_1}\, dS}$; see (6.14). Hence, $\tau_{nn} + c_\tau = P_{z_1}\tau_{nn}$, where the projection P_{z_1} is defined according to (6.2).

On the solutions of problem (6.13), we have

$$\lambda u_n = \mathbf{B}_\Gamma^{-1}P_{z_1}\tau_{nn} + Cz_1. \tag{6.16}$$

By (6.15), the constant C can be found from the condition

$$\int_\Gamma (\mathbf{B}_\Gamma^{-1}P_{z_1}\tau_{nn} + Cz_1)\, dS = 0. \tag{6.17}$$

We multiply (6.16) by $\overline{\tau_{nn}}$ and integrate over Γ: $\lambda\int_\Gamma u_n\overline{\tau_{nn}}\, dS = \int_\Gamma(\mathbf{B}_\Gamma^{-1}P_{z_1}\tau_{nn} + Cz_1)\overline{\tau_{nn}}\, dS$. Under conditions from (6.13), using the Green formula (5.1), we obtain $\int_\Gamma u_n\overline{\tau_{nn}}\, dS = \mu E_\Omega[\mathbf{u}]$. On the other hand, by (6.17) and the obvious equality $\int_\Gamma z_1\overline{P_{z_1}\tau_{nn}}\, dS = 0$, we have $\int_\Gamma(\mathbf{B}_\Gamma^{-1}P_{z_1}\tau_{nn} + Cz_1)\overline{\tau_{nn}}\, dS = \int_\Gamma(\mathbf{B}_\Gamma^{-1}P_{z_1}\tau_{nn})\overline{P_{z_1}\tau_{nn}}\, dS$.

Let us now consider the case where $Z_B = \{0\}$. Then on the solutions of problem (6.13) we have

$$\lambda u_n = \mathbf{B}_\Gamma^{-1}(\tau_{nn} + c_\tau). \tag{6.18}$$

By condition (6.15), from (6.18) it follows that

$$\int_\Gamma \mathbf{B}_\Gamma^{-1}(\tau_{nn} + c_\tau)\, dS = 0, \tag{6.19}$$

which is equivalent to the relation $\int_\Gamma(\tau_{nn} + c_\tau)\overline{z_0}\, dS = 0$, $z_0 := \mathbf{B}_\Gamma^{-1}1$. Note that, in the case under consideration, we have $\int_\Gamma \overline{z_0}\, dS = \int_\Gamma B_\Gamma z_0\overline{z_0}\, dS \ne 0$. It follows that the constant c_τ can be found from the condition $c_\tau = -\frac{\int_\Gamma \tau_{nn}\overline{z_0}\, dS}{\int_\Gamma \overline{z_0}\, dS}$. Then $\tau_{nn} + c_\tau = P_{z_0}\tau_{nn}$. We multiply (6.18) by $\overline{\tau_{nn}}$ and integrate over Γ: $\lambda\int_\Gamma u_n\overline{\tau_{nn}}\, dS = \int_\Gamma(\mathbf{B}_\Gamma^{-1}P_{z_0}\tau_{nn})\overline{\tau_{nn}}\, dS$. As before, from the Green formula it follows that $\int_\Gamma u_n\overline{\tau_{nn}}\, dS = \mu E_\Omega[\mathbf{u}]$. On the other hand, by (6.19), we have $\int_\Gamma(\mathbf{B}_\Gamma^{-1}P_{z_0}\tau_{nn})\overline{\tau_{nn}}\, dS = \int_\Gamma(\mathbf{B}_\Gamma^{-1}P_{z_0}\tau_{nn})\overline{P_{z_0}\tau_{nn}}\, dS$.

As a result, we made sure that problem (6.13) is equivalent to the variational problem on the spectrum of the form ratio

$$\frac{\int_\Gamma (\mathbf{B}_\Gamma^{-1} P_z \tau_{nn}) \overline{P_z \tau_{nn}}\, dS}{\mu E_\Omega[\mathbf{u}]}, \quad \{\mathbf{u}, p\} \in \mathcal{H}_1^\tau. \tag{6.20}$$

Here $z = z_1$ if $Z_B \neq \{0\}$, and $z = z_0$ if $Z_B = \{0\}$. By Proposition 6.2, the form $E_\Omega[\mathbf{u}]$ defines an equivalent norm in $\mathcal{H}_1^\tau$.

Note that the comparison functions in (6.20) must satisfy the equation $-\mu\Delta\mathbf{u} + \nabla p = 0$, as well as the condition that the tangential stresses on Γ are equal to zero (in contrast to the spectral problem for ratio (5.4), in which these conditions were natural). The reason is that the numerator of ratio (6.20) depends on τ_{nn}, and thus on the function p, and the denominator depends only on $\mathbf{u}$ (while ratio (5.4) depends only on $\mathbf{u}$). Therefore, the function p must be linked to $\mathbf{u}$. At a certain step of solving the problem, the above "links" will be removed after p is expressed in terms of $\mathbf{u}$.

Let us check that the form in the numerator of ratio (6.20) is compact[5] in $\mathcal{H}_1^\tau$. By the homeomorphism theorem, the operator B_Γ is a homeomorphism of the following pairs of spaces:

$H^2_{\mathrm{bc}}(\Gamma) := H^2(\Gamma) \cap H^1_0(\Gamma) \cap (L_2(\Gamma) \ominus Z_B) \to L_2(\Gamma) \ominus Z_B, \quad H^1_{\mathrm{bc}}(\Gamma) := H^1_0(\Gamma) \cap (L_2(\Gamma) \ominus Z_B) \to H^{-1}_Z(\Gamma).$

Here $H^{-1}_Z(\Gamma)$ denotes the dual space to $H^1_{\mathrm{bc}}(\Gamma)$ with respect to the $(L_2(\Gamma) \ominus Z_B)$-duality.

Using the interpolation theory (see [28]), we see that B_Γ is a homeomorphism of the spaces $\big[H^2_{\mathrm{bc}}(\Gamma), H^1_{\mathrm{bc}}(\Gamma)\big]_{1/2} \to \big[L_2(\Gamma) \ominus Z_B, H^{-1}_Z(\Gamma)\big]_{1/2}$. Here by $[\mathfrak{H}_1, \mathfrak{H}_2]_{1/2}$ we denote the intermediate space between the Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, where $\mathfrak{H}_1 \subset \mathfrak{H}_2$, $\mathfrak{H}_1$ is dense in $\mathfrak{H}_2$ and continuously embedded in it; see [28, § 1.2.1].

Using the fact that $\big[H^2(\Gamma), H^1(\Gamma)\big]_{1/2} = H^{3/2}(\Gamma)$, it can be shown that $\big[H^2_{\mathrm{bc}}(\Gamma), H^1_{\mathrm{bc}}(\Gamma)\big]_{1/2} = H^{3/2}(\Gamma) \cap H^1_0(\Gamma) \cap (L_2(\Gamma) \ominus Z_B) =: H^{3/2}_{\mathrm{bc}}(\Gamma)$.

Theorem 1.12.4 from [28] states that $\big[L_2(\Gamma), H^{-1}(\Gamma)\big]_{1/2} = (H^{1/2}_{00}(\Gamma))'$, where $H^{-1}(\Gamma) = (H^1_0(\Gamma))'$. Taking this fact into account, it is easy to show that $\big[L_2(\Gamma) \ominus Z_B, H^{-1}_Z(\Gamma)\big]_{1/2} = H^{-1/2}_{\mathrm{bc}}(\Gamma)$, where $H^{-1/2}_{\mathrm{bc}}(\Gamma)$ is the dual space to $H^{1/2}_{\mathrm{bc}}(\Gamma) := H^{1/2}_{00}(\Gamma) \cap (L_2(\Gamma) \ominus Z_B)$ with respect to the $(L_2(\Gamma) \ominus Z_B)$-duality. The space $H^{-1/2}_{\mathrm{bc}}(\Gamma)$ can be identified with $\big\{\varphi \in (H^{1/2}_{00}(\Gamma))' : (\varphi, z_1) = 0\big\}$. Thus, B_Γ is a homeomorphism of the following pair of spaces:

$$B_\Gamma : H^{3/2}_{\mathrm{bc}}(\Gamma) \to H^{-1/2}_{\mathrm{bc}}(\Gamma). \tag{6.21}$$

Since $H^{3/2}_{\mathrm{bc}}(\Gamma)$ is compactly embedded into $H^{1/2}_{\mathrm{bc}}(\Gamma)$, then the operator $\mathbf{B}_\Gamma^{-1}$ is compact from $H^{-1/2}_{\mathrm{bc}}(\Gamma)$ to $H^{1/2}_{\mathrm{bc}}(\Gamma)$. Together with Proposition 6.4, this shows that the form in the numerator of (6.20) is compact in $\mathcal{H}_1^\tau$.

Let $b \in C^\infty(\overline{\Gamma})$, $z \in H^{1/2}_{00}(\Gamma)$, and $\int_\Gamma z\, dS \neq 0$. Denote $\mathcal{B}_{\Gamma,z}[\varphi] := \mathrm{Re} \int_\Gamma b(\mathbf{x})(\mathbf{B}_\Gamma^{-1} P_z \varphi)\overline{P_z \varphi}\, dS(\mathbf{x})$. Instead of (6.20) we consider the form ratio of more general form

$$\pm \frac{\mathcal{B}_{\Gamma,z}[\tau_{nn}]}{\mu E_\Omega[\mathbf{u}]}, \quad \{\mathbf{u}, p\} \in \mathcal{H}_1^\tau. \tag{6.22}$$

Theorem 6.1 follows directly from the following theorem.

Theorem 6.2. *Suppose that Ω satisfies Condition* 4.1. *Let $b(\mathbf{x})$ be a smooth real-valued function on $\overline{\Gamma}$. Then the spectrum counting functions for ratio* (6.22) *satisfy the following asymptotics for $\lambda \to +0$:*

$$N_\pm(\lambda,\ (6.22)) \sim \lambda^{-2} \frac{\mu^2}{\pi \sigma^2} \int_\Gamma b_\pm^2(\mathbf{x})\, dS(\mathbf{x}). \tag{6.23}$$

We will prove Theorem 6.2, using the same scheme as for the previous problems.

[5]Note that in the case of the third boundary condition on γ, the corresponding form is not even bounded in $\mathcal{H}_1^\tau$; see [38, § 8].

6.4. Estimates of the spectrum of the ratio (6.22).

Lemma 6.1. *Let $b(\mathbf{x})$ be a smooth real-valued function on $\overline{\Gamma}$. We have $\Delta_2^{\pm}$ (6.22) $\le C\|b\|^2_{L_2(\Gamma)}$, where C does not depend on the function b.*

Proof. By Proposition 6.4 and the Korn inequality (5.6), $E_\Omega[\mathbf{u}] \ge C\,\|P_z\tau_{nn}\|^2_{(H^{1/2}_{00}(\Gamma))'}$, $\{\mathbf{u},p\}\in\mathcal{H}_1$, $C>0$. By Lemma 2.2, this implies that the functions $N_\pm(\lambda,$ (6.22)) do not exceed the spectrum counting functions for the ratio

$$\pm C\frac{\operatorname{Re}\int_\Gamma b(\mathbf{B}_\Gamma^{-1}\psi)\overline{\psi}\,dS}{\|\psi\|^2_{(H^{1/2}_{00}(\Gamma))'}},\quad \psi\in (H^{1/2}_{00}(\Gamma))'. \tag{6.24}$$

We substitute $\mathbf{B}_\Gamma^{-1}\psi = f$ in (6.24). Then (see (6.21)) $f\in H^{3/2}(\Gamma)\cap H^1_0(\Gamma)$ and $\|f\|_{H^{3/2}(\Gamma)}\le C\|\psi\|_{(H^{1/2}_{00}(\Gamma))'}$. Applying Lemma 2.2, we obtain that the functions $N_\pm(\lambda,$ (6.24)) do not exceed the spectrum counting functions for the ratio

$$\pm C\frac{\operatorname{Re}\int_\Gamma bf\overline{B_\Gamma f}\,dS}{\|f\|^2_{H^{3/2}(\Gamma)}},\quad f\in H^{3/2}(\Gamma)\cap H^1_0(\Gamma). \tag{6.25}$$

Integrating by parts in the numerator of (6.25) and discarding the lower order terms, we obtain that the values $\Delta_2^{\pm}$ (6.25) do not exceed the similar values for the ratio

$$\pm C\frac{\int_\Gamma b|\nabla_\Gamma f|^2\,dS}{\|f\|^2_{H^{3/2}(\Gamma)}},\quad f\in H^{3/2}(\Gamma). \tag{6.26}$$

Substituting $\nabla_\Gamma f =: \mathbf{g}$ and using Lemma 2.8, we arrive at the estimate $\Delta_2^{\pm}$ (6.26) $\le C\|b\|^2_{L_2(\Gamma)}$. $\square$

Lemmas 6.1 and 2.5 allow us, when calculating the principal term of the spectral asymptotics of ratio (6.22), to consider only the case where $b\in C_0^\infty(\Gamma)$.

6.5. Comparison with the problems in a smooth domain. Let us fix a real-valued function $z_2\in C_0^\infty(\Gamma)$ such that $\int_\Gamma z_2\,dS = 1$. According to Lemma 2.6 and Corollary 6.1, the values $\Delta_2^{\pm}$ (6.22) and $\delta_2^{\pm}$ (6.22) coincide with the similar values for the ratio

$$\pm\frac{\mathcal{B}_{\Gamma,z_2}[\tau_{nn}]}{\mu E_\Omega[\mathbf{u}]},\quad \{\mathbf{u},p\}\in\mathcal{H}_1^\tau,\quad P_z\tau_{nn} = P_{z_2}\tau_{nn}. \tag{6.27}$$

We have

$$\Delta_2^{\pm}\ (6.22) = \Delta_2^{\pm}\ (6.27),\quad \delta_2^{\pm}\ (6.22) = \delta_2^{\pm}\ (6.27). \tag{6.28}$$

Thus, let $b\in C_0^\infty(\Gamma)$. Suppose that $\widetilde{\Omega}$ is a bounded domain with smooth boundary such that $\widetilde{\Omega}\subset\Omega$, $\operatorname{supp} b$ and $\operatorname{supp} z_2$ lie strictly inside the set $\partial\widetilde{\Omega}\cap\Gamma$. Let $\widetilde{b}\in C^\infty(\partial\widetilde{\Omega})$ be the function that is equal to $b(\mathbf{x})$ on $\partial\widetilde{\Omega}\cap\Gamma$ and equal to zero outside this set. Similarly, we define a function $\widetilde{z}_2\in C^\infty(\partial\widetilde{\Omega})$. Let $d(\mathbf{x})$ be a smooth positive function on $\partial\widetilde{\Omega}$. Consider the differential expression $B_{\partial\widetilde{\Omega}} := -\sigma\Delta_{\partial\widetilde{\Omega}} + d(\mathbf{x})$, where $\Delta_{\partial\widetilde{\Omega}}$ is the Laplace–Beltrami operator on $\partial\widetilde{\Omega}$. The operator $\mathbf{B}_{\partial\widetilde{\Omega}}$ (cf. Sec. 4.3) is given by the expression $B_{\partial\widetilde{\Omega}}$ on the domain $H^2(\partial\widetilde{\Omega})$. The inverse operator $\mathbf{B}^{-1}_{\partial\widetilde{\Omega}}$ is a pseudodifferential operator on $\partial\widetilde{\Omega}$ of order (-2). By $P_{\widetilde{z}_2}$ we denote a projection in $L_2(\partial\widetilde{\Omega})$ acting as follows: $(P_{\widetilde{z}_2}f)(\mathbf{x}) = f(\mathbf{x}) - \int_{\partial\widetilde{\Omega}} f(\mathbf{y})\widetilde{z}_2(\mathbf{y})\,dS(\mathbf{y})$, $\mathbf{x}\in\partial\widetilde{\Omega}$.

Let $\boldsymbol{\nu}(\mathbf{x})$ be the unit inner normal vector to $\partial\widetilde{\Omega}$. On $\partial\widetilde{\Omega}$ we define a function $\widetilde{\tau}_{\nu\nu} = \widetilde{\tau}_{\nu\nu}(\mathbf{u},p)$ (which is similar to τ_{nn} on Γ). We have $\widetilde{\tau}_{\nu\nu}(\mathbf{x}) = \langle\tau(\mathbf{x})\boldsymbol{\nu}(\mathbf{x}),\boldsymbol{\nu}(\mathbf{x})\rangle$, $\mathbf{x}\in\partial\widetilde{\Omega}$. Note that $\widetilde{\tau}_{\nu\nu}(\mathbf{x}) = \tau_{nn}(\mathbf{x})$ for $\mathbf{x}\in\partial\widetilde{\Omega}\cap\Gamma$. We put $\widetilde{\mathcal{B}}[\varphi] := \operatorname{Re}\int_{\partial\widetilde{\Omega}}\widetilde{b}(\mathbf{x})\big(\mathbf{B}^{-1}_{\partial\widetilde{\Omega}}P_{\widetilde{z}_2}\varphi\big)\overline{P_{\widetilde{z}_2}\varphi}\,dS(\mathbf{x})$.

Consider the form ratio

$$\pm\frac{\widetilde{\mathcal{B}}[\widetilde{\tau}_{\nu\nu}]}{\mu E_\Omega[\mathbf{u}]},\quad \{\mathbf{u},p\}\in\mathcal{H}_1^\tau. \tag{6.29}$$

The following statement is an analogue of Lemma 4.2.

Lemma 6.2. *We have*

$$\Delta_2^{\pm}\ (6.27) = \Delta_2^{\pm}\ (6.29), \quad \delta_2^{\pm}\ (6.27) = \delta_2^{\pm}\ (6.29). \tag{6.30}$$

Proof. According to Lemma 2.6, the values $\Delta_2^{\pm}$ (6.29), $\delta_2^{\pm}$ (6.29) will not change if we consider the ratio on the subspace of finite codimension in $\mathcal{H}_1^{\tau}$ distinguished by the condition $P_z\tau_{nn} = P_{z_2}\tau_{nn}$. Denote $\Lambda[\{\mathbf{u},p\}] = \mathcal{B}_{\Gamma,z_2}[\tau_{nn}] - \widetilde{\mathcal{B}}[\widetilde{\tau}_{\nu\nu}]$. Consider the form ratio

$$\pm\frac{\Lambda[\{\mathbf{u},p\}]}{\mu E_{\Omega}[\mathbf{u}]}, \quad \{\mathbf{u},p\} \in \mathcal{H}_1^{\tau}, \quad P_z\tau_{nn} = P_{z_2}\tau_{nn}. \tag{6.31}$$

By Lemma 2.4, relations (6.30) will be proved as soon as it is shown that Δ_2^{+} (6.31) $= \Delta_2^{-}$ (6.31) $= 0$.

Let $\{\mathbf{u},p\} \in \mathcal{H}_1^{\tau}$ and let $P_z\tau_{nn} = P_{z_2}\tau_{nn}$. Denote $f = \mathbf{B}_{\Gamma}^{-1}P_{z_2}\tau_{nn}$ and $g = \mathbf{B}_{\partial\widetilde{\Omega}}^{-1}P_{\widetilde{z}_2}\widetilde{\tau}_{\nu\nu}$. Suppose that $\widetilde{\Gamma}$ is an open set lying strictly inside $\Gamma \cap \partial\widetilde{\Omega}$, $\operatorname{supp} b \subset \widetilde{\Gamma}$, $\operatorname{supp} z_2 \subset \widetilde{\Gamma}$. We can assume that $\widetilde{\Gamma}$ is a two-dimensional surface with sufficiently smooth boundary. Note that $P_{z_2}\tau_{nn}(\mathbf{x}) = P_{\widetilde{z}_2}\widetilde{\tau}_{\nu\nu}(\mathbf{x})$ for $\mathbf{x} \in \widetilde{\Gamma}$. Consequently, $(B_{\Gamma}f)(\mathbf{x}) = (B_{\partial\widetilde{\Omega}}g)(\mathbf{x})$ for $\mathbf{x} \in \widetilde{\Gamma}$. We transform the form $\Lambda[\{\mathbf{u},p\}]$ (cf. the proof of Lemma 4.2):

$$\Lambda[\{\mathbf{u},p\}] = \operatorname{Re}\left(\int_{\Gamma} bf\overline{B_{\Gamma}f}\,dS - \int_{\partial\widetilde{\Omega}} \widetilde{b}g\overline{B_{\partial\widetilde{\Omega}}g}\,dS\right)$$
$$= \operatorname{Re}\int_{\widetilde{\Gamma}} \left(-\sigma(\Delta b)(f-g)\overline{g} - 2\sigma\nabla b\cdot(\nabla f - \nabla g)\overline{g} + b(d-h)f\overline{g}\right)\,dS.$$

It follows that $\Lambda[\{\mathbf{u},p\}] = \operatorname{Re}\int_{\widetilde{\Gamma}}(B'f + B''g)\overline{g}\,dS$, where B' and B'' are first-order differential operators.

We have

$$E_{\Omega}[\mathbf{u}] \geq C\left(\|g\|^2_{H^{3/2}(\partial\widetilde{\Omega})} + \|f\|^2_{H^{3/2}(\Gamma)}\right), \quad C > 0. \tag{6.32}$$

Indeed, the inequality $\|f\|^2_{H^{3/2}(\Gamma)} \leq CE_{\Omega}[\mathbf{u}]$ was obtained in the proof of Lemma 6.1. A similar inequality for $\|g\|^2_{H^{3/2}(\partial\widetilde{\Omega})}$ is only easier to check, because $\partial\widetilde{\Omega} \in C^{\infty}$.

From (6.32) it follows that $E_{\Omega}[\mathbf{u}] \geq C\big(\|g\|^2_{H^{3/2}(\widetilde{\Gamma})} + \|B'f + B''g\|^2_{H^{1/2}(\widetilde{\Gamma})}\big)$, $C > 0$. By Lemma 2.2, the functions $N_{\pm}(\lambda,\ (6.31))$ do not exceed the spectrum counting functions for the ratio

$$\pm\frac{\operatorname{Re}\int_{\widetilde{\Gamma}}\psi\overline{g}\,dS}{\mu C\big(\|\psi\|^2_{H^{1/2}(\widetilde{\Gamma})} + \|g\|^2_{H^{3/2}(\widetilde{\Gamma})}\big)}, \quad \{\psi,g\} \in H^{1/2}(\widetilde{\Gamma}) \oplus H^{3/2}(\widetilde{\Gamma}). \tag{6.33}$$

Applying Lemma 2.9, we see that $N_{\pm}(\lambda,\ (6.33)) = O(\lambda^{-1})$, hence, $\Delta_2^{\pm}$ (6.33) $= 0$. Then $\Delta_2^{\pm}$ (6.31) $= 0$. $\square$

Now, our goal is to compare ratio (6.29) with a similar ratio of forms given in a smooth domain $\widetilde{\Omega}$. Direct making such a comparison is hampered by the presence of constraints: the equation $-\mu\Delta\mathbf{u} + \nabla p = 0$ in Ω and the boundary condition $\boldsymbol{\tau}_{tn} = 0$ on Γ. Let us transform the numerator of ratio (6.29) in order to "remove" these constraints (see Lemma 6.3 below). For $\{\mathbf{u},p\} \in \mathcal{H}_1$ denote $\boldsymbol{\varphi} = \mathbf{u}|_{\partial\widetilde{\Omega}}$. Then we have $\{\mathbf{u},p\} = \mathcal{G}\boldsymbol{\varphi}$ in $\widetilde{\Omega}$. The operator $\mathcal{G}$ was defined in Sec. 5.4. From the properties of the Boutet de Monvel algebra it follows that

$$\widetilde{\tau}_{\nu\nu}(\mathcal{G}\boldsymbol{\varphi}) = \mathcal{T}\boldsymbol{\varphi} + \text{Const}, \tag{6.34}$$

where $\mathcal{T}$ is a first-order pseudodifferential operator on $\partial\widetilde{\Omega}$. Relation (6.34) contains an arbitrary constant associated with the fact that the function $p = p(\boldsymbol{\varphi})$ is defined up to an arbitrary constant. Locally in a neighborhood U of some point $\mathbf{x}_0 \in \partial\widetilde{\Omega}$ we choose an orthogonal curvilinear coordinate system so that the coordinate lines for the third coordinate at the point $\mathbf{x} \in \partial\widetilde{\Omega}$ are directed along the normal $\boldsymbol{\nu}(\mathbf{x})$, and the corresponding Lamé coefficient on $\partial\widetilde{\Omega}$ is equal to 1. In this coordinate system, the principal symbol of the pseudodifferential operator $\mathcal{T}$ is a row $t^{\circ}(\mathbf{x},\boldsymbol{\xi}) = \{t_1^{\circ}(\mathbf{x},\boldsymbol{\xi}), t_2^{\circ}(\mathbf{x},\boldsymbol{\xi}), t_3^{\circ}(\mathbf{x},\boldsymbol{\xi})\}$ with the entries

$$t_j^{\circ}(\mathbf{x},\boldsymbol{\xi}) = -Y_4^{(j)}(\mathbf{x},\boldsymbol{\xi},0) + 2\mu\Big(\frac{d}{dt}Y_3^{(j)}(\mathbf{x},\boldsymbol{\xi},t)\Big)\Big|_{t=0}, \quad \mathbf{x} \in U, \quad \boldsymbol{\xi} \perp \boldsymbol{\nu}(\mathbf{x}),$$

where the functions $Y^{(j)}$ are defined in (5.30). Calculation shows that $t_1^\circ(\mathbf{x},\boldsymbol{\xi}) = t_2^\circ(\mathbf{x},\boldsymbol{\xi}) = 0$, $t_3^\circ(\mathbf{x},\boldsymbol{\xi}) = -2\mu|\boldsymbol{\xi}|$. From (6.34) it follows that

$$\widetilde{\mathcal{B}}[\widetilde{\tau}_{\nu\nu}] = \operatorname{Re} \int_{\partial\widetilde{\Omega}} \widetilde{b}\big(\mathbf{B}_{\partial\widetilde{\Omega}}^{-1}\mathcal{T}\boldsymbol{\varphi}\big)\overline{\mathcal{T}\boldsymbol{\varphi}}\, dS + \mathcal{D}[\boldsymbol{\varphi}], \tag{6.35}$$

where $\mathcal{D}$ is a finite-rank form. Let $\mathcal{R}$ be a matrix pseudodifferential operator on $\partial\widetilde{\Omega}$ of order zero corresponding to the first term in the right-hand side of (6.35). The principal symbol of the pseudodifferential operator $\mathcal{R}$ is given by

$$r^\circ(\mathbf{x},\boldsymbol{\xi}) = \frac{\widetilde{b}(\mathbf{x})}{\sigma|\boldsymbol{\xi}|^2}\big(t^\circ(\mathbf{x},\boldsymbol{\xi})\big)^+ t^\circ(\mathbf{x},\boldsymbol{\xi}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4\mu^2\sigma^{-1}\widetilde{b}(\mathbf{x}) \end{pmatrix}. \tag{6.36}$$

We put

$$J_S^1(\Omega,\mathcal{L}) = \big\{\mathbf{u} \in J_S^1(\Omega) : -\mu\Delta\mathbf{u} + \nabla p = 0 \text{ for some } p \in L_2(\Omega)\big\},$$
$$\widetilde{J}_S^1(\Omega,\mathcal{L}) = \big\{\mathbf{u} \in J_S^1(\Omega,\mathcal{L}) :\ \boldsymbol{\tau}_{tn}(\mathbf{u}) = 0 \text{ on } \Gamma\big\}.$$

Consider the ratio

$$\pm\frac{(\mathcal{R}\boldsymbol{\varphi},\boldsymbol{\varphi})_{L_2(\partial\widetilde{\Omega})}}{\mu E_\Omega[\mathbf{u}]}, \quad \mathbf{u} \in \widetilde{J}_S^1(\Omega,\mathcal{L}), \quad \boldsymbol{\varphi} := \mathbf{u}\big|_{\partial\widetilde{\Omega}}. \tag{6.37}$$

By Lemma 2.4, representation (6.35) implies that

$$\Delta_2^\pm\ (6.29) = \Delta_2^\pm\ (6.37), \quad \delta_2^\pm\ (6.29) = \delta_2^\pm\ (6.37). \tag{6.38}$$

Let us show that the principal term of the spectral asymptotics will not change if we remove a part of constraints and replace (6.37) by the ratio

$$\pm\frac{(\mathcal{R}\boldsymbol{\varphi},\boldsymbol{\varphi})_{L_2(\partial\widetilde{\Omega})}}{\mu E_\Omega[\mathbf{u}]}, \quad \mathbf{u} \in J_S^1(\Omega), \quad \boldsymbol{\varphi} := \mathbf{u}\big|_{\partial\widetilde{\Omega}}. \tag{6.39}$$

Lemma 6.3. *We have*

$$\Delta_2^\pm\ (6.37) = \Delta_2^\pm\ (6.39), \quad \delta_2^\pm\ (6.37) = \delta_2^\pm\ (6.39). \tag{6.40}$$

Proof. Denote $J_0^1(\Omega) := J^1(\Omega) \cap H_0^1(\Omega)$. We have $J_S^1(\Omega) = J_0^1(\Omega) \oplus^E J_S^1(\Omega,\mathcal{L})$, where the orthogonal sum is understood in the sense of the inner product $E_\Omega[\mathbf{u},\mathbf{v}]$.

Next, we put $\widehat{J}_S^1(\Omega,\mathcal{L}) = \big\{\mathbf{u} \in J_S^1(\Omega,\mathcal{L}) : u_n = 0 \text{ on } \Gamma\big\}$. Proposition 6.6 shows that $J_S^1(\Omega,\mathcal{L}) = \widehat{J}_S^1(\Omega,\mathcal{L}) \oplus \widetilde{J}_S^1(\Omega,\mathcal{L})$. Thus,

$$J_S^1(\Omega) = J_0^1(\Omega) \oplus \widehat{J}_S^1(\Omega,\mathcal{L}) \oplus \widetilde{J}_S^1(\Omega,\mathcal{L}). \tag{6.41}$$

We put $(\widehat{\mathcal{R}}\boldsymbol{\varphi})(\mathbf{x}) = 4\mu^2\sigma^{-1}\widetilde{b}(\mathbf{x})\varphi_\nu(\mathbf{x})\boldsymbol{\nu}(\mathbf{x})$, $\mathbf{x} \in \partial\widetilde{\Omega}$, where φ_ν is the normal component of $\boldsymbol{\varphi}$ on $\partial\widetilde{\Omega}$. Then $\widehat{\mathcal{R}}$ is a matrix pseudodifferential operator on $\partial\widetilde{\Omega}$ of order zero with the principal symbol $r^\circ(\mathbf{x},\boldsymbol{\xi})$. Therefore, $(\mathcal{R} - \widehat{\mathcal{R}})$ is a pseudodifferential operator of order less than or equal to (-1). Consider the ratio

$$\pm\frac{((\mathcal{R} - \widehat{\mathcal{R}})\boldsymbol{\varphi},\boldsymbol{\varphi})_{L_2(\partial\widetilde{\Omega})}}{\mu E_\Omega[\mathbf{u}]}, \quad \mathbf{u} \in J_S^1(\Omega), \quad \boldsymbol{\varphi} := \mathbf{u}\big|_{\partial\widetilde{\Omega}}. \tag{6.42}$$

Using the inequality $E_\Omega[\mathbf{u}] \geq C\|\boldsymbol{\varphi}\|^2_{H^{1/2}(\partial\widetilde{\Omega})}$, $C > 0$, by Lemma 2.2 we obtain that the functions $N_\pm(\lambda,\ (6.42))$ do not exceed the spectrum counting functions for the ratio

$$\pm\frac{((\mathcal{R} - \widehat{\mathcal{R}})\boldsymbol{\varphi},\boldsymbol{\varphi})_{L_2(\partial\widetilde{\Omega})}}{\mu C\|\boldsymbol{\varphi}\|^2_{H^{1/2}(\partial\widetilde{\Omega})}}, \quad \boldsymbol{\varphi} \in H^{1/2}(\partial\widetilde{\Omega};\mathbb{C}^3). \tag{6.43}$$

Since $N_\pm(\lambda,\ (6.43)) = O(\lambda^{-1})$, then $\Delta_2^\pm\ (6.43) = 0$. Hence, by Lemma 2.4 it follows that the values $\Delta_2^\pm\ (6.39)$ and $\delta_2^\pm\ (6.39)$ coincide with the similar values for the ratio

$$\pm\frac{(\widehat{\mathcal{R}}\boldsymbol{\varphi},\boldsymbol{\varphi})_{L_2(\partial\widetilde{\Omega})}}{\mu E_\Omega[\mathbf{u}]}, \quad \mathbf{u} \in J_S^1(\Omega), \quad \boldsymbol{\varphi} := \mathbf{u}\big|_{\partial\widetilde{\Omega}}, \tag{6.44}$$

and the values $\Delta_2^\pm$ (6.37), $\delta_2^\pm$ (6.37) coincide with the similar values for the ratio

$$\pm\frac{(\widehat{\mathcal{R}}\boldsymbol{\varphi},\boldsymbol{\varphi})_{L_2(\partial\widetilde{\Omega})}}{\mu E_\Omega[\mathbf{u}]}, \quad \mathbf{u}\in \widetilde{J}^1_S(\Omega,\mathcal{L}), \quad \boldsymbol{\varphi}:=\mathbf{u}\big|_{\partial\widetilde{\Omega}}. \tag{6.45}$$

Note that $\widehat{\mathcal{R}}\boldsymbol{\varphi}=0$ for $\mathbf{u}\in J^1_0(\Omega)\oplus\widehat{J}^1_S(\Omega,\mathcal{L})$. Therefore, from (6.41) it follows that $N_\pm(\lambda,\ (6.44)) = N_\pm(\lambda,\ (6.45))$. In view of the above, this implies relations (6.40). ☐

From Lemma 6.3, Lemma 6.2, and relations (6.28), (6.38) it follows that

$$\Delta_2^\pm\ (6.22) = \Delta_2^\pm\ (6.39), \quad \delta_2^\pm\ (6.22) = \delta_2^\pm\ (6.39). \tag{6.46}$$

Now, we compare ratio (6.39) with the form ratios in $\widetilde{\Omega}$. To estimate $\Delta_2^\pm$ (6.39) from above, consider the ratio

$$\pm\frac{(\mathcal{R}\boldsymbol{\varphi},\boldsymbol{\varphi})_{L_2(\partial\widetilde{\Omega})}}{\mu E_{\widetilde{\Omega}}[\mathbf{u}]+C\|\mathbf{u}\|^2_{L_2(\widetilde{\Omega})}}, \quad \mathbf{u}\in J^1(\widetilde{\Omega}), \quad \boldsymbol{\varphi}:=\mathbf{u}\big|_{\partial\widetilde{\Omega}}. \tag{6.47}$$

Here the constant C is so large that the form in denominator of (6.47) defines an equivalent norm in $H^1(\widetilde{\Omega})$. We have

$$\Delta_2^\pm\ (6.39) \le \Delta_2^\pm\ (6.47). \tag{6.48}$$

The proof of inequality (6.48) is similar to the proof of inequality (5.10): one should use Lemma 2.2, in which $\mathcal{S}$ is the restriction operator.

To estimate $N_\pm(\lambda,\ (6.39))$ from below, we consider the ratio

$$\pm\frac{(\mathcal{R}\boldsymbol{\varphi},\boldsymbol{\varphi})_{L_2(\partial\widetilde{\Omega})}}{\mu E_{\widetilde{\Omega}}[\mathbf{u}]}, \quad \mathbf{u}\in J^1_{\widetilde{S}}(\widetilde{\Omega}), \quad \boldsymbol{\varphi}:=\mathbf{u}\big|_{\partial\widetilde{\Omega}}. \tag{6.49}$$

Here $\widetilde{S}=\partial\widetilde{\Omega}\setminus\widetilde{\Gamma}$ and $J^1_{\widetilde{S}}(\widetilde{\Omega})=\{\mathbf{u}\in J^1(\widetilde{\Omega}):\ \mathbf{u}|_{\widetilde{S}}=0\}$. By analogy with the proof of inequality (5.12), using Lemma 2.2 and the operator of extension by zero, we obtain that

$$N_\pm(\lambda,\ (6.49)) \le N_\pm(\lambda,\ (6.39)). \tag{6.50}$$

6.6. Asymptotic formulas in the smooth case. Using (6.46), (6.48), and (6.50), we conclude that the spectral asymptotics for ratio (6.22) will be found as soon as the following statement is proved.

Lemma 6.4. *For $\lambda\to+0$ we have*

$$N_\pm(\lambda,\ (6.47)) \sim N_\pm(\lambda,\ (6.49)) \sim \lambda^{-2}\frac{\mu^2}{\pi\sigma^2}\int_{\partial\widetilde{\Omega}}\widetilde{b}^2_\pm(\mathbf{x})\,dS(\mathbf{x}). \tag{6.51}$$

Proof. By analogy with the proof of Lemma 5.2, it is easy to show that the problem on the spectrum of ratio (6.47) is equivalent (in the sense of the spectral asymptotics) to the problem on the spectrum of the ratio

$$\pm\frac{(\mathcal{R}\boldsymbol{\varphi},\boldsymbol{\varphi})_{L_2(\partial\widetilde{\Omega})}}{\mu(\mathcal{E}\boldsymbol{\varphi},\boldsymbol{\varphi})_{L_2(\partial\widetilde{\Omega})}}, \quad \boldsymbol{\varphi}\in H^{1/2}(\partial\widetilde{\Omega};\mathbb{C}^3), \tag{6.52}$$

and the problem on the spectrum of ratio (6.49) is equivalent (in the sense of the spectral asymptotics) to the problem of the spectrum of the ratio

$$\pm\frac{(\mathcal{R}\boldsymbol{\varphi},\boldsymbol{\varphi})_{L_2(\partial\widetilde{\Omega})}}{\mu(\mathcal{E}\boldsymbol{\varphi},\boldsymbol{\varphi})_{L_2(\partial\widetilde{\Omega})}}, \quad \boldsymbol{\varphi}\in H^{1/2}(\partial\widetilde{\Omega};\mathbb{C}^3), \quad \boldsymbol{\varphi}|_{\widetilde{S}}=0. \tag{6.53}$$

The pseudodifferential operator $\mathcal{E}$ is defined in Sec. 5.4. As before, by changing the lower order terms in $\mathcal{E}$, we assume that inequality (5.19) holds. Consider the algebraic problem

$$r^\circ(\mathbf{x},\boldsymbol{\xi})\mathbf{z}=\lambda\mu e^\circ(\mathbf{x},\boldsymbol{\xi})\mathbf{z}, \quad \mathbf{z}\in\mathbb{C}^3. \tag{6.54}$$

Lemma 2.10 and Lemma 2.11 imply the following asymptotic formulas for $\lambda\to+0$:

$$N_\pm(\lambda,\ (6.52)) \sim \frac{1}{(2\pi)^2}\int_{\partial\widetilde{\Omega}}dS(\mathbf{x})\int_{\boldsymbol{\xi}\perp\boldsymbol{\nu}(\mathbf{x})}d\boldsymbol{\xi}\,n_\pm(\lambda,\mathbf{x},\boldsymbol{\xi};\ (6.54)), \tag{6.55}$$

$$N_\pm(\lambda,\ (6.53)) \sim \frac{1}{(2\pi)^2}\int_{\widetilde{\Gamma}}dS(\mathbf{x})\int_{\boldsymbol{\xi}\perp\boldsymbol{\nu}(\mathbf{x})}d\boldsymbol{\xi}\,n_\pm(\lambda,\mathbf{x},\boldsymbol{\xi};\ (6.54)). \tag{6.56}$$

According to the expressions for $r^\circ(\mathbf{x},\boldsymbol{\xi})$, $e^\circ(\mathbf{x},\boldsymbol{\xi})$ (see (5.31) and (6.36)), we have

$$n_\pm(\lambda,\mathbf{x},\boldsymbol{\xi};\ (6.54)) = \begin{cases} 1, & \lambda < \frac{2\mu\widetilde{b}_\pm(\mathbf{x})}{\sigma|\boldsymbol{\xi}|}, \\ 0, & \lambda \geq \frac{2\mu\widetilde{b}_\pm(\mathbf{x})}{\sigma|\boldsymbol{\xi}|}. \end{cases}$$

Calculating by formulas (6.55) and (6.56), we get (6.51). □

Relations (6.46), (6.48), (6.50), and Lemma 6.4 imply (6.23) in the case where $b \in C_0^\infty(\Gamma)$. By closure, the asymptotics is valid for any $b \in C^\infty(\overline{\Gamma})$. Theorem 6.2, and Theorem 6.1 with it, are proved.

6.7. Oscillations of a system of the capillary viscous fluids. Theorem 5.1 allows generalization to the case of oscillations of a system of the capillary viscous fluids, partially or completely filling a vessel. For definiteness, consider the case of complete filling. We restrict ourselves to the statement of the problem and formulation of the result.

Suppose that a domain $\Omega \subset \mathbb{R}^3$ satisfies Condition 4.3. We consider the following boundary-value problem for a system of vector-valued functions $\{\mathbf{u}_j(\mathbf{x})\}$, $j = 1,\dots,k+1$, and a system of scalar functions $\{p_j(\mathbf{x})\}$, $j = 1,\dots,k+1$:

$$\begin{gathered} -\mu_j\Delta\mathbf{u}_j + \nabla p_j = 0, \quad \operatorname{div}\mathbf{u}_j = 0 \ \text{ in } \Omega_j, \quad j = 1,\dots,k+1, \\ \mathbf{u}_j = 0 \ \text{ on } S_j, \quad j = 1,\dots,k+1, \\ \mathbf{u}_j = \mathbf{u}_{j+1}, \quad \boldsymbol{\tau}_{tn}(\mathbf{u}_j) = \boldsymbol{\tau}_{tn}(\mathbf{u}_{j+1}) \ \text{ on } \Gamma_j, \quad j = 1,\dots,k, \\ \left(-\sigma_j\Delta_{\Gamma_j} + h_j(\mathbf{x})\right) u_{jn} = \lambda^{-1}\left(\tau_{nn}^{(j)} - \tau_{nn}^{(j+1)} + c_j\right) \quad \text{on } \Gamma_j, \quad j = 1,\dots,k. \end{gathered} \tag{6.57}$$

Here $u_{jn}(\mathbf{x}) := \langle \mathbf{u}_j(\mathbf{x}), \mathbf{n}_j(\mathbf{x})\rangle$, $\tau_{nn}^{(j)} := \tau_{nn}(\mathbf{u}_j, p_j)$; $\mu_j > 0$, $\sigma_j > 0$ are constants; $h_j \in C^\infty(\overline{\Gamma_j})$ are real-valued functions. The constants c_j are searched along with the solution.

By $\mathbf{B}_j$ we denote the operator in $L_2(\Gamma_j)$ given by the expression $-\sigma_j\Delta_{\Gamma_j} + h_j(\mathbf{x})$ on the domain $\operatorname{Dom}\mathbf{B}_j = H^2(\Gamma_j) \cap H_0^1(\Gamma_j)$.

Let us give a variational statement of problem (6.57) in the case where the operators $\mathbf{B}_j$ are invertible (the general case can be considered by analogy with Sec. 6.2). Let $z_j = \mathbf{B}_j^{-1}1$ and let P_{z_j} be a projection in $L_2(\Gamma_j)$ defined similarly to (6.2).

Problem (6.57) is equivalent to the problem on the spectrum of the form ratio

$$\begin{gathered} \pm\frac{\sum_{j=1}^k \int_{\Gamma_j}\left(\mathbf{B}_j^{-1}P_{z_j}\left(\tau_{nn}^{(j)} - \tau_{nn}^{(j+1)}\right)\right)\overline{P_{z_j}\left(\tau_{nn}^{(j)} - \tau_{nn}^{(j+1)}\right)}\,dS}{\sum_{j=1}^{k+1}\mu_j E_{\Omega_j}[\mathbf{u}_j]}, \\ \mathbf{u}_j \in H^1(\Omega_j;\mathbb{C}^3), \quad p_j \in L_2(\Omega_j), \quad j = 1,\dots,k+1, \\ -\mu_j\Delta\mathbf{u}_j + \nabla p_j = 0, \quad \operatorname{div}\mathbf{u}_j = 0 \ \text{ in } \Omega_j, \quad \mathbf{u}_j = 0 \ \text{ on } S_j, \quad j = 1,\dots,k+1, \\ \mathbf{u}_j = \mathbf{u}_{j+1}, \quad \boldsymbol{\tau}_{tn}(\mathbf{u}_j) = \boldsymbol{\tau}_{tn}(\mathbf{u}_{j+1}) \ \text{ on } \Gamma_j, \quad j = 1,\dots,k. \end{gathered} \tag{6.58}$$

Proposition 6.8. *Under the above assumptions, the spectrum counting function for ratio* (6.58) *satisfies the following asymptotics for* $\lambda \to +0$:

$$N(\lambda,\ (6.58)) \sim \lambda^{-2}\sum_{j=1}^k (\mu_j + \mu_{j+1})^2\frac{\operatorname{meas}\Gamma_j}{\pi\sigma_j^2}.$$

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