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HOMOGENIZATION OF THE HIGHER-ORDER PARABOLIC EQUATIONS WITH PERIODIC COEFFICIENTS

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Abstract. In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider a wide class of matrix elliptic operators $\mathcal{A}_\varepsilon$ of order $2p$ (where $p \geq 2$) with periodic rapidly oscillating coefficients (depending on $\mathbf{x}/\varepsilon$). Here $\varepsilon > 0$ is a small parameter. We study the behavior of the operator exponential $e^{-\mathcal{A}_\varepsilon \tau}$ for $\tau > 0$ and small ε. It is shown that the operator $e^{-\mathcal{A}_\varepsilon \tau}$ converges as $\varepsilon \to 0$ in the operator norm in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the exponential $e^{-\mathcal{A}^0 \tau}$ of the effective operator $\mathcal{A}^0$. Also we obtain an approximation of the operator exponential $e^{-\mathcal{A}_\varepsilon \tau}$ in the norm of operators acting from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the Sobolev space $H^p(\mathbb{R}^d;\mathbb{C}^n)$. We derive error estimates for these approximations depending on two parameters: ε and τ. For a fixed $\tau > 0$, the errors are of the sharp order $O(\varepsilon)$. The results are applied to study the behavior of the solution of the Cauchy problem for the parabolic equation $\partial_\tau \mathbf{u}_\varepsilon(\mathbf{x},\tau) = -(\mathcal{A}_\varepsilon \mathbf{u}_\varepsilon)(\mathbf{x},\tau) + \mathbf{F}(\mathbf{x},\tau)$ in $\mathbb{R}^d$.

Keywords: homogenization, higher-order parabolic equations, periodic coefficients, operator exponential, approximation.

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1. Introduction

The paper concerns homogenization theory for periodic differential operators (DOs). This theory studies the behavior of solutions of differential equations with periodic rapidly oscillating coefficients. It is a wide area of theoretical and applied science. An extensive literature is devoted to homogenization problems; first of all, we mention the monographs [1, 2, 14, 26].

1.1. Operator error estimates for elliptic and parabolic homogenization problems in $\mathbb{R}^d$.

In a series of papers [3–6] by Birman and Suslina, an *operator-theoretic approach* to homogenization problems was suggested and developed. By this approach, a wide class of matrix second-order self-adjoint DOs acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ and admitting a factorization of the form

$$A = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}), \quad \mathbf{D} = -i\nabla, \quad \operatorname{ord} b(\mathbf{D}) = 1, \tag{1.1}$$

was studied. Here $g(\mathbf{x})$ is a bounded and uniformly positive definite matrix-valued function of size $m \times m$, periodic with respect to some lattice $\Gamma \subset \mathbb{R}^d$. Let Ω be the elementary cell of the lattice Γ. Next, $b(\mathbf{D})$ is an $(m \times n)$-matrix homogeneous first-order DO. It is assumed that $m \geq n$ and the symbol $b(\boldsymbol{\xi})$ has rank n for any $0 \neq \boldsymbol{\xi} \in \mathbb{R}^d$. Under these assumptions, the operator A is strongly elliptic. The simplest example of operator (1.1) is the scalar elliptic operator $A = -\operatorname{div} g(\mathbf{x})\nabla$ (the acoustics operator); the operator of elasticity theory also can be written in form (1.1). These and other examples were considered in [3, 5, 6] in details.

Let $\varepsilon > 0$ be a small parameter. For any Γ-periodic function $\varphi(\mathbf{x})$, we denote $\varphi^\varepsilon(\mathbf{x}) := \varphi(\varepsilon^{-1}\mathbf{x})$. Consider the operator $A_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})$, whose coefficients oscillate rapidly for $\varepsilon \to 0$.

In [3], it was shown that, as $\varepsilon \to 0$, the resolvent $(A_\varepsilon + I)^{-1}$ converges in the operator norm in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the resolvent of the *effective operator* $A^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$. Here g^0 is a constant positive matrix called the *effective matrix*. The following estimate was obtained:

$$\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq C\varepsilon, \quad \varepsilon > 0. \tag{1.2}$$

In [4, 5], a more accurate approximation of the resolvent $(A_\varepsilon + I)^{-1}$ in the operator norm in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ with an error of order $O(\varepsilon^2)$ was found. In [6], an approximation of the resolvent $(A_\varepsilon + I)^{-1}$ in the norm of operators acting from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the Sobolev space $H^1(\mathbb{R}^d;\mathbb{C}^n)$ with the following error estimate was obtained:

$$\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1} - \varepsilon K(\varepsilon)\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leq C\varepsilon, \quad \varepsilon > 0. \tag{1.3}$$

Here $K(\varepsilon)$ is the so-called *corrector*; this operator contains a rapidly oscillating factor and thus depends on ε; we have $\|K(\varepsilon)\|_{L_2\to H^1} = O(\varepsilon^{-1})$.

The operator-theoretic approach was also applied to parabolic problems of homogenization theory. In [18, 19], it was shown that for fixed $\tau > 0$ and $\varepsilon \to 0$ the operator exponential $e^{-A_\varepsilon \tau}$ converges in the operator norm in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the exponential of the effective operator A^0. The following estimate holds:

$$\|e^{-A_\varepsilon \tau} - e^{-A^0 \tau}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \frac{C\varepsilon}{\tau^{\frac{1}{2}} + \varepsilon}, \quad \varepsilon > 0, \ \tau \geq 0. \tag{1.4}$$

A more accurate approximation of the exponential $e^{-A_\varepsilon \tau}$ in the operator norm in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ with an error of order $O(\varepsilon^2)$ was found in the paper [23]. In [20], approximation of the operator exponential $e^{-A_\varepsilon \tau}$ in the norm of operators acting from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the Sobolev space $H^1(\mathbb{R}^d;\mathbb{C}^n)$ with the following error estimate was obtained:

$$\|e^{-A_\varepsilon \tau} - e^{-A^0 \tau} - \varepsilon K(\varepsilon,\tau)\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leq \frac{C\varepsilon}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2}} + \varepsilon)}, \quad \varepsilon > 0, \ \tau > 0. \tag{1.5}$$

Here $K(\varepsilon,\tau)$ is the corresponding corrector.

Estimates (1.2)–(1.5) are sharp with respect to the order of the parameter ε; the constants are controlled explicitly in terms of the problem data. Such results are called the *operator error estimates* in homogenization theory. The method in [3–6, 18–20, 23] is based on the scaling transformation, the direct integral expansion for the operator A (on the basis of the Floquet–Bloch theory), and the analytic perturbation theory. It was found that the resolvent and the exponential for A_ε can be approximated in terms of the threshold characteristics of the operator A at the bottom of the spectrum. In this sense, the homogenization procedure is a manifestation of the *spectral threshold effect*.

We also mention paper [21], in which analogs of estimates (1.2), (1.3) for the resolvent $(A_\varepsilon - \zeta I)^{-1}$ at an arbitrary regular point $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$ were obtained (i.e., two-parametric error estimates depending on ε and ζ).

A different approach to obtaining operator error estimates (the so-called *modified method of the first-order approximation* or the *shift method*) was suggested by Zhikov [25] and developed by him together with Pastukhova [27, 28]. In the mentioned papers, the operator error estimates were obtained for the acoustics and elasticity operators. For further results, see the survey [29].

The homogenization problem for periodic elliptic DOs of higher even order is of particular interest. In papers [24] by Veniaminov and [8] by Kukushkin and Suslina, the operator-theoretic approach was developed for such operators. In [24], an operator of the form $\mathcal{B}_\varepsilon = (\mathbf{D}^p)^* g^\varepsilon(\mathbf{x})\mathbf{D}^p$ was studied; here $g(\mathbf{x})$ is a symmetric uniformly positive definite tensor of order $2p$ that is periodic with respect to the lattice Γ. For $p = 2$, such an operator arises in the theory of elastic plates (see [26]). In [24], the following analog of estimate (1.2) was obtained:

$$\|(\mathcal{B}_\varepsilon + I)^{-1} - (\mathcal{B}^0 + I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq C\varepsilon, \quad \varepsilon > 0.$$

In [8], a more general class of higher-order DOs similar to operators (1.1) was studied:

$$\widehat{\mathcal{A}} = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}), \quad \operatorname{ord} b(\mathbf{D}) = p \geq 2.$$

Here $g(\mathbf{x})$ is a bounded and uniformly positive definite matrix-valued function of size $m \times m$ that is periodic with respect to the lattice Γ, and $b(\mathbf{D})$ is an $(m \times n)$-matrix homogeneous DO of order p. Assume that $p \geq 2$. It is supposed that $m \geq n$ and the symbol $b(\boldsymbol{\xi})$ has rank n for any $0 \neq \boldsymbol{\xi} \in \mathbb{R}^d$. Under these assumptions, the operator $\widehat{\mathcal{A}}$ is strongly elliptic. Let $\widehat{\mathcal{A}}_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})$. In [8], the behavior of the resolvent $(\widehat{\mathcal{A}}_\varepsilon - \zeta I)^{-1}$ at an arbitrary regular point $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$ was studied. It was shown that, as $\varepsilon \to 0$, the resolvent $(\widehat{\mathcal{A}}_\varepsilon - \zeta I)^{-1}$ converges in the operator norm in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ to the resolvent of the *effective operator* $\widehat{\mathcal{A}}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$. The following error estimate was obtained:

$$\|(\widehat{\mathcal{A}}_\varepsilon - \zeta I)^{-1} - (\widehat{\mathcal{A}}^0 - \zeta I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq C_1(\zeta)\varepsilon, \quad \varepsilon > 0.$$

Also, approximation of the resolvent in the norm of operators acting from $L_2(\mathbb{R}^d; \mathbb{C}^n)$ to the Sobolev space $H^p(\mathbb{R}^d; \mathbb{C}^n)$ with the following error estimate was found:

$$\|(\widehat{\mathcal{A}}_\varepsilon - \zeta I)^{-1} - (\widehat{\mathcal{A}}^0 - \zeta I)^{-1} - \varepsilon^p \mathcal{K}(\varepsilon, \zeta)\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)} \leq C_2(\zeta)\varepsilon, \quad \varepsilon > 0.$$

Moreover, the dependence of the values $C_1(\zeta)$, $C_2(\zeta)$ on the parameter ζ was tracked, so that the estimates were two-parametric.

We also mention recent papers [15–17], in which a more accurate approximation of the resolvent $(\widehat{\mathcal{A}}_\varepsilon + I)^{-1}$ in the operator norm in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ with an error of order $O(\varepsilon^{2p})$ was found by using the operator-theoretic approach.

The shift method was applied to higher-order elliptic operators in papers [10–13] by Pastukhova.

1.2. Statement of the problem. Main results. In this paper, we continue to study homogenization problems for the operators $\widehat{\mathcal{A}}_\varepsilon$ of the order $2p$ in $\mathbb{R}^d$ described above. We also study more general operators of the form

$$\mathcal{A}_\varepsilon = (f^\varepsilon)^* \widehat{\mathcal{A}}_\varepsilon f^\varepsilon = (f^\varepsilon(\mathbf{x}))^* b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) f^\varepsilon(\mathbf{x}), \tag{1.6}$$

where $f(\mathbf{x})$ is a periodic matrix-valued function of size $n \times n$ such that $f, f^{-1} \in L_\infty(\mathbb{R}^d)$.

Our goal is to approximate the operator exponential $e^{-\mathcal{A}_\varepsilon \tau}$ for fixed $\tau > 0$ and small ε in different operator norms.

Our *first main result* is the following: it is shown that for a fixed $\tau > 0$ and $\varepsilon \to 0$ the operator $e^{-\widehat{\mathcal{A}}_\varepsilon \tau}$ converges in the operator norm in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ to the operator $e^{-\widehat{\mathcal{A}}^0 \tau}$. The following error estimate is proved:

$$\left\| e^{-\widehat{\mathcal{A}}_\varepsilon \tau} - e^{-\widehat{\mathcal{A}}^0 \tau} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \frac{C\varepsilon}{\tau^{\frac{1}{2p}} + \varepsilon}, \quad \varepsilon > 0, \ \tau \geq 0. \tag{1.7}$$

This inequality is a generalization of estimate (1.4) to the case of higher-order operators. It turned out that for more general operators of form (1.6), a similar result is true for the "sandwiched" operator exponential $f^\varepsilon e^{-\mathcal{A}_\varepsilon \tau} (f^\varepsilon)^*$:

$$\left\| f^\varepsilon e^{-\mathcal{A}_\varepsilon \tau} (f^\varepsilon)^* - f_0 e^{-\mathcal{A}^0 \tau} f_0 \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \frac{C\varepsilon}{\tau^{\frac{1}{2p}} + \varepsilon}, \quad \varepsilon > 0, \ \tau \geq 0.$$

Here f_0 is some constant matrix (see (9.1)) and $\mathcal{A}^0 = f_0 \widehat{\mathcal{A}}^0 f_0$.

Next, we distinguish a condition on the operator, under which these results can be improved. Under this condition, instead of (1.7) we have

$$\left\| e^{-\widehat{\mathcal{A}}_\varepsilon \tau} - e^{-\widehat{\mathcal{A}}^0 \tau} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \frac{C\varepsilon^2}{\tau^{\frac{1}{p}} + \varepsilon^2}, \quad \varepsilon > 0, \ \tau \geq 0.$$

The condition mentioned above is formulated in terms of the spectral characteristics of the operator at the bottom of the spectrum; it is automatically satisfied for the scalar operator $\widehat{\mathcal{A}}_\varepsilon$ (i.e., in the case where $n = 1$) with real-valued coefficients. Thus, a new effect characteristic for higher-order operators is discovered: in the "scalar real" case it is possible to approximate the exponential of the operator $\widehat{\mathcal{A}}_\varepsilon$ by the exponential of the effective operator with error $O(\varepsilon^2)$ without taking any correctors into account. There is no such effect for the second-order operators. Note that the same effect for higher-order operators is also observed when approximating the resolvent or the exponential of the form $e^{-i\widehat{\mathcal{A}}_\varepsilon \tau}$; see [12, 15, 16, 22].

The same improvement occurs for the operator $f^\varepsilon e^{-\mathcal{A}_\varepsilon \tau} (f^\varepsilon)^*$.

The *second main result is the following*: approximation for the exponential $e^{-\widehat{\mathcal{A}}_\varepsilon \tau}$ in the "energy" norm is found. The following estimates are proved:

$$\left\| \mathbf{D}^p \left(e^{-\widehat{\mathcal{A}}_\varepsilon \tau} - e^{-\widehat{\mathcal{A}}^0 \tau} - \varepsilon^p \widehat{\mathcal{K}}(\varepsilon, \tau) \right) \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \frac{C\varepsilon}{\tau^{\frac{1}{2}} (\tau^{\frac{1}{2p}} + \varepsilon)}, \quad \varepsilon > 0, \ \tau > 0,$$

$$\left\| e^{-\widehat{\mathcal{A}}_\varepsilon \tau} - e^{-\widehat{\mathcal{A}}^0 \tau} - \varepsilon^p \widehat{\mathcal{K}}(\varepsilon, \tau) \right\|_{L_2(\mathbb{R}^d) \to H^p(\mathbb{R}^d)} \leq \frac{C(1 + \tau^{-\frac{1}{2}})\varepsilon}{\tau^{\frac{1}{2p}} + \varepsilon}, \quad \varepsilon > 0, \ \tau > 0.$$

Also, approximation for the operator $g^\varepsilon b(\mathbf{D}) e^{-\widehat{\mathcal{A}}_\varepsilon \tau}$ (corresponding to the "flux") in the $(L_2 \to L_2)$-norm is found. Here $\widehat{\mathcal{K}}(\varepsilon, \tau)$ is the corresponding corrector. It contains a rapidly oscillating factor and therefore depends on ε; we have $\|\widehat{\mathcal{K}}(\varepsilon, \tau)\|_{L_2 \to H^p} = O(\varepsilon^{-p})$. Similar results are obtained for the sandwiched exponential $f^\varepsilon e^{-\mathcal{A}_\varepsilon \tau} (f^\varepsilon)^*$.

The results formulated in operator terms are then applied to study the behavior of the solution of the Cauchy problem for the parabolic equation. Let $\mathbf{u}_\varepsilon(\mathbf{x}, \tau)$ be the solution of the problem

$$\begin{aligned} &\frac{\partial \mathbf{u}_\varepsilon(\mathbf{x}, \tau)}{\partial \tau} = -b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) \mathbf{u}_\varepsilon(\mathbf{x}, \tau) + \mathbf{F}(\mathbf{x}, \tau), \quad \mathbf{x} \in \mathbb{R}^d, \ \tau > 0, \\ &\mathbf{u}_\varepsilon(\mathbf{x}, 0) = \boldsymbol{\phi}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d. \end{aligned}$$

We have

$$\mathbf{u}_\varepsilon(\cdot,\tau) = e^{-\widehat{\mathcal{A}}_\varepsilon \tau}\boldsymbol{\phi} + \int\limits_0^\tau e^{-\widehat{\mathcal{A}}_\varepsilon(\tau-\widetilde{\tau})}\mathbf{F}(\cdot,\widetilde{\tau})\,d\widetilde{\tau}.$$

Based on this representation, we derive the results on approximations of the solution $\mathbf{u}_\varepsilon$ from the results on approximation of the operator exponential.

It turns out that for a fixed $\tau > 0$ the solution converges in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the solution $\mathbf{u}_0$ of the homogenized problem

$$\frac{\partial \mathbf{u}_0(\mathbf{x},\tau)}{\partial \tau} = -b(\mathbf{D})^* g^0 b(\mathbf{D})\mathbf{u}_0(\mathbf{x},\tau) + \mathbf{F}(\mathbf{x},\tau), \quad \mathbf{x}\in\mathbb{R}^d,\ \tau>0,$$
$$\mathbf{u}_0(\mathbf{x},0) = \boldsymbol{\phi}(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^d.$$

An error estimate, as well as approximation of the solution in the norm of $H^p(\mathbb{R}^d;\mathbb{C}^n)$, are obtained.

A more general Cauchy problem is also considered (see (15.31) below); the results for it can be deduced from the results about the behavior of the operator $f^\varepsilon e^{-\mathcal{A}_\varepsilon \tau}(f^\varepsilon)^*$.

1.3. Method. We rely on the operator-theoretic approach. Let us explain the method by the example of the proof of estimate (1.7). By the scaling transformation, we have

$$\left\| e^{-\widehat{\mathcal{A}}_\varepsilon \tau} - e^{-\widehat{\mathcal{A}}^0 \tau}\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} = \left\| e^{-\widehat{\mathcal{A}}\varepsilon^{-2p}\tau} - e^{-\widehat{\mathcal{A}}^0\varepsilon^{-2p}\tau}\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}.$$

Therefore, estimate (1.7) is equivalent to the inequality

$$\left\| e^{-\widehat{\mathcal{A}}\tau} - e^{-\widehat{\mathcal{A}}^0 \tau}\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \frac{C}{\tau^{\frac{1}{2p}}+1}, \quad \tau \ge 0.$$

Next, we apply the Floquet–Bloch theory. Using the unitary Gelfand transform, we expand the operator $\widehat{\mathcal{A}}$ with periodic coefficients in the direct integral of the operators $\widehat{\mathcal{A}}(\mathbf{k})$ acting in the space $L_2(\Omega;\mathbb{C}^n)$ and depending on the parameter $\mathbf{k}$ (the *quasimomentum*). The operator $\widehat{\mathcal{A}}(\mathbf{k})$ is given by the differential expression $b(\mathbf{D}+\mathbf{k})^* g(\mathbf{x}) b(\mathbf{D}+\mathbf{k})$ with periodic boundary conditions. The family $\widehat{\mathcal{A}}(\mathbf{k})$ is an analytic operator family with discrete spectrum. We put $\mathbf{k} = t\boldsymbol{\theta}$, where $t = |\mathbf{k}|$ and $\boldsymbol{\theta} = \dfrac{\mathbf{k}}{|\mathbf{k}|}$, and study the family $\widehat{\mathcal{A}}(\mathbf{k}) =: A(t,\boldsymbol{\theta})$ by means of the analytic perturbation theory with respect to the one-dimensional parameter t. The role of the unperturbed operator is played by $\widehat{\mathcal{A}}(0)$. The point $\lambda_0 = 0$ is an isolated eigenvalue of the operator $\widehat{\mathcal{A}}(0)$ of multiplicity n; the corresponding eigenspace $\mathfrak{N}$ consists of constant vector-valued functions in $L_2(\Omega;\mathbb{C}^n)$. Then for $t \le t^0$ the operator $A(t,\boldsymbol{\theta})$ has exactly n eigenvalues (counting multiplicities) on the interval $[0,\delta]$, while the interval $(\delta, 3\delta)$ is free of the spectrum. The numbers δ and t^0 are controlled explicitly. It turns out that only the selected part of the spectrum is essential for our problem. By the analytic perturbation theory, there exist real-analytic (in t) branches of the eigenvalues $\lambda_l(t,\boldsymbol{\theta})$ and the branches of the eigenelements $\varphi_l(t,\boldsymbol{\theta})$ of the operator $A(t,\boldsymbol{\theta})$, $l = 1,\dots,n$. The set $\{\varphi_l(t,\boldsymbol{\theta})\}$, $l = 1,\dots,n$, forms an orthonormal basis in the eigenspace of the operator $A(t,\boldsymbol{\theta})$ corresponding to the interval $[0,\delta]$. These analytic branches satisfy the following power series expansions:

$$\lambda_l(t,\boldsymbol{\theta}) = \gamma_l(\boldsymbol{\theta})t^{2p} + \mu_l(\boldsymbol{\theta})t^{2p+1} + \dots, \quad \gamma_l(\boldsymbol{\theta}) > 0, \quad l = 1,\dots,n,$$
$$\varphi_l(t,\boldsymbol{\theta}) = \omega_l(\boldsymbol{\theta}) + t\varphi_l^{(1)}(\boldsymbol{\theta}) + \dots, \quad l = 1,\dots,n.$$

The set $\{\omega_l(\boldsymbol{\theta})\}, l = 1,\dots,n$, forms an orthonormal basis in the subspace $\mathfrak{N}$. The coefficients of these power series expansions are called the *threshold characteristics* of the operator family $A(t,\boldsymbol{\theta})$ at the bottom of the spectrum. In their terms, the so-called *spectral germ* $S(\boldsymbol{\theta})$ is defined. This is a self-adjoint operator in the n-dimensional space $\mathfrak{N}$ such that

$$S(\boldsymbol{\theta})\omega_l(\boldsymbol{\theta}) = \gamma_l(\boldsymbol{\theta})\omega_l(\boldsymbol{\theta}), \quad l = 1,\dots,n.$$

Our main technical result is approximation of the exponential $e^{-A(t,\boldsymbol{\theta})\tau}$ in terms of the spectral germ:

$$\left\| e^{-A(t,\boldsymbol{\theta})\tau} - e^{-t^{2p}S(\boldsymbol{\theta})\tau}P \right\|_{L_2(\Omega)\to L_2(\Omega)} \le \frac{C}{\tau^{\frac{1}{2p}}+1}, \quad \tau > 0, \quad t \le t^0.$$

Here P is the orthogonal projection of the space $L_2(\Omega;\mathbb{C}^n)$ onto $\mathfrak{N}$. It is possible to calculate the germ: we have

$$S(\boldsymbol{\theta}) = b(\boldsymbol{\theta})^* g^0 b(\boldsymbol{\theta}),$$

where g^0 is the effective matrix. It turns out that the effective operator has the same spectral germ. This gives us the opportunity to switch to the approximation of the exponential $e^{-A(t,\boldsymbol{\theta})\tau}$ by $e^{-A^0(t,\boldsymbol{\theta})\tau}$. As a result, we arrive at the required estimate (1.7).

It is convenient to study the family $A(t,\boldsymbol{\theta})$, which is a polynomial operator pencil, within the framework of an abstract operator-theoretic scheme proposed in [8, 24]. In the abstract setting, we study a polynomial pencil of operators of the form $A(t) = X(t)^*X(t)$, where $X(t) = \sum_{j=0}^{p} t^j X_j$, acting in some Hilbert space. A pencil $A(t)$ models the family $\widehat{\mathcal{A}}(\mathbf{k}) = A(t,\boldsymbol{\theta})$, but the parameter $\boldsymbol{\theta}$ in the abstract setting is absent. In Chap. 1, we further develop the abstract scheme and obtain the required approximations of the operator exponential $e^{-A(t)\tau}$.

1.4. Plan of the paper. The paper consists of three chapters. Chap. 1 (Secs. 2–4) contains the necessary abstract operator-theoretic material. Here the main results are obtained on the abstract level. Chap. 2 (Secs. 5–11) is devoted to the study of periodic DOs acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. In Sec. 5, the class of operators $\mathcal{A}$ is introduced, the lattices and the Gelfand transform are described. Sec. 6 is devoted to the expansion of the operator $\mathcal{A}$ in the direct integral of the operators $\mathcal{A}(\mathbf{k})$. In Sec. 7, the effective characteristics of the operator $\widehat{\mathcal{A}}$ (the case where $f = \mathbf{1}$) are described, the effective operator is introduced. In Sec. 8, using the abstract results, we find an approximation of the exponential $e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau}$. In Sec. 9, we consider a more general family $\mathcal{A}(\mathbf{k})$ and find an approximation for the sandwiched exponential $fe^{-\mathcal{A}(\mathbf{k})\tau}f^*$. In Sec. 10, by direct integral expansion, we deduce approximation of the exponential $e^{-\widehat{\mathcal{A}}\tau}$ from the results of Sec. 8. Similarly, in Sec. 11 we derive approximation for the sandwiched exponential $fe^{-\mathcal{A}\tau}f^*$ from the results of Sec. 9. Chap. 3 (Secs. 12–15) is devoted to homogenization problems. The operators $\mathcal{A}_\varepsilon$ and $\widehat{\mathcal{A}}_\varepsilon$, and also the scaling transformation are described in short Sec. 12. In Secs. 13 and 14, we deduce main results on approximations for the exponential $e^{-\widehat{\mathcal{A}}_\varepsilon\tau}$ and the sandwiched exponential $f^\varepsilon e^{-\mathcal{A}_\varepsilon\tau}(f^\varepsilon)^*$. Finally, in Sec. 15, we apply the results to homogenization of solutions of the Cauchy problem. In Appendix (Sec. 16), we discuss another way to obtain the results on homogenization of the operator exponential.

1.5. Notation. Let $\mathfrak{H}$ and $\mathfrak{H}_*$ be complex separable Hilbert spaces. The symbols $(\cdot,\cdot)_\mathfrak{H}$ and $\|\cdot\|_\mathfrak{H}$ stand for the inner product and the norm in $\mathfrak{H}$, respectively; the symbol $\|\cdot\|_{\mathfrak{H}\to\mathfrak{H}_*}$ denotes the norm of a linear continuous operator from $\mathfrak{H}$ to $\mathfrak{H}_*$. Sometimes we omit the indices. By $I = I_\mathfrak{H}$ we denote the identity operator in $\mathfrak{H}$. If $X : \mathfrak{H} \to \mathfrak{H}_*$ is a linear operator, then $\operatorname{Dom} X$ denotes its domain and $\operatorname{Ker} X$ denotes its kernel. If $\mathfrak{N}$ is a subspace of $\mathfrak{H}$, then the symbol $\mathfrak{N}^\perp$ denotes its orthogonal complement. If P is the orthogonal projection of the space $\mathfrak{H}$ onto $\mathfrak{N}$, then $P^\perp = I - P$ is the orthogonal projection onto $\mathfrak{N}^\perp$.

The inner product and the norm in $\mathbb{C}^n$ are denoted by $\langle\cdot,\cdot\rangle$ and $|\cdot|$, respectively; $\mathbf{1} = \mathbf{1}_n$ is the unit matrix of size $n \times n$. If a is a matrix of size $m \times n$, then $|a|$ denotes the norm of a viewed as an operator from $\mathbb{C}^n$ to $\mathbb{C}^m$.

The classes L_q of $\mathbb{C}^n$-valued functions in a domain $\mathcal{O} \subset \mathbb{R}^d$ are denoted by $L_q(\mathcal{O};\mathbb{C}^n)$, $1 \le q \le \infty$. The Sobolev classes of order s of $\mathbb{C}^n$-valued functions in a domain $\mathcal{O}$ are denoted by $H^s(\mathcal{O};\mathbb{C}^n)$, $s \in \mathbb{R}$. If $n = 1$, we write simply $L_q(\mathcal{O})$, $H^s(\mathcal{O})$, but sometimes we use such simple notation also for the spaces of vector-valued or matrix-valued functions.

We denote $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, $D_j = -i\partial_j = -i\partial/\partial x_j$, $j = 1, \dots, n$; $\mathbf{D} = -i\nabla = (D_1, \dots, D_d)$. If $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}_+^d$ is a multiindex and $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{R}^d$, then $|\alpha| := \alpha_1 + \dots + \alpha_d$, $\mathbf{k}^\alpha := k_1^{\alpha_1} \cdots k_d^{\alpha_d}$, $\mathbf{D}^\alpha := D_1^{\alpha_1} \cdots D_d^{\alpha_d}$. For two multiindices α, β, we write $\beta \leq \alpha$ if $\beta_j \leq \alpha_j$, $j = 1, \dots, d$; the multibinomial coefficients are defined by $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d}$. We use the notation $\mathbb{R}_+ = [0, \infty)$. By $C, \mathcal{C}, \mathfrak{C}, c, \mathfrak{c}$ we denote various constants in estimates.

Chapter 1

ABSTRACT OPERATOR-THEORETIC SCHEME

2. Abstract Scheme for Polynomial Operator Pencils

In this section, we briefly describe the results of the abstract scheme borrowed from [8, 17, 24].

2.1. Polynomial pencils of the form $A(t) = X(t)^* X(t)$. Let $\mathfrak{H}$ and $\mathfrak{H}_*$ be complex separable Hilbert spaces. Consider the following family of self-adjoint operators:

$$A(t) = X(t)^* X(t) : \mathfrak{H} \to \mathfrak{H}, \quad t \in \mathbb{R}, \tag{2.1}$$

where $X(t) : \mathfrak{H} \to \mathfrak{H}_*$ is a polynomial operator pencil of the form

$$X(t) = \sum_{j=0}^{p} t^j X_j, \quad t \in \mathbb{R}, \ p \in \mathbb{N}, \ p \geq 2.$$

The case where $p = 1$ has been studied in details in [3, 4, 6]. We assume the following about the operators $X_j : \mathfrak{H} \to \mathfrak{H}_*$, $j = 0, \dots, p$. Suppose that X_0 is *densely defined and closed*, X_p is *bounded*, and the domains of the operators $X_0, \dots, X_p$ satisfy the following additional condition.

Condition 2.1. *Suppose that*

$$\operatorname{Dom} X(t) = \operatorname{Dom} X_0 \subset \operatorname{Dom} X_j \subset \operatorname{Dom} X_p = \mathfrak{H}, \quad j = 1, \dots, p-1, \ t \in \mathbb{R}.$$

Moreover, suppose that the operators X_j with $j = 1, \dots, p-1$ are subordinate to the operator X_0.

Condition 2.2. *For $j = 0, \dots, p-1$ and any $u \in \operatorname{Dom} X_0$ we have*

$$\|X_j u\|_{\mathfrak{H}_*} \leq C_0 \|X_0 u\|_{\mathfrak{H}_*}, \tag{2.2}$$

where C_0 is some constant (*obviously, $C_0 \geq 1$*).

From the above assumptions it follows that the operator $X(t)$ is *closed* on the domain $\operatorname{Dom} X(t) = \operatorname{Dom} X_0$ if $|t| \leq (2(p-1)C_0)^{-1}$. Condition 2.2 also yields the following relation for the kernels of the operators X_j:

$$\operatorname{Ker} X_0 \subset \operatorname{Ker} X_j, \quad j = 1, \dots, p-1.$$

Operator (2.1) is generated by the following nonnegative and closed quadratic form in $\mathfrak{H}$:

$$a(t)[u, u] = \|X(t)u\|^2_{\mathfrak{H}_*}, \quad u \in \operatorname{Dom} X_0.$$

We denote: $A(0) = X_0^* X_0 =: A_0$;

$$\mathfrak{N} := \operatorname{Ker} A_0 = \operatorname{Ker} X_0; \quad \mathfrak{N}_* := \operatorname{Ker} X_0^*.$$

Let P and P_* be the orthogonal projections of the space $\mathfrak{H}$ onto $\mathfrak{N}$ and of the space $\mathfrak{H}_*$ onto $\mathfrak{N}_*$, respectively. We impose the following condition.

Condition 2.3. $\lambda_0 = 0$ *is an isolated point of the spectrum of the operator A_0, and*

$$n := \dim \mathfrak{N} < \infty; \quad n \leq n_* := \dim \mathfrak{N}_* \leq \infty.$$

Let $F(t,\sigma)$ be the spectral projection of the operator $A(t)$ for the interval $[0,\sigma]$. Denote $\mathfrak{F}(t,\sigma) := F(t,\sigma)\mathfrak{H}$. We fix a number $\delta > 0$ such that

$$\delta \le \min\left\{\frac{d^0}{36}, \frac{1}{4}\right\}, \tag{2.3}$$

where d^0 is the distance from the point $\lambda_0 = 0$ to the rest of the spectrum of the operator A_0. Next, we choose a positive number t^0 such that

$$t^0 \le \sqrt{\delta}(C^\circ)^{-1}, \quad \text{where} \;\; C^\circ = \max\{(p-1)C_0, \|X_p\|\}, \tag{2.4}$$

and C_0 is the constant from Condition 2.2. Note that $t^0 \le 1/2$. Automatically, the operator $X(t)$ is closed for $|t| \le t^0$, because $t^0 \le (2(p-1)C_0)^{-1}$. In [24, Proposition 3.10] it was checked that

$$F(t,\delta) = F(t,3\delta), \quad \operatorname{rank} F(t,\delta) = n, \quad |t| \le t^0. \tag{2.5}$$

This means that for $|t| \le t^0$ the operator $A(t)$ has exactly n eigenvalues (counting multiplicities) at the interval $[0,\delta]$, while the interval $(\delta, 3\delta)$ is free of the spectrum. For brevity, denote

$$F(t) := F(t,\delta); \quad \mathfrak{F}(t) := \mathfrak{F}(t,\delta).$$

2.2. The operators Z, R, and S. We put $\mathcal{D} = \operatorname{Dom} X_0 \cap \mathfrak{N}^\perp$. Note that the set $\mathcal{D}$ with the inner product

$$(f_1, f_2)_{\mathcal{D}} = (X_0 f_1, X_0 f_2)_{\mathfrak{H}_*}, \quad f_1, f_2 \in \mathcal{D},$$

is a Hilbert space.

Let $v \in \mathfrak{H}_*$. Consider the equation $X_0^*(X_0\psi - v) = 0$ for the element $\psi \in \mathcal{D}$ understood in the weak sense:

$$(X_0\psi, X_0\zeta)_{\mathfrak{H}_*} = (v, X_0\zeta)_{\mathfrak{H}_*}, \quad \forall \zeta \in \mathcal{D}. \tag{2.6}$$

The right-hand side in (2.6) is an antilinear continuous functional of $\zeta \in \mathcal{D}$. Consequently, by the Riesz theorem, there exists a unique solution $\psi \in \mathcal{D}$, and $\|X_0\psi\|_{\mathfrak{H}_*} \le \|v\|_{\mathfrak{H}_*}$. Now, let $\omega \in \mathfrak{N}$ and $v = -X_p\omega$. In this case, the solution of Eq. (2.6) is denoted by $\psi(\omega)$. We define a bounded operator $Z : \mathfrak{H} \to \mathcal{D}$ by the relation

$$Zu = \psi(Pu), \quad u \in \mathfrak{H}. \tag{2.7}$$

From the definition of Z it follows that $PZ = 0$ and $Z^*P = 0$. In [8, (1.11)], it was checked that

$$\|Z\|_{\mathfrak{H}\to\mathfrak{H}} \le \frac{\|X_p\|}{6\sqrt{\delta}}. \tag{2.8}$$

Next, we define a bounded operator $R : \mathfrak{N} \to \mathfrak{N}_*$ by

$$R\omega = X_0\psi(\omega) + X_p\omega, \quad \omega \in \mathfrak{N}. \tag{2.9}$$

Another representation for the operator R is given by $R = P_* X_p|_{\mathfrak{N}}$. A self-adjoint operator $S = R^*R : \mathfrak{N} \to \mathfrak{N}$ is called the *spectral germ* of the operator family $A(t)$ at $t = 0$. In other words,

$$S = P X_p^* P_* X_p|_{\mathfrak{N}}.$$

Clearly, we have

$$\|S\| \le \|X_p\|^2. \tag{2.10}$$

The germ S is called *nondegenerate* if $\operatorname{Ker} S = \{0\}$.

2.3. Analytic branches of eigenvalues and eigenvectors. According to the general analytic perturbation theory (see [7]), for $|t| \le t^0$ there exist real-analytic functions $\lambda_j(t)$ (the branches of eigenvalues) and real-analytic $\mathfrak{H}$-valued functions $\varphi_j(t)$ (the branches of eigenvectors) such that

$$A(t)\varphi_j(t) = \lambda_j(t)\varphi_j(t), \quad j = 1, \dots, n, \quad |t| \le t^0,$$

and the set $\{\varphi_j(t)\}_{j=1}^n$ forms an orthonormal basis in $\mathfrak{F}(t)$. It $t_* \le t^0$ is sufficiently small, then for $|t| \le t_*$ we have the convergent power series expansions (see [24, Theorem 3.15])

$$\lambda_j(t) = \gamma_j t^{2p} + \mu_j t^{2p+1} + \dots, \quad \gamma_j \ge 0, \ \mu_j \in \mathbb{R}, \ j = 1, \dots, n, \tag{2.11}$$

$$\varphi_j(t) = \omega_j + \varphi_j^{(1)} t + \varphi_j^{(2)} t^2 + \dots, \quad j = 1, \dots, n. \tag{2.12}$$

The set $\{\omega_j\}_{j=1}^n$ forms an orthonormal basis in $\mathfrak{N}$. The numbers γ_j and the vectors ω_j are eigenvalues and eigenvectors of the spectral germ S, i.e.,

$$S\omega_j = \gamma_j \omega_j, \quad j = 1, \dots, n. \tag{2.13}$$

This allows us to write down the following representations for the operators P, SP, $F(t)$, and $A(t)F(t)$:

$$P = \sum_{j=1}^{n} (\cdot, \omega_j)\omega_j, \tag{2.14}$$

$$SP = \sum_{j=1}^{n} \gamma_j (\cdot, \omega_j)\omega_j, \tag{2.15}$$

$$F(t) = \sum_{j=1}^{n} (\cdot, \varphi_j(t))\varphi_j(t), \quad |t| \le t^0, \tag{2.16}$$

$$A(t)F(t) = \sum_{j=1}^{n} \lambda_j(t)(\cdot, \varphi_j(t))\varphi_j(t), \quad |t| \le t^0. \tag{2.17}$$

Comparing (2.11), (2.12), and (2.14)–(2.17), we obtain the following power series expansions for $|t| \le t_*$ (see [17, Sec. 1.3] for details):

$$F(t) = P + tF_1 + \dots,$$
$$A(t)F(t) = t^{2p} SP + t^{2p+1} G + \dots$$

2.4. Threshold approximations. In [24, Sec. 4.2] (see also [8, Sec. 2.2] and [17, Sec. 3]), for $|t| \le t^0$ approximations for the operators $F(t)$ and $A(t)F(t)$ in terms of the operators S and P (the so-called *threshold approximations*) have been found. We present them here in the form of a theorem. In what follows, by $c(p)$ we denote various constants depending only on p.

Theorem 2.1. *Suppose that $A(t)$ is the operator family introduced in Sec.* 2.1. *Let $\delta > 0$ be a fixed number subject to* (2.3). *Let $F(t) = F(t, \delta)$ be the spectral projection of the operator $A(t)$ for the interval $[0, \delta]$, and let P be the orthogonal projection of $\mathfrak{H}$ onto the subspace $\mathfrak{N} = \operatorname{Ker} A(0)$. Let $S : \mathfrak{N} \to \mathfrak{N}$ be the spectral germ of the family $A(t)$ at $t = 0$. Then we have*

$$\|F(t) - P\|_{\mathfrak{H} \to \mathfrak{H}} \le C_1 |t|, \quad |t| \le t^0, \tag{2.18}$$

$$\|A(t)F(t) - t^{2p} SP\|_{\mathfrak{H} \to \mathfrak{H}} \le C_2 |t|^{2p+1}, \quad |t| \le t^0, \tag{2.19}$$

where the number t^0 satisfies condition (2.4). *The constants C_1, C_2 are given by $C_1 = c(p) C_T$ and $C_2 = c(p) C_T^{2p+1}$, where*

$$C_T := p\, C_0^2 + \|X_p\|^2 \delta^{-1}, \tag{2.20}$$

and C_0 is the constant from (2.2).

We also need more accurate approximations for the operators $F(t)$ and $A(t)F(t)$ obtained in [17, Theorem 3.2].

Theorem 2.2. *Suppose that the assumptions of Theorem* 2.1 *are satisfied. We put*

$$F_p := ZP + PZ^*, \tag{2.21}$$

$$G := (RP)^* X_1 Z + (X_1 Z)^* RP, \tag{2.22}$$

where the operators Z and R are defined in Sec. 2.2. *In terms of expansions* (2.11) *and* (2.12), *the operator G has the form*

$$G = \sum_{j=1}^{n} \mu_j (\cdot, \omega_j)_{\mathfrak{H}} \omega_j + \sum_{j=1}^{n} \gamma_j \left((\cdot, \varphi_j^{(1)})_{\mathfrak{H}} \omega_j + (\cdot, \omega_j)_{\mathfrak{H}} \varphi_j^{(1)} \right).$$

We have

$$\|F(t) - P\| \leq C_3 |t|^p, \quad |t| \leq t^0, \tag{2.23}$$

$$\|A(t)F(t) - t^{2p} SP - t^{2p+1} G\| \leq C_4 t^{2p+2}, \quad |t| \leq t^0. \tag{2.24}$$

The following representation and estimate are true:

$$F(t) = P + t^p F_p + F_*(t), \tag{2.25}$$

$$\|F_*(t)\|_{\mathfrak{H} \to \mathfrak{H}} \leq C_5 |t|^{p+1}, \quad |t| \leq t^0.$$

The constants C_3, C_4, C_5 are given by

$$C_3 = c(p) C_T^p, \quad C_4 = c(p) C_T^{2p+2}, \quad C_5 = c(p) C_T^{p+1},$$

where C_T is defined by (2.20).

We also need the estimate

$$\|A(t)F(t)\|_{\mathfrak{H} \to \mathfrak{H}} \leq t^{2p} \|S\| + C_2 |t|^{2p+1} \leq C_6 t^{2p}, \quad |t| \leq t^0,$$

where $C_6 = \|X_p\|^2 + C_2$. It follows directly from (2.10), (2.19), and the inequality $t^0 \leq 1$. Thus, for $|t| \leq t^0$ the eigenvalues $\lambda_j(t)$ of the operator $A(t)$ satisfy the inequalities $\lambda_j(t) \leq C_6 t^{2p}$, $j = 1, \ldots, n$. Hence,

$$\|A(t)^{1/2} F(t)\|_{\mathfrak{H} \to \mathfrak{H}} \leq \sqrt{C_6} |t|^p, \quad |t| \leq t^0. \tag{2.26}$$

Moreover, we have

$$\|A(t)^{1/2} F_*(t)\|_{\mathfrak{H} \to \mathfrak{H}} \leq C_7 |t|^{p+1}, \quad |t| \leq t^0; \quad C_7 = c(p) C_T^{p+1}. \tag{2.27}$$

(See [8, (3.52)] and the proof of [17, Theorem 3.2].)

3. Approximation of the Operator Exponential $e^{-A(t)\tau}$

3.1. The principal term of approximation. Our goal in this section is to approximate the operator $e^{-A(t)\tau}$ for large values of $\tau \geq 0$ in terms of the spectral germ S. As compared with the assumptions of Sec. 2, we need an additional assumption.

Condition 3.1. *There exists a constant $c_* > 0$ such that for $|t| \leq t^0$ we have*

$$A(t) \geq c_* t^{2p} I. \tag{3.1}$$

Condition 3.1 is equivalent to the following inequalities for the eigenvalues $\lambda_j(t)$ of the operator $A(t)$:

$$\lambda_j(t) \geq c_* t^{2p}, \quad j = 1, \ldots, n, \quad |t| \leq t^0. \tag{3.2}$$

By (2.11), from (3.2) it follows that $\gamma_j \geq c_*$. By (2.13), this implies

$$S \geq c_* I_{\mathfrak{N}}. \tag{3.3}$$

This ensures that the spectral germ is nondegenerate. Further considerations are carried out under the assumption that $|t| \le t^0$. Obviously,

$$e^{-A(t)\tau} = e^{-A(t)\tau}F(t) + e^{-A(t)\tau}F(t)^{\perp}. \tag{3.4}$$

By (2.5), the operator $F(t)^{\perp}$ is the spectral projection of the operator $A(t)$ for the interval $[3\delta, +\infty)$, whence

$$\|e^{-A(t)\tau}F(t)^{\perp}\|_{\mathfrak{H}\to\mathfrak{H}} \le e^{-3\delta\tau}, \ \tau \ge 0. \tag{3.5}$$

Obviously,

$$e^{-A(t)\tau}F(t) = Pe^{-A(t)\tau}F(t) + P^{\perp}e^{-A(t)\tau}F(t). \tag{3.6}$$

From the relation $P^{\perp}F(t) = (F(t) - P)F(t)$ and (3.1) it follows that

$$\|P^{\perp}e^{-A(t)\tau}F(t)\|_{\mathfrak{H}\to\mathfrak{H}} = \|(F(t) - P)e^{-A(t)\tau}F(t)\|_{\mathfrak{H}\to\mathfrak{H}} \le e^{-c_* t^{2p}\tau}\|F(t) - P\|. \tag{3.7}$$

We put

$$\Sigma(t,\tau) := Pe^{-A(t)\tau}F(t) - e^{-t^{2p}S\tau}P, \tag{3.8}$$

$$\mathcal{E}(t,\tau) := e^{t^{2p}S\tau}\Sigma(t,\tau) = e^{t^{2p}S\tau}Pe^{-A(t)\tau}F(t) - P. \tag{3.9}$$

Differentiating (3.9) with respect to τ, we obtain

$$\mathcal{E}'(t,\tau) := \frac{d\mathcal{E}(t,\tau)}{d\tau} = e^{t^{2p}S\tau}P(t^{2p}SP - A(t)F(t))e^{-A(t)\tau}F(t).$$

Hence,

$$\begin{aligned}\mathcal{E}(t,\tau) &= \mathcal{E}(t,0) + \int_0^{\tau} \mathcal{E}'(t,\widetilde{\tau})\,d\widetilde{\tau} \\ &= PF(t) - P - \int_0^{\tau} e^{t^{2p}S\widetilde{\tau}}P(A(t)F(t) - t^{2p}SP)e^{-A(t)\widetilde{\tau}}F(t)\,d\widetilde{\tau}.\end{aligned}$$

Then

$$\begin{aligned}\Sigma(t,\tau) &= e^{-t^{2p}S\tau}\mathcal{E}(t,\tau) \\ &= e^{-t^{2p}S\tau}P(F(t) - P) - \int_0^{\tau} e^{-t^{2p}S(\tau-\widetilde{\tau})}P(A(t)F(t) - t^{2p}SP)e^{-A(t)\widetilde{\tau}}F(t)\,d\widetilde{\tau}.\end{aligned}$$

Together with (3.1) and (3.3), this implies

$$\|\Sigma(t,\tau)\|_{\mathfrak{H}\to\mathfrak{H}} \le e^{-c_* t^{2p}\tau}\left(\|F(t) - P\| + \tau\|A(t)F(t) - t^{2p}SP\|\right). \tag{3.10}$$

Combining (3.6)–(3.8) and (3.10), we arrive at the inequality

$$\|e^{-A(t)\tau}F(t) - e^{-t^{2p}S\tau}P\|_{\mathfrak{H}\to\mathfrak{H}} \le e^{-c_* t^{2p}\tau}\left(2\|F(t) - P\| + \tau\|A(t)F(t) - t^{2p}SP\|\right). \tag{3.11}$$

Together with (2.18) and (2.19), this yields

$$\|e^{-A(t)\tau}F(t) - e^{-t^{2p}S\tau}P\|_{\mathfrak{H}\to\mathfrak{H}} \le \left(2C_1|t| + C_2|t|^{2p+1}\tau\right)e^{-c_* t^{2p}\tau}. \tag{3.12}$$

We put $\alpha = |t|^p\sqrt{c_*\tau}$. Then estimate (3.12) takes the form

$$\|e^{-A(t)\tau}F(t) - e^{-t^{2p}S\tau}P\|_{\mathfrak{H}\to\mathfrak{H}} \le \tau^{-\frac{1}{2p}}\Phi_1(\alpha), \quad \tau > 0,$$

where

$$\Phi_1(\alpha) := \left(2C_1 c_*^{-\frac{1}{2p}}\alpha^{\frac{1}{p}} + C_2 c_*^{-1-\frac{1}{2p}}\alpha^{2+\frac{1}{p}}\right)e^{-\alpha^2}.$$

Estimating the maximum of the function $\Phi_1(\alpha)$, $\alpha \geq 0$, we arrive at the inequality

$$\|e^{-A(t)\tau} F(t) - e^{-t^{2p} S\tau} P\|_{\mathfrak{H}\to\mathfrak{H}} \leq C_8^\circ \tau^{-\frac{1}{2p}}, \quad \tau > 0, \tag{3.13}$$

where

$$C_8^\circ = 2C_1 c_*^{-\frac{1}{2p}} + C_2 c_*^{-1-\frac{1}{2p}}. \tag{3.14}$$

It is easily seen that $e^{-3\delta\tau} \leq (3\delta\tau)^{-\frac{1}{2p}}$. Combining this with (3.4), (3.5), and (3.13), we obtain

$$\|e^{-A(t)\tau} - e^{-t^{2p} S\tau} P\|_{\mathfrak{H}\to\mathfrak{H}} \leq C_8^\circ \tau^{-\frac{1}{2p}} + e^{-3\delta\tau} \leq \left(C_8^\circ + (3\delta)^{-\frac{1}{2p}}\right) \tau^{-\frac{1}{2p}}, \quad \tau > 0. \tag{3.15}$$

For $\tau \geq 1$ we use (3.15), and for $0 < \tau < 1$ we estimate the left-hand side of (3.15) by 2. Combining these estimates and putting

$$C_8 := 2 \max \left\{2, C_8^\circ + (3\delta)^{-\frac{1}{2p}}\right\}, \tag{3.16}$$

we arrive at the following result.

Theorem 3.1. *Suppose that $A(t)$ is the operator family introduced in Sec.* 2.1. *Suppose that Condition* 3.1 *is satisfied. Let P be the orthogonal projection of the space $\mathfrak{H}$ onto the subspace $\mathfrak{N} = \operatorname{Ker} A(0)$, and let $S : \mathfrak{N} \to \mathfrak{N}$ be the spectral germ of the family $A(t)$ at $t = 0$. Then we have*

$$\|e^{-A(t)\tau} - e^{-t^{2p} S\tau} P\|_{\mathfrak{H}\to\mathfrak{H}} \leq \frac{C_8}{\tau^{\frac{1}{2p}} + 1}, \quad \tau \geq 0, \quad |t| \leq t^0. \tag{3.17}$$

The number t^0 satisfies condition (2.4). *The constant C_8 is defined according to* (3.14), (3.16) *and depends only on p, δ, the constant C_0 from* (2.2), $\|X_p\|$, *and c_*.*

In the case where $G = 0$, the result can be improved.

Theorem 3.2. *Suppose that the assumptions of Theorem* 3.1 *are satisfied. Let G be the operator* (2.22). *Suppose that $G = 0$. Then we have*

$$\|e^{-A(t)\tau} - e^{-t^{2p} S\tau} P\|_{\mathfrak{H}\to\mathfrak{H}} \leq \frac{C_9}{\tau^{\frac{1}{p}} + 1}, \quad \tau \geq 0, \quad |t| \leq t^0. \tag{3.18}$$

The constant C_9 is defined below according to (3.20), (3.22) *and depends only on p, δ, the constant C_0 from* (2.2), $\|X_p\|$, *and c_*.*

Proof. Using inequality (3.11), estimates (2.23), (2.24), and the condition $G = 0$, we obtain

$$\|e^{-A(t)\tau} F(t) - e^{-t^{2p} S\tau} P\|_{\mathfrak{H}\to\mathfrak{H}} \leq \left(2C_3|t|^p + C_4 t^{2p+2}\tau\right) e^{-c_* t^{2p}\tau} \leq \left(2C_3 t^2 + C_4 t^{2p+2}\tau\right) e^{-c_* t^{2p}\tau}.$$

In the last passage we have taken into account that $|t|^p \leq t^2$ for $|t| \leq t^0$, since $p \geq 2$ and $t^0 \leq 1$.

Putting again $\alpha = |t|^p \sqrt{c_*\tau}$, we rewrite the resulting inequality in the form

$$\|e^{-A(t)\tau} F(t) - e^{-t^{2p} S\tau} P\|_{\mathfrak{H}\to\mathfrak{H}} \leq \tau^{-\frac{1}{p}} \Phi_2(\alpha), \quad \tau > 0,$$

where

$$\Phi_2(\alpha) := \left(2C_3 c_*^{-\frac{1}{p}} \alpha^{\frac{2}{p}} + C_4 c_*^{-1-\frac{1}{p}} \alpha^{2+\frac{2}{p}}\right) e^{-\alpha^2}.$$

Estimating the maximum of the function $\Phi_2(\alpha)$, $\alpha \geq 0$, we arrive at the inequality

$$\|e^{-A(t)\tau} F(t) - e^{-t^{2p} S\tau} P\|_{\mathfrak{H}\to\mathfrak{H}} \leq C_9^\circ \tau^{-\frac{1}{p}}, \quad \tau > 0, \tag{3.19}$$

where

$$C_9^\circ = 2C_3 c_*^{-\frac{1}{p}} + C_4 c_*^{-1-\frac{1}{p}}. \tag{3.20}$$

Using (3.4), (3.5), (3.19) and taking into account that $e^{-3\delta\tau} \leq (3\delta\tau)^{-\frac{1}{p}}$, we obtain

$$\|e^{-A(t)\tau} - e^{-t^{2p} S\tau} P\|_{\mathfrak{H}\to\mathfrak{H}} \leq C_9^\circ \tau^{-\frac{1}{p}} + e^{-3\delta\tau} \leq \left(C_9^\circ + (3\delta)^{-\frac{1}{p}}\right) \tau^{-\frac{1}{p}}, \quad \tau > 0. \tag{3.21}$$

For $\tau \geq 1$ we apply (3.21), and for $0 < \tau < 1$ we estimate the left-hand side of (3.21) by 2. Putting

$$C_9 := 2\max\left\{2, C_9^\circ + (3\delta)^{-\frac{1}{p}}\right\}, \tag{3.22}$$

we arrive at (3.18). □

3.2. Approximation of the operator exponential in the "energy" norm. In this subsection, we obtain another approximation of the operator $e^{-A(t)\tau}$ (in the "energy" norm). Namely, we study the operator $A(t)^{1/2}e^{-A(t)\tau}$. Our goal is to prove the following theorem.

Theorem 3.3. *Let $A(t)$ be the operator family* (2.1) *satisfying the assumptions of Sec.* 2.1 *and also Condition* 3.1. *Let P be the orthogonal projection of the space $\mathfrak{H}$ onto the subspace $\mathfrak{N} = \operatorname{Ker} A(0)$. Let Z be operator* (2.7), *and let S be the spectral germ of the family $A(t)$ at $t = 0$. Then for $\tau > 0$ and $|t| \leq t^0$ we have*

$$\|A(t)^{1/2}\left(e^{-A(t)\tau} - (I + t^p Z)e^{-t^{2p}S\tau}P\right)\|_{\mathfrak{H}\to\mathfrak{H}} \leq C_{10}\tau^{-\frac{1}{2p}-\frac{1}{2}}. \tag{3.23}$$

The number t^0 is subject to condition (2.4). *The constant C_{10} is defined below according to* (3.30), (3.37), (3.38) *and depends only on p, δ, the constant C_0 from* (2.2), $\|X_p\|$, *and c_*.*

Proof. We put

$$\mathfrak{A}(t,\tau) := A(t)^{1/2}e^{-A(t)\tau}, \tag{3.24}$$

and represent this operator as

$$\mathfrak{A}(t,\tau) = \mathfrak{A}(t,\tau)F(t)^\perp + \mathfrak{A}(t,\tau)F(t)(F(t) - P) + F(t)\mathfrak{A}(t,\tau)P. \tag{3.25}$$

By the spectral theorem, taking (2.5) into account and using the elementary inequality $x^{\frac{1}{2p}+\frac{1}{2}}e^{-x} \leq 1$ for $x \geq 0$, we have

$$\|\mathfrak{A}(t,\tau)F(t)^\perp\|_{\mathfrak{H}\to\mathfrak{H}} \leq \sup_{\lambda \geq 3\delta} \lambda^{\frac{1}{2}}e^{-\lambda\tau} \leq (3\delta)^{-\frac{1}{2p}}\tau^{-\frac{1}{2p}-\frac{1}{2}}, \quad \tau > 0,\ |t| \leq t^0. \tag{3.26}$$

Next, by (3.2), for $\tau > 0$ and $0 < |t| \leq t^0$ we have

$$\|\mathfrak{A}(t,\tau)F(t)\|_{\mathfrak{H}\to\mathfrak{H}} \leq \sup_{1\leq j\leq n}\left(\lambda_j(t)^{\frac{1}{2}}e^{-\lambda_j(t)\tau}\right) \leq \tau^{-\frac{1}{2p}-\frac{1}{2}}\sup_{1\leq j\leq n}\lambda_j(t)^{-\frac{1}{2p}} \leq c_*^{-\frac{1}{2p}}|t|^{-1}\tau^{-\frac{1}{2p}-\frac{1}{2}}.$$

Together with (2.18), this implies

$$\|\mathfrak{A}(t,\tau)F(t)(F(t) - P)\|_{\mathfrak{H}\to\mathfrak{H}} \leq C_1 c_*^{-\frac{1}{2p}}\tau^{-\frac{1}{2p}-\frac{1}{2}}, \quad \tau > 0,\ |t| \leq t^0. \tag{3.27}$$

The last term in the right-hand side of (3.25) can be represented in the form

$$F(t)\mathfrak{A}(t,\tau)P = A(t)^{1/2}F(t)(e^{-A(t)\tau}F(t) - e^{-t^{2p}S\tau}P)P + A(t)^{1/2}F(t)e^{-t^{2p}S\tau}P. \tag{3.28}$$

From (2.26) and (3.12) it follows that for $\tau > 0$ and $|t| \leq t^0$

$$\begin{aligned}&\|A(t)^{1/2}F(t)(e^{-A(t)\tau}F(t) - e^{-t^{2p}S\tau}P)P\|_{\mathfrak{H}\to\mathfrak{H}}\\ &\leq \sqrt{C_6}|t|^p\left(2C_1|t| + C_2|t|^{2p+1}\tau\right)e^{-c_*t^{2p}\tau} = \tau^{-\frac{1}{2p}-\frac{1}{2}}\Phi_3(\alpha),\end{aligned}$$

where $\alpha = |t|^p\sqrt{c_*\tau}$ and

$$\Phi_3(\alpha) := \sqrt{C_6}\left(2C_1 c_*^{-\frac{1}{2}-\frac{1}{2p}}\alpha^{1+\frac{1}{p}} + C_2 c_*^{-\frac{3}{2}-\frac{1}{2p}}\alpha^{3+\frac{1}{p}}\right)e^{-\alpha^2}.$$

Estimating the maximum of the function $\Phi_3(\alpha)$, $\alpha \geq 0$, we arrive at the estimate

$$\|A(t)^{1/2}F(t)(e^{-A(t)\tau}F(t) - e^{-t^{2p}S\tau}P)P\|_{\mathfrak{H}\to\mathfrak{H}} \leq C'_{10}\tau^{-\frac{1}{2p}-\frac{1}{2}}, \tag{3.29}$$

where

$$C'_{10} = \sqrt{C_6}\left(2C_1 c_*^{-\frac{1}{2}-\frac{1}{2p}} + C_2 c_*^{-\frac{3}{2}-\frac{1}{2p}}\right). \tag{3.30}$$

From (3.24)–(3.29) it follows that for $\tau > 0$ and $|t| \le t^0$

$$\begin{aligned} &\|A(t)^{1/2}\left(e^{-A(t)\tau} - (I + t^p Z)e^{-t^{2p} S\tau} P\right)\|_{\mathfrak{H}\to\mathfrak{H}} \\ &\qquad \le \tau^{-\frac{1}{2p}-\frac{1}{2}}\left((3\delta)^{-\frac{1}{2p}} + C_1 c_*^{-\frac{1}{2p}} + C'_{10}\right) + \|A(t)^{1/2}(F(t) - P - t^p Z)e^{-t^{2p}S\tau}P\|_{\mathfrak{H}\to\mathfrak{H}}. \end{aligned} \tag{3.31}$$

Denote

$$\Xi(t,\tau) = e^{-t^{2p}S\tau}P. \tag{3.32}$$

It remains to estimate the norm of the operator

$$\mathcal{I}(t,\tau) := A(t)^{1/2}(F(t) - P - t^p Z)\Xi(t,\tau). \tag{3.33}$$

By (2.21), (2.25), and the identity $Z^*P = 0$, we have $(F(t) - P - t^p Z)P = F_*(t)P$, whence

$$\mathcal{I}(t,\tau) = A(t)^{1/2}F_*(t)\Xi(t,\tau). \tag{3.34}$$

From (3.3) it follows that

$$\|\Xi(t,\tau)\|_{\mathfrak{H}\to\mathfrak{H}} = \|e^{-t^{2p}S\tau}P\|_{\mathfrak{H}\to\mathfrak{H}} \le e^{-c_* t^{2p}\tau}. \tag{3.35}$$

To estimate operator (3.34), we use estimates (2.27) and (3.35). Taking into account the elementary inequality $x^{\frac{1}{2p}+\frac{1}{2}}e^{-x} \le 1$ for $x \ge 0$, we have

$$\|\mathcal{I}(t,\tau)\|_{\mathfrak{H}\to\mathfrak{H}} \le C_7|t|^{p+1}e^{-c_* t^{2p}\tau} \le C''_{10}\tau^{-\frac{1}{2p}-\frac{1}{2}}, \quad \tau > 0,\ |t| \le t^0, \tag{3.36}$$

where

$$C''_{10} = C_7 c_*^{-\frac{1}{2p}-\frac{1}{2}}. \tag{3.37}$$

As a result, combining relations (3.31)–(3.33) and (3.36), we arrive at the required estimate (3.23) with the constant

$$C_{10} = (3\delta)^{-\frac{1}{2p}} + C_1 c_*^{-\frac{1}{2p}} + C'_{10} + C''_{10}. \tag{3.38}$$

□

Remark 3.1. We have tracked the dependence of the constants in estimates on the problem data. Below, when applying abstract results to differential operators, the following is essential. The constants C_8, C_9, C_{10} from Theorems 3.1, 3.2, 3.3, after a possible overestimation, become polynomials in the variables C_0, $\|X_p\|$, $\delta^{-\frac{1}{2p}}$, $c_*^{-\frac{1}{2p}}$ with positive coefficients depending only on p.

4. The Operator Family of the Form $A(t) = M^*\widehat{A}(t)M$. Approximation of the Sandwiched Operator Exponential

4.1. The operator of the form $A(t) = M^*\widehat{A}(t)M$.

Let $\widehat{\mathfrak{H}}$ be yet another complex separable Hilbert space. Suppose that $M : \mathfrak{H} \to \widehat{\mathfrak{H}}$ is an isomorphism.

Let $\widehat{X}(t) : \widehat{\mathfrak{H}} \to \mathfrak{H}_*$ be an operator pencil of the form

$$\widehat{X}(t) = \sum_{j=1}^{p} t^j \widehat{X}_j, \quad t \in \mathbb{R},$$

satisfying all the assumptions of Sec. 2.1. The space $\mathfrak{H}_*$ remains the same. Assume that

$$M\,\mathrm{Dom}\,X_j = \mathrm{Dom}\,\widehat{X}_j, \quad X_j = \widehat{X}_j M, \quad j = 0, 1, \dots, p. \tag{4.1}$$

Then $X(t) = \widehat{X}(t)M$. In the space $\widehat{\mathfrak{H}}$, we consider the family of self-adjoint operators $\widehat{A}(t) = \widehat{X}(t)^*\widehat{X}(t)$. Then

$$A(t) = M^*\widehat{A}(t)M.$$

Below all the objects corresponding to the family $\widehat{A}(t)$ are marked by "hat". Note that $\widehat{\mathfrak{N}} = M\mathfrak{N}$, $\widehat{n} = n$, and $\widehat{\mathfrak{N}}_* = \mathfrak{N}_*$.

In the space $\widehat{\mathfrak{H}}$, consider the operator

$$Q := (MM^*)^{-1} : \widehat{\mathfrak{H}} \to \widehat{\mathfrak{H}}. \tag{4.2}$$

The operator Q is bounded and positive definite together with Q^{-1}. Let $Q_{\widehat{\mathfrak{N}}}$ be the block of the operator Q in the subspace $\widehat{\mathfrak{N}} = \operatorname{Ker} \widehat{X}_0$:

$$Q_{\widehat{\mathfrak{N}}} := \widehat{P} Q|_{\widehat{\mathfrak{N}}}. \tag{4.3}$$

Obviously, $Q_{\widehat{\mathfrak{N}}}$ is an isomorphism in $\widehat{\mathfrak{N}}$.

It is easily seen that the orthogonal projection P of the space $\mathfrak{H}$ onto the subspace $\mathfrak{N}$ and the orthogonal projection $\widehat{P}$ of the space $\widehat{\mathfrak{H}}$ onto $\widehat{\mathfrak{N}}$ satisfy the following relation:

$$P = M^{-1}(Q_{\widehat{\mathfrak{N}}})^{-1}\widehat{P}(M^*)^{-1}; \tag{4.4}$$

cf. [19, Proposition 1.2], where this identity was checked in the case where $p = 1$.

For $\widehat{A}(t)$ we define the operator $\widehat{Z}$ by analogy with (2.7). For each $\widehat{\omega} \in \widehat{\mathfrak{N}}$, define the solution $\widehat{\psi} = \widehat{\psi}(\widehat{\omega})$ of the problem

$$\widehat{X}_0^*(\widehat{X}_0\widehat{\psi} + \widehat{X}_p\widehat{\omega}) = 0, \quad \widehat{\psi} \perp \widehat{\mathfrak{N}}.$$

Then

$$\widehat{Z}\widehat{u} = \widehat{\psi}(\widehat{P}\widehat{u}), \quad \widehat{u} \in \widehat{\mathfrak{H}}. \tag{4.5}$$

Along with $\widehat{Z}$, we need the operator $\widehat{Z}_Q$. Let $\widehat{\omega} \in \widehat{\mathfrak{N}}$ and let $\widehat{\psi}_Q = \widehat{\psi}_Q(\widehat{\omega})$ be the solution of the problem

$$\widehat{X}_0^*(\widehat{X}_0\widehat{\psi}_Q + \widehat{X}_p\widehat{\omega}) = 0, \quad Q\widehat{\psi}_Q \perp \widehat{\mathfrak{N}}.$$

The operator $\widehat{Z}_Q$ is defined by the relation

$$\widehat{Z}_Q\widehat{u} = \widehat{\psi}_Q(\widehat{P}\widehat{u}), \quad \widehat{u} \in \widehat{\mathfrak{H}}.$$

Clearly, $\widehat{\psi}_Q = \widehat{\psi} + \widehat{\omega}_Q$, where the element $\widehat{\omega}_Q \in \widehat{\mathfrak{N}}$ is determined from the condition $Q\widehat{\psi}_Q \perp \widehat{\mathfrak{N}}$. Then

$$\widehat{\omega}_Q = -(Q_{\widehat{\mathfrak{N}}})^{-1}\widehat{P}(Q\widehat{\psi}).$$

Thus,

$$\widehat{Z}_Q = \widehat{Z} - (Q_{\widehat{\mathfrak{N}}})^{-1}\widehat{P}Q\widehat{Z}. \tag{4.6}$$

In particular, this implies that

$$\widehat{X}_0\widehat{Z}_Q = \widehat{X}_0\widehat{Z}, \quad \widehat{X}_1\widehat{Z}_Q = \widehat{X}_1\widehat{Z}. \tag{4.7}$$

We have taken into account that $\widehat{\mathfrak{N}} = \operatorname{Ker} \widehat{X}_0 \subset \operatorname{Ker} \widehat{X}_1$.

It is easily seen that

$$\widehat{Z}_Q = MZM^{-1}\widehat{P}, \tag{4.8}$$

where Z is given by (2.7); cf. [4, Lemma 6.1]. The operator $\widehat{R}$ for the family $\widehat{A}(t)$ is defined by analogy with (2.9):

$$\widehat{R} := (\widehat{X}_0\widehat{Z} + \widehat{X}_p)|_{\widehat{\mathfrak{N}}} = (\widehat{X}_0\widehat{Z}_Q + \widehat{X}_p)|_{\widehat{\mathfrak{N}}}.$$

Here the second relation follows from (4.7). The operator R defined by (2.9) and the operator $\widehat{R}$ satisfy the following relation:

$$R = \widehat{R}M|_{\mathfrak{N}}. \tag{4.9}$$

Finally, the spectral germ $\widehat{S} := \widehat{R}^*\widehat{R} : \widehat{\mathfrak{N}} \to \widehat{\mathfrak{N}}$ of the operator family $\widehat{A}(t)$ and the germ S of the family $A(t)$ satisfy

$$S = PM^*\widehat{S}M|_{\mathfrak{N}}. \tag{4.10}$$

For $\widehat{A}(t)$, we introduce the operator $\widehat{G}$ by analogy with (2.22). Then, by (4.7),

$$\widehat{G} := (\widehat{R}\widehat{P})^*\widehat{X}_1\widehat{Z} + (\widehat{X}_1\widehat{Z})^*\widehat{R}\widehat{P} = (\widehat{R}\widehat{P})^*\widehat{X}_1\widehat{Z}_Q + (\widehat{X}_1\widehat{Z}_Q)^*\widehat{R}\widehat{P}. \tag{4.11}$$

By (4.1), (4.8), and (4.9), the operator G defined by (2.22) and operator (4.11) satisfy the following relation:

$$G = PM^*\widehat{G}MP. \tag{4.12}$$

Remark 4.1. Since the operator G acts nontrivially only in the subspace $\mathfrak{N}$, using (4.4) and (4.12), it is easy to check that the relations $G = 0$ and $\widehat{G} = 0$ are equivalent.

4.2. Approximation of the sandwiched operator exponential $Me^{-A(t)\tau}M^*$. Under the assumptions of Sec. 4.1, we find an approximation of the operator

$$Me^{-A(t)\tau}M^* = Me^{-M^*\widehat{A}(t)M}M^* : \widehat{\mathfrak{H}} \to \widehat{\mathfrak{H}}$$

in terms of the spectral germ $\widehat{S}$ of the family $\widehat{A}(t)$ and the isomorphism M.

Let $Q_{\widehat{\mathfrak{N}}}$ be operator (4.3). We put

$$M_0 = (Q_{\widehat{\mathfrak{N}}})^{-1/2} : \widehat{\mathfrak{N}} \to \widehat{\mathfrak{N}}. \tag{4.13}$$

Let S be the spectral germ of the family $A(t)$. Using (4.4) and (4.10), it is easy to check that

$$Me^{-t^{2p}S\tau}PM^* = M_0 e^{-t^{2p}M_0\widehat{S}M_0\tau}M_0\widehat{P} : \widehat{\mathfrak{N}} \to \widehat{\mathfrak{N}}; \tag{4.14}$$

cf. [19, Proposition 2.3], where this identity was checked in the case where $p = 1$.

Theorem 4.1. *Let $A(t)$ and $\widehat{A}(t)$ be the operator families satisfying the assumptions of Sec. 4.1. Suppose that Condition 3.1 is satisfied. Let $\widehat{P}$ be the orthogonal projection of the space $\widehat{\mathfrak{H}}$ onto the subspace $\widehat{\mathfrak{N}} = \operatorname{Ker}\widehat{A}(0)$, and let $\widehat{S} : \widehat{\mathfrak{N}} \to \widehat{\mathfrak{N}}$ be the spectral germ of the family $\widehat{A}(t)$ at $t = 0$. Let M_0 be the operator defined by (4.13). Then we have*

$$\left\| Me^{-A(t)\tau}M^* - M_0 e^{-t^{2p}M_0\widehat{S}M_0\tau}M_0\widehat{P} \right\|_{\widehat{\mathfrak{H}}\to\widehat{\mathfrak{H}}} \le \frac{C_8\|M\|^2}{\tau^{\frac{1}{2p}} + 1}, \quad \tau \ge 0, \quad |t| \le t^0. \tag{4.15}$$

The number t^0 is subject to condition (2.4). The constant C_8 is defined according to (3.14), (3.16) and depends only on p, δ, the constant C_0 from (2.2), $\|X_p\|$, and c_*.

Proof. Multiplying the operators under the norm sign in (3.17) by M from the left and by M^* from the right, we obtain

$$\left\| Me^{-A(t)\tau}M^* - Me^{-t^{2p}S\tau}PM^* \right\|_{\widehat{\mathfrak{H}}\to\widehat{\mathfrak{H}}} \le \frac{C_8\|M\|^2}{\tau^{\frac{1}{2p}} + 1}, \quad \tau \ge 0, \quad |t| \le t^0.$$

Together with identity (4.14), this implies the required inequality (4.15). ☐

Similarly, from Theorem 3.2 and Remark 4.1 we deduce the following result.

Theorem 4.2. *Suppose that the assumptions of Theorem 4.1 are satisfied. Let $\widehat{G}$ be the operator defined by (4.11). Suppose that $\widehat{G} = 0$. Then we have*

$$\left\| Me^{-A(t)\tau}M^* - M_0 e^{-t^{2p}M_0\widehat{S}M_0\tau}M_0\widehat{P} \right\|_{\widehat{\mathfrak{H}}\to\widehat{\mathfrak{H}}} \le \frac{C_9\|M\|^2}{\tau^{\frac{1}{p}} + 1}, \quad \tau \ge 0, \quad |t| \le t^0.$$

The number t^0 is subject to condition (2.4). The constant C_9 is defined according to (3.20), (3.22) and depends only on p, δ, the constant C_0 from (2.2), $\|X_p\|$, and c_*.

Next, Theorem 3.3 yields the following result.

Theorem 4.3. *Suppose that the assumptions of Theorem 4.1 are satisfied. Let $\widehat{Z}$ be the operator defined by (4.5). Then we have*

$$\left\| \widehat{A}(t)^{1/2} \left(Me^{-A(t)\tau}M^* - (I + t^p\widehat{Z})M_0 e^{-t^{2p}M_0\widehat{S}M_0\tau}M_0\widehat{P} \right) \right\|_{\widehat{\mathfrak{H}}\to\widehat{\mathfrak{H}}} \le C_{11}\tau^{-\frac{1}{2p}-\frac{1}{2}}, \quad \tau > 0, \quad |t| \le t^0. \tag{4.16}$$

The number t^0 is subject to condition (2.4)*. The constant C_{11} depends only on p, δ, $\widehat{\delta}$, the constant C_0 from* (2.2)*, $\|\widehat{X}_p\|$, $\|X_p\|$, c_*, $\|M\|$, $\|M^{-1}\|$.*

Proof. By (4.8) and (4.14),

$$\begin{aligned}
&\left\| \widehat{A}(t)^{1/2} \left(M e^{-A(t)\tau} M^* - (I + t^p \widehat{Z}_Q) M_0 e^{-t^{2p} M_0 \widehat{S} M_0 \tau} M_0 \widehat{P} \right) \right\|_{\widehat{\mathfrak{H}} \to \widehat{\mathfrak{H}}} \\
&= \left\| \widehat{X}(t) M \left(e^{-A(t)\tau} - (I + t^p Z) e^{-t^{2p} S \tau} P \right) M^* \right\|_{\widehat{\mathfrak{H}} \to \widehat{\mathfrak{H}}} \\
&\leq \|M\| \left\| A(t)^{1/2} \left(e^{-A(t)\tau} - (I + t^p Z) e^{-t^{2p} S \tau} P \right) \right\|_{\widehat{\mathfrak{H}} \to \widehat{\mathfrak{H}}}.
\end{aligned}$$

Combining this with estimate (3.23), we obtain

$$\left\| \widehat{A}(t)^{1/2} \left(M e^{-A(t)\tau} M^* - (I + t^p \widehat{Z}_Q) M_0 e^{-t^{2p} M_0 \widehat{S} M_0 \tau} M_0 \widehat{P} \right) \right\|_{\widehat{\mathfrak{H}} \to \widehat{\mathfrak{H}}} \leq C_{10} \|M\| \tau^{-\frac{1}{2p} - \frac{1}{2}}, \quad \tau > 0, \ |t| \leq t^0. \tag{4.17}$$

It remains to show that, within the margin of the error, it is possible to replace $\widehat{Z}_Q$ by $\widehat{Z}$ in the left-hand side of (4.17). By (4.6), we have

$$\widehat{X}(t)(\widehat{Z}_Q - \widehat{Z}) = t^p \widehat{X}_p (\widehat{Z}_Q - \widehat{Z}) = -t^p \widehat{X}_p (Q_{\widehat{\mathfrak{N}}})^{-1} \widehat{P} Q \widehat{Z}.$$

We have taken into account that $\widehat{\mathfrak{N}} \subset \operatorname{Ker} \widehat{X}_j$, $j = 0, \dots, p-1$. Hence,

$$\begin{aligned}
&\left\| \widehat{A}(t)^{1/2} \left(t^p (\widehat{Z}_Q - \widehat{Z}) M_0 e^{-t^{2p} M_0 \widehat{S} M_0 \tau} M_0 \widehat{P} \right) \right\|_{\widehat{\mathfrak{H}} \to \widehat{\mathfrak{H}}} \\
&= \left\| \widehat{X}(t) \left(t^p (\widehat{Z}_Q - \widehat{Z}) M_0 e^{-t^{2p} M_0 \widehat{S} M_0 \tau} M_0 \widehat{P} \right) \right\|_{\widehat{\mathfrak{H}} \to \widehat{\mathfrak{H}}} \\
&= t^{2p} \left\| \widehat{X}_p (Q_{\widehat{\mathfrak{N}}})^{-1} \widehat{P} Q \widehat{Z} M_0 e^{-t^{2p} M_0 \widehat{S} M_0 \tau} M_0 \widehat{P} \right\|_{\widehat{\mathfrak{H}} \to \widehat{\mathfrak{H}}} \\
&\leq \|\widehat{X}_p\| \|M\|^4 \|M^{-1}\|^2 \|\widehat{Z}\| \, t^{2p} e^{-c_* t^{2p} \tau}, \quad \tau > 0, \ |t| \leq t^0.
\end{aligned}$$

In the last passage, we used identity (4.14), inequality (3.3), and also the estimate $\|(Q_{\widehat{\mathfrak{N}}})^{-1} \widehat{P}\| \leq \|M\|^2$, which follows from the identity $(Q_{\widehat{\mathfrak{N}}})^{-1} \widehat{P} = M P M^*$ (see (4.4)). According to (2.8), we have $\|\widehat{Z}\| \leq \dfrac{1}{6} \widehat{\delta}^{-\frac{1}{2}} \|\widehat{X}_p\|$, where $\widehat{\delta}$ is the analog of δ for the operator $\widehat{A}(t)$. Now, using the elementary inequality $x^{\frac{1}{2p} + \frac{1}{2}} e^{-x} \leq 1$ for $x \geq 0$, we obtain

$$\left\| \widehat{A}(t)^{1/2} \left(t^p (\widehat{Z}_Q - \widehat{Z}) M_0 e^{-t^{2p} M_0 \widehat{S} M_0 \tau} M_0 \widehat{P} \right) \right\|_{\widehat{\mathfrak{H}} \to \widehat{\mathfrak{H}}} \leq C_{12} \tau^{-\frac{1}{2p} - \frac{1}{2}}, \quad \tau > 0, \ |t| \leq t^0, \tag{4.18}$$

where

$$C_{12} = \frac{1}{6} \widehat{\delta}^{-\frac{1}{2}} \|\widehat{X}_p\|^2 \|M\|^4 \|M^{-1}\|^2 c_*^{-\frac{1}{2p} - \frac{1}{2}}. \tag{4.19}$$

We took into account that $t^0 \leq 1$.

Relations (4.17) and (4.18) imply the required estimate (4.16) with the constant

$$C_{11} = C_{10} \|M\| + C_{12}. \tag{4.20}$$

□

Remark 4.2. From Remark 3.1 and relations (4.19), (4.20) it follows that, after possible overestimation, the constant C_{11} from Theorem 4.3 becomes a polynomial in the variables C_0, $\|\widehat{X}_p\|$, $\|X_p\|$, $\widehat{\delta}^{-\frac{1}{2}}$, $\delta^{-\frac{1}{2p}}$, $c_*^{-\frac{1}{2p}}$, $\|M\|$, $\|M^{-1}\|$ with positive coefficients depending only on p.

CHAPTER 2

PERIODIC DIFFERENTIAL OPERATORS IN $\mathbb{R}^d$. APPROXIMATION OF THE OPERATOR EXPONENTIAL

5. Periodic Differential Operators in $L_2(\mathbb{R}^d; \mathbb{C}^n)$

5.1. Factorized operators of order $2p$ in $\mathbb{R}^d$. In the space $L_2(\mathbb{R}^d; \mathbb{C}^n)$, we consider differential operators formally given by

$$\mathcal{A} = f(\mathbf{x})^* b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}) f(\mathbf{x}). \tag{5.1}$$

Here $g(\mathbf{x})$ is a uniformly positive definite and bounded measurable matrix-valued function of size $m \times m$ (in general, $g(\mathbf{x})$ is a Hermitian matrix with complex entries):

$$c' \mathbf{1}_m \leq g(\mathbf{x}) \leq c'' \mathbf{1}_m, \quad \mathbf{x} \in \mathbb{R}^d; \quad 0 < c' \leq c'' < \infty. \tag{5.2}$$

A matrix-valued function $f(\mathbf{x})$ of size $n \times n$ (with complex entries) is assumed to be bounded together with the inverse matrix:

$$f, f^{-1} \in L_\infty(\mathbb{R}^d). \tag{5.3}$$

An operator $b(\mathbf{D})$ is given by

$$b(\mathbf{D}) = \sum_{|\beta|=p} b_\beta \mathbf{D}^\beta, \tag{5.4}$$

where b_β are $(m \times n)$-matrices with constant (in general, complex) entries. Assume that $m \geq n$ and the symbol

$$b(\boldsymbol{\xi}) := \sum_{|\beta|=p} b_\beta \boldsymbol{\xi}^\beta, \quad \boldsymbol{\xi} \in \mathbb{R}^d,$$

of the operator $b(\mathbf{D})$ has maximal rank, i.e.,

$$\operatorname{rank} b(\boldsymbol{\xi}) = n, \quad 0 \neq \boldsymbol{\xi} \in \mathbb{R}^d.$$

The last condition is equivalent to the estimates

$$\alpha_0 \mathbf{1}_n \leq b(\boldsymbol{\theta})^* b(\boldsymbol{\theta}) \leq \alpha_1 \mathbf{1}_n, \quad \boldsymbol{\theta} \in \mathbb{S}^{d-1}; \quad 0 < \alpha_0 \leq \alpha_1 < \infty, \tag{5.5}$$

with some positive constants α_0, α_1. Without loss of generality, we assume that the norms of the matrices b_β do not exceed the constant $\sqrt{\alpha_1}$:

$$|b_\beta| \leq \sqrt{\alpha_1}, \quad |\beta| = p. \tag{5.6}$$

The precise definition of the operator $\mathcal{A}$ is given in terms of the quadratic form. From conditions (5.2) it follows that the matrix $g(\mathbf{x})$ can be written in a factorized form:

$$g(\mathbf{x}) = h(\mathbf{x})^* h(\mathbf{x}),$$

where h, $h^{-1} \in L_\infty(\mathbb{R}^d)$. For instance, we can put $h(\mathbf{x}) = g(\mathbf{x})^{1/2}$.

Consider the operator $\mathcal{X} : L_2(\mathbb{R}^d; \mathbb{C}^n) \to L_2(\mathbb{R}^d; \mathbb{C}^m)$ acting as follows:

$$(\mathcal{X}\mathbf{u})(\mathbf{x}) = h(\mathbf{x}) b(\mathbf{D})(f(\mathbf{x})\mathbf{u}(\mathbf{x})), \quad \operatorname{Dom} \mathcal{X} = \{\mathbf{u} \in L_2(\mathbb{R}^d; \mathbb{C}^n) : \ f\mathbf{u} \in H^p(\mathbb{R}^d; \mathbb{C}^n)\},$$

and the quadratic form

$$a[\mathbf{u}, \mathbf{u}] = \|\mathcal{X}\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \langle g(\mathbf{x}) b(\mathbf{D}) f(\mathbf{x})\mathbf{u}(\mathbf{x}), b(\mathbf{D}) f(\mathbf{x})\mathbf{u}(\mathbf{x}) \rangle \, d\mathbf{x}, \quad \mathbf{u} \in \operatorname{Dom} a = \operatorname{Dom} \mathcal{X}. \tag{5.7}$$

Let us check that

$$c_0 \int_{\mathbb{R}^d} |\mathbf{D}^p (f(\mathbf{x})\mathbf{u}(\mathbf{x}))|^2 \, d\mathbf{x} \leq a[\mathbf{u}, \mathbf{u}] \leq c_1 \int_{\mathbb{R}^d} |\mathbf{D}^p (f(\mathbf{x})\mathbf{u}(\mathbf{x}))|^2 \, d\mathbf{x}, \quad \mathbf{u} \in \operatorname{Dom} \mathcal{X}, \tag{5.8}$$

where $|\mathbf{D}^p\mathbf{v}(\mathbf{x})|^2 := \sum_{|\beta|=p} |\mathbf{D}^\beta\mathbf{v}(\mathbf{x})|^2$. First, by the Parseval identity,

$$\|g^{-1}\|^{-1}_{L_\infty} \int_{\mathbb{R}^d} |b(\boldsymbol{\xi})\widehat{\mathbf{v}}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \leq a[\mathbf{u},\mathbf{u}] \leq \|g\|_{L_\infty} \int_{\mathbb{R}^d} |b(\boldsymbol{\xi})\widehat{\mathbf{v}}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi}, \quad \mathbf{v} = f\mathbf{u} \in H^p(\mathbb{R}^d;\mathbb{C}^n),$$

where $\widehat{\mathbf{v}}(\boldsymbol{\xi})$ is the Fourier image of the function $\mathbf{v}(\mathbf{x})$. Combining this with (5.5), we see that

$$\alpha_0\|g^{-1}\|^{-1}_{L_\infty} \int_{\mathbb{R}^d} |\boldsymbol{\xi}|^{2p}|\widehat{\mathbf{v}}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \leq a[\mathbf{u},\mathbf{u}] \leq \alpha_1\|g\|_{L_\infty} \int_{\mathbb{R}^d} |\boldsymbol{\xi}|^{2p}|\widehat{\mathbf{v}}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi}, \quad \mathbf{v} = f\mathbf{u} \in H^p(\mathbb{R}^d;\mathbb{C}^n). \quad (5.9)$$

Finally, using the elementary inequalities

$$c'_p \sum_{|\beta|=p} |\boldsymbol{\xi}^\beta|^2 \leq |\boldsymbol{\xi}|^{2p} \leq c''_p \sum_{|\beta|=p} |\boldsymbol{\xi}^\beta|^2, \quad \boldsymbol{\xi} \in \mathbb{R}^d, \quad (5.10)$$

where the constants c'_p and c''_p depend only on d and p, we arrive at the required relations (5.8) with the constants

$$c_0 = c'_p\alpha_0\|g^{-1}\|^{-1}_{L_\infty}, \quad c_1 = c''_p\alpha_1\|g\|_{L_\infty}. \quad (5.11)$$

Consequently, form (5.7) is closed and nonnegative. By definition, $\mathcal{A}$ is a self-adjoint operator in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ corresponding to form (5.7).

5.2. The lattices Γ and $\widetilde{\Gamma}$. In what follows, the functions g, h, and f are assumed to be periodic with respect to some lattice $\Gamma \subset \mathbb{R}^d$. Let $\mathbf{a}_1,\dots,\mathbf{a}_d$ be a basis in $\mathbb{R}^d$ generating the lattice Γ, i.e.,

$$\Gamma = \Big\{\mathbf{a} \in \mathbb{R}^d : \mathbf{a} = \sum_{j=1}^d \nu_j\mathbf{a}_j,\ \nu_j \in \mathbb{Z}\Big\},$$

and let Ω be the elementary cell of the lattice Γ:

$$\Omega = \Big\{\mathbf{x} \in \mathbb{R}^d :\ \mathbf{x} = \sum_{j=1}^d \kappa_j\mathbf{a}_j,\ 0 < \kappa_j < 1\Big\}. \quad (5.12)$$

The basis $\mathbf{b}_1,\dots,\mathbf{b}_d$ in $\mathbb{R}^d$ dual to $\mathbf{a}_1,\dots,\mathbf{a}_d$ is defined by the relations $\langle\mathbf{b}_j,\mathbf{a}_l\rangle = 2\pi\delta_{jl}$. The lattice

$$\widetilde{\Gamma} = \Big\{\mathbf{b} \in \mathbb{R}^d : \mathbf{b} = \sum_{j=1}^d \zeta_j\mathbf{b}_j,\ \zeta_j \in \mathbb{Z}\Big\}$$

generated by this basis is called the *dual* lattice to Γ. The cell of the lattice $\widetilde{\Gamma}$ can be defined similarly to (5.12), however, it is more convenient to consider the central Brillouin zone of the dual lattice:

$$\widetilde{\Omega} := \Big\{\mathbf{k} \in \mathbb{R}^d : |\mathbf{k}| < |\mathbf{k} - \mathbf{b}|,\ 0 \neq \mathbf{b} \in \widetilde{\Gamma}\Big\}. \quad (5.13)$$

The domain $\widetilde{\Omega}$ is a fundamental domain of the lattice $\widetilde{\Gamma}$. Denote $|\Omega| = \operatorname{meas}\Omega$, $|\widetilde{\Omega}| = \operatorname{meas}\widetilde{\Omega}$, and note that $|\Omega||\widetilde{\Omega}| = (2\pi)^d$. Let r_0 be the radius of the ball inscribed in $\operatorname{clos}\widetilde{\Omega}$. Note that

$$|\mathbf{k} + \mathbf{b}| \geq r_0, \quad \mathbf{k} \in \operatorname{clos}\widetilde{\Omega}, \quad 0 \neq \mathbf{b} \in \widetilde{\Gamma}. \quad (5.14)$$

We put $\mathcal{B}(r) := \big\{\mathbf{k} \in \mathbb{R}^d : |\mathbf{k}| \leq r\big\}$, $r > 0$. Let $\widetilde{H}^s(\Omega;\mathbb{C}^n)$ be the subspace of functions from $H^s(\Omega;\mathbb{C}^n)$ whose Γ-periodic extension to $\mathbb{R}^d$ belongs to $H^s_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{C}^n)$.

The discrete Fourier transform $\{\widehat{\mathbf{v}}_\mathbf{b}\}_{\mathbf{b}\in\widetilde{\Gamma}} \mapsto \mathbf{v}$ is associated with the lattice Γ:

$$\mathbf{v}(\mathbf{x}) = |\Omega|^{-1/2} \sum_{\mathbf{b}\in\widetilde{\Gamma}} \widehat{\mathbf{v}}_\mathbf{b} e^{i\langle\mathbf{b},\mathbf{x}\rangle}, \quad \mathbf{x} \in \Omega.$$

This transform is a unitary mapping of $l_2(\widetilde{\Gamma};\mathbb{C}^n)$ onto $L_2(\Omega;\mathbb{C}^n)$:

$$\int_\Omega |\mathbf{v}(\mathbf{x})|^2\,d\mathbf{x} = \sum_{\mathbf{b}\in\widetilde{\Gamma}} |\widehat{\mathbf{v}}_{\mathbf{b}}|^2.$$

5.3. The Gelfand transform. Initially, the *Gelfand transform* $\mathcal{U}$ is defined on functions of the Schwartz class $\mathcal{S}(\mathbb{R}^d;\mathbb{C}^n)$ by the following relation:

$$\widetilde{\mathbf{v}}(\mathbf{k},\mathbf{x}) = (\mathcal{U}\mathbf{v})(\mathbf{k},\mathbf{x}) = |\widetilde{\Omega}|^{-1/2}\sum_{\mathbf{a}\in\Gamma} e^{-i\langle\mathbf{k},\mathbf{x}+\mathbf{a}\rangle}\mathbf{v}(\mathbf{x}+\mathbf{a}),\quad \mathbf{v}\in\mathcal{S}(\mathbb{R}^d;\mathbb{C}^n),\quad \mathbf{x},\mathbf{k}\in\mathbb{R}^d.$$

Next, $\mathcal{U}$ extends by continuity up to the unitary mapping:

$$\mathcal{U}: L_2(\mathbb{R}^d;\mathbb{C}^n)\to \int_{\widetilde{\Omega}} \oplus L_2(\Omega;\mathbb{C}^n)\,d\mathbf{k} =: \mathcal{K}. \tag{5.15}$$

The relation $\mathbf{v}\in H^p(\mathbb{R}^d;\mathbb{C}^n)$ is equivalent to the fact that $\widetilde{\mathbf{v}}(\mathbf{k},\cdot)\in\widetilde{H}^p(\Omega;\mathbb{C}^n)$ for almost every $\mathbf{k}\in\widetilde{\Omega}$ and

$$\int_{\widetilde{\Omega}}\int_\Omega \left(|\,(\mathbf{D}+\mathbf{k})^p\,\widetilde{\mathbf{v}}(\mathbf{k},\mathbf{x})|^2 + |\widetilde{\mathbf{v}}(\mathbf{k},\mathbf{x})|^2\right)\,d\mathbf{x}\,d\mathbf{k} < \infty.$$

Under the Gelfand transform, the operator of multiplication by a bounded Γ-periodic function in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ turns into multiplication by the same function on the fibers of the direct integral $\mathcal{K}$ from (5.15). Action of the operator $b(\mathbf{D})$ on the function $\mathbf{v}\in H^p(\mathbb{R}^d;\mathbb{C}^n)$ turns into action of the operator $b(\mathbf{D}+\mathbf{k})$ on the function $\widetilde{\mathbf{v}}(\mathbf{k},\cdot)\in\widetilde{H}^p(\Omega;\mathbb{C}^n)$.

6. Direct Integral Expansion for the Operator $\mathcal{A}$. The Operators $\mathcal{A}(\mathbf{k})$

6.1. The forms $a(\mathbf{k})$ and the operators $\mathcal{A}(\mathbf{k})$. We put $\mathfrak{H} = L_2(\Omega;\mathbb{C}^n)$ and $\mathfrak{H}_* = L_2(\Omega;\mathbb{C}^m)$. Consider the operator $\mathcal{X}(\mathbf{k}):\mathfrak{H}\to\mathfrak{H}_*$, $\mathbf{k}\in\mathbb{R}^d$, given by

$$\begin{gathered}(\mathcal{X}(\mathbf{k})\mathbf{u})(\mathbf{x}) = h(\mathbf{x})b(\mathbf{D}+\mathbf{k})(f(\mathbf{x})\mathbf{u}(\mathbf{x})),\\ \operatorname{Dom}\mathcal{X}(\mathbf{k}) = \{\mathbf{u}\in L_2(\Omega;\mathbb{C}^n):\ f\mathbf{u}\in\widetilde{H}^p(\Omega;\mathbb{C}^n)\} =: \mathfrak{D}.\end{gathered} \tag{6.1}$$

Consider the quadratic form

$$a(\mathbf{k})[\mathbf{u},\mathbf{u}] = \|\mathcal{X}(\mathbf{k})\mathbf{u}\|^2_{L_2(\Omega)} = \int_\Omega \langle g(\mathbf{x})b(\mathbf{D}+\mathbf{k})f(\mathbf{x})\mathbf{u}(\mathbf{x}), b(\mathbf{D}+\mathbf{k})f(\mathbf{x})\mathbf{u}(\mathbf{x})\rangle\,d\mathbf{x},\quad \mathbf{u}\in\mathfrak{D}. \tag{6.2}$$

Using the discrete Fourier transform and relations (5.2) and (5.5), it is easily seen that

$$\alpha_0\|g^{-1}\|^{-1}_{L_\infty} a_*(\mathbf{k})[\mathbf{u},\mathbf{u}] \le a(\mathbf{k})[\mathbf{u},\mathbf{u}] \le \alpha_1\|g\|_{L_\infty} a_*(\mathbf{k})[\mathbf{u},\mathbf{u}],\quad \mathbf{u}\in\mathfrak{D}, \tag{6.3}$$

for any $\mathbf{k}\in\mathbb{R}^d$, where

$$a_*(\mathbf{k})[\mathbf{u},\mathbf{u}] = \sum_{\mathbf{b}\in\widetilde{\Gamma}} |\mathbf{b}+\mathbf{k}|^{2p}|\widehat{\mathbf{v}}_{\mathbf{b}}|^2,\quad \mathbf{v} = f\mathbf{u}\in\widetilde{H}^p(\Omega;\mathbb{C}^n). \tag{6.4}$$

Combining this with (5.10), we obtain

$$c_0\int_\Omega |(\mathbf{D}+\mathbf{k})^p(f\mathbf{u})|^2\,d\mathbf{x} \le a(\mathbf{k})[\mathbf{u},\mathbf{u}] \le c_1\int_\Omega |(\mathbf{D}+\mathbf{k})^p(f\mathbf{u})|^2\,d\mathbf{x},\quad \mathbf{u}\in\mathfrak{D},$$

where the constants c_0, c_1 are defined by (5.11). Hence, the operator $\mathcal{X}(\mathbf{k})$ is closed, and the form (6.2) is closed and nonnegative. The self-adjoint operator in $L_2(\Omega;\mathbb{C}^n)$ corresponding to the form $a(\mathbf{k})$ is denoted by $\mathcal{A}(\mathbf{k})$. Formally, we can write

$$\mathcal{A}(\mathbf{k}) = f(\mathbf{x})^* b(\mathbf{D}+\mathbf{k})^* g(\mathbf{x}) b(\mathbf{D}+\mathbf{k}) f(\mathbf{x}). \tag{6.5}$$

6.2. The direct integral for the operator $\mathcal{A}$. The operators $\mathcal{A}(\mathbf{k})$ allow us to partially diagonalize the operator $\mathcal{A}$ in the direct integral $\mathcal{K}$ (see (5.15)). Let $\widetilde{\mathbf{u}} = \mathcal{U}\mathbf{u}$, $\mathbf{u} \in \operatorname{Dom} a$. Then

$$\widetilde{\mathbf{u}}(\mathbf{k}, \cdot) \in \operatorname{Dom} a(\mathbf{k}) = \mathfrak{D} \text{ for almost every } \mathbf{k} \in \widetilde{\Omega}, \tag{6.6}$$

$$a[\mathbf{u}, \mathbf{u}] = \int_{\widetilde{\Omega}} a(\mathbf{k})[\widetilde{\mathbf{u}}(\mathbf{k}, \cdot), \widetilde{\mathbf{u}}(\mathbf{k}, \cdot)]\, d\mathbf{k}. \tag{6.7}$$

On the contrary, if $\widetilde{\mathbf{u}} \in \mathcal{K}$ satisfies (6.6) and the integral in (6.7) converges, then $\mathbf{u} \in \operatorname{Dom} a$ and (6.7) is satisfied. Thus, under the action of the Gelfand transform, the operator $\mathcal{A}$ turns into multiplication by the operator-valued function $\mathcal{A}(\mathbf{k})$, $\mathbf{k} \in \widetilde{\Omega}$, in the direct integral $\mathcal{K}$. All this can be briefly expressed by the following relation:

$$\mathcal{U}\mathcal{A}\mathcal{U}^{-1} = \int_{\widetilde{\Omega}} \oplus \mathcal{A}(\mathbf{k})\, d\mathbf{k}. \tag{6.8}$$

6.3. Incorporation of the operators $\mathcal{A}(\mathbf{k})$ in the abstract scheme. For $\mathbf{k} \in \mathbb{R}^d$ we put

$$\mathbf{k} = t\boldsymbol{\theta}, \quad t = |\mathbf{k}|, \quad \boldsymbol{\theta} = \frac{\mathbf{k}}{|\mathbf{k}|},$$

and consider t as the perturbation parameter. At the same time, all constructions will depend on the parameter $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$, and we should take care about the uniformity of estimates with respect to this parameter.

Applying the method described in Sec. 2, we put $\mathfrak{H} = L_2(\Omega; \mathbb{C}^n)$ and $\mathfrak{H}_* = L_2(\Omega; \mathbb{C}^m)$. According to (5.4) and (6.1), we have

$$\begin{aligned}\mathcal{X}(\mathbf{k}) &= h \sum_{|\beta|=p} b_\beta (\mathbf{D} + \mathbf{k})^\beta f = h \sum_{|\beta|=p} b_\beta \sum_{\gamma \le \beta} \binom{\beta}{\gamma} \mathbf{k}^{\beta-\gamma} \mathbf{D}^\gamma f \\ &= h \sum_{|\beta|=p} b_\beta \sum_{\gamma \le \beta} \binom{\beta}{\gamma} t^{|\beta-\gamma|} \boldsymbol{\theta}^{\beta-\gamma} \mathbf{D}^\gamma f.\end{aligned}$$

Hence, the operator $\mathcal{X}(\mathbf{k})$ can be written as

$$\mathcal{X}(\mathbf{k}) = X(t, \boldsymbol{\theta}) = X_0 + \sum_{j=1}^{p} t^j X_j(\boldsymbol{\theta}).$$

Here the operator

$$X_0 = h \sum_{|\beta|=p} b_\beta \mathbf{D}^\beta f = h b(\mathbf{D}) f$$

is closed on the domain

$$\operatorname{Dom} X_0 = \{\mathbf{u} \in L_2(\Omega; \mathbb{C}^n) : \ f\mathbf{u} \in \widetilde{H}^p(\Omega; \mathbb{C}^n)\} = \mathfrak{D}, \tag{6.9}$$

the "intermediate" operators $X_j(\boldsymbol{\theta})$, $j = 1, \dots, p-1$, are given by

$$X_j(\boldsymbol{\theta}) = h \sum_{|\beta|=p} b_\beta \sum_{\gamma \le \beta,\ |\gamma|=p-j} \binom{\beta}{\gamma} \boldsymbol{\theta}^{\beta-\gamma} \mathbf{D}^\gamma f \tag{6.10}$$

on the domains

$$\operatorname{Dom} X_j(\boldsymbol{\theta}) = \{\mathbf{u} \in L_2(\Omega; \mathbb{C}^n) : \ f\mathbf{u} \in \widetilde{H}^{p-j}(\Omega; \mathbb{C}^n)\}, \tag{6.11}$$

and the operator

$$X_p(\boldsymbol{\theta}) = h \sum_{|\beta|=p} b_\beta \boldsymbol{\theta}^\beta f = h b(\boldsymbol{\theta}) f$$

is bounded from $L_2(\Omega; \mathbb{C}^n)$ to $L_2(\Omega; \mathbb{C}^m)$.

From (6.9) and (6.11) it follows that Condition 2.1 is satisfied:

$$\operatorname{Dom} X_0 \subset \operatorname{Dom} X_j(\boldsymbol{\theta}) \subset \operatorname{Dom} X_p(\boldsymbol{\theta}) = L_2(\Omega;\mathbb{C}^n), \quad j=1,\dots,p-1.$$

By (5.5),

$$\|X_p(\boldsymbol{\theta})\| \le \alpha_1^{\frac12}\|g\|_{L_\infty}^{\frac12}\|f\|_{L_\infty}. \tag{6.12}$$

It is easily seen that the kernel of the operator X_0 is given by

$$\mathfrak{N} := \operatorname{Ker} X_0 = \{\mathbf{u}\in L_2(\Omega;\mathbb{C}^n) : f(\mathbf{x})\mathbf{u}(\mathbf{x}) = \mathbf{c}\in\mathbb{C}^n\}, \tag{6.13}$$

whence $\dim\mathfrak{N} = n$; cf. [8, Proposition 5.1], where this fact was checked in the case where $f = \mathbf{1}_n$.

Let $n_* = \dim\operatorname{Ker} X_0^*$. The relation $m \ge n$ implies that $n_* \ge n$. Moreover, since

$$\mathfrak{N}_* = \operatorname{Ker} X_0^* = \{\mathbf{q}\in L_2(\Omega;\mathbb{C}^m) :\ h^*\mathbf{q}\in \widetilde{H}^p(\Omega;\mathbb{C}^m) :\ b(\mathbf{D})^*(h^*\mathbf{q}) = 0\},$$

then the following alternative is implemented: either $n_* = \infty$ (if $m > n$), or $n_* = n$ (if $m = n$).

From [8, Proposition 5.2] it follows that Condition 2.2 is satisfied, namely, for $j = 1,\dots,p-1$ we have

$$\|X_j(\boldsymbol{\theta})\mathbf{u}\|_{L_2(\Omega)} \le \widetilde{C}_j\|X_0\mathbf{u}\|_{L_2(\Omega)}, \quad \mathbf{u}\in\mathfrak{D}, \tag{6.14}$$

where

$$\widetilde{C}_j = \alpha_1^{\frac12}\alpha_0^{-\frac12}\|g\|_{L_\infty}^{\frac12}\|g^{-1}\|_{L_\infty}^{\frac12}(2r_0)^{-j}\Bigl(\sum_{|\beta|=p}\ \sum_{\gamma\le\beta,\ |\gamma|=p-j}\binom{\beta}{\gamma}\Bigr). \tag{6.15}$$

Note that the constants $\widetilde{C}_j$ do not depend on the parameter $\boldsymbol{\theta}\in\mathbb{S}^{d-1}$, but depend only on d, p, j, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, α_0, α_1, and r_0.

By the compactness of the embedding of $\operatorname{Dom} a(0) = \mathfrak{D}$ in $L_2(\Omega;\mathbb{C}^n)$, the spectrum of the operator $\mathcal{A}(0)$ is discrete. The point $\lambda_0 = 0$ is an isolated eigenvalue of the operator $\mathcal{A}(0)$ of multiplicity n, the corresponding eigenspace $\mathfrak{N}$ is given by (6.13).

Using variational arguments, by the lower estimate (6.3) and (6.4), it is easy to estimate the distance d^0 from the point $\lambda_0 = 0$ to the rest of the spectrum of $\mathcal{A}(0)$:

$$d^0 \ge \alpha_0\|g^{-1}\|_{L_\infty}^{-1}\|f^{-1}\|_{L_\infty}^{-2}(2r_0)^{2p}. \tag{6.16}$$

Cf. [3, Chap. 2, Sec. 2.2], where such estimate was obtained for the second-order operators, and also [8, (5.17)], where the case $f = \mathbf{1}_n$ was considered.

Following the abstract scheme, we fix a positive number $\delta \le \min\left\{\frac{d^0}{36},\frac14\right\}$. Taking (6.16) into account, we put

$$\delta = \min\left\{\frac{\alpha_0 r_0^{2p}}{4\|g^{-1}\|_{L_\infty}\|f^{-1}\|_{L_\infty}^2},\frac14\right\}. \tag{6.17}$$

Inequalities (6.14) allow us to choose the constant C_0 from (2.2) as follows:

$$C_0 = \max\{1,\widetilde{C}_1,\dots,\widetilde{C}_{p-1}\}, \tag{6.18}$$

where the constants $\widetilde{C}_j$ are defined by (6.15). The constant C_0 depends only on d, p, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, α_0, α_1, and r_0. (Note that the constant C_0 does not depend on f.)

The constant $C^\circ = \max\{(p-1)C_0, \|X_p(\boldsymbol{\theta})\|\}$ (see (2.4)) now depends on $\boldsymbol{\theta}$. Taking (6.12) into account (and overestimating the constant), we take the value

$$C^\circ = \max\left\{(p-1)C_0, \alpha_1^{\frac12}\|g\|_{L_\infty}^{\frac12}\|f\|_{L_\infty}\right\},$$

which does not depend on $\boldsymbol{\theta}$. According to (2.4), we put

$$t^0 = \sqrt{\delta}(C^\circ)^{-1} = \frac{\sqrt{\delta}}{\max\left\{(p-1)C_0, \alpha_1^{\frac{1}{2}}\|g\|_{L_\infty}^{\frac{1}{2}}\|f\|_{L_\infty}\right\}}. \tag{6.19}$$

Note that $t^0 \le 1$, since $(p-1)C_0 \ge 1$ and $\delta \le 1$. Hence,

$$t^0 \le (t^0)^{\frac{1}{p}} \le \delta^{\frac{1}{2p}}\alpha_1^{-\frac{1}{2p}}\|g\|_{L_\infty}^{-\frac{1}{2p}}\|f\|_{L_\infty}^{-\frac{1}{p}} \le 4^{-\frac{1}{2p}}\alpha_0^{\frac{1}{2p}}\alpha_1^{-\frac{1}{2p}}\|g\|_{L_\infty}^{-\frac{1}{2p}}\|g^{-1}\|_{L_\infty}^{-\frac{1}{2p}}\|f\|_{L_\infty}^{-\frac{1}{p}}\|f^{-1}\|_{L_\infty}^{-\frac{1}{p}} r_0 < r_0.$$

In the last passage, we have used the obvious inequalities $\alpha_0 \le \alpha_1$, $\|g\|_{L_\infty}\|g^{-1}\|_{L_\infty} \ge 1$, and $\|f\|_{L_\infty}\|f^{-1}\|_{L_\infty} \ge 1$. Thus, the ball $\mathcal{B}(t^0)$ lies inside the ball $\mathcal{B}(r_0)$ and is thus entirely contained in $\widetilde{\Omega}$.

6.4. Nondegeneracy of the spectral germ. From the lower estimate (6.3) and (6.4), taking (5.13) into account, we deduce that

$$a(\mathbf{k})[\mathbf{u},\mathbf{u}] \ge \alpha_0\|g^{-1}\|_{L_\infty}^{-1}\|f^{-1}\|_{L_\infty}^{-2}|\mathbf{k}|^{2p}\|\mathbf{u}\|_{L_2(\Omega)}^2, \quad \mathbf{u} \in \mathfrak{D}, \quad \mathbf{k} \in \operatorname{clos}\widetilde{\Omega}.$$

Thus,

$$\mathcal{A}(\mathbf{k}) \ge c_*|\mathbf{k}|^{2p}I, \quad \mathbf{k} \in \operatorname{clos}\widetilde{\Omega}, \tag{6.20}$$

where

$$c_* = \alpha_0\|g^{-1}\|_{L_\infty}^{-1}\|f^{-1}\|_{L_\infty}^{-2}. \tag{6.21}$$

This verifies that Condition 3.1 is fulfilled.

Thus, we have made sure that the operator family $\mathcal{A}(\mathbf{k}) =: A(t,\boldsymbol{\theta})$ satisfies all the assumptions of the abstract scheme. It is essential that δ, t^0, and c_* do not depend on $\boldsymbol{\theta}$ (see (6.17), (6.19), (6.21)).

Now the analytic (in t) branches of the eigenvalues $\lambda_j(t,\boldsymbol{\theta})$ and the analytic branches of the eigenfunctions $\varphi_j(t,\boldsymbol{\theta})$, $j = 1,\dots,n$, $|t| \le t^0$, of the operator family $A(t,\boldsymbol{\theta})$ (see Sec. 2.3) depend on $\boldsymbol{\theta}$. From (6.20) it follows that

$$\lambda_j(t,\boldsymbol{\theta}) \ge c_* t^{2p}, \quad j = 1,\dots,n, \quad t = |\mathbf{k}| \le t^0. \tag{6.22}$$

Expansions (2.11), (2.12) take the form

$$\lambda_j(t,\boldsymbol{\theta}) = \gamma_j(\boldsymbol{\theta})t^{2p} + \mu_j(\boldsymbol{\theta})t^{2p+1} + \dots, \quad j = 1,\dots,n, \tag{6.23}$$

$$\varphi_j(t,\boldsymbol{\theta}) = \omega_j(\boldsymbol{\theta}) + \varphi_j^{(1)}(\boldsymbol{\theta})t + \dots, \quad j = 1,\dots,n.$$

From (6.22) and (6.23) it follows that $\gamma_j(\boldsymbol{\theta}) \ge c_*$, $j = 1,\dots,n$. This implies (see (2.13)) that the germ $S(\boldsymbol{\theta})$ of the family $A(t,\boldsymbol{\theta})$ is nondegenerate independently of $\boldsymbol{\theta}$ and

$$S(\boldsymbol{\theta}) \ge c_* I_{\mathfrak{N}}. \tag{6.24}$$

7. The Effective Characteristics in the Case where $f = \mathbf{1}_n$

7.1. The operator $\widehat{\mathcal{A}}$. The case where $f = \mathbf{1}_n$ is basic for us. In this case, we agree to mark all objects by "hat". Then the operator $\widehat{\mathcal{A}}$ acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ is given by

$$\widehat{\mathcal{A}} = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}), \tag{7.1}$$

and the corresponding operators $\widehat{\mathcal{A}}(\mathbf{k}) = \widehat{A}(t,\boldsymbol{\theta})$ acting in $L_2(\Omega;\mathbb{C}^n)$ take the form

$$\widehat{\mathcal{A}}(\mathbf{k}) = b(\mathbf{D}+\mathbf{k})^* g(\mathbf{x}) b(\mathbf{D}+\mathbf{k}) \tag{7.2}$$

(with periodic boundary conditions).

If $f = \mathbf{1}_n$, kernel (6.13) consists of constant vector-valued functions:

$$\widehat{\mathfrak{N}} = \{\mathbf{u} \in L_2(\Omega;\mathbb{C}^n) : \mathbf{u}(\mathbf{x}) = \mathbf{c} \in \mathbb{C}^n\}. \tag{7.3}$$

The orthogonal projection of the space $L_2(\Omega;\mathbb{C}^n)$ onto the subspace $\widehat{\mathfrak{N}}$ acts as averaging over the cell:

$$\widehat{P}\mathbf{u} = |\Omega|^{-1} \int\limits_{\Omega} \mathbf{u}(\mathbf{x})\, d\mathbf{x}, \quad \mathbf{u} \in L_2(\Omega;\mathbb{C}^n). \tag{7.4}$$

Now parameters (6.17), (6.19), and (6.21) take the form

$$\widehat{\delta} = \min\left\{ \frac{\alpha_0 r_0^{2p}}{4\|g^{-1}\|_{L_\infty}}, \frac{1}{4} \right\}, \tag{7.5}$$

$$\widehat{t}^0 = \frac{(\widehat{\delta})^{\frac{1}{2}}}{\max\left\{ (p-1)C_0, \alpha_1^{\frac{1}{2}} \|g\|_{L_\infty}^{\frac{1}{2}} \right\}}, \tag{7.6}$$

$$\widehat{c}_* = \alpha_0 \|g^{-1}\|_{L_\infty}^{-1}. \tag{7.7}$$

By (6.20) with $f = \mathbf{1}_n$, we have

$$\widehat{\mathcal{A}}(\mathbf{k}) \geq \widehat{c}_* |\mathbf{k}|^{2p} I, \quad \mathbf{k} \in \operatorname{clos} \widetilde{\Omega}. \tag{7.8}$$

7.2. The operators $\widehat{Z}(\boldsymbol{\theta})$, $\widehat{R}(\boldsymbol{\theta})$, and $\widehat{S}(\boldsymbol{\theta})$. For the operator family $\widehat{A}(t,\boldsymbol{\theta})$, the operators Z, R, and S introduced in abstract terms in Sec. 2.2 depend on the parameter $\boldsymbol{\theta}$. They were constructed in [8, Sec. 5.3].

To describe these operators, we introduce the matrix-valued function $\Lambda(\mathbf{x})$ of size $n \times m$, which is a Γ-periodic solution of the problem

$$b(\mathbf{D})^* g(\mathbf{x}) \left(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m \right) = 0, \quad \int\limits_{\Omega} \Lambda(\mathbf{x})\, d\mathbf{x} = 0. \tag{7.9}$$

Here the equation is understood in the weak sense: for each $\mathbf{C} \in \mathbb{C}^m$ we have $\Lambda\mathbf{C} \in \widetilde{H}^p(\Omega;\mathbb{C}^n)$ and

$$\int\limits_{\Omega} \langle g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x})\mathbf{C} + \mathbf{C}), b(\mathbf{D})\boldsymbol{\eta}(\mathbf{x}) \rangle\, d\mathbf{x} = 0, \quad \boldsymbol{\eta} \in \widetilde{H}^p(\Omega;\mathbb{C}^n).$$

According to [8, Sec. 5.3]), the operator $\widehat{Z}(\boldsymbol{\theta})$ takes the form

$$\widehat{Z}(\boldsymbol{\theta}) = [\Lambda] b(\boldsymbol{\theta}) \widehat{P}, \tag{7.10}$$

where $[\Lambda]$ is the operator of multiplication by the matrix-valued function $\Lambda(\mathbf{x})$. The operator $\widehat{R}(\boldsymbol{\theta})$ is given by

$$\widehat{R}(\boldsymbol{\theta}) = [h(b(\mathbf{D})\Lambda + \mathbf{1}_m)] b(\boldsymbol{\theta})|_{\widehat{\mathfrak{N}}}. \tag{7.11}$$

Then (see [8, Sec. 5.3]) the spectral germ $\widehat{S}(\boldsymbol{\theta}) = \widehat{R}(\boldsymbol{\theta})^* \widehat{R}(\boldsymbol{\theta})$ acts in the subspace $\widehat{\mathfrak{N}}$ (see (7.3)) and is represented as

$$\widehat{S}(\boldsymbol{\theta}) = b(\boldsymbol{\theta})^* g^0 b(\boldsymbol{\theta}), \quad \boldsymbol{\theta} \in \mathbb{S}^{d-1}. \tag{7.12}$$

Here g^0 is the so-called *effective matrix.* The constant matrix g^0 of size $m \times m$ is defined by

$$g^0 = |\Omega|^{-1} \int\limits_{\Omega} \widetilde{g}(\mathbf{x})\, d\mathbf{x}, \quad \widetilde{g}(\mathbf{x}) := g(\mathbf{x}) \left(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m \right). \tag{7.13}$$

It turns out that the effective matrix g^0 is positive definite. This implies once again that the spectral germ $\widehat{S}(\boldsymbol{\theta})$ is nondegenerate, which has been already discussed in Sec. 6.4.

Let us mention some properties of the effective matrix; see [8, Propositions 5.3–5.5].

Proposition 7.1. *Let g^0 be the effective matrix defined by* (7.13). *Denote*

$$\overline{g} = |\Omega|^{-1} \int_\Omega g(\mathbf{x})\, d\mathbf{x}, \quad \underline{g} = \left(|\Omega|^{-1} \int_\Omega g(\mathbf{x})^{-1}\, d\mathbf{x} \right)^{-1}.$$

We have

$$\underline{g} \leq g^0 \leq \overline{g}. \tag{7.14}$$

In the case where $m = n$ we have $g^0 = \underline{g}$.

Estimates (7.14) are known in homogenization theory for particular DOs as the Voigt–Reuss bracketing. Inequalities (7.14) imply the following estimates for the norms of the matrices g^0 and $(g^0)^{-1}$:

$$|g^0| \leq \|g\|_{L_\infty}, \quad |(g^0)^{-1}| \leq \|g^{-1}\|_{L_\infty}. \tag{7.15}$$

Proposition 7.2.

1°. *Let $\mathbf{g}_k(\mathbf{x})$, $k = 1, \dots, m$, be the columns of the matrix $g(\mathbf{x})$. The identity $g^0 = \overline{g}$ is equivalent to the relations*

$$b(\mathbf{D})^* \mathbf{g}_k(\mathbf{x}) = 0, \quad k = 1, \dots, m. \tag{7.16}$$

2°. *Let $\mathbf{l}_k(\mathbf{x})$, $k = 1, \dots, m$, be the columns of the matrix $g(\mathbf{x})^{-1}$. The identity $g^0 = \underline{g}$ is equivalent to the representations*

$$\mathbf{l}_k(\mathbf{x}) = \mathbf{l}_k^0 + b(\mathbf{D})\mathbf{v}_k(\mathbf{x}), \quad \mathbf{l}_k^0 \in \mathbb{C}^m, \ \mathbf{v}_k \in \widetilde{H}^p(\Omega; \mathbb{C}^n); \quad k = 1, \dots, m. \tag{7.17}$$

Remark 7.1. In the case where $g^0 = \underline{g}$, the matrix $\widetilde{g}(\mathbf{x})$ is constant: $\widetilde{g}(\mathbf{x}) = g^0 = \underline{g}$.

Below we will need the following estimates for the periodic solution Λ of problem (7.9), which can be easily checked:

$$\|hb(\mathbf{D})\Lambda\|_{L_2(\Omega)} \leq |\Omega|^{\frac{1}{2}} \|g\|_{L_\infty}^{\frac{1}{2}}, \tag{7.18}$$

$$\|\Lambda\|_{L_2(\Omega)} \leq |\Omega|^{\frac{1}{2}} \alpha_0^{-\frac{1}{2}} (2r_0)^{-p} \|g\|_{L_\infty}^{\frac{1}{2}} \|g^{-1}\|_{L_\infty}^{\frac{1}{2}} =: |\Omega|^{\frac{1}{2}} C_\Lambda, \tag{7.19}$$

$$\|\Lambda\|_{H^p(\Omega)} \leq |\Omega|^{\frac{1}{2}} \alpha_0^{-\frac{1}{2}} \|g\|_{L_\infty}^{\frac{1}{2}} \|g^{-1}\|_{L_\infty}^{\frac{1}{2}} \Big(\sum_{|\beta| \leq p} (2r_0)^{-2(p-|\beta|)} \Big)^{\frac{1}{2}} =: |\Omega|^{\frac{1}{2}} \widetilde{C}_\Lambda. \tag{7.20}$$

7.3. The effective operator.

Using (7.12) and the homogeneity of the symbol $b(\mathbf{k})$, we have

$$\widehat{S}(\mathbf{k}) := t^{2p} \widehat{S}(\boldsymbol{\theta}) = b(\mathbf{k})^* g^0 b(\mathbf{k}), \quad \mathbf{k} \in \mathbb{R}^d. \tag{7.21}$$

Expression (7.21) is the symbol of the DO

$$\widehat{\mathcal{A}}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D}), \quad \operatorname{Dom} \widehat{\mathcal{A}}^0 = H^{2p}(\mathbb{R}^d; \mathbb{C}^n), \tag{7.22}$$

which is called the *effective operator* for the operator $\widehat{\mathcal{A}}$.

Let $\widehat{\mathcal{A}}^0(\mathbf{k})$ be the operator family in $L_2(\Omega; \mathbb{C}^n)$ corresponding to the operator $\widehat{\mathcal{A}}^0$. Then $\widehat{\mathcal{A}}^0(\mathbf{k})$ is given by the differential expression $b(\mathbf{D}+\mathbf{k})^* g^0 b(\mathbf{D}+\mathbf{k})$ on the domain $\widetilde{H}^{2p}(\Omega; \mathbb{C}^n)$. Similarly to (7.8), using (7.15), we check that

$$\widehat{\mathcal{A}}^0(\mathbf{k}) \geq \widehat{c}_* |\mathbf{k}|^{2p} I, \quad \mathbf{k} \in \operatorname{clos} \widetilde{\Omega}. \tag{7.23}$$

By (7.4) and (7.21),

$$t^{2p} \widehat{S}(\boldsymbol{\theta}) \widehat{P} = \widehat{S}(\mathbf{k}) \widehat{P} = \widehat{\mathcal{A}}^0(\mathbf{k}) \widehat{P}. \tag{7.24}$$

7.4. The operator $\widehat{G}(\boldsymbol{\theta})$. For the operator family $\widehat{A}(t,\boldsymbol{\theta})$, the operator G that is defined in abstract terms by (2.22) depends on the parameter $\boldsymbol{\theta}$ and takes the form

$$\widehat{G}(\boldsymbol{\theta}) = (\widehat{R}(\boldsymbol{\theta})\widehat{P})^*\widehat{X}_1(\boldsymbol{\theta})\widehat{Z}(\boldsymbol{\theta}) + (\widehat{X}_1(\boldsymbol{\theta})\widehat{Z}(\boldsymbol{\theta}))^*\widehat{R}(\boldsymbol{\theta})\widehat{P}.$$

Denote by $B_1(\boldsymbol{\theta};\mathbf{D})$ the DO of order $p-1$ such that $\widehat{X}_1(\boldsymbol{\theta}) = hB_1(\boldsymbol{\theta};\mathbf{D})$ (see (6.10) with $f=\mathbf{1}_n$). Then

$$B_1(\boldsymbol{\theta};\mathbf{D}) = \sum_{|\beta|=p} b_\beta \sum_{\gamma\le\beta:|\gamma|=p-1} \binom{\beta}{\gamma}\boldsymbol{\theta}^{\beta-\gamma}\mathbf{D}^\gamma.$$

Using (7.10) and (7.11), we obtain

$$\widehat{G}(\boldsymbol{\theta}) = b(\boldsymbol{\theta})^* g^{(1)}(\boldsymbol{\theta}) b(\boldsymbol{\theta})\widehat{P}, \tag{7.25}$$

where $g^{(1)}(\boldsymbol{\theta})$ is the Hermitian matrix of size $m\times m$ given by

$$g^{(1)}(\boldsymbol{\theta}) = |\Omega|^{-1}\int_\Omega (\widetilde{g}(\mathbf{x})^* B_1(\boldsymbol{\theta};\mathbf{D})\Lambda(\mathbf{x}) + (B_1(\boldsymbol{\theta};\mathbf{D})\Lambda(\mathbf{x}))^*\widetilde{g}(\mathbf{x}))\,d\mathbf{x}. \tag{7.26}$$

We distinguish some cases where the operator (7.25) is equal to zero (see [22, Proposition 3.3]).

Proposition 7.3.

1°. *Suppose that relations* (7.16) *are satisfied. Then* $\Lambda(\mathbf{x}) = 0$, *whence* $g^{(1)}(\boldsymbol{\theta}) = 0$ *and* $\widehat{G}(\boldsymbol{\theta}) = 0$ *for any* $\boldsymbol{\theta}\in\mathbb{S}^{d-1}$.

2°. *Suppose that representations* (7.17) *are satisfied. Then* $g^{(1)}(\boldsymbol{\theta}) = 0$ *and* $\widehat{G}(\boldsymbol{\theta}) = 0$ *for any* $\boldsymbol{\theta}\in\mathbb{S}^{d-1}$.

3°. *Let* $n=1$. *Suppose that the matrices* $g(\mathbf{x})$ *and* b_β, $|\beta|=p$, *have real entries. Then* $\widehat{G}(\boldsymbol{\theta}) = 0$ *for any* $\boldsymbol{\theta}\in\mathbb{S}^{d-1}$.

Remark 7.2. In the general case, the operator $\widehat{G}(\boldsymbol{\theta})$ may be nonzero. In particular, it is easy to give examples of the operator $\mathcal{A} = b(\mathbf{D})^*g(\mathbf{x})b(\mathbf{D})$ in the case where $n=1$ and $g(\mathbf{x})$ is a Hermitian matrix with complex entries such that the corresponding operator $\widehat{G}(\boldsymbol{\theta})$ is not zero.

8. Approximation of the Operator Exponential $e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau}$

8.1. Approximation of the operator exponential $e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau}$. The principal term. We apply Theorems 3.1, 3.2, 3.3 to the operator family $\widehat{A}(t,\boldsymbol{\theta}) = \widehat{\mathcal{A}}(\mathbf{k})$. By (7.24),

$$e^{-t^{2p}\widehat{S}(\boldsymbol{\theta})\tau}\widehat{P} = e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau}\widehat{P}. \tag{8.1}$$

It remains to specify the constants in estimates. The constants $\widehat{\delta}$, C_0, $\widehat{t}^0$, and $\widehat{c}_*$ are defined by (7.5), (6.18), (7.6), (7.7) and do not depend on $\boldsymbol{\theta}$. They depend only on the following set of parameters:

$$d,\ p,\ \alpha_0,\ \alpha_1,\ \|g\|_{L_\infty},\ \|g^{-1}\|_{L_\infty},\ \text{the parameters of the lattice } \Gamma, \tag{8.2}$$

which for brevity is called the *problem data*. Next, according to Remark 3.1, the constants $\widehat{C}_8$, $\widehat{C}_9$, $\widehat{C}_{10}$ from Theorems 3.1, 3.2, 3.3 (as applied to $\widehat{A}(t,\boldsymbol{\theta})$) are majorated by the polynomials of the variables C_0, $\|\widehat{X}_p\|$, $\widehat{\delta}^{-\frac{1}{2p}}$, $\widehat{c}_*^{-\frac{1}{2p}}$ with positive coefficients depending only on p. Now the operator $\widehat{X}_p$ depends on $\boldsymbol{\theta}$, but its norm is estimated by the value $\alpha_1^{\frac12}\|g\|_{L_\infty}^{\frac12}$, which is independent of $\boldsymbol{\theta}$; see (6.12) with $f=\mathbf{1}_n$. Thus, after possible overestimation, the constants $\widehat{C}_8$, $\widehat{C}_9$, $\widehat{C}_{10}$ (for the operator family $\widehat{A}(t,\boldsymbol{\theta})$) depend only on parameters (8.2).

Applying Theorem 3.1 to the operator family $\widehat{A}(t,\boldsymbol{\theta}) = \widehat{\mathcal{A}}(\mathbf{k})$ and using (8.1), we obtain the inequality

$$\left\| e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau} - e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau}\widehat{P}\right\|_{L_2(\Omega)\to L_2(\Omega)} \le \frac{\widehat{C}_8}{\tau^{\frac{1}{2p}}+1}, \quad \tau\ge 0, \quad |\mathbf{k}|\le\widehat{t}^0. \tag{8.3}$$

Let us show that, within the margin of the admissible error, we can get rid of the projection $\widehat{P}$ in (8.3) (replace $\widehat{P}$ by the identity operator). From (6.24) (with $f = \mathbf{1}_n$), (7.12), and the homogeneity of the symbol $b(\boldsymbol{\xi})$ it follows that

$$b(\boldsymbol{\xi})^* g^0 b(\boldsymbol{\xi}) \geq \widehat{c}_* |\boldsymbol{\xi}|^{2p} \mathbf{1}_n, \quad \boldsymbol{\xi} \in \mathbb{R}^d. \tag{8.4}$$

Hence, using the discrete Fourier transform, by (5.14) and the elementary estimate $(x^{\frac{1}{2p}} + 1)e^{-x} \leq 2$ for $x \geq 0$, we obtain

$$\begin{aligned} \left\| e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau}(I - \widehat{P}) \right\|_{L_2(\Omega)\to L_2(\Omega)} &\leq \sup_{0\neq \mathbf{b}\in\widetilde{\Gamma}} e^{-\widehat{c}_*|\mathbf{b}+\mathbf{k}|^{2p}\tau} \leq e^{-\widehat{c}_* r_0^{2p}\tau} \\ &\leq \frac{2}{(\widehat{c}_* r_0^{2p}\tau)^{\frac{1}{2p}} + 1} \leq \frac{2\max\{1, \widehat{c}_*^{\,-\frac{1}{2p}} r_0^{-1}\}}{\tau^{\frac{1}{2p}} + 1}, \quad \tau \geq 0, \quad \mathbf{k} \in \operatorname{clos}\widetilde{\Omega}. \end{aligned} \tag{8.5}$$

Now from (8.3) and (8.5) it follows that

$$\left\| e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau} - e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \right\|_{L_2(\Omega)\to L_2(\Omega)} \leq \frac{\widehat{C}_8'}{\tau^{\frac{1}{2p}} + 1}, \quad \tau \geq 0, \quad |\mathbf{k}| \leq \widehat{t}^0, \tag{8.6}$$

where $\widehat{C}_8' = \widehat{C}_8 + 2\max\left\{1, \widehat{c}_*^{\,-\frac{1}{2p}} r_0^{-1}\right\}$.

For $|\mathbf{k}| > \widehat{t}^0$ the estimate is trivial. By (7.8) and (7.23),

$$\left\| e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau} - e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \right\|_{L_2(\Omega)\to L_2(\Omega)} \leq 2e^{-\widehat{c}_*|\mathbf{k}|^{2p}\tau} \leq 2e^{-\widehat{c}_*(\widehat{t}^0)^{2p}\tau}, \quad \tau \geq 0, \quad \mathbf{k} \in \operatorname{clos}\widetilde{\Omega} \setminus \mathcal{B}(\widehat{t}^0).$$

Combining this with the elementary estimate $(x^{\frac{1}{2p}} + 1)e^{-x} \leq 2$ for $x \geq 0$, we obtain

$$\begin{aligned} \left\| e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau} - e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \right\|_{L_2(\Omega)\to L_2(\Omega)} &\leq \frac{4}{(\widehat{c}_*(\widehat{t}^0)^{2p}\tau)^{\frac{1}{2p}} + 1} \leq \frac{4\max\{1, \widehat{c}_*^{\,-\frac{1}{2p}}(\widehat{t}^0)^{-1}\}}{\tau^{\frac{1}{2p}} + 1}, \\ &\qquad \tau \geq 0, \quad \mathbf{k} \in \operatorname{clos}\widetilde{\Omega} \setminus \mathcal{B}(\widehat{t}^0). \end{aligned} \tag{8.7}$$

As a result, estimates (8.6) and (8.7) imply the following result.

Theorem 8.1. *Suppose that $\widehat{\mathcal{A}}(\mathbf{k})$ is the operator family of form* (7.2) *and $\widehat{\mathcal{A}}^0(\mathbf{k})$ is the effective operator family defined in Sec.* 7.3. *Then we have*

$$\left\| e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau} - e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \right\|_{L_2(\Omega)\to L_2(\Omega)} \leq \frac{\widehat{\mathcal{C}}_1}{\tau^{\frac{1}{2p}} + 1}, \quad \tau \geq 0, \quad \mathbf{k} \in \operatorname{clos}\widetilde{\Omega}. \tag{8.8}$$

The constant $\widehat{\mathcal{C}}_1$ depends only on the problem data (8.2).

Under the additional assumption that $\widehat{G}(\boldsymbol{\theta}) \equiv 0$, we can apply Theorem 3.2 to the operator family $\widehat{A}(t, \boldsymbol{\theta}) = \widehat{\mathcal{A}}(\mathbf{k})$. This yields

$$\left\| e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau} - e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau}\widehat{P} \right\|_{L_2(\Omega)\to L_2(\Omega)} \leq \frac{\widehat{C}_9}{\tau^{\frac{1}{p}} + 1}, \quad \tau \geq 0, \quad |\mathbf{k}| \leq \widehat{t}^0. \tag{8.9}$$

By analogy with (8.5),

$$\left\| e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau}(I - \widehat{P}) \right\|_{L_2(\Omega)\to L_2(\Omega)} \leq \frac{2\max\{1, \widehat{c}_*^{\,-\frac{1}{p}} r_0^{-2}\}}{\tau^{\frac{1}{p}} + 1}, \quad \tau \geq 0, \quad \mathbf{k} \in \operatorname{clos}\widetilde{\Omega}. \tag{8.10}$$

Next, by analogy with (8.7), we have

$$\left\| e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau} - e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \right\|_{L_2(\Omega)\to L_2(\Omega)} \leq \frac{4\max\{1, \widehat{c}_*^{\,-\frac{1}{p}}(\widehat{t}^0)^{-2}\}}{\tau^{\frac{1}{p}} + 1}, \quad \tau \geq 0, \quad \mathbf{k} \in \operatorname{clos}\widetilde{\Omega} \setminus \mathcal{B}(\widehat{t}^0). \tag{8.11}$$

Combining estimates (8.9)–(8.11), we arrive at the following result.

Theorem 8.2. *Suppose that the assumptions of Theorem* 8.1 *are satisfied. Let* $\widehat{G}(\boldsymbol{\theta})$ *be the operator defined by* (7.25), (7.26)*. Suppose that* $\widehat{G}(\boldsymbol{\theta}) = 0$ *for any* $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$*. Then we have*

$$\left\| e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau} - e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \right\|_{L_2(\Omega)\to L_2(\Omega)} \leq \frac{\widehat{\mathcal{C}}_2}{\tau^{\frac{1}{p}} + 1}, \quad \tau \geq 0, \quad \mathbf{k} \in \operatorname{clos} \widetilde{\Omega}. \tag{8.12}$$

The constant $\widehat{\mathcal{C}}_2$ *depends only on the problem data* (8.2).

8.2. Approximation of the operator exponential $e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau}$ **in the "energy" norm.** Now, we apply Theorem 3.3 to the operator family $\widehat{A}(t,\boldsymbol{\theta}) = \widehat{\mathcal{A}}(\mathbf{k})$. By (7.4), (7.10), and (7.24),

$$\left(I + t^p \widehat{Z}(\boldsymbol{\theta})\right) e^{-t^{2p}\widehat{S}(\boldsymbol{\theta})\tau} \widehat{P} = \left(I + [\Lambda] b(\mathbf{k}) \widehat{P}\right) e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \widehat{P} = (I + [\Lambda] b(\mathbf{D}+\mathbf{k}))\, e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \widehat{P}.$$

Using this identity and applying Theorem 3.3, we obtain

$$\left\| \widehat{\mathcal{A}}(\mathbf{k})^{1/2} \left(e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau} - (I + [\Lambda] b(\mathbf{D}+\mathbf{k}))\, e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \widehat{P} \right) \right\|_{L_2(\Omega)\to L_2(\Omega)} \leq \widehat{C}_{10} \tau^{-\frac{1}{2}-\frac{1}{2p}}, \quad \tau > 0, \ |\mathbf{k}| \leq \widehat{t}^{\,0}. \tag{8.13}$$

Next, we show that, within the margin of the admissible error, the operator $\widehat{\mathcal{A}}(\mathbf{k})^{1/2} e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \widehat{P}$ in (8.13) can be replaced by $\widehat{\mathcal{A}}(\mathbf{k})^{1/2} e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau}$. Similarly to (8.5), using (5.5), (5.14), (8.4), and the elementary estimate $x^{\frac{1}{2}+\frac{1}{2p}} e^{-x} \leq 1$ for $x \geq 0$, we obtain

$$\begin{aligned} &\left\| \widehat{\mathcal{A}}(\mathbf{k})^{1/2} e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} (I - \widehat{P}) \right\|_{L_2(\Omega)\to L_2(\Omega)} = \left\| h b(\mathbf{D}+\mathbf{k}) e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} (I - \widehat{P}) \right\|_{L_2(\Omega)\to L_2(\Omega)} \\ &\leq \alpha_1^{\frac{1}{2}} \|g\|_{L_\infty}^{\frac{1}{2}} \sup_{0 \neq \mathbf{b} \in \widetilde{\Gamma}} |\mathbf{b}+\mathbf{k}|^p e^{-\widehat{c}_* |\mathbf{b}+\mathbf{k}|^{2p}\tau} \leq \alpha_1^{\frac{1}{2}} \|g\|_{L_\infty}^{\frac{1}{2}} \widehat{c}_*^{\,-\frac{1}{2}-\frac{1}{2p}} r_0^{-1} \tau^{-\frac{1}{2}-\frac{1}{2p}}, \quad \tau > 0, \quad \mathbf{k} \in \operatorname{clos} \widetilde{\Omega}. \end{aligned} \tag{8.14}$$

Together with (8.13) this implies

$$\left\| \widehat{\mathcal{A}}(\mathbf{k})^{1/2} \left(e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau} - \left(I + [\Lambda] b(\mathbf{D}+\mathbf{k}) \widehat{P}\right) e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \right) \right\|_{L_2(\Omega)\to L_2(\Omega)} \leq \widehat{\mathcal{C}}_3' \tau^{-\frac{1}{2}-\frac{1}{2p}}, \quad \tau > 0, \ |\mathbf{k}| \leq \widehat{t}^{\,0}, \tag{8.15}$$

where $\widehat{\mathcal{C}}_3' = \widehat{C}_{10} + \alpha_1^{\frac{1}{2}} \|g\|_{L_\infty}^{\frac{1}{2}} \widehat{c}_*^{\,-\frac{1}{2}-\frac{1}{2p}} r_0^{-1}$.

For $|\mathbf{k}| > \widehat{t}^{\,0}$ estimates are trivial: each term under the norm sign in (8.15) is estimated separately. By the spectral theorem and inequality (7.8),

$$\left\| \widehat{\mathcal{A}}(\mathbf{k})^{1/2} e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau} \right\|_{L_2(\Omega)\to L_2(\Omega)} \leq \sup_{\lambda \geq \widehat{c}_* (\widehat{t}^{\,0})^{2p}} \sqrt{\lambda} e^{-\lambda\tau} \leq \widehat{c}_*^{\,-\frac{1}{2p}} (\widehat{t}^{\,0})^{-1} \tau^{-\frac{1}{2}-\frac{1}{2p}}, \quad \tau > 0, \ \mathbf{k} \in \operatorname{clos} \widetilde{\Omega} \setminus \mathcal{B}(\widehat{t}^{\,0}). \tag{8.16}$$

Similarly, using (7.23), we obtain

$$\left\| \widehat{\mathcal{A}}^0(\mathbf{k})^{1/2} e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \right\|_{L_2(\Omega)\to L_2(\Omega)} \leq \widehat{c}_*^{\,-\frac{1}{2p}} (\widehat{t}^{\,0})^{-1} \tau^{-\frac{1}{2}-\frac{1}{2p}}, \quad \tau > 0, \ \mathbf{k} \in \operatorname{clos} \widetilde{\Omega} \setminus \mathcal{B}(\widehat{t}^{\,0}).$$

Combining this with (7.15), we see that

$$\begin{aligned} &\left\| \widehat{\mathcal{A}}(\mathbf{k})^{1/2} e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \right\|_{L_2(\Omega)\to L_2(\Omega)} = \left\| h b(\mathbf{D}+\mathbf{k}) e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \right\| \\ &\leq \|g\|_{L_\infty}^{\frac{1}{2}} \|g^{-1}\|_{L_\infty}^{\frac{1}{2}} \left\| (g^0)^{1/2} b(\mathbf{D}+\mathbf{k}) e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \right\| = \|g\|_{L_\infty}^{\frac{1}{2}} \|g^{-1}\|_{L_\infty}^{\frac{1}{2}} \left\| \widehat{\mathcal{A}}^0(\mathbf{k})^{1/2} e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \right\| \\ &\leq \|g\|_{L_\infty}^{\frac{1}{2}} \|g^{-1}\|_{L_\infty}^{\frac{1}{2}} \widehat{c}_*^{\,-\frac{1}{2p}} (\widehat{t}^{\,0})^{-1} \tau^{-\frac{1}{2}-\frac{1}{2p}}, \quad \tau > 0, \ \mathbf{k} \in \operatorname{clos} \widetilde{\Omega} \setminus \mathcal{B}(\widehat{t}^{\,0}). \end{aligned} \tag{8.17}$$

It remains to estimate the operator

$$\widehat{\mathcal{A}}(\mathbf{k})^{1/2} [\Lambda] b(\mathbf{D}+\mathbf{k}) e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \widehat{P} = \left(\widehat{\mathcal{A}}(\mathbf{k})^{1/2} [\Lambda] \widehat{P}_m \right) \left(b(\mathbf{D}+\mathbf{k}) e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \widehat{P} \right). \tag{8.18}$$

Here $\widehat{P}_m$ is the orthogonal projection of $\mathfrak{H}_* = L_2(\Omega;\mathbb{C}^m)$ onto the subspace of constants. From (5.5), (7.18), and (7.19) it follows that

$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}[\Lambda]\widehat{P}_m\|_{L_2(\Omega)\to L_2(\Omega)} = |\Omega|^{-\frac{1}{2}}\|hb(\mathbf{D}+\mathbf{k})\Lambda\|_{L_2(\Omega)} \le \|g\|_{L_\infty}^{\frac{1}{2}}\left(1+\alpha_1^{\frac{1}{2}}r_1^p C_\Lambda\right), \tag{8.19}$$

where $2r_1 = \operatorname{diam}\widetilde{\Omega}$. Similarly to (8.17), we have

$$\begin{aligned}\left\|b(\mathbf{D}+\mathbf{k})e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau}\right\| &\le \|g^{-1}\|_{L_\infty}^{\frac{1}{2}}\left\|(g^0)^{1/2}b(\mathbf{D}+\mathbf{k})e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau}\right\| \\ &\le \|g^{-1}\|_{L_\infty}^{\frac{1}{2}}\widehat{c}_*^{\,-\frac{1}{2p}}(\widehat{t}^0)^{-1}\tau^{-\frac{1}{2}-\frac{1}{2p}}, \quad \tau>0,\ \mathbf{k}\in\operatorname{clos}\widetilde{\Omega}\setminus\mathcal{B}(\widehat{t}^0).\end{aligned} \tag{8.20}$$

Combining (8.18)–(8.20), we obtain

$$\begin{aligned}\left\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}[\Lambda]b(\mathbf{D}+\mathbf{k})e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau}\widehat{P}\right\| \le \|g\|_{L_\infty}^{\frac{1}{2}}\|g^{-1}\|_{L_\infty}^{\frac{1}{2}}\left(1+\alpha_1^{\frac{1}{2}}r_1^pC_\Lambda\right)\widehat{c}_*^{\,-\frac{1}{2p}}(\widehat{t}^0)^{-1}\tau^{-\frac{1}{2}-\frac{1}{2p}}, \\ \tau>0,\ \mathbf{k}\in\operatorname{clos}\widetilde{\Omega}\setminus\mathcal{B}(\widehat{t}^0).\end{aligned} \tag{8.21}$$

As a result, from (8.15)–(8.17) and (8.21) it follows that

$$\left\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\left(e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau} - \left(I+[\Lambda]b(\mathbf{D}+\mathbf{k})\widehat{P}\right)e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau}\right)\right\|_{L_2(\Omega)\to L_2(\Omega)} \le \widehat{\mathcal{C}}_3''\tau^{-\frac{1}{2}-\frac{1}{2p}}, \quad \tau>0,\ \mathbf{k}\in\operatorname{clos}\widetilde{\Omega}, \tag{8.22}$$

where

$$\widehat{\mathcal{C}}_3'' = \max\left\{\widehat{\mathcal{C}}_3', \widehat{c}_*^{\,-\frac{1}{2p}}(\widehat{t}^0)^{-1}\left(1+\|g\|_{L_\infty}^{\frac{1}{2}}\|g^{-1}\|_{L_\infty}^{\frac{1}{2}}\left(2+\alpha_1^{\frac{1}{2}}r_1^pC_\Lambda\right)\right)\right\}.$$

Inequality (8.22) is interesting for $\tau \ge 1$. For $0<\tau<1$ trivial estimates give the best order: each term under the norm sign in (8.22) is estimated separately. By the spectral theorem,

$$\left\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau}\right\|_{L_2(\Omega)\to L_2(\Omega)} \le \sup_{\lambda\ge 0}\lambda^{1/2}e^{-\lambda\tau} \le \tau^{-\frac{1}{2}}, \quad \tau>0,\ \mathbf{k}\in\operatorname{clos}\widetilde{\Omega}. \tag{8.23}$$

Similarly, by (7.15), we have

$$\begin{aligned}&\left\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau}\right\|_{L_2(\Omega)\to L_2(\Omega)} = \left\|hb(\mathbf{D}+\mathbf{k})e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau}\right\|_{L_2(\Omega)\to L_2(\Omega)} \\ &\le \|g\|_{L_\infty}^{\frac{1}{2}}\|g^{-1}\|_{L_\infty}^{\frac{1}{2}}\left\|\widehat{\mathcal{A}}^0(\mathbf{k})^{1/2}e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau}\right\|_{L_2(\Omega)\to L_2(\Omega)} \le \|g\|_{L_\infty}^{\frac{1}{2}}\|g^{-1}\|_{L_\infty}^{\frac{1}{2}}\tau^{-\frac{1}{2}}, \quad \tau>0,\ \mathbf{k}\in\operatorname{clos}\widetilde{\Omega}.\end{aligned} \tag{8.24}$$

To estimate the operator $\widehat{\mathcal{A}}(\mathbf{k})^{1/2}[\Lambda]b(\mathbf{D}+\mathbf{k})e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau}\widehat{P}$, we apply (8.18), (8.19), and the inequality

$$\begin{aligned}\left\|b(\mathbf{D}+\mathbf{k})e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau}\right\|_{L_2(\Omega)\to L_2(\Omega)} \le \|g^{-1}\|_{L_\infty}^{\frac{1}{2}}\left\|\widehat{\mathcal{A}}^0(\mathbf{k})^{1/2}e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau}\right\|_{L_2(\Omega)\to L_2(\Omega)} \le \|g^{-1}\|_{L_\infty}^{\frac{1}{2}}\tau^{-\frac{1}{2}}, \\ \tau>0,\ \mathbf{k}\in\operatorname{clos}\widetilde{\Omega}.\end{aligned} \tag{8.25}$$

We obtain

$$\begin{aligned}\left\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}[\Lambda]b(\mathbf{D}+\mathbf{k})e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau}\widehat{P}\right\|_{L_2(\Omega)\to L_2(\Omega)} \le \|g\|_{L_\infty}^{\frac{1}{2}}\|g^{-1}\|_{L_\infty}^{\frac{1}{2}}\left(1+\alpha_1^{\frac{1}{2}}r_1^pC_\Lambda\right)\tau^{-\frac{1}{2}}, \\ \tau>0,\ \mathbf{k}\in\operatorname{clos}\widetilde{\Omega}.\end{aligned} \tag{8.26}$$

Together with (8.23) and (8.24) this implies

$$\left\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\left(e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau} - \left(I+[\Lambda]b(\mathbf{D}+\mathbf{k})\widehat{P}\right)e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau}\right)\right\|_{L_2(\Omega)\to L_2(\Omega)} \le \widehat{\mathcal{C}}_3'''\tau^{-\frac{1}{2}}, \quad \tau>0,\ \mathbf{k}\in\operatorname{clos}\widetilde{\Omega}, \tag{8.27}$$

where

$$\widehat{\mathcal{C}}_3''' = 1+\|g\|_{L_\infty}^{\frac{1}{2}}\|g^{-1}\|_{L_\infty}^{\frac{1}{2}}\left(2+\alpha_1^{\frac{1}{2}}r_1^pC_\Lambda\right).$$

Applying (8.22) for $\tau \geq 1$ and (8.27) for $0 < \tau < 1$, we arrive at the inequality

$$\left\| \widehat{\mathcal{A}}(\mathbf{k})^{1/2} \left(e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau} - \left(I + [\Lambda] b(\mathbf{D}+\mathbf{k}) \widehat{P} \right) e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \right) \right\|_{L_2(\Omega)\to L_2(\Omega)} \leq \frac{2\max\{\widehat{\mathcal{C}}_3'', \widehat{\mathcal{C}}_3'''\}}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2p}}+1)}, \quad \tau > 0,\ \mathbf{k} \in \operatorname{clos}\widetilde{\Omega}. \tag{8.28}$$

Let us summarize the results.

Theorem 8.3. *Suppose that the assumptions of Theorem* 8.1 *are satisfied. Let* Λ *be the* Γ*-periodic solution of problem* (7.9)*, and let* $\widehat{P}$ *be projection* (7.4)*. Then for* $\tau > 0$ *and* $\mathbf{k} \in \operatorname{clos}\widetilde{\Omega}$ *we have*

$$\left\| \widehat{\mathcal{A}}(\mathbf{k})^{1/2} \left(e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau} - \left(I + [\Lambda] b(\mathbf{D}+\mathbf{k}) \widehat{P} \right) e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \right) \right\|_{L_2(\Omega)\to L_2(\Omega)} \leq \frac{\widehat{\mathcal{C}}_3}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2p}}+1)}, \tag{8.29}$$

$$\left\| \widehat{\mathcal{A}}(\mathbf{k})^{1/2} \left(e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau} - \left(I + [\Lambda] b(\mathbf{D}+\mathbf{k}) \right) e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \widehat{P} \right) \right\|_{L_2(\Omega)\to L_2(\Omega)} \leq \frac{\widehat{\mathcal{C}}_3}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2p}}+1)}. \tag{8.30}$$

The constant $\widehat{\mathcal{C}}_3$ *depends only on the problem data* (8.2)*.*

Proof. Inequality (8.28) implies (8.29) with the constant $\widehat{\mathcal{C}}_3 = 2\max\{\widehat{\mathcal{C}}_3'', \widehat{\mathcal{C}}_3'''\}$.

Inequality (8.30) is proved by analogy with the proof of estimate (8.28): it follows from (8.13), (8.16), (8.17), (8.21), (8.23), (8.24), and (8.26). □

We also need to estimate the operator $e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau} - \left(I + [\Lambda] b(\mathbf{D}+\mathbf{k}) \widehat{P} \right) e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau}$ in the operator norm in $L_2(\Omega; \mathbb{C}^n)$.

Proposition 8.1. *Suppose that the assumptions of Theorem* 8.3 *are satisfied.*

1°. *For* $\tau > 0$ *and* $\mathbf{k} \in \operatorname{clos}\widetilde{\Omega}$ *we have*

$$\left\| e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau} - \left(I + [\Lambda] b(\mathbf{D}+\mathbf{k}) \widehat{P} \right) e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \right\|_{L_2(\Omega)\to L_2(\Omega)} \leq \frac{\widehat{\mathcal{C}}_4}{\tau^{\frac{1}{2p}}+1}. \tag{8.31}$$

The constant $\widehat{\mathcal{C}}_4$ *depends only on the problem data* (8.2)*.*

2°. *In addition, suppose that* $\widehat{G}(\boldsymbol{\theta}) \equiv 0$*. Then for* $\tau > 0$ *and* $\mathbf{k} \in \operatorname{clos}\widetilde{\Omega}$ *we have*

$$\left\| e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau} - \left(I + [\Lambda] b(\mathbf{D}+\mathbf{k}) \widehat{P} \right) e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \right\|_{L_2(\Omega)\to L_2(\Omega)} \leq \frac{\widehat{\mathcal{C}}_5}{\tau^{\frac{1}{p}}+1}. \tag{8.32}$$

The constant $\widehat{\mathcal{C}}_5$ *depends only on the problem data* (8.2)*.*

Proof. Similarly to (8.18), we have

$$[\Lambda] b(\mathbf{D}+\mathbf{k}) e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \widehat{P} = \left([\Lambda] \widehat{P}_m \right) \left(b(\mathbf{D}+\mathbf{k}) e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \widehat{P} \right). \tag{8.33}$$

By (7.19),

$$\| [\Lambda] \widehat{P}_m \|_{L_2(\Omega)\to L_2(\Omega)} = |\Omega|^{-\frac{1}{2}} \|\Lambda\|_{L_2(\Omega)} \leq C_\Lambda. \tag{8.34}$$

Combining this with (8.25) and (8.33), we deduce

$$\| [\Lambda] b(\mathbf{D}+\mathbf{k}) e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \widehat{P} \|_{L_2(\Omega)\to L_2(\Omega)} \leq C_\Lambda \|g^{-1}\|_{L_\infty}^{\frac{1}{2}} \tau^{-\frac{1}{2}} \leq \frac{2 C_\Lambda \|g^{-1}\|_{L_\infty}^{\frac{1}{2}}}{\tau^{\frac{1}{2p}}+1}, \quad \tau \geq 1,\ \mathbf{k} \in \operatorname{clos}\widetilde{\Omega}. \tag{8.35}$$

For $0 < \tau < 1$, using the identity $b(\mathbf{D}+\mathbf{k}) e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \widehat{P} = b(\mathbf{k}) e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \widehat{P}$ and (5.5), we obtain

$$\| b(\mathbf{D}+\mathbf{k}) e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \widehat{P} \|_{L_2(\Omega)\to L_2(\Omega)} \leq \alpha_1^{\frac{1}{2}} r_1^p \leq \frac{2\alpha_1^{\frac{1}{2}} r_1^p}{\tau^{\frac{1}{2p}}+1}, \quad 0 < \tau < 1, \quad \mathbf{k} \in \operatorname{clos}\widetilde{\Omega}. \tag{8.36}$$

From (8.33), (8.34), and (8.36) it follows that

$$\|[\Lambda]b(\mathbf{D}+\mathbf{k})e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau}\widehat{P}\|_{L_2(\Omega)\to L_2(\Omega)} \le \frac{2C_\Lambda \alpha_1^{\frac{1}{2}} r_1^p}{\tau^{\frac{1}{2p}}+1}, \quad 0<\tau<1, \quad \mathbf{k}\in \operatorname{clos}\widetilde{\Omega}. \tag{8.37}$$

As a result, relations (8.8), (8.35), and (8.37) imply the required estimate (8.31) with the constant

$$\widehat{\mathcal{C}}_4 = \widehat{\mathcal{C}}_1 + 2C_\Lambda \max\left\{\|g^{-1}\|_{L_\infty}^{\frac{1}{2}}, \alpha_1^{\frac{1}{2}} r_1^p\right\}.$$

To check statement 2°, we should estimate operator (8.33) differently. Instead of (8.35) we use the inequality

$$\|[\Lambda]b(\mathbf{D}+\mathbf{k})e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau}\widehat{P}\|_{L_2(\Omega)\to L_2(\Omega)} \le C_\Lambda \|g^{-1}\|_{L_\infty}^{\frac{1}{2}} \tau^{-\frac{1}{2}} \le \frac{2C_\Lambda \|g^{-1}\|_{L_\infty}^{\frac{1}{2}}}{\tau^{\frac{1}{p}}+1}, \quad \tau\ge 1, \ \mathbf{k}\in \operatorname{clos}\widetilde{\Omega}. \tag{8.38}$$

Instead of (8.37) we apply the estimate

$$\|[\Lambda]b(\mathbf{D}+\mathbf{k})e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau}\widehat{P}\|_{L_2(\Omega)\to L_2(\Omega)} \le C_\Lambda \alpha_1^{\frac{1}{2}} r_1^p \le \frac{2C_\Lambda \alpha_1^{\frac{1}{2}} r_1^p}{\tau^{\frac{1}{p}}+1}, \quad 0<\tau<1, \quad \mathbf{k}\in \operatorname{clos}\widetilde{\Omega}. \tag{8.39}$$

As a result, inequality (8.12) (which is valid under the condition $\widehat{G}(\boldsymbol{\theta})\equiv 0$) and (8.38), (8.39) imply estimate (8.32) with the constant $\widehat{\mathcal{C}}_5 = \widehat{\mathcal{C}}_2 + 2C_\Lambda \max\left\{\|g^{-1}\|_{L_\infty}^{\frac{1}{2}}, \alpha_1^{\frac{1}{2}} r_1^p\right\}$. □

9. Approximation of the Operator $fe^{-\mathcal{A}(\mathbf{k})\tau}f^*$

9.1. Incorporation of the operator family $\mathcal{A}(\mathbf{k})$ in the framework of Sec. 4. We return to consideration of the operator family (6.5) in the general case where $f \ne \mathbf{1}_n$. Now the assumptions of Sec. 4 are satisfied with $\mathfrak{H} = \widehat{\mathfrak{H}} = L_2(\Omega;\mathbb{C}^n)$ and $\mathfrak{H}_* = L_2(\Omega;\mathbb{C}^m)$. The role of the operator $\widehat{A}(t)$ is played by $\widehat{A}(t,\boldsymbol{\theta}) = \widehat{\mathcal{A}}(\mathbf{k})$ and the role of the operator $A(t)$ is played by $A(t,\boldsymbol{\theta}) = \mathcal{A}(\mathbf{k})$ (where $\mathbf{k} = t\boldsymbol{\theta}$). The isomorphism M acts as the operator of multiplication by the matrix-valued function $f(\mathbf{x})$. The operator Q (see (4.2)) is the operator of multiplication by the matrix-valued function

$$Q(\mathbf{x}) := (f(\mathbf{x})f(\mathbf{x})^*)^{-1}.$$

By (5.3), the matrix-valued function $Q(\mathbf{x})$ is positive definite and bounded. The block $Q_{\widehat{\mathfrak{N}}}$ of the operator Q in the subspace $\widehat{\mathfrak{N}}$ (see (7.3)) is the operator of multiplication by the constant matrix

$$\overline{Q} = (\underline{ff^*})^{-1} = |\Omega|^{-1} \int_\Omega (f(\mathbf{x})f(\mathbf{x})^*)^{-1}\, d\mathbf{x}.$$

Next, M_0 (see (4.13)) is the operator of multiplication by the constant matrix

$$f_0 = (\overline{Q})^{-1/2} = (\underline{ff^*})^{1/2}. \tag{9.1}$$

Obviously, we have

$$|f_0| \le \|f\|_{L_\infty}, \quad |f_0^{-1}| \le \|f^{-1}\|_{L_\infty}. \tag{9.2}$$

Let $\widehat{\mathcal{A}}^0$ be the effective operator for $\widehat{\mathcal{A}}$; see (7.22). We put

$$\mathcal{A}^0 := f_0 \widehat{\mathcal{A}}^0 f_0 = f_0 b(\mathbf{D})^* g^0 b(\mathbf{D}) f_0. \tag{9.3}$$

Let $\mathcal{A}^0(\mathbf{k})$ be the corresponding operator family in $L_2(\Omega;\mathbb{C}^n)$. Then

$$\mathcal{A}^0(\mathbf{k}) = f_0 \widehat{\mathcal{A}}^0(\mathbf{k}) f_0 = f_0 b(\mathbf{D}+\mathbf{k})^* g^0 b(\mathbf{D}+\mathbf{k}) f_0 \tag{9.4}$$

(with periodic boundary conditions). From (7.23), (9.2), and the relation $c_* = \widehat{c}_* \|f^{-1}\|_{L_\infty}^{-2}$ it follows that

$$\mathcal{A}^0(\mathbf{k}) \ge c_* |\mathbf{k}|^{2p} I, \quad \mathbf{k}\in \operatorname{clos}\widetilde{\Omega}. \tag{9.5}$$

9.2. Approximation of the operator $fe^{-\mathcal{A}(\mathbf{k})\tau}f^*$. The principal term. We apply Theorems 4.1, 4.2, and 4.3 to the operator family $A(t,\boldsymbol{\theta})=\mathcal{A}(\mathbf{k})$. By (7.24),

$$t^{2p}f_0\widehat{S}(\boldsymbol{\theta})f_0\widehat{P}=f_0\widehat{\mathcal{A}}^0(\mathbf{k})f_0\widehat{P}=\mathcal{A}^0(\mathbf{k})\widehat{P},$$

whence

$$f_0e^{-t^{2p}f_0\widehat{S}(\boldsymbol{\theta})f_0\tau}f_0\widehat{P}=f_0e^{-\mathcal{A}^0(\mathbf{k})\tau}f_0\widehat{P}. \tag{9.6}$$

It remains to specify the constants in estimates. The constants δ, C_0, t^0, and c_* are defined by (6.17), (6.18), (6.19), (6.21) and do not depend on $\boldsymbol{\theta}$. They depend only on the following set of parameters:

$$d,\ p,\ \alpha_0,\ \alpha_1,\ \|g\|_{L_\infty},\ \|g^{-1}\|_{L_\infty},\ \|f\|_{L_\infty},\ \|f^{-1}\|_{L_\infty},\ \text{the parameters of the lattice }\Gamma, \tag{9.7}$$

which for brevity is called the *problem data.* Next, according to Remarks 3.1 and 4.2, the constants from Theorems 4.1, 4.2, 4.3 (as applied to $A(t,\boldsymbol{\theta})$) are majorated by polynomials of the variables C_0, $\|\widehat{X}_p\|$, $\|X_p\|$, $\widehat{\delta}^{-\frac{1}{2}}$, $\delta^{-\frac{1}{2p}}$, and $c_*^{-\frac{1}{2p}}$ with positive coefficients depending only on p. Now the operators $\widehat{X}_p$, X_p depend on $\boldsymbol{\theta}$, but their norms are estimated by the values $\alpha_1^{\frac{1}{2}}\|g\|_{L_\infty}^{\frac{1}{2}}$ and $\alpha_1^{\frac{1}{2}}\|g\|_{L_\infty}^{\frac{1}{2}}\|f\|_{L_\infty}$, respectively, which do not depend on $\boldsymbol{\theta}$; see (6.12). Thus, after possible overestimation, the constants from Theorems 4.1, 4.2, 4.3 (applied to the operator family $A(t,\boldsymbol{\theta})$) depend only on parameters (9.7).

Applying Theorem 4.1 to the operator family $A(t,\boldsymbol{\theta})=\mathcal{A}(\mathbf{k})$ and taking identity (9.6) into account, we obtain the inequality

$$\left\|fe^{-\mathcal{A}(\mathbf{k})\tau}f^*-f_0e^{-\mathcal{A}^0(\mathbf{k})\tau}f_0\widehat{P}\right\|_{L_2(\Omega)\to L_2(\Omega)}\le\frac{C_8\|f\|^2_{L_\infty}}{\tau^{\frac{1}{2p}}+1},\quad \tau\ge0,\quad |\mathbf{k}|\le t^0. \tag{9.8}$$

Let us show that, within the margin of the admissible error, the projection $\widehat{P}$ in (9.8) can be replaced by the identity operator. From (8.4) and (9.2) it follows that

$$f_0b(\boldsymbol{\xi})^*g^0b(\boldsymbol{\xi})f_0\ge c_*|\boldsymbol{\xi}|^{2p}\mathbf{1}_n,\quad \boldsymbol{\xi}\in\mathbb{R}^d. \tag{9.9}$$

Hence, using the discrete Fourier transform, similarly to (8.5), we obtain

$$\begin{aligned}\left\|f_0e^{-\mathcal{A}^0(\mathbf{k})\tau}f_0(I-\widehat{P})\right\|_{L_2(\Omega)\to L_2(\Omega)}&\le\|f\|^2_{L_\infty}\sup_{0\ne\mathbf{b}\in\widetilde{\Gamma}}e^{-c_*|\mathbf{b}+\mathbf{k}|^{2p}\tau}\le\|f\|^2_{L_\infty}e^{-c_*r_0^{2p}\tau}\\ &\le\frac{2\|f\|^2_{L_\infty}}{(c_*r_0^{2p}\tau)^{\frac{1}{2p}}+1}\le\frac{2\|f\|^2_{L_\infty}\max\{1,c_*^{-\frac{1}{2p}}r_0^{-1}\}}{\tau^{\frac{1}{2p}}+1},\quad\tau\ge0,\quad\mathbf{k}\in\operatorname{clos}\widetilde{\Omega}.\end{aligned} \tag{9.10}$$

Now, from (9.8) and (9.10) it follows that

$$\left\|fe^{-\mathcal{A}(\mathbf{k})\tau}f^*-f_0e^{-\mathcal{A}^0(\mathbf{k})\tau}f_0\right\|_{L_2(\Omega)\to L_2(\Omega)}\le\frac{C_8'\|f\|^2_{L_\infty}}{\tau^{\frac{1}{2p}}+1},\quad\tau\ge0,\quad|\mathbf{k}|\le t^0, \tag{9.11}$$

where $C_8'=C_8+2\max\{1,c_*^{-\frac{1}{2p}}r_0^{-1}\}$.

For $|\mathbf{k}|>t^0$ the estimate is trivial. By analogy with (8.7), from (6.20), (9.2), and (9.5) it follows that

$$\begin{aligned}\left\|fe^{-\mathcal{A}(\mathbf{k})\tau}f^*-f_0e^{-\mathcal{A}^0(\mathbf{k})\tau}f_0\right\|_{L_2(\Omega)\to L_2(\Omega)}&\le2\|f\|^2_{L_\infty}e^{-c_*|\mathbf{k}|^{2p}\tau}\le2\|f\|^2_{L_\infty}e^{-c_*(t^0)^{2p}\tau}\\ &\le\frac{4\|f\|^2_{L_\infty}\max\{1,c_*^{-\frac{1}{2p}}(t^0)^{-1}\}}{\tau^{\frac{1}{2p}}+1},\quad\tau\ge0,\quad\mathbf{k}\in\operatorname{clos}\widetilde{\Omega}\setminus\mathcal{B}(t^0).\end{aligned} \tag{9.12}$$

As a result, estimates (9.11) and (9.12) imply the following statement.

Theorem 9.1. *Let $\mathcal{A}(\mathbf{k})$ be the operator family of form* (6.5)*. Suppose that the matrix f_0 is given by* (9.1) *and the operator family $\mathcal{A}^0(\mathbf{k})$ is defined by* (9.4)*. Then we have*

$$\left\| f e^{-\mathcal{A}(\mathbf{k})\tau} f^* - f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0 \right\|_{L_2(\Omega)\to L_2(\Omega)} \le \frac{\mathcal{C}_1}{\tau^{\frac{1}{2p}}+1}, \quad \tau \ge 0, \quad \mathbf{k} \in \operatorname{clos} \widetilde{\Omega}. \tag{9.13}$$

The constant $\mathcal{C}_1$ depends only on the problem data (9.7)*.*

Similarly, from Theorem 4.2 we deduce the following result.

Theorem 9.2. *Suppose that the assumptions of Theorem* 9.1 *are satisfied. Let $\widehat{G}(\boldsymbol{\theta})$ be the operator defined according to* (7.25), (7.26)*. Suppose that $\widehat{G}(\boldsymbol{\theta}) = 0$ for any $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. Then we have*

$$\left\| f e^{-\mathcal{A}(\mathbf{k})\tau} f^* - f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0 \right\|_{L_2(\Omega)\to L_2(\Omega)} \le \frac{\mathcal{C}_2}{\tau^{\frac{1}{p}}+1}, \quad \tau \ge 0, \quad \mathbf{k} \in \operatorname{clos} \widetilde{\Omega}. \tag{9.14}$$

The constant $\mathcal{C}_2$ depends only on the problem data (9.7)*.*

9.3. Approximation of the operator $f e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau} f^*$ in the "energy" norm. Now, we apply Theorem 4.3 to the operator family $A(t,\boldsymbol{\theta}) = \mathcal{A}(\mathbf{k})$. By (7.4), (7.10), and (9.6),

$$\begin{aligned} &\left(I + t^p \widehat{Z}(\boldsymbol{\theta})\right) f_0 e^{-t^{2p} f_0 \widehat{S}(\boldsymbol{\theta}) f_0 \tau} f_0 \widehat{P} = \left(I + [\Lambda] b(\mathbf{k}) \widehat{P}\right) f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0 \widehat{P} \\ &= (I + [\Lambda] b(\mathbf{D}+\mathbf{k})) f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0 \widehat{P}. \end{aligned}$$

Using this identity and applying Theorem 4.3, we obtain

$$\begin{aligned} \left\| \widehat{\mathcal{A}}(\mathbf{k})^{1/2} \left(f e^{-\mathcal{A}(\mathbf{k})\tau} f^* - (I + [\Lambda] b(\mathbf{D}+\mathbf{k})) f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0 \widehat{P} \right) \right\|_{L_2(\Omega)\to L_2(\Omega)} &\le C_{11} \tau^{-\frac{1}{2}-\frac{1}{2p}}, \\ \tau > 0,\ |\mathbf{k}| \le t^0. \end{aligned} \tag{9.15}$$

Next, we show that, within the margin of the admissible error, the operator $\widehat{\mathcal{A}}(\mathbf{k})^{1/2} f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0 \widehat{P}$ in (9.15) can be replaced by $\widehat{\mathcal{A}}(\mathbf{k})^{1/2} f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0$. Similarly to (8.14), from (5.5), (9.2), and (9.9), it follows that

$$\begin{aligned} &\left\| \widehat{\mathcal{A}}(\mathbf{k})^{1/2} f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0 (I - \widehat{P}) \right\|_{L_2(\Omega)\to L_2(\Omega)} = \left\| h b(\mathbf{D}+\mathbf{k}) f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0 (I - \widehat{P}) \right\|_{L_2(\Omega)\to L_2(\Omega)} \\ &\le \alpha_1^{\frac{1}{2}} \|g\|_{L_\infty}^{\frac{1}{2}} \|f\|_{L_\infty}^2 \sup_{0\ne \mathbf{b}\in\widetilde{\Gamma}} |\mathbf{b}+\mathbf{k}|^p e^{-c_* |\mathbf{b}+\mathbf{k}|^{2p}\tau} \le \alpha_1^{\frac{1}{2}} \|g\|_{L_\infty}^{\frac{1}{2}} \|f\|_{L_\infty}^2 c_*^{-\frac{1}{2}-\frac{1}{2p}} r_0^{-1} \tau^{-\frac{1}{2}-\frac{1}{2p}} \end{aligned}$$

for $\tau > 0$ and $\mathbf{k} \in \operatorname{clos} \widetilde{\Omega}$. Together with (9.15), this implies

$$\begin{aligned} \left\| \widehat{\mathcal{A}}(\mathbf{k})^{1/2} \left(f e^{-\mathcal{A}(\mathbf{k})\tau} f^* - \left(I + [\Lambda] b(\mathbf{D}+\mathbf{k}) \widehat{P}\right) f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0 \right) \right\|_{L_2(\Omega)\to L_2(\Omega)} &\le \mathcal{C}_3' \tau^{-\frac{1}{2}-\frac{1}{2p}}, \\ \tau > 0,\ |\mathbf{k}| \le t^0, \end{aligned} \tag{9.16}$$

where $\mathcal{C}_3' = C_{11} + \alpha_1^{\frac{1}{2}} \|g\|_{L_\infty}^{\frac{1}{2}} \|f\|_{L_\infty}^2 c_*^{-\frac{1}{2}-\frac{1}{2p}} r_0^{-1}$.

Estimates for $|\mathbf{k}| > t^0$ are trivial: each term under the norm sign in (9.16) is estimated separately. By the spectral theorem and estimate (6.20),

$$\begin{aligned} &\left\| \widehat{\mathcal{A}}(\mathbf{k})^{1/2} f e^{-\mathcal{A}(\mathbf{k})\tau} f^* \right\|_{L_2(\Omega)\to L_2(\Omega)} = \left\| h b(\mathbf{D}+\mathbf{k}) f e^{-\mathcal{A}(\mathbf{k})\tau} f^* \right\| = \left\| \mathcal{A}(\mathbf{k})^{1/2} e^{-\mathcal{A}(\mathbf{k})\tau} f^* \right\| \\ &\le \|f\|_{L_\infty} \sup_{\lambda \ge c_* (t^0)^{2p}} \sqrt{\lambda} e^{-\lambda\tau} \le \|f\|_{L_\infty} c_*^{-\frac{1}{2p}} (t^0)^{-1} \tau^{-\frac{1}{2}-\frac{1}{2p}}, \quad \tau > 0,\ \mathbf{k} \in \operatorname{clos}\widetilde{\Omega} \setminus \mathcal{B}(t^0). \end{aligned} \tag{9.17}$$

By analogy with (8.17), using (9.5), we have

$$\begin{aligned}
&\left\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2} f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0\right\|_{L_2(\Omega)\to L_2(\Omega)} = \left\|hb(\mathbf{D}+\mathbf{k}) f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0\right\| \\
&\le \|g\|_{L_\infty}^{\frac12} \|g^{-1}\|_{L_\infty}^{\frac12} \left\|\mathcal{A}^0(\mathbf{k})^{1/2} e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0\right\| \\
&\le \|g\|_{L_\infty}^{\frac12} \|g^{-1}\|_{L_\infty}^{\frac12} \|f\|_{L_\infty} c_*^{-\frac{1}{2p}} (t^0)^{-1} \tau^{-\frac12-\frac{1}{2p}}, \quad \tau>0,\ \mathbf{k}\in \operatorname{clos}\widetilde{\Omega}\setminus \mathcal{B}(t^0).
\end{aligned} \tag{9.18}$$

It remains to estimate the operator

$$\widehat{\mathcal{A}}(\mathbf{k})^{1/2}[\Lambda] b(\mathbf{D}+\mathbf{k}) f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0 \widehat{P} = \left(\widehat{\mathcal{A}}(\mathbf{k})^{1/2}[\Lambda]\widehat{P}_m\right)\left(b(\mathbf{D}+\mathbf{k}) f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0 \widehat{P}\right). \tag{9.19}$$

Similarly to (9.18),

$$\begin{aligned}
&\left\|b(\mathbf{D}+\mathbf{k}) f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0\right\| \le \|g^{-1}\|_{L_\infty}^{\frac12} \left\|\mathcal{A}^0(\mathbf{k})^{1/2} e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0\right\| \\
&\le \|g^{-1}\|_{L_\infty}^{\frac12} \|f\|_{L_\infty} c_*^{-\frac{1}{2p}} (t^0)^{-1} \tau^{-\frac12-\frac{1}{2p}}, \quad \tau>0,\ \mathbf{k}\in \operatorname{clos}\widetilde{\Omega}\setminus \mathcal{B}(t^0).
\end{aligned} \tag{9.20}$$

Combining (8.19), (9.19), and (9.20), we obtain

$$\begin{aligned}
\left\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}[\Lambda] b(\mathbf{D}+\mathbf{k}) f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0 \widehat{P}\right\| &\le \|g\|_{L_\infty}^{\frac12} \|g^{-1}\|_{L_\infty}^{\frac12} \|f\|_{L_\infty} \left(1+\alpha_1^{\frac12} r_1^p C_\Lambda\right) c_*^{-\frac{1}{2p}} (t^0)^{-1} \tau^{-\frac12-\frac{1}{2p}}, \\
&\qquad \tau>0,\ \mathbf{k}\in \operatorname{clos}\widetilde{\Omega}\setminus \mathcal{B}(t^0).
\end{aligned} \tag{9.21}$$

As a result, from (9.16)–(9.18) and (9.21) it follows that

$$\begin{aligned}
&\left\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\left(f e^{-\mathcal{A}(\mathbf{k})\tau} f^* - \left(I+[\Lambda] b(\mathbf{D}+\mathbf{k})\widehat{P}\right) f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0\right)\right\|_{L_2(\Omega)\to L_2(\Omega)} \le \mathcal{C}_3'' \tau^{-\frac12-\frac{1}{2p}}, \\
&\qquad \tau>0,\ \mathbf{k}\in \operatorname{clos}\widetilde{\Omega},
\end{aligned} \tag{9.22}$$

where

$$\mathcal{C}_3'' = \max\left\{\mathcal{C}_3', \|f\|_{L_\infty} c_*^{-\frac{1}{2p}} (t^0)^{-1} \left(1+\|g\|_{L_\infty}^{\frac12}\|g^{-1}\|_{L_\infty}^{\frac12}\left(2+\alpha_1^{\frac12} r_1^p C_\Lambda\right)\right)\right\}.$$

Inequality (9.22) is interesting for $\tau \ge 1$. For $0<\tau<1$ trivial estimates have better order. By analogy with (9.17), using the spectral theorem, we obtain

$$\begin{aligned}
&\left\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2} f e^{-\mathcal{A}(\mathbf{k})\tau} f^*\right\|_{L_2(\Omega)\to L_2(\Omega)} = \left\|\mathcal{A}(\mathbf{k})^{1/2} e^{-\mathcal{A}(\mathbf{k})\tau} f^*\right\|_{L_2(\Omega)\to L_2(\Omega)} \\
&\le \|f\|_{L_\infty} \sup_{\lambda\ge 0} \lambda^{1/2} e^{-\lambda\tau} \le \|f\|_{L_\infty} \tau^{-\frac12}, \quad \tau>0,\ \mathbf{k}\in\operatorname{clos}\widetilde{\Omega}.
\end{aligned} \tag{9.23}$$

Similarly, by (7.15) and (9.2),

$$\begin{aligned}
&\left\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2} f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0\right\|_{L_2(\Omega)\to L_2(\Omega)} \le \|g\|_{L_\infty}^{\frac12} \|g^{-1}\|_{L_\infty}^{\frac12} \left\|\mathcal{A}^0(\mathbf{k})^{1/2} e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0\right\|_{L_2(\Omega)\to L_2(\Omega)} \\
&\le \|g\|_{L_\infty}^{\frac12} \|g^{-1}\|_{L_\infty}^{\frac12} \|f\|_{L_\infty} \tau^{-\frac12}, \quad \tau>0,\ \mathbf{k}\in\operatorname{clos}\widetilde{\Omega}.
\end{aligned} \tag{9.24}$$

To estimate the operator $\widehat{\mathcal{A}}(\mathbf{k})^{1/2}[\Lambda] b(\mathbf{D}+\mathbf{k}) f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0 \widehat{P}$, we apply (9.19), (8.19), and the inequality

$$\begin{aligned}
&\left\|b(\mathbf{D}+\mathbf{k}) f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0\right\|_{L_2(\Omega)\to L_2(\Omega)} \le \|g^{-1}\|_{L_\infty}^{\frac12} \left\|\mathcal{A}^0(\mathbf{k})^{1/2} e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0\right\|_{L_2(\Omega)\to L_2(\Omega)} \\
&\le \|g^{-1}\|_{L_\infty}^{\frac12} \|f\|_{L_\infty} \tau^{-\frac12}, \quad \tau>0,\ \mathbf{k}\in\operatorname{clos}\widetilde{\Omega}.
\end{aligned} \tag{9.25}$$

Hence,

$$\left\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}[\Lambda]b(\mathbf{D}+\mathbf{k})f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau}f_0\widehat{P}\right\|_{L_2(\Omega)\to L_2(\Omega)} \le \|g\|_{L_\infty}^{\frac{1}{2}}\|g^{-1}\|_{L_\infty}^{\frac{1}{2}}\|f\|_{L_\infty}\left(1+\alpha_1^{\frac{1}{2}}r_1^p C_\Lambda\right)\tau^{-\frac{1}{2}}, \quad \tau>0,\ \mathbf{k}\in\operatorname{clos}\widetilde{\Omega}. \tag{9.26}$$

Together with (9.23) and (9.24) this implies

$$\left\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\left(fe^{-\mathcal{A}(\mathbf{k})\tau}f^* - \left(I+[\Lambda]b(\mathbf{D}+\mathbf{k})\widehat{P}\right)f_0e^{-\mathcal{A}^0(\mathbf{k})\tau}f_0\right)\right\|_{L_2(\Omega)\to L_2(\Omega)} \le \mathcal{C}_3'''\tau^{-\frac{1}{2}}, \quad \tau>0,\ \mathbf{k}\in\operatorname{clos}\widetilde{\Omega}, \tag{9.27}$$

where

$$\mathcal{C}_3''' = \|f\|_{L_\infty} + \|g\|_{L_\infty}^{\frac{1}{2}}\|g^{-1}\|_{L_\infty}^{\frac{1}{2}}\|f\|_{L_\infty}\left(2+\alpha_1^{\frac{1}{2}}r_1^p C_\Lambda\right).$$

Combining (9.22) for $\tau\ge 1$ and (9.27) for $0<\tau<1$, we arrive at the inequality

$$\left\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\left(fe^{-\mathcal{A}(\mathbf{k})\tau}f^* - \left(I+[\Lambda]b(\mathbf{D}+\mathbf{k})\widehat{P}\right)f_0e^{-\mathcal{A}^0(\mathbf{k})\tau}f_0\right)\right\|_{L_2(\Omega)\to L_2(\Omega)} \le \frac{2\max\{\mathcal{C}_3'',\mathcal{C}_3'''\}}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2p}}+1)}, \quad \tau>0,\ \mathbf{k}\in\operatorname{clos}\widetilde{\Omega}. \tag{9.28}$$

Let us summarize the results.

Theorem 9.3. *Suppose that the assumptions of Theorem* 9.1 *are satisfied. Let* Λ *be the* Γ*-periodic solution of problem* (7.9), *and let* $\widehat{P}$ *be projection* (7.4). *Then for* $\tau>0$ *and* $\mathbf{k}\in\operatorname{clos}\widetilde{\Omega}$ *we have*

$$\left\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\left(fe^{-\mathcal{A}(\mathbf{k})\tau}f^* - \left(I+[\Lambda]b(\mathbf{D}+\mathbf{k})\widehat{P}\right)f_0e^{-\mathcal{A}^0(\mathbf{k})\tau}f_0\right)\right\|_{L_2(\Omega)\to L_2(\Omega)} \le \frac{\mathcal{C}_3}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2p}}+1)}, \tag{9.29}$$

$$\left\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\left(fe^{-\mathcal{A}(\mathbf{k})\tau}f^* - (I+[\Lambda]b(\mathbf{D}+\mathbf{k}))f_0e^{-\mathcal{A}^0(\mathbf{k})\tau}f_0\widehat{P}\right)\right\|_{L_2(\Omega)\to L_2(\Omega)} \le \frac{\mathcal{C}_3}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2p}}+1)}. \tag{9.30}$$

The constant $\mathcal{C}_3$ *depends only on the problem data* (9.7).

Proof. Inequality (9.28) implies (9.29) with the constant $\mathcal{C}_3 = 2\max\{\mathcal{C}_3'',\mathcal{C}_3'''\}$.

Inequality (9.30) is checked similarly to the proof of estimate (9.28): it follows from (9.15), (9.17), (9.18), (9.21), (9.23), (9.24), and (9.26). □

We also need to estimate the operator $fe^{-\mathcal{A}(\mathbf{k})\tau}f^* - \left(I+[\Lambda]b(\mathbf{D}+\mathbf{k})\widehat{P}\right)f_0e^{-\mathcal{A}^0(\mathbf{k})\tau}f_0$ in the operator norm in $L_2(\Omega;\mathbb{C}^n)$.

Proposition 9.1. *Suppose that the assumptions of Theorem* 9.3 *are satisfied.*

1°. *For* $\tau>0$ *and* $\mathbf{k}\in\operatorname{clos}\widetilde{\Omega}$ *we have*

$$\left\|fe^{-\mathcal{A}(\mathbf{k})\tau}f^* - \left(I+[\Lambda]b(\mathbf{D}+\mathbf{k})\widehat{P}\right)f_0e^{-\mathcal{A}^0(\mathbf{k})\tau}f_0\right\|_{L_2(\Omega)\to L_2(\Omega)} \le \frac{\mathcal{C}_4}{\tau^{\frac{1}{2p}}+1}. \tag{9.31}$$

The constant $\mathcal{C}_4$ *depends only on the problem data* (9.7).

2°. *In addition, suppose that* $G(\boldsymbol{\theta})\equiv 0$. *Then for* $\tau>0$ *and* $\mathbf{k}\in\operatorname{clos}\widetilde{\Omega}$ *we have*

$$\left\|fe^{-\mathcal{A}(\mathbf{k})\tau}f^* - \left(I+[\Lambda]b(\mathbf{D}+\mathbf{k})\widehat{P}\right)f_0e^{-\mathcal{A}^0(\mathbf{k})\tau}f_0\right\|_{L_2(\Omega)\to L_2(\Omega)} \le \frac{\mathcal{C}_5}{\tau^{\frac{1}{p}}+1}. \tag{9.32}$$

The constant $\mathcal{C}_5$ *depends only on the problem data* (9.7).

Proof. Similarly to (9.19), we have

$$[\Lambda]b(\mathbf{D}+\mathbf{k})f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0 \widehat{P} = \left([\Lambda]\widehat{P}_m\right)\left(b(\mathbf{D}+\mathbf{k})f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0 \widehat{P}\right). \tag{9.33}$$

Together with (8.34) and (9.25), this implies

$$\|[\Lambda]b(\mathbf{D}+\mathbf{k})f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0 \widehat{P}\|_{L_2(\Omega)\to L_2(\Omega)} \le C_\Lambda \|g^{-1}\|^{\frac{1}{2}}_{L_\infty} \|f\|_{L_\infty} \tau^{-\frac{1}{2}} \le \frac{2C_\Lambda \|g^{-1}\|^{\frac{1}{2}}_{L_\infty}\|f\|_{L_\infty}}{\tau^{\frac{1}{2p}}+1}, \quad \tau \ge 1, \quad \mathbf{k} \in \operatorname{clos}\widetilde{\Omega}. \tag{9.34}$$

For $0 < \tau < 1$, using the identity $b(\mathbf{D}+\mathbf{k})f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0 \widehat{P} = b(\mathbf{k})f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0 \widehat{P}$ and relations (5.5), (9.2), we obtain

$$\|b(\mathbf{D}+\mathbf{k})f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0 \widehat{P}\|_{L_2(\Omega)\to L_2(\Omega)} \le \alpha_1^{\frac{1}{2}} r_1^p \|f\|^2_{L_\infty} \le \frac{2\alpha_1^{\frac{1}{2}} r_1^p \|f\|^2_{L_\infty}}{\tau^{\frac{1}{2p}}+1}, \quad 0<\tau<1, \ \mathbf{k} \in \operatorname{clos}\widetilde{\Omega}. \tag{9.35}$$

From (8.34), (9.33), (9.35) it follows that

$$\|[\Lambda]b(\mathbf{D}+\mathbf{k})f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0 \widehat{P}\|_{L_2(\Omega)\to L_2(\Omega)} \le \frac{2C_\Lambda \alpha_1^{\frac{1}{2}} r_1^p \|f\|^2_{L_\infty}}{\tau^{\frac{1}{2p}}+1}, \quad 0<\tau<1, \quad \mathbf{k} \in \operatorname{clos}\widetilde{\Omega}. \tag{9.36}$$

As a result, relations (9.13), (9.34), and (9.36) imply the required estimate (9.31) with the constant

$$\mathcal{C}_4 = \mathcal{C}_1 + 2C_\Lambda \max\left\{\|g^{-1}\|^{\frac{1}{2}}_{L_\infty}\|f\|_{L_\infty}, \alpha_1^{\frac{1}{2}} r_1^p \|f\|^2_{L_\infty}\right\}.$$

To check statement 2°, we should estimate operator (9.33) differently. Instead of (9.34), we use the inequality

$$\|[\Lambda]b(\mathbf{D}+\mathbf{k})f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0 \widehat{P}\|_{L_2(\Omega)\to L_2(\Omega)} \le C_\Lambda \|g^{-1}\|^{\frac{1}{2}}_{L_\infty} \|f\|_{L_\infty} \tau^{-\frac{1}{2}} \le \frac{2C_\Lambda \|g^{-1}\|^{\frac{1}{2}}_{L_\infty}\|f\|_{L_\infty}}{\tau^{\frac{1}{p}}+1}, \quad \tau \ge 1, \quad \mathbf{k} \in \operatorname{clos}\widetilde{\Omega}. \tag{9.37}$$

Instead of (9.36), we apply the estimate

$$\|[\Lambda]b(\mathbf{D}+\mathbf{k})f_0 e^{-\mathcal{A}^0(\mathbf{k})\tau} f_0 \widehat{P}\|_{L_2(\Omega)\to L_2(\Omega)} \le C_\Lambda \alpha_1^{\frac{1}{2}} r_1^p \|f\|^2_{L_\infty} \le \frac{2C_\Lambda \alpha_1^{\frac{1}{2}} r_1^p \|f\|^2_{L_\infty}}{\tau^{\frac{1}{p}}+1}, \quad 0<\tau<1, \quad \mathbf{k} \in \operatorname{clos}\widetilde{\Omega}. \tag{9.38}$$

As a result, estimate (9.14) (which is valid under the condition $G(\boldsymbol{\theta}) \equiv 0$) and (9.37), (9.38) imply inequality (9.32) with the constant $\mathcal{C}_5 = \mathcal{C}_2 + 2C_\Lambda \max\left\{\|g^{-1}\|^{\frac{1}{2}}_{L_\infty}\|f\|_{L_\infty}, \alpha_1^{\frac{1}{2}} r_1^p \|f\|^2_{L_\infty}\right\}$. □

10. Approximation of the Operator Exponential $e^{-\widehat{\mathcal{A}}\tau}$

10.1. Approximation of the operator exponential $e^{-\widehat{\mathcal{A}}\tau}$ in the operator norm in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. Consider the operator $\widehat{\mathcal{A}}$ of form (7.1) acting in the space $L_2(\mathbb{R}^d;\mathbb{C}^n)$. By expansion (6.8) (with $f = \mathbf{1}_n$), we have

$$e^{-\widehat{\mathcal{A}}\tau} = \mathcal{U}^{-1}\left(\int_{\widetilde{\Omega}} \oplus e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau}\, d\mathbf{k}\right)\mathcal{U}. \tag{10.1}$$

Let $\widehat{\mathcal{A}}^0$ be the effective operator (7.22). The operator $e^{-\widehat{\mathcal{A}}^0\tau}$ satisfies expansion similar to (10.1). Hence,

$$\left\| e^{-\widehat{\mathcal{A}}\tau} - e^{-\widehat{\mathcal{A}}^0\tau} \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} = \operatorname*{ess\text{-}sup}_{\mathbf{k}\in\widetilde{\Omega}} \left\| e^{-\widehat{\mathcal{A}}(\mathbf{k})\tau} - e^{-\widehat{\mathcal{A}}^0(\mathbf{k})\tau} \right\|_{L_2(\Omega)\to L_2(\Omega)}.$$

Combining this with Theorems 8.1 and 8.2, we obtain the following results.

Theorem 10.1. *Let $\widehat{\mathcal{A}}$ be operator (7.1) and $\widehat{\mathcal{A}}^0$ be the effective operator (7.22). Then we have*

$$\left\| e^{-\widehat{\mathcal{A}}\tau} - e^{-\widehat{\mathcal{A}}^0\tau} \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \frac{\widehat{\mathcal{C}}_1}{\tau^{\frac{1}{2p}}+1}, \quad \tau \ge 0.$$

The constant $\widehat{\mathcal{C}}_1$ depends only on the problem data (8.2).

Theorem 10.2. *Suppose that the assumptions of Theorem 10.1 are satisfied. Let $\widehat{G}(\boldsymbol{\theta})$ be the operator defined according to (7.25), (7.26). Suppose that $\widehat{G}(\boldsymbol{\theta}) = 0$ for any $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. Then we have*

$$\left\| e^{-\widehat{\mathcal{A}}\tau} - e^{-\widehat{\mathcal{A}}^0\tau} \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \frac{\widehat{\mathcal{C}}_2}{\tau^{\frac{1}{p}}+1}, \quad \tau \ge 0.$$

The constant $\widehat{\mathcal{C}}_2$ depends only on the problem data (8.2).

10.2. Approximation of the operator exponential $e^{-\widehat{\mathcal{A}}\tau}$ in the energy norm. Now, we obtain approximation of the exponential $e^{-\widehat{\mathcal{A}}\tau}$ in the "energy" norm relying on Theorem 8.3 and expansion (10.1). Recall that, under the Gelfand transform, the operator $b(\mathbf{D})$ expands into the direct integral of the operators $b(\mathbf{D}+\mathbf{k})$, and the operator of multiplication by the periodic matrix-valued function $\Lambda(\mathbf{x})$ turns into multiplication by the same function on the fibers of the direct integral $\mathcal{K}$ (see (5.15)). We also need the operator $\Pi := \mathcal{U}^{-1}[\widehat{P}]\mathcal{U}$ acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. Here $[\widehat{P}]$ is the operator in $\mathcal{K}$ acting on the fibers as the operator $\widehat{P}$. It is easily seen (cf. [5, (6.8)]) that Π is the pseudodifferential operator with the symbol $\chi_{\widetilde{\Omega}}(\boldsymbol{\xi})$, where $\chi_{\widetilde{\Omega}}$ is the characteristic function of the set $\widetilde{\Omega}$, i.e.,

$$(\Pi\mathbf{u})(\mathbf{x}) = (2\pi)^{-d/2} \int_{\widetilde{\Omega}} e^{i\langle \mathbf{x},\boldsymbol{\xi}\rangle} \widehat{\mathbf{u}}(\boldsymbol{\xi})\, d\boldsymbol{\xi}. \tag{10.2}$$

It follows that the operator $\widehat{\mathcal{A}}^{1/2}\left(e^{-\widehat{\mathcal{A}}\tau} - (I + [\Lambda]b(\mathbf{D})\Pi)\, e^{-\widehat{\mathcal{A}}^0\tau}\right)$ expands into the direct integral of the operators standing under the norm sign in (8.29), the operator $\widehat{\mathcal{A}}^{1/2}\left(e^{-\widehat{\mathcal{A}}\tau} - (I + [\Lambda]b(\mathbf{D}))\, e^{-\widehat{\mathcal{A}}^0\tau}\Pi\right)$ expands into the direct integral of the operators from (8.30). Combining this with Theorem 8.3, we deduce the following result.

Theorem 10.3. *Let $\widehat{\mathcal{A}}$ be operator (7.1) and let $\widehat{\mathcal{A}}^0$ be the effective operator (7.22). Let Λ be the Γ-periodic solution of problem (7.9). Suppose that the matrix-valued function $\widetilde{g}$ is defined by (7.13). Let Π be operator (10.2). Then we have*

$$\left\| \widehat{\mathcal{A}}^{1/2}\left(e^{-\widehat{\mathcal{A}}\tau} - (I + [\Lambda]b(\mathbf{D})\Pi)\, e^{-\widehat{\mathcal{A}}^0\tau}\right) \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \frac{\widehat{\mathcal{C}}_3}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2p}}+1)}, \quad \tau > 0, \tag{10.3}$$

$$\left\| g b(\mathbf{D}) e^{-\widehat{\mathcal{A}}\tau} - \widetilde{g} b(\mathbf{D}) e^{-\widehat{\mathcal{A}}^0\tau}\Pi \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \frac{\widehat{\mathcal{C}}_6}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2p}}+1)}, \quad \tau > 0. \tag{10.4}$$

The constants $\widehat{\mathcal{C}}_3$ and $\widehat{\mathcal{C}}_6$ depend only on the problem data (8.2).

Proof. Inequality (10.3) follows from (8.29) and the direct integral expansion.

Now, let us check inequality (10.4). By the direct integral expansion, from (8.30) it follows that

$$\left\| \widehat{\mathcal{A}}^{1/2} \left(e^{-\widehat{\mathcal{A}}\tau} - (I + [\Lambda] b(\mathbf{D}))\, e^{-\widehat{\mathcal{A}}^0 \tau} \Pi \right) \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \frac{\widehat{\mathcal{C}}_3}{\tau^{\frac{1}{2}} (\tau^{\frac{1}{2p}} + 1)}, \quad \tau > 0.$$

This estimate can be rewritten as

$$\left\| g^{1/2} b(\mathbf{D}) \left(e^{-\widehat{\mathcal{A}}\tau} - (I + [\Lambda] b(\mathbf{D}))\, e^{-\widehat{\mathcal{A}}^0 \tau} \Pi \right) \right\|_{L_2((\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \frac{\widehat{\mathcal{C}}_3}{\tau^{\frac{1}{2}} (\tau^{\frac{1}{2p}} + 1)}, \quad \tau > 0,$$

which implies

$$\left\| g b(\mathbf{D}) e^{-\widehat{\mathcal{A}}\tau} - g b(\mathbf{D})\, (I + [\Lambda] b(\mathbf{D}))\, e^{-\widehat{\mathcal{A}}^0 \tau} \Pi \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \frac{\widehat{\mathcal{C}}_3 \|g\|_{L_\infty}^{\frac{1}{2}}}{\tau^{\frac{1}{2}} (\tau^{\frac{1}{2p}} + 1)}, \quad \tau > 0. \tag{10.5}$$

Using (5.4) and (7.13), we have

$$g b(\mathbf{D})\, (I + [\Lambda] b(\mathbf{D}))\, e^{-\widehat{\mathcal{A}}^0 \tau} \Pi = \widetilde{g} b(\mathbf{D}) e^{-\widehat{\mathcal{A}}^0 \tau} \Pi + g \sum_{|\beta| = p} b_\beta \sum_{\gamma \leq \beta : |\gamma| \geq 1} \binom{\beta}{\gamma} (\mathbf{D}^{\beta - \gamma} \Lambda) \mathbf{D}^\gamma b(\mathbf{D}) e^{-\widehat{\mathcal{A}}^0 \tau} \Pi. \tag{10.6}$$

Denote the second term on the right by $\mathcal{G}(\tau)$. Obviously, $\mathcal{G}(\tau)$ can be written as

$$\mathcal{G}(\tau) = g \sum_{|\beta| = p} b_\beta \sum_{\gamma \leq \beta : |\gamma| \geq 1} \binom{\beta}{\gamma} [\mathbf{D}^{\beta - \gamma} \Lambda] \Pi_m \mathbf{D}^\gamma b(\mathbf{D}) e^{-\widehat{\mathcal{A}}^0 \tau} \Pi, \tag{10.7}$$

where Π_m is the pseudodifferential operator with the symbol $\chi_{\widetilde{\Omega}}(\boldsymbol{\xi})$ in $L_2(\mathbb{R}^d; \mathbb{C}^m)$. The operator $[\mathbf{D}^{\beta - \gamma} \Lambda] \Pi_m$ is unitarily equivalent to the direct integral of the operators $[\mathbf{D}^{\beta - \gamma} \Lambda] \widehat{P}_m$, whence

$$\left\| [\mathbf{D}^{\beta - \gamma} \Lambda] \Pi_m \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} = \left\| [\mathbf{D}^{\beta - \gamma} \Lambda] \widehat{P}_m \right\|_{L_2(\Omega) \to L_2(\Omega)} = |\Omega|^{-\frac{1}{2}} \|\mathbf{D}^{\beta - \gamma} \Lambda\|_{L_2(\Omega)} \leq \widetilde{C}_\Lambda. \tag{10.8}$$

We have taken estimate (7.20) into account. Next, by the Fourier transform, using (5.5), (8.4), (10.2), and the elementary inequality $x^{\frac{1}{2} + \frac{1}{2p}} e^{-x} \leq 1$ for $x \geq 0$, we obtain

$$\left\| \mathbf{D}^\gamma b(\mathbf{D}) e^{-\widehat{\mathcal{A}}^0 \tau} \Pi \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \alpha_1^{\frac{1}{2}} \sup_{\boldsymbol{\xi} \in \widetilde{\Omega}} |\boldsymbol{\xi}|^{p + |\gamma|} e^{-\widehat{c}_* |\boldsymbol{\xi}|^{2p} \tau} \leq \alpha_1^{\frac{1}{2}} r_1^{|\gamma| - 1} \widehat{c}_*^{\,-\frac{1}{2} - \frac{1}{2p}} \tau^{-\frac{1}{2} - \frac{1}{2p}}, \quad |\gamma| \geq 1, \ \tau > 0. \tag{10.9}$$

Similarly, using estimate $x^{\frac{1}{2}} e^{-x} \leq 1$ for $x \geq 0$, we arrive at the inequality

$$\left\| \mathbf{D}^\gamma b(\mathbf{D}) e^{-\widehat{\mathcal{A}}^0 \tau} \Pi \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \alpha_1^{\frac{1}{2}} r_1^{|\gamma|} \widehat{c}_*^{\,-\frac{1}{2}} \tau^{-\frac{1}{2}}, \quad \tau > 0. \tag{10.10}$$

Applying (10.9) for $\tau \geq 1$ and (10.10) for $0 < \tau < 1$, we find that

$$\left\| \mathbf{D}^\gamma b(\mathbf{D}) e^{-\widehat{\mathcal{A}}^0 \tau} \Pi \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \frac{\widehat{\mathcal{C}}^{(\gamma)}}{\tau^{\frac{1}{2}} (\tau^{\frac{1}{2p}} + 1)}, \quad \tau > 0, \tag{10.11}$$

where

$$\widehat{\mathcal{C}}^{(\gamma)} = 2 \alpha_1^{\frac{1}{2}} \max \left\{ r_1^{|\gamma| - 1} \widehat{c}_*^{\,-\frac{1}{2} - \frac{1}{2p}}, r_1^{|\gamma|} \widehat{c}_*^{\,-\frac{1}{2}} \right\}.$$

Now, relations (10.7), (10.8), (10.11), and (5.6) imply that

$$\|\mathcal{G}(\tau)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \frac{\widehat{\mathcal{C}}_6'}{\tau^{\frac{1}{2}} (\tau^{\frac{1}{2p}} + 1)}, \quad \tau > 0,$$

where

$$\widehat{\mathcal{C}}_6' = c(p, d) \alpha_1^{\frac{1}{2}} \|g\|_{L_\infty} \widetilde{C}_\Lambda \max_{1 \leq |\gamma| \leq p} \widehat{\mathcal{C}}^{(\gamma)}.$$

Together with (10.5) and (10.6), this yields the required estimate (10.4) with the constant $\widehat{\mathcal{C}}_6 = \widehat{\mathcal{C}}_3 \|g\|_{L_\infty}^{\frac{1}{2}} + \widehat{\mathcal{C}}_6'$. ☐

By the direct integral expansion, Proposition 8.1 implies the following statement.

Proposition 10.1. *Suppose that the assumptions of Theorem* 10.3 *are satisfied.*

1°. *We have*

$$\left\| e^{-\widehat{\mathcal{A}}\tau} - (I + [\Lambda] b(\mathbf{D})\Pi)\, e^{-\widehat{\mathcal{A}}^0\tau} \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \frac{\widehat{\mathcal{C}}_4}{\tau^{\frac{1}{2p}}+1}, \quad \tau > 0. \tag{10.12}$$

The constant $\widehat{\mathcal{C}}_4$ *depends only on the problem data* (8.2).

2°. *In addition, suppose that* $\widehat{G}(\boldsymbol{\theta}) \equiv 0$. *Then we have*

$$\left\| e^{-\widehat{\mathcal{A}}\tau} - (I + [\Lambda] b(\mathbf{D})\Pi)\, e^{-\widehat{\mathcal{A}}^0\tau} \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \frac{\widehat{\mathcal{C}}_5}{\tau^{\frac{1}{p}}+1}, \quad \tau > 0. \tag{10.13}$$

The constant $\widehat{\mathcal{C}}_5$ *depends only on the problem data* (8.2).

10.3. Removing of the operator Π in the corrector for $\tau \geq 1$. Now, we show that for $\tau \geq 1$ the operator Π in estimates (10.3), (10.4), (10.12), (10.13) can be removed (i.e., in the margin of the admissible error, the operator Π can be replaced by the identity operator).

Proposition 10.2. *For any $s \geq 0$ and $\tau > 0$ the operator $b(\mathbf{D})e^{-\widehat{\mathcal{A}}^0\tau}(I - \Pi)$ is continuous from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to $H^s(\mathbb{R}^d;\mathbb{C}^m)$, and we have*

$$\left\| b(\mathbf{D}) e^{-\widehat{\mathcal{A}}^0\tau}(I-\Pi) \right\|_{L_2(\mathbb{R}^d)\to H^s(\mathbb{R}^d)} \leq \widehat{\mathfrak{C}}^{(s)} \tau^{-\frac{1}{2}-\frac{s}{2p}}, \quad \tau > 0. \tag{10.14}$$

The constant $\widehat{\mathfrak{C}}^{(s)}$ *depends only on s, p, α_0, α_1, $\|g^{-1}\|_{L_\infty}$, and r_0.*

Proof. We need the elementary inequality

$$x^{\frac{1}{2}+\frac{s}{2p}} e^{-x} \leq c(s,p), \quad x \geq 0, \quad c(s,p) = q^q e^{-q},\ q = \frac{1}{2} + \frac{s}{2p}. \tag{10.15}$$

By the Fourier transform, using (5.5), (8.4), (10.2), and (10.15), we obtain

$$\begin{aligned}
\left\| b(\mathbf{D}) e^{-\widehat{\mathcal{A}}^0\tau}(I-\Pi) \right\|_{L_2(\mathbb{R}^d)\to H^s(\mathbb{R}^d)} &\leq \alpha_1^{\frac{1}{2}} \sup_{\boldsymbol{\xi}\in\mathbb{R}^d} (1+|\boldsymbol{\xi}|^2)^{\frac{s}{2}} (1 - \chi_{\widetilde{\Omega}}(\boldsymbol{\xi})) |\boldsymbol{\xi}|^p e^{-\widehat{c}_*|\boldsymbol{\xi}|^{2p}\tau} \\
&\leq c(s,p)\alpha_1^{\frac{1}{2}} \widehat{c}_*^{\,-\frac{1}{2}-\frac{s}{2p}} \tau^{-\frac{1}{2}-\frac{s}{2p}} \sup_{|\boldsymbol{\xi}|\geq r_0} (1+|\boldsymbol{\xi}|^{-2})^{\frac{s}{2}} \leq \widehat{\mathfrak{C}}^{(s)} \tau^{-\frac{1}{2}-\frac{s}{2p}}, \quad \tau > 0,
\end{aligned}$$

where $\widehat{\mathfrak{C}}^{(s)} = c(s,p)\alpha_1^{\frac{1}{2}} \widehat{c}_*^{\,-\frac{1}{2}-\frac{s}{2p}} (1 + r_0^{-2})^{\frac{s}{2}}$. ☐

Proposition 10.3.

1°. *Let $s > p + d/2$ and $\tau > 0$. Then the operator $\widehat{\mathcal{A}}^{1/2}[\Lambda]$ is continuous from $H^s(\mathbb{R}^d;\mathbb{C}^m)$ to $L_2(\mathbb{R}^d;\mathbb{C}^n)$, and*

$$\left\| \widehat{\mathcal{A}}^{1/2}[\Lambda] \right\|_{H^s(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \mathfrak{C}_1^{(s)}, \quad \tau > 0. \tag{10.16}$$

The constant $\mathfrak{C}_1^{(s)}$ *depends only on s and the data* (8.2).

2°. *Let $s > d/2$ and $\tau > 0$. Then the operators $[\Lambda]$ and $[\widetilde{g}]$ are continuous from $H^s(\mathbb{R}^d;\mathbb{C}^m)$ to $L_2(\mathbb{R}^d;\mathbb{C}^n)$, and*

$$\|[\Lambda]\|_{H^s(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \mathfrak{C}_2^{(s)}, \quad \tau > 0, \tag{10.17}$$

$$\|[\widetilde{g}]\|_{H^s(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \mathfrak{C}_3^{(s)}, \quad \tau > 0. \tag{10.18}$$

The constants $\mathfrak{C}_2^{(s)}$ *and* $\mathfrak{C}_3^{(s)}$ *depend only on* s *and data* (8.2).

Proof. We start with the proof of statement 1°. By (5.4) and (5.6),

$$\left\|\widehat{\mathcal{A}}^{1/2}[\Lambda]\right\|_{H^s(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} = \|hb(\mathbf{D})[\Lambda]\|_{H^s(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \sqrt{\alpha_1}\|g\|_{L_\infty}^{\frac{1}{2}} \sum_{|\beta|=p} \left\|\mathbf{D}^\beta[\Lambda]\right\|_{H^s(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}. \tag{10.19}$$

Next, let $\mathbf{u} \in H^s(\mathbb{R}^d;\mathbb{C}^m)$ with some $s > p + d/2$. Then

$$\mathbf{D}^\beta(\Lambda(\mathbf{x})\mathbf{u}(\mathbf{x})) = \sum_{\gamma\le\beta} \binom{\beta}{\gamma} (\mathbf{D}^{\beta-\gamma}\Lambda(\mathbf{x}))\mathbf{D}^\gamma\mathbf{u}(\mathbf{x}). \tag{10.20}$$

We estimate the L_2-norm of each term in (10.20) separately:

$$\begin{aligned}\int_{\mathbb{R}^d} |\mathbf{D}^{\beta-\gamma}\Lambda(\mathbf{x})|^2|\mathbf{D}^\gamma\mathbf{u}(\mathbf{x})|^2\,d\mathbf{x} &= \sum_{\mathbf{a}\in\Gamma}\int_{\Omega+\mathbf{a}} |\mathbf{D}^{\beta-\gamma}\Lambda(\mathbf{x})|^2|\mathbf{D}^\gamma\mathbf{u}(\mathbf{x})|^2\,d\mathbf{x}\\ &\le \int_\Omega |\mathbf{D}^{\beta-\gamma}\Lambda(\mathbf{x})|^2\,d\mathbf{x}\sum_{\mathbf{a}\in\Gamma}\|\mathbf{D}^\gamma\mathbf{u}\|^2_{L_\infty(\Omega+\mathbf{a})}.\end{aligned} \tag{10.21}$$

We have taken into account that the matrix-valued function Λ is periodic. Next, we use the continuous embedding $H^{s-p}(\Omega;\mathbb{C}^m) \subset L_\infty(\Omega;\mathbb{C}^m)$; let $c_{s-p}(\Omega)$ be the norm of the corresponding embedding operator. Then

$$\|\mathbf{D}^\gamma\mathbf{u}\|_{L_\infty(\Omega+\mathbf{a})} \le c_{s-p}(\Omega)\,\|\mathbf{D}^\gamma\mathbf{u}\|_{H^{s-p}(\Omega+\mathbf{a})} \le c_{s-p}(\Omega)\,\|\mathbf{u}\|_{H^s(\Omega+\mathbf{a})}, \quad \mathbf{a}\in\Gamma,\ |\gamma|\le p. \tag{10.22}$$

From (10.21), (10.22), and (7.20) it follows that

$$\int_{\mathbb{R}^d} |\mathbf{D}^{\beta-\gamma}\Lambda(\mathbf{x})|^2|\mathbf{D}^\gamma\mathbf{u}(\mathbf{x})|^2\,d\mathbf{x} \le c^2_{s-p}(\Omega)\,\|\Lambda\|^2_{H^p(\Omega)}\,\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)} \le c^2_{s-p}(\Omega)|\Omega|\widetilde{C}^2_\Lambda\,\|\mathbf{u}\|^2_{H^s(\mathbb{R}^d)},$$

$$\gamma\le\beta,\ |\beta| = p.$$

Comparing this with (10.19) and (10.20), we arrive at the required inequality (10.16).

Statement 2° is proved similarly (and even simpler). Let $\mathbf{u} \in H^s(\mathbb{R}^d;\mathbb{C}^m)$ with some $s > d/2$. By analogy with (10.21),

$$\int_{\mathbb{R}^d} |\Lambda(\mathbf{x})|^2|\mathbf{u}(\mathbf{x})|^2\,d\mathbf{x} \le \int_\Omega |\Lambda(\mathbf{x})|^2\,d\mathbf{x}\sum_{\mathbf{a}\in\Gamma}\|\mathbf{u}\|^2_{L_\infty(\Omega+\mathbf{a})},$$

$$\int_{\mathbb{R}^d} |\widetilde{g}(\mathbf{x})|^2|\mathbf{u}(\mathbf{x})|^2\,d\mathbf{x} \le \int_\Omega |\widetilde{g}(\mathbf{x})|^2\,d\mathbf{x}\sum_{\mathbf{a}\in\Gamma}\|\mathbf{u}\|^2_{L_\infty(\Omega+\mathbf{a})}.$$

Hence, using the embedding $H^s(\Omega;\mathbb{C}^m) \subset L_\infty(\Omega;\mathbb{C}^m)$ and estimates (7.18), (7.19), we obtain the required inequalities (10.17) and (10.18). □

Now, we deduce the following result from Theorem 10.3 and Propositions 10.2, 10.3.

Theorem 10.4. *Let* $\widehat{\mathcal{A}}$ *be operator* (7.1) *and let* $\widehat{\mathcal{A}}^0$ *be the effective operator* (7.22). *Suppose that* Λ *is the* Γ*-periodic solution of problem* (7.9) *and* $\widetilde{g}$ *is the matrix-valued function defined by* (7.13). *Then we have*

$$\left\|\widehat{\mathcal{A}}^{1/2}\left(e^{-\widehat{\mathcal{A}}\tau} - (I + [\Lambda]b(\mathbf{D}))\,e^{-\widehat{\mathcal{A}}^0\tau}\right)\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \widehat{\mathcal{C}}^\circ_3\tau^{-\frac{1}{2}-\frac{1}{2p}}, \quad \tau\ge 1, \tag{10.23}$$

$$\left\|gb(\mathbf{D})e^{-\widehat{\mathcal{A}}\tau} - \widetilde{g}b(\mathbf{D})e^{-\widehat{\mathcal{A}}^0\tau}\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \widehat{\mathcal{C}}^\circ_6\tau^{-\frac{1}{2}-\frac{1}{2p}}, \quad \tau\ge 1. \tag{10.24}$$

The constants $\widehat{\mathcal{C}}^\circ_3$ *and* $\widehat{\mathcal{C}}^\circ_6$ *depend only on the problem data* (8.2).

Proof. We start with the proof of estimate (10.23). From (10.14) and (10.16) it follows that

$$\left\| \widehat{\mathcal{A}}^{1/2}[\Lambda]b(\mathbf{D})e^{-\widehat{\mathcal{A}}^0\tau}(I-\Pi)\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \widehat{\mathfrak{C}}^{(s)}\mathfrak{C}_1^{(s)}\tau^{-\frac{1}{2}-\frac{s}{2p}}, \quad \tau>0, \tag{10.25}$$

for any $s>p+d/2$. We fix $s=p+d/2+1$ and note that $\tau^{-\frac{1}{2}-\frac{s}{2p}} \leq \tau^{-\frac{1}{2}-\frac{1}{2p}}$ for $\tau \geq 1$. Then (10.3) and (10.25) imply (10.23).

Similarly, relations (10.14) and (10.18) yield

$$\left\| [\widetilde{g}]b(\mathbf{D})e^{-\widehat{\mathcal{A}}^0\tau}(I-\Pi)\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \widehat{\mathfrak{C}}^{(s)}\mathfrak{C}_3^{(s)}\tau^{-\frac{1}{2}-\frac{s}{2p}}, \quad \tau>0, \tag{10.26}$$

for any $s>d/2$. We fix $s=d/2+1$ and note that $\tau^{-\frac{1}{2}-\frac{s}{2p}} \leq \tau^{-\frac{1}{2}-\frac{1}{2p}}$ for $\tau\geq 1$. Then from (10.4) and (10.26) we deduce the required estimate (10.24). □

Similarly, Propositions 10.1, 10.2, and 10.3 imply the following statement.

Proposition 10.4. *Suppose that the assumptions of Theorem* 10.4 *are satisfied.*

1°. *We have*

$$\left\| e^{-\widehat{\mathcal{A}}\tau} - (I+[\Lambda]b(\mathbf{D}))\, e^{-\widehat{\mathcal{A}}^0\tau}\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \widehat{\mathcal{C}}_4^{\circ}\tau^{-\frac{1}{2p}}, \quad \tau\geq 1.$$

The constant $\widehat{\mathcal{C}}_4^{\circ}$ *depends only on the problem data* (8.2).

2°. *In addition, assume that* $\widehat{G}(\boldsymbol{\theta})\equiv 0$. *Then we have*

$$\left\| e^{-\widehat{\mathcal{A}}\tau} - (I+[\Lambda]b(\mathbf{D}))\, e^{-\widehat{\mathcal{A}}^0\tau}\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \widehat{\mathcal{C}}_5^{\circ}\tau^{-\frac{1}{p}}, \quad \tau\geq 1.$$

The constant $\widehat{\mathcal{C}}_5^{\circ}$ *depends only on the problem data* (8.2).

11. Approximation of the Operator $fe^{-\mathcal{A}\tau}f^*$

11.1. Approximation of the sandwiched exponential $fe^{-\mathcal{A}\tau}f^*$ in the operator norm in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. Now, we consider the operator $\mathcal{A}$ of form (5.1) acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. Similarly to Sec. 10.1, using the direct integral expansion (6.8), we deduce the following results from Theorems 9.1 and 9.2.

Theorem 11.1. *Let* $\mathcal{A}$ *be the operator* (5.1) *and let* $\mathcal{A}^0$ *be the operator* (9.3). *Suppose that the matrix* f_0 *is given by* (9.1). *Then we have*

$$\left\| fe^{-\mathcal{A}\tau}f^* - f_0e^{-\mathcal{A}^0\tau}f_0\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \frac{\mathcal{C}_1}{\tau^{\frac{1}{2p}}+1}, \quad \tau\geq 0.$$

The constant $\mathcal{C}_1$ *depends only on the problem data* (9.7).

Theorem 11.2. *Suppose that the assumptions of Theorem* 11.1 *are satisfied. Let* $\widehat{G}(\boldsymbol{\theta})$ *be the operator defined according to* (7.25) *and* (7.26). *Suppose that* $\widehat{G}(\boldsymbol{\theta})=0$ *for any* $\boldsymbol{\theta}\in\mathbb{S}^{d-1}$. *Then we have*

$$\left\| fe^{-\mathcal{A}\tau}f^* - f_0e^{-\mathcal{A}^0\tau}f_0\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \frac{\mathcal{C}_2}{\tau^{\frac{1}{p}}+1}, \quad \tau\geq 0.$$

The constant $\mathcal{C}_2$ *depends only on the problem data* (9.7).

11.2. Approximation of the operator $fe^{-\mathcal{A}\tau}f^*$ in the energy norm.

Theorem 11.3. *Suppose that the assumptions of Theorem* 11.1 *are satisfied. Let* Λ *be the* Γ*-periodic solution of problem* (7.9)*, and let* $\widetilde{g}$ *be the matrix-valued function defined by* (7.13)*. Let* Π *be operator* (10.2)*. Then we have*

$$\left\| \widehat{\mathcal{A}}^{1/2} \left(fe^{-\mathcal{A}\tau}f^* - (I + [\Lambda]b(\mathbf{D})\Pi)\, f_0 e^{-\mathcal{A}^0\tau} f_0 \right) \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \frac{\mathcal{C}_3}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2p}}+1)}, \quad \tau > 0, \tag{11.1}$$

$$\left\| gb(\mathbf{D})fe^{-\mathcal{A}\tau}f^* - \widetilde{g}b(\mathbf{D})f_0 e^{-\mathcal{A}^0\tau} f_0\Pi \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \frac{\mathcal{C}_6}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2p}}+1)}, \quad \tau > 0. \tag{11.2}$$

The constants $\mathcal{C}_3$ *and* $\mathcal{C}_6$ *depend only on the problem data* (9.7)*.*

Proof. Inequality (11.1) follows from (9.29) with the help of the direct integral expansion.
By the direct integral expansion, (9.30) implies that

$$\left\| \widehat{\mathcal{A}}^{1/2} \left(fe^{-\mathcal{A}\tau}f^* - (I + [\Lambda]b(\mathbf{D}))\, f_0 e^{-\mathcal{A}^0\tau} f_0\Pi \right) \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \frac{\mathcal{C}_3}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2p}}+1)}, \quad \tau > 0.$$

Similarly to the proof of estimate (10.4), this yields estimate (11.2); see the proof of Theorem 10.3. □

Using the direct integral expansion, we deduce the following statement from Proposition 9.1.

Proposition 11.1. *Suppose that the assumptions of Theorem* 11.3 *are satisfied.*

1°. *We have*

$$\left\| fe^{-\mathcal{A}\tau}f^* - (I + [\Lambda]b(\mathbf{D})\Pi)\, f_0 e^{-\mathcal{A}^0\tau} f_0 \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \frac{\mathcal{C}_4}{\tau^{\frac{1}{2p}}+1}, \quad \tau > 0. \tag{11.3}$$

The constant $\mathcal{C}_4$ *depends only on the problem data* (9.7)*.*

2°. *In addition, suppose that* $\widehat{G}(\boldsymbol{\theta}) \equiv 0$*. Then we have*

$$\left\| fe^{-\mathcal{A}\tau}f^* - (I + [\Lambda]b(\mathbf{D})\Pi)\, f_0 e^{-\mathcal{A}^0\tau} f_0 \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \frac{\mathcal{C}_5}{\tau^{\frac{1}{p}}+1}, \quad \tau > 0. \tag{11.4}$$

The constant $\mathcal{C}_5$ *depends only on the problem data* (9.7)*.*

11.3. Removing of the operator Π in the corrector for $\tau \ge 1$. Now, we show that for $\tau \ge 1$ it is possible to remove the operator Π in estimates (11.1)–(11.4).

Proposition 11.2. *For any* $s \ge 0$ *and* $\tau > 0$ *the operator* $b(\mathbf{D})f_0 e^{-\mathcal{A}^0\tau} f_0(I-\Pi)$ *is continuous from* $L_2(\mathbb{R}^d;\mathbb{C}^n)$ *to* $H^s(\mathbb{R}^d;\mathbb{C}^m)$*, and*

$$\left\| b(\mathbf{D})f_0 e^{-\mathcal{A}^0\tau} f_0(I-\Pi) \right\|_{L_2(\mathbb{R}^d)\to H^s(\mathbb{R}^d)} \le \mathfrak{C}^{(s)}\tau^{-\frac{1}{2}-\frac{s}{2p}}, \quad \tau > 0. \tag{11.5}$$

The constant $\mathfrak{C}^{(s)}$ *depends only on* s, p, α_0, α_1, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, *and* r_0.

Proof. By the Fourier transform, using (5.5), (9.2), (9.9), (10.2), and (10.15), we obtain

$$\begin{aligned}\left\| b(\mathbf{D})f_0 e^{-\mathcal{A}^0\tau} f_0(I-\Pi) \right\|_{L_2(\mathbb{R}^d)\to H^s(\mathbb{R}^d)} &\le \alpha_1^{\frac{1}{2}}\|f\|^2_{L_\infty} \sup_{\boldsymbol{\xi}\in\mathbb{R}^d} (1+|\boldsymbol{\xi}|^2)^{\frac{s}{2}}(1-\chi_{\widetilde{\Omega}}(\boldsymbol{\xi}))|\boldsymbol{\xi}|^p e^{-c_*|\boldsymbol{\xi}|^{2p}\tau} \\ &\le c(s,p)\alpha_1^{\frac{1}{2}}\|f\|^2_{L_\infty} c_*^{-\frac{1}{2}-\frac{s}{2p}}\tau^{-\frac{1}{2}-\frac{s}{2p}} \sup_{|\boldsymbol{\xi}|\ge r_0}(1+|\boldsymbol{\xi}|^{-2})^{\frac{s}{2}} \le \mathfrak{C}^{(s)}\tau^{-\frac{1}{2}-\frac{s}{2p}}, \quad \tau > 0,\end{aligned}$$

where $\mathfrak{C}^{(s)} = c(s,p)\alpha_1^{\frac{1}{2}}\|f\|^2_{L_\infty} c_*^{-\frac{1}{2}-\frac{s}{2p}}(1+r_0^{-2})^{\frac{s}{2}}$. □

Next, we deduce the following result from Theorem 11.3 and Propositions 10.3, 11.2.

Theorem 11.4. *Let $\mathcal{A}$ be operator* (5.1) *and let $\mathcal{A}^0$ be operator* (9.3). *Suppose that f_0 is the matrix given by* (9.1). *Let Λ be the Γ-periodic solution of problem* (7.9), *and let $\widetilde{g}$ be the matrix-valued function defined by* (7.13). *Then we have*

$$\left\| \widehat{\mathcal{A}}^{1/2} \left(f e^{-\mathcal{A}\tau} f^* - (I + [\Lambda] b(\mathbf{D}))\, f_0 e^{-\mathcal{A}^0\tau} f_0 \right) \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \mathcal{C}_3^\circ \tau^{-\frac{1}{2}-\frac{1}{2p}}, \quad \tau \geq 1, \tag{11.6}$$

$$\left\| g b(\mathbf{D}) f e^{-\mathcal{A}\tau} f^* - \widetilde{g} b(\mathbf{D}) f_0 e^{-\mathcal{A}^0\tau} f_0 \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \mathcal{C}_6^\circ \tau^{-\frac{1}{2}-\frac{1}{2p}}, \quad \tau \geq 1. \tag{11.7}$$

The constants $\mathcal{C}_3^\circ$ and $\mathcal{C}_6^\circ$ depend only on the problem data (9.7).

Proof. We start with the proof of estimate (11.6). Relations (10.16) and (11.5) imply that

$$\left\| \widehat{\mathcal{A}}^{1/2} [\Lambda] b(\mathbf{D}) f_0 e^{-\mathcal{A}^0\tau} f_0 (I - \Pi) \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \mathfrak{C}^{(s)} \mathfrak{C}_1^{(s)} \tau^{-\frac{1}{2}-\frac{s}{2p}}, \quad \tau > 0, \tag{11.8}$$

for any $s > p + d/2$. We fix $s = p + d/2 + 1$ and note that $\tau^{-\frac{1}{2}-\frac{s}{2p}} \leq \tau^{-\frac{1}{2}-\frac{1}{2p}}$ for $\tau \geq 1$. Then (11.1) and (11.8) yield (11.6).

Similarly, from (10.18) and (11.5) it follows that

$$\left\| [\widetilde{g}] b(\mathbf{D}) f_0 e^{-\mathcal{A}^0\tau} f_0 (I - \Pi) \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \mathfrak{C}^{(s)} \mathfrak{C}_3^{(s)} \tau^{-\frac{1}{2}-\frac{s}{2p}}, \quad \tau > 0, \tag{11.9}$$

for any $s > d/2$. We fix $s = d/2 + 1$ and note that $\tau^{-\frac{1}{2}-\frac{s}{2p}} \leq \tau^{-\frac{1}{2}-\frac{1}{2p}}$ for $\tau \geq 1$. Then (11.2) and (11.9) imply the required estimate (11.7). □

Similarly, we deduce the following result from Propositions 10.3, 11.1, and 11.2.

Proposition 11.3. *Suppose that the assumptions of Theorem* 11.4 *are satisfied.*

1°. *We have*

$$\left\| f e^{-\mathcal{A}\tau} f^* - (I + [\Lambda] b(\mathbf{D}))\, f_0 e^{-\mathcal{A}^0\tau} f_0 \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \mathcal{C}_4^\circ \tau^{-\frac{1}{2p}}, \quad \tau \geq 1.$$

The constant $\mathcal{C}_4^\circ$ depends only on the problem data (9.7).

2°. *In addition, suppose that $\widehat{G}(\boldsymbol{\theta}) \equiv 0$. Then we have*

$$\left\| f e^{-\mathcal{A}\tau} f^* - (I + [\Lambda] b(\mathbf{D}))\, f_0 e^{-\mathcal{A}^0\tau} f_0 \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \mathcal{C}_5^\circ \tau^{-\frac{1}{p}}, \quad \tau \geq 1.$$

The constant $\mathcal{C}_5^\circ$ depends only on the problem data (9.7).

Chapter 3

HOMOGENIZATION OF PARABOLIC EQUATIONS

12. The Operator $\mathcal{A}_\varepsilon$. The Scaling Transformation

12.1. The operators $\mathcal{A}_\varepsilon$ and $\widehat{\mathcal{A}}_\varepsilon$. We proceed to the problems of homogenization in the small period limit for periodic DOs in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. For any Γ-periodic function $\varphi(\mathbf{x})$ in $\mathbb{R}^d$ denote

$$\varphi^\varepsilon(\mathbf{x}) := \varphi(\varepsilon^{-1}\mathbf{x}), \quad \varepsilon > 0.$$

In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider the operator $\mathcal{A}_\varepsilon$, $\varepsilon > 0$, formally given by

$$\mathcal{A}_\varepsilon = (f^\varepsilon(\mathbf{x}))^* b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) f^\varepsilon(\mathbf{x}), \quad \varepsilon > 0. \tag{12.1}$$

As usual, the precise definition of the operator $\mathcal{A}_\varepsilon$ is given in terms of the corresponding closed quadratic form

$$a_\varepsilon[\mathbf{u},\mathbf{u}] = \int\limits_{\mathbb{R}^d} \langle g^\varepsilon(\mathbf{x}) b(\mathbf{D}) f^\varepsilon(\mathbf{x})\mathbf{u}(\mathbf{x}), b(\mathbf{D}) f^\varepsilon(\mathbf{x})\mathbf{u}(\mathbf{x})\rangle \, d\mathbf{x}, \quad f^\varepsilon \mathbf{u} \in H^p(\mathbb{R}^d;\mathbb{C}^n).$$

This form satisfies the following estimates similar to (5.9), (5.8):

$$\alpha_0 \|g^{-1}\|^{-1}_{L_\infty} \int\limits_{\mathbb{R}^d} |\boldsymbol{\xi}|^{2p} |\widehat{\mathbf{v}}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \le a_\varepsilon[\mathbf{u},\mathbf{u}] \le \alpha_1 \|g\|_{L_\infty} \int\limits_{\mathbb{R}^d} |\boldsymbol{\xi}|^{2p} |\widehat{\mathbf{v}}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi}, \ \ \mathbf{v} = f^\varepsilon \mathbf{u} \in H^p(\mathbb{R}^d;\mathbb{C}^n), \tag{12.2}$$

$$c_0 \int\limits_{\mathbb{R}^d} |\mathbf{D}^p \mathbf{v}(\mathbf{x})|^2 \, d\mathbf{x} \le a_\varepsilon[\mathbf{u},\mathbf{u}] \le c_1 \int\limits_{\mathbb{R}^d} |\mathbf{D}^p \mathbf{v}(\mathbf{x})|^2 \, d\mathbf{x}, \quad \mathbf{v} = f^\varepsilon \mathbf{u} \in H^p(\mathbb{R}^d;\mathbb{C}^n). \tag{12.3}$$

In the case where $f = \mathbf{1}_n$, operator (12.1) is denoted by $\widehat{\mathcal{A}}_\varepsilon$:

$$\widehat{\mathcal{A}}_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}), \quad \varepsilon > 0. \tag{12.4}$$

For small ε the coefficients of operators (12.1) and (12.4) oscillate rapidly. Our goal in Chap. 3 is to approximate the operators $e^{-\widehat{\mathcal{A}}_\varepsilon \tau}$ and $f^\varepsilon e^{-\mathcal{A}_\varepsilon \tau}(f^\varepsilon)^*$ for small ε and to apply the results to study the behavior of the solutions of the Cauchy problem for parabolic equations with rapidly oscillating coefficients.

12.2. The scaling transformation. Let T_ε be the unitary scaling transformation in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ given by

$$(T_\varepsilon \mathbf{u})(\mathbf{x}) := \varepsilon^{d/2} \mathbf{u}(\varepsilon \mathbf{x}).$$

It is easy to check the following identities:

$$\mathcal{A}_\varepsilon = \varepsilon^{-2p} T_\varepsilon^* \mathcal{A} T_\varepsilon, \quad \widehat{\mathcal{A}}_\varepsilon = \varepsilon^{-2p} T_\varepsilon^* \widehat{\mathcal{A}} T_\varepsilon, \tag{12.5}$$

where $\mathcal{A}$ and $\widehat{\mathcal{A}}$ are operators (5.1) and (7.1), respectively. By (12.5), we have

$$e^{-\mathcal{A}_\varepsilon \tau} = T_\varepsilon^* e^{-\mathcal{A}\varepsilon^{-2p}\tau} T_\varepsilon, \quad e^{-\widehat{\mathcal{A}}_\varepsilon \tau} = T_\varepsilon^* e^{-\widehat{\mathcal{A}}\varepsilon^{-2p}\tau} T_\varepsilon. \tag{12.6}$$

Similar identities are valid for operators (9.3) and (7.22):

$$e^{-\mathcal{A}^0 \tau} = T_\varepsilon^* e^{-\mathcal{A}^0\varepsilon^{-2p}\tau} T_\varepsilon, \quad e^{-\widehat{\mathcal{A}}^0 \tau} = T_\varepsilon^* e^{-\widehat{\mathcal{A}}^0\varepsilon^{-2p}\tau} T_\varepsilon. \tag{12.7}$$

These relations allow us to deduce the results on homogenization of the operator exponential from the results of Secs. 10 and 11.

13. Homogenization of the Operator Exponential $e^{-\widehat{\mathcal{A}}_\varepsilon \tau}$

13.1. Approximation of the operator $e^{-\widehat{\mathcal{A}}_\varepsilon \tau}$ in the operator norm in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. Using (12.6), (12.7), and the fact that T_ε is unitary, we have

$$\left\| e^{-\widehat{\mathcal{A}}_\varepsilon \tau} - e^{-\widehat{\mathcal{A}}^0 \tau} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} = \left\| e^{-\widehat{\mathcal{A}}\varepsilon^{-2p} \tau} - e^{-\widehat{\mathcal{A}}^0 \varepsilon^{-2p}\tau} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)}. \tag{13.1}$$

Applying Theorems 10.1, 10.2 (with τ replaced by $\varepsilon^{-2p}\tau$) and using identity (13.1), we obtain the following results.

Theorem 13.1. *Let $\widehat{\mathcal{A}}_\varepsilon$ be operator* (12.4) *and let $\widehat{\mathcal{A}}^0$ be the effective operator* (7.22)*. Then for $\tau \ge 0$ and $\varepsilon > 0$ we have*

$$\left\| e^{-\widehat{\mathcal{A}}_\varepsilon \tau} - e^{-\widehat{\mathcal{A}}^0 \tau} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \le \frac{\widehat{\mathcal{C}}_1 \varepsilon}{\tau^{\frac{1}{2p}} + \varepsilon}. \tag{13.2}$$

The constant $\widehat{\mathcal{C}}_1$ depends only on the problem data (8.2).

Theorem 13.2. *Let $\widehat{\mathcal{A}}_\varepsilon$ be operator* (12.4) *and let $\widehat{\mathcal{A}}^0$ be the effective operator* (7.22). *Suppose that $\widehat{G}(\boldsymbol{\theta})$ is the operator defined according to* (7.25), (7.26). *Assume that $\widehat{G}(\boldsymbol{\theta}) = 0$ for any $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. Then for $\tau \geq 0$ and $\varepsilon > 0$ we have*

$$\left\| e^{-\widehat{\mathcal{A}}_\varepsilon \tau} - e^{-\widehat{\mathcal{A}}^0 \tau} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \frac{\widehat{\mathcal{C}}_2 \varepsilon^2}{\tau^{\frac{1}{p}} + \varepsilon^2}. \tag{13.3}$$

The constant $\widehat{\mathcal{C}}_2$ depends only on the problem data (8.2).

Theorem 13.2 and Proposition 7.3 directly imply the following corollary.

Corollary 13.1. *Suppose that at least one of the following conditions is satisfied:*

1°. *Relations* (7.16) *are valid, i.e., $g^0 = \overline{g}$.*
2°. *Representations* (7.17) *hold, i.e., $g^0 = \underline{g}$.*
3°. *$n = 1$ and the matrices $g(\mathbf{x})$, b_β, $|\beta| = p$, have real entries.*

Then estimate (13.3) *holds.*

Remark 13.1. Corollary 13.1 demonstrates a new effect which is typical for higher-order operators: for the scalar operator $\widehat{\mathcal{A}}_\varepsilon$ with real-valued coefficients, the exponential is approximated by the exponential of the effective operator in the operator norm in L_2 with error $O(\varepsilon^2)$ without taking into account any correctors. There is no such effect for the second-order operators.

13.2. Approximation of the operator $e^{-\widehat{\mathcal{A}}_\varepsilon \tau}$ in the energy norm. Now, using Theorems 10.3, 10.4 and Propositions 10.1, 10.4, we obtain approximation for the exponential $e^{-\widehat{\mathcal{A}}_\varepsilon \tau}$ in the norm of operators acting from $L_2(\mathbb{R}^d; \mathbb{C}^n)$ to $H^p(\mathbb{R}^d; \mathbb{C}^n)$, and also approximation of the operator $g^\varepsilon b(\mathbf{D}) e^{-\widehat{\mathcal{A}}_\varepsilon \tau}$ (corresponding to the "flux") in the norm of operators acting from $L_2(\mathbb{R}^d; \mathbb{C}^n)$ to $L_2(\mathbb{R}^d; \mathbb{C}^m)$.

In addition to (12.5)–(12.7), we need the identities

$$b(\mathbf{D}) = \varepsilon^{-p} T_\varepsilon^* b(\mathbf{D}) T_\varepsilon, \quad [\Lambda^\varepsilon] = T_\varepsilon^* [\Lambda] T_\varepsilon, \quad [\widetilde{g}^\varepsilon] = T_\varepsilon^* [\widetilde{g}] T_\varepsilon. \tag{13.4}$$

Let Π_ε be the pseudodifferential operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ with the symbol $\chi_{\widetilde{\Omega}/\varepsilon}(\boldsymbol{\xi})$, i.e.,

$$(\Pi_\varepsilon \mathbf{u})(\mathbf{x}) = (2\pi)^{-d/2} \int\limits_{\widetilde{\Omega}/\varepsilon} e^{i\langle \mathbf{x}, \boldsymbol{\xi} \rangle} \widehat{\mathbf{u}}(\boldsymbol{\xi})\, d\boldsymbol{\xi}. \tag{13.5}$$

Operators (10.2) and (13.5) satisfy the following relation:

$$\Pi_\varepsilon = T_\varepsilon^* \Pi T_\varepsilon. \tag{13.6}$$

We put

$$\widehat{\mathcal{K}}(\varepsilon; \tau) := [\Lambda^\varepsilon] b(\mathbf{D}) e^{-\widehat{\mathcal{A}}^0 \tau} \Pi_\varepsilon. \tag{13.7}$$

Operator (13.7) is called a *corrector*; it is a continuous mapping of $L_2(\mathbb{R}^d; \mathbb{C}^n)$ into $H^p(\mathbb{R}^d; \mathbb{C}^n)$.

Theorem 13.3. *Let $\widehat{\mathcal{A}}_\varepsilon$ be operator* (12.4) *and let $\widehat{\mathcal{A}}^0$ be the effective operator* (7.22). *Suppose that Π_ε is operator* (13.5), *$\widehat{\mathcal{K}}(\varepsilon; \tau)$ is given by* (13.7), *and $\widetilde{g}$ is the matrix-valued function defined by* (7.13). *Then for $\tau > 0$ and $\varepsilon > 0$ we have*

$$\left\| \mathbf{D}^p \left(e^{-\widehat{\mathcal{A}}_\varepsilon \tau} - e^{-\widehat{\mathcal{A}}^0 \tau} - \varepsilon^p \widehat{\mathcal{K}}(\varepsilon; \tau) \right) \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \frac{\widehat{\mathcal{C}}_7 \varepsilon}{\tau^{\frac{1}{2}} (\tau^{\frac{1}{2p}} + \varepsilon)}, \tag{13.8}$$

$$\left\| g^\varepsilon b(\mathbf{D}) e^{-\widehat{\mathcal{A}}_\varepsilon \tau} - \widetilde{g}^\varepsilon b(\mathbf{D}) e^{-\widehat{\mathcal{A}}^0 \tau} \Pi_\varepsilon \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \frac{\widehat{\mathcal{C}}_6 \varepsilon}{\tau^{\frac{1}{2}} (\tau^{\frac{1}{2p}} + \varepsilon)}. \tag{13.9}$$

The constants $\widehat{\mathcal{C}}_6$ and $\widehat{\mathcal{C}}_7$ depend only on the problem data (8.2).

Proof. We start with the proof of unequality (13.8). By (12.7), (13.4), and (13.6),

$$\varepsilon^p \widehat{\mathcal{K}}(\varepsilon;\tau) = T_\varepsilon^* [\Lambda] b(\mathbf{D}) e^{-\widehat{\mathcal{A}}^0 \varepsilon^{-2p}\tau} \Pi T_\varepsilon. \tag{13.10}$$

Together with (12.5)–(12.7), this implies

$$\begin{aligned} &\left\| \widehat{\mathcal{A}}_\varepsilon^{1/2} \left(e^{-\widehat{\mathcal{A}}_\varepsilon \tau} - e^{-\widehat{\mathcal{A}}^0 \tau} - \varepsilon^p \widehat{\mathcal{K}}(\varepsilon;\tau)\right)\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ &= \varepsilon^{-p} \left\| \widehat{\mathcal{A}}^{1/2} \left(e^{-\widehat{\mathcal{A}}\varepsilon^{-2p}\tau} - e^{-\widehat{\mathcal{A}}^0 \varepsilon^{-2p}\tau} - [\Lambda] b(\mathbf{D}) e^{-\widehat{\mathcal{A}}^0 \varepsilon^{-2p}\tau}\Pi\right)\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}. \end{aligned} \tag{13.11}$$

Now, from (10.3) (with τ replaced by $\varepsilon^{-2p}\tau$) and (13.11) it follows that

$$\left\| \widehat{\mathcal{A}}_\varepsilon^{1/2} \left(e^{-\widehat{\mathcal{A}}_\varepsilon \tau} - e^{-\widehat{\mathcal{A}}^0 \tau} - \varepsilon^p \widehat{\mathcal{K}}(\varepsilon;\tau)\right)\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \frac{\widehat{\mathcal{C}}_3 \varepsilon}{\tau^{\frac12}(\tau^{\frac{1}{2p}}+\varepsilon)}. \tag{13.12}$$

Combining (13.12) and the lower estimate (12.3) (with $f = \mathbf{1}_n$), we obtain the required inequality (13.8) with the constant $\widehat{\mathcal{C}}_7 = c_0^{-\frac12}\widehat{\mathcal{C}}_3$.

We proceed to the proof of estimate (13.9). Similarly to (13.11), we have

$$\begin{aligned} &\left\| g^\varepsilon b(\mathbf{D}) e^{-\widehat{\mathcal{A}}_\varepsilon \tau} - \widetilde{g}^\varepsilon b(\mathbf{D}) e^{-\widehat{\mathcal{A}}^0 \tau}\Pi_\varepsilon \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ &= \varepsilon^{-p} \left\| g b(\mathbf{D}) e^{-\widehat{\mathcal{A}}\varepsilon^{-2p}\tau} - \widetilde{g} b(\mathbf{D}) e^{-\widehat{\mathcal{A}}^0 \varepsilon^{-2p}\tau}\Pi \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}. \end{aligned}$$

Together with (10.4) (with τ replaced by $\varepsilon^{-2p}\tau$), this implies estimate (13.9). ☐

Next, we find approximation of the exponential $e^{-\widehat{\mathcal{A}}_\varepsilon \tau}$ in the $(L_2 \to H^p)$-norm.

Theorem 13.4. *Suppose that the assumptions of Theorem* 13.3 *are satisfied.*

1°. *For $\tau > 0$ and $\varepsilon > 0$ we have*

$$\left\| e^{-\widehat{\mathcal{A}}_\varepsilon \tau} - e^{-\widehat{\mathcal{A}}^0 \tau} - \varepsilon^p \widehat{\mathcal{K}}(\varepsilon;\tau)\right\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)} \le \frac{\widehat{\mathcal{C}}_8 (1+\tau^{-\frac12})\varepsilon}{\tau^{\frac{1}{2p}}+\varepsilon}. \tag{13.13}$$

The constant $\widehat{\mathcal{C}}_8$ depends only on the problem data (8.2).

2°. *In addition, suppose that $\widehat{G}(\boldsymbol{\theta}) \equiv 0$. Then for $\tau > 0$ and $\varepsilon > 0$ we have*

$$\left\| e^{-\widehat{\mathcal{A}}_\varepsilon \tau} - e^{-\widehat{\mathcal{A}}^0 \tau} - \varepsilon^p \widehat{\mathcal{K}}(\varepsilon;\tau)\right\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)} \le \widehat{\mathcal{C}}_9 \left(\frac{\varepsilon}{\tau^{\frac12}(\tau^{\frac{1}{2p}}+\varepsilon)} + \frac{\varepsilon^2}{\tau^{\frac1p}+\varepsilon^2}\right). \tag{13.14}$$

The constant $\widehat{\mathcal{C}}_9$ depends only on the problem data (8.2).

Proof. By (12.6), (12.7), and (13.10),

$$\begin{aligned} &\left\| e^{-\widehat{\mathcal{A}}_\varepsilon \tau} - e^{-\widehat{\mathcal{A}}^0 \tau} - \varepsilon^p \widehat{\mathcal{K}}(\varepsilon;\tau)\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ &= \left\| e^{-\widehat{\mathcal{A}}\varepsilon^{-2p}\tau} - e^{-\widehat{\mathcal{A}}^0 \varepsilon^{-2p}\tau} - [\Lambda] b(\mathbf{D}) e^{-\widehat{\mathcal{A}}^0 \varepsilon^{-2p}\tau}\Pi\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}. \end{aligned} \tag{13.15}$$

Now, from (10.12) (with τ replaced by $\varepsilon^{-2p}\tau$) and (13.15) it follows that

$$\left\| e^{-\widehat{\mathcal{A}}_\varepsilon \tau} - e^{-\widehat{\mathcal{A}}^0 \tau} - \varepsilon^p \widehat{\mathcal{K}}(\varepsilon;\tau)\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \frac{\widehat{\mathcal{C}}_4 \varepsilon}{\tau^{\frac{1}{2p}}+\varepsilon}. \tag{13.16}$$

Since $(1+|\boldsymbol{\xi}|^2)^p \le 2^{p-1}(1+|\boldsymbol{\xi}|^{2p})$, then the lower estimate (12.2) (with $f = \mathbf{1}_n$) for any function $\mathbf{u} \in H^p(\mathbb{R}^d;\mathbb{C}^n)$ implies

$$\begin{aligned}\|\mathbf{u}\|^2_{H^p(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} (1+|\boldsymbol{\xi}|^2)^p |\widehat{\mathbf{u}}(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi} \le 2^{p-1} \int_{\mathbb{R}^d} (1+|\boldsymbol{\xi}|^{2p}) |\widehat{\mathbf{u}}(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi} \\ &\le 2^{p-1}\left(\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} + \alpha_0^{-1}\|g^{-1}\|_{L_\infty} \|\widehat{\mathcal{A}}_\varepsilon^{1/2}\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}\right).\end{aligned} \tag{13.17}$$

Combining this with (13.12) and (13.16), we obtain

$$\left\| e^{-\widehat{\mathcal{A}}_\varepsilon \tau} - e^{-\widehat{\mathcal{A}}^0 \tau} - \varepsilon^p \widehat{\mathcal{K}}(\varepsilon;\tau)\right\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)} \le 2^{\frac{p-1}{2}} \left(\widehat{\mathcal{C}}_4 + \widehat{\mathcal{C}}_3 \alpha_0^{-\frac{1}{2}} \|g^{-1}\|^{\frac{1}{2}}_{L_\infty} \tau^{-\frac{1}{2}}\right) \frac{\varepsilon}{\tau^{\frac{1}{2p}} + \varepsilon}.$$

This yields the required estimate (13.13) with the constant

$$\widehat{\mathcal{C}}_8 = 2^{\frac{p-1}{2}} \max\left\{ \widehat{\mathcal{C}}_4, \widehat{\mathcal{C}}_3 \alpha_0^{-\frac{1}{2}} \|g^{-1}\|^{\frac{1}{2}}_{L_\infty}\right\}.$$

We proceed to the proof of statement 2°. Suppose that $\widehat{G}(\boldsymbol{\theta}) \equiv 0$. Then estimate (10.13) holds, which together with (13.15) implies

$$\left\| e^{-\widehat{\mathcal{A}}_\varepsilon \tau} - e^{-\widehat{\mathcal{A}}^0 \tau} - \varepsilon^p \widehat{\mathcal{K}}(\varepsilon;\tau)\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \frac{\widehat{\mathcal{C}}_5 \varepsilon^2}{\tau^{\frac{1}{p}} + \varepsilon^2}.$$

Combining this with (13.12) and (13.17), we obtain estimate (13.14) with the constant

$$\widehat{\mathcal{C}}_9 = 2^{\frac{p-1}{2}} \max\left\{ \widehat{\mathcal{C}}_5, \widehat{\mathcal{C}}_3 \alpha_0^{-\frac{1}{2}} \|g^{-1}\|^{\frac{1}{2}}_{L_\infty}\right\}.$$

□

Similarly to the proof of Theorem 13.3, we deduce the following result from Theorem 10.4.

Theorem 13.5. *Suppose that the assumptions of Theorem* 13.3 *are satisfied. Denote*

$$\widehat{\mathcal{K}}^0(\varepsilon;\tau) := [\Lambda^\varepsilon] b(\mathbf{D}) e^{-\widehat{\mathcal{A}}^0 \tau}.$$

Then we have

$$\left\| \mathbf{D}^p \left(e^{-\widehat{\mathcal{A}}_\varepsilon \tau} - e^{-\widehat{\mathcal{A}}^0 \tau} - \varepsilon^p \widehat{\mathcal{K}}^0(\varepsilon;\tau)\right)\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \widehat{\mathcal{C}}_7^\circ \varepsilon \tau^{-\frac{1}{2}-\frac{1}{2p}}, \quad \varepsilon > 0,\ \tau \ge \varepsilon^{2p},$$

$$\left\| g^\varepsilon b(\mathbf{D}) e^{-\widehat{\mathcal{A}}_\varepsilon \tau} - \widetilde{g}^\varepsilon b(\mathbf{D}) e^{-\widehat{\mathcal{A}}^0 \tau}\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \widehat{\mathcal{C}}_6^\circ \varepsilon \tau^{-\frac{1}{2}-\frac{1}{2p}}, \quad \varepsilon > 0,\ \tau \ge \varepsilon^{2p}.$$

The constants $\widehat{\mathcal{C}}_6^\circ$ and $\widehat{\mathcal{C}}_7^\circ$ depend only on the problem data (8.2)*.*

Similarly to the proof of Theorem 13.4, we deduce the following statement from Theorem 10.4 and Proposition 10.4.

Theorem 13.6. *Suppose that the assumptions of Theorem* 13.5 *are satisfied.*

1°. *We have*

$$\left\| e^{-\widehat{\mathcal{A}}_\varepsilon \tau} - e^{-\widehat{\mathcal{A}}^0 \tau} - \varepsilon^p \widehat{\mathcal{K}}^0(\varepsilon;\tau)\right\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)} \le \widehat{\mathcal{C}}_8^\circ \varepsilon \tau^{-\frac{1}{2p}}, \quad \varepsilon > 0,\ \tau \ge \varepsilon^{2p}.$$

The constant $\widehat{\mathcal{C}}_8^\circ$ depends only on the problem data (8.2)*.*

2°. *In addition, suppose that $\widehat{G}(\boldsymbol{\theta}) \equiv 0$. Then we have*

$$\left\| e^{-\widehat{\mathcal{A}}_\varepsilon \tau} - e^{-\widehat{\mathcal{A}}^0 \tau} - \varepsilon^p \widehat{\mathcal{K}}^0(\varepsilon;\tau)\right\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)} \le \widehat{\mathcal{C}}_9^\circ \left(\varepsilon \tau^{-\frac{1}{2}-\frac{1}{2p}} + \varepsilon^2 \tau^{-\frac{1}{p}}\right), \quad \varepsilon > 0,\ \tau \ge \varepsilon^{2p}.$$

The constant $\widehat{\mathcal{C}}_9^\circ$ depends only on the problem data (8.2)*.*

13.3. Special cases. Let us focus on the special cases.

Suppose that relations (7.16) hold, i.e., $g^0 = \overline{g}$. In this case we have $\Lambda(\mathbf{x}) = 0$, whence corrector (13.7) is equal to zero. Moreover, in this case $\widehat{G}(\boldsymbol{\theta}) \equiv 0$. Then Theorems 13.3 and 13.4 (2°) imply the following result.

Corollary 13.2. *Suppose that relations* (7.16) *are satisfied, i.e.,* $g^0 = \overline{g}$. *Then for* $\tau > 0$ *and* $\varepsilon > 0$ *we have*

$$\left\| \mathbf{D}^p \left(e^{-\widehat{\mathcal{A}}_\varepsilon \tau} - e^{-\widehat{\mathcal{A}}^0 \tau} \right) \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \frac{\widehat{\mathcal{C}}_7 \varepsilon}{\tau^{\frac{1}{2}} (\tau^{\frac{1}{2p}} + \varepsilon)},$$

$$\left\| e^{-\widehat{\mathcal{A}}_\varepsilon \tau} - e^{-\widehat{\mathcal{A}}^0 \tau} \right\|_{L_2(\mathbb{R}^d) \to H^p(\mathbb{R}^d)} \leq \widehat{\mathcal{C}}_9 \left(\frac{\varepsilon}{\tau^{\frac{1}{2}} (\tau^{\frac{1}{2p}} + \varepsilon)} + \frac{\varepsilon^2}{\tau^{\frac{1}{p}} + \varepsilon^2} \right).$$

Next, consider the case where relations (7.17) are satisfied, i.e., $g^0 = \underline{g}$. Then $\widetilde{g}(\mathbf{x}) = g^0$. Theorem 10.3 implies the following statement.

Corollary 13.3. *Suppose that relations* (7.17) *are satisfied, i.e.,* $g^0 = \underline{g}$. *Then for* $\tau > 0$ *and* $\varepsilon > 0$ *we have*

$$\left\| g^\varepsilon b(\mathbf{D}) e^{-\widehat{\mathcal{A}}_\varepsilon \tau} - g^0 b(\mathbf{D}) e^{-\widehat{\mathcal{A}}^0 \tau} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \frac{\widehat{\mathcal{C}}'_6 \varepsilon}{\tau^{\frac{1}{2}} (\tau^{\frac{1}{2p}} + \varepsilon)}. \tag{13.18}$$

The constant $\widehat{\mathcal{C}}'_6$ *depends only on the problem data* (8.2).

Proof. Since $\widetilde{g}(\mathbf{x}) = g^0$, then inequality (10.4) takes the form

$$\left\| g b(\mathbf{D}) e^{-\widehat{\mathcal{A}} \tau} - g^0 b(\mathbf{D}) e^{-\widehat{\mathcal{A}}^0 \tau} \Pi \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \frac{\widehat{\mathcal{C}}_6}{\tau^{\frac{1}{2}} (\tau^{\frac{1}{2p}} + 1)}, \quad \tau > 0. \tag{13.19}$$

Similarly to the proof of Proposition 10.2, using (5.5), (8.4), (10.2), and the estimate $x^{\frac{1}{2} + \frac{1}{2p}} e^{-x} \leq 1$ for $x \geq 0$, we have

$$\left\| g^0 b(\mathbf{D}) e^{-\widehat{\mathcal{A}}^0 \widetilde{\tau}} (I - \Pi) \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \|g\|_{L_\infty} \alpha_1^{\frac{1}{2}} \sup_{|\boldsymbol{\xi}| \geq r_0} |\boldsymbol{\xi}|^p e^{-\widehat{c}_* |\boldsymbol{\xi}|^{2p} \widetilde{\tau}} \leq \|g\|_{L_\infty} \alpha_1^{\frac{1}{2}} \widehat{c}_*^{\,-\frac{1}{2} - \frac{1}{2p}} r_0^{-1} \widetilde{\tau}^{-\frac{1}{2} - \frac{1}{2p}}.$$

Using the estimate $x^{\frac{1}{2}} e^{-x} \leq 1$ for $x \geq 0$, we obtain

$$\left\| g^0 b(\mathbf{D}) e^{-\widehat{\mathcal{A}}^0 \widetilde{\tau}} (I - \Pi) \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \|g\|_{L_\infty} \alpha_1^{\frac{1}{2}} \widehat{c}_*^{\,-\frac{1}{2}} \widetilde{\tau}^{-\frac{1}{2}}.$$

Applying the first inequality for $\widetilde{\tau} \geq 1$ and the second one for $0 < \widetilde{\tau} < 1$, we arrive at the estimate

$$\left\| g^0 b(\mathbf{D}) e^{-\widehat{\mathcal{A}}^0 \widetilde{\tau}} (I - \Pi) \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \frac{\widehat{\mathcal{C}}''_6}{\widetilde{\tau}^{\frac{1}{2}} (\widetilde{\tau}^{\frac{1}{2p}} + 1)},$$

where $\widehat{\mathcal{C}}''_6 = 2\|g\|_{L_\infty} \alpha_1^{\frac{1}{2}} \max \left\{ \widehat{c}_*^{\,-\frac{1}{2} - \frac{1}{2p}} r_0^{-1}, \widehat{c}_*^{\,-\frac{1}{2}} \right\}$. Together with (13.19), this implies

$$\left\| g b(\mathbf{D}) e^{-\widehat{\mathcal{A}} \widetilde{\tau}} - g^0 b(\mathbf{D}) e^{-\widehat{\mathcal{A}}^0 \widetilde{\tau}} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \frac{\widehat{\mathcal{C}}'_6}{\widetilde{\tau}^{\frac{1}{2}} (\widetilde{\tau}^{\frac{1}{2p}} + 1)}, \quad \widetilde{\tau} > 0,$$

where $\widehat{\mathcal{C}}'_6 = \widehat{\mathcal{C}}_6 + \widehat{\mathcal{C}}''_6$. Putting $\widetilde{\tau} = \varepsilon^{-2p} \tau$ and applying the scaling transformation, we arrive at the required estimate (13.18). $\square$

14. Homogenization of the Operator $f^\varepsilon e^{-\mathcal{A}_\varepsilon \tau}(f^\varepsilon)^*$

14.1. Approximation of the operator $f^\varepsilon e^{-\mathcal{A}_\varepsilon \tau}(f^\varepsilon)^*$ in the operator norm in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. Using (12.6), (12.7), and the fact that T_ε is unitary, we obtain

$$\left\| f^\varepsilon e^{-\mathcal{A}_\varepsilon \tau}(f^\varepsilon)^* - f_0 e^{-\mathcal{A}^0 \tau} f_0 \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} = \left\| f e^{-\mathcal{A}\varepsilon^{-2p}\tau} f^* - f_0 e^{-\mathcal{A}^0\varepsilon^{-2p}\tau} f_0 \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}. \tag{14.1}$$

Applying Theorems 11.1, 11.2 (with τ replaced by $\varepsilon^{-2p}\tau$) and using identity (14.1), we arrive at the following results.

Theorem 14.1. *Let $\mathcal{A}_\varepsilon$ be operator* (12.1) *and let $\mathcal{A}^0$ be operator* (9.3). *Suppose that f_0 is the matrix defined by* (9.1). *Then for $\tau \geq 0$ and $\varepsilon > 0$ we have*

$$\left\| f^\varepsilon e^{-\mathcal{A}_\varepsilon \tau}(f^\varepsilon)^* - f_0 e^{-\mathcal{A}^0 \tau} f_0 \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \frac{\mathcal{C}_1 \varepsilon}{\tau^{\frac{1}{2p}} + \varepsilon}.$$

The constant $\mathcal{C}_1$ depends only on the problem data (9.7).

Theorem 14.2. *Suppose that the assumptions of Theorem* 14.1 *are satisfied. Let $\widehat{G}(\boldsymbol{\theta})$ be the operator defined according to* (7.25), (7.26). *Suppose that $\widehat{G}(\boldsymbol{\theta}) = 0$ for any $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. Then for $\tau \geq 0$ and $\varepsilon > 0$ we have*

$$\left\| f^\varepsilon e^{-\mathcal{A}_\varepsilon \tau}(f^\varepsilon)^* - f_0 e^{-\mathcal{A}^0 \tau} f_0 \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \frac{\mathcal{C}_2 \varepsilon^2}{\tau^{\frac{1}{p}} + \varepsilon^2}. \tag{14.2}$$

The constant $\mathcal{C}_2$ depends only on the problem data (9.7).

Theorem 14.2 and Proposition 7.3 directly imply the following corollary.

Corollary 14.1. *Suppose that at least one of the following conditions is satisfied:*

1°. *Relations* (7.16) *hold, i.e., $g^0 = \overline{g}$.*

2°. *Representations* (7.17) *are valid, i.e., $g^0 = \underline{g}$.*

3°. *$n = 1$ and the matrices $g(\mathbf{x})$, b_β, $|\beta| = p$, have real entries.*

Then estimate (14.2) *is satisfied.*

14.2. Approximation of the operator $f^\varepsilon e^{-\mathcal{A}_\varepsilon \tau}(f^\varepsilon)^*$ in the energy norm. Applying Theorems 11.3, 11.4 and Propositions 11.1, 11.3, we obtain approximation of the operator $f^\varepsilon e^{-\mathcal{A}_\varepsilon \tau}(f^\varepsilon)^*$ in the norm of operators acting from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to $H^p(\mathbb{R}^d;\mathbb{C}^n)$, and also approximation of the operator $g^\varepsilon b(\mathbf{D}) f^\varepsilon e^{-\mathcal{A}_\varepsilon \tau}(f^\varepsilon)^*$ in the norm of operators acting from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to $L_2(\mathbb{R}^d;\mathbb{C}^m)$.

We introduce the corrector

$$\mathcal{K}(\varepsilon;\tau) := [\Lambda^\varepsilon] b(\mathbf{D}) f_0 e^{-\mathcal{A}^0 \tau} f_0 \Pi_\varepsilon. \tag{14.3}$$

Using the scaling transformation, we deduce the following result from Theorem 11.3; its proof is completely similar to the proof of Theorem 13.3.

Theorem 14.3. *Let $\mathcal{A}_\varepsilon$ be operator* (12.1) *and let $\mathcal{A}^0$ be operator* (9.3). *Suppose that f_0 is the matrix defined by* (9.1). *Let Π_ε be operator* (13.5). *Suppose that $\mathcal{K}(\varepsilon;\tau)$ is operator* (14.3) *and $\widetilde{g}$ is the matrix-valued function defined by* (7.13). *Then for $\tau > 0$ and $\varepsilon > 0$ we have*

$$\left\| \mathbf{D}^p \left(f^\varepsilon e^{-\mathcal{A}_\varepsilon \tau}(f^\varepsilon)^* - f_0 e^{-\mathcal{A}^0 \tau} f_0 - \varepsilon^p \mathcal{K}(\varepsilon;\tau) \right) \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \frac{\mathcal{C}_7 \varepsilon}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2p}} + \varepsilon)},$$

$$\left\| g^\varepsilon b(\mathbf{D}) f^\varepsilon e^{-\mathcal{A}_\varepsilon \tau}(f^\varepsilon)^* - \widetilde{g}^\varepsilon b(\mathbf{D}) f_0 e^{-\mathcal{A}^0 \tau} f_0 \Pi_\varepsilon \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq \frac{\mathcal{C}_6 \varepsilon}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2p}} + \varepsilon)}.$$

The constants $\mathcal{C}_6$ and $\mathcal{C}_7$ depend only on the problem data (9.7).

Similarly, Theorem 11.3 and Proposition 11.1 imply the following result; its proof is analogous to the proof of Theorem 13.4.

Theorem 14.4. *Suppose that the assumptions of Theorem* 14.3 *are satisfied.*

1°. *For* $\tau > 0$ *and* $\varepsilon > 0$ *we have*

$$\left\| f^\varepsilon e^{-\mathcal{A}_\varepsilon \tau} (f^\varepsilon)^* - f_0 e^{-\mathcal{A}^0 \tau} f_0 - \varepsilon^p \mathcal{K}(\varepsilon;\tau) \right\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)} \le \frac{\mathcal{C}_8 (1+\tau^{-\frac{1}{2}})\varepsilon}{\tau^{\frac{1}{2p}} + \varepsilon}.$$

The constant $\mathcal{C}_8$ *depends only on the problem data* (9.7)*.*

2°. *In addition, suppose that* $\widehat{G}(\boldsymbol{\theta}) \equiv 0$*. Then for* $\tau > 0$ *and* $\varepsilon > 0$ *we have*

$$\left\| f^\varepsilon e^{-\mathcal{A}_\varepsilon \tau} (f^\varepsilon)^* - f_0 e^{-\mathcal{A}^0 \tau} f_0 - \varepsilon^p \mathcal{K}(\varepsilon;\tau) \right\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)} \le \mathcal{C}_9 \left(\frac{\varepsilon}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2p}} + \varepsilon)} + \frac{\varepsilon^2}{\tau^{\frac{1}{p}} + \varepsilon^2} \right).$$

The constant $\mathcal{C}_9$ *depends only on the problem data* (9.7)*.*

In turn, by the scaling transformation, Theorem 11.4 implies the following result.

Theorem 14.5. *Suppose that the assumptions of Theorem* 14.3 *are satisfied. Denote*

$$\mathcal{K}^0(\varepsilon;\tau) := [\Lambda^\varepsilon] b(\mathbf{D}) f_0 e^{-\mathcal{A}^0 \tau} f_0.$$

Then we have

$$\left\| \mathbf{D}^p \left(f^\varepsilon e^{-\mathcal{A}_\varepsilon \tau} (f^\varepsilon)^* - f_0 e^{-\mathcal{A}^0 \tau} f_0 - \varepsilon^p \mathcal{K}^0(\varepsilon;\tau) \right) \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \mathcal{C}_7^\circ \varepsilon \tau^{-\frac{1}{2}-\frac{1}{2p}}, \quad \varepsilon > 0,\ \tau \ge \varepsilon^{2p},$$

$$\left\| g^\varepsilon b(\mathbf{D}) f^\varepsilon e^{-\mathcal{A}_\varepsilon \tau} (f^\varepsilon)^* - \widetilde{g}^\varepsilon b(\mathbf{D}) f_0 e^{-\mathcal{A}^0 \tau} f_0 \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \mathcal{C}_6^\circ \varepsilon \tau^{-\frac{1}{2}-\frac{1}{2p}}, \quad \varepsilon > 0,\ \tau \ge \varepsilon^{2p}.$$

The constants $\mathcal{C}_6^\circ$ *and* $\mathcal{C}_7^\circ$ *depend only on the problem data* (9.7)*.*

Theorem 11.4 and Proposition 11.3 imply the following statement.

Theorem 14.6. *Suppose that the assumptions of Theorem* 14.5 *are satisfied.*

1°. *We have*

$$\left\| f^\varepsilon e^{-\mathcal{A}_\varepsilon \tau} (f^\varepsilon)^* - f_0 e^{-\mathcal{A}^0 \tau} f_0 - \varepsilon^p \mathcal{K}^0(\varepsilon;\tau) \right\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)} \le \mathcal{C}_8^\circ \varepsilon \tau^{-\frac{1}{2p}}, \quad \varepsilon > 0,\ \tau \ge \varepsilon^{2p}.$$

The constant $\mathcal{C}_8^\circ$ *depends only on the problem data* (9.7)*.*

2°. *In addition, assume that* $\widehat{G}(\boldsymbol{\theta}) \equiv 0$*. Then we have*

$$\left\| f^\varepsilon e^{-\mathcal{A}_\varepsilon \tau} (f^\varepsilon)^* - f_0 e^{-\mathcal{A}^0 \tau} f_0 - \varepsilon^p \mathcal{K}^0(\varepsilon;\tau) \right\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)} \le \mathcal{C}_9^\circ \left(\varepsilon \tau^{-\frac{1}{2}-\frac{1}{2p}} + \varepsilon^2 \tau^{-\frac{1}{p}} \right), \quad \varepsilon > 0,\ \tau \ge \varepsilon^{2p}.$$

The constant $\mathcal{C}_9^\circ$ *depends only on the problem data* (9.7)*.*

14.3. Special cases.

Let us focus on the special cases.

Suppose that relations (7.16) are valid, i.e., $g^0 = \overline{g}$. In this case we have $\Lambda(\mathbf{x}) = 0$, whence the corrector (14.3) is equal to zero and $\widehat{G}(\boldsymbol{\theta}) \equiv 0$. Theorems 14.3 and 14.4 imply the following result.

Corollary 14.2. *Suppose that relations* (7.16) *hold, i.e.,* $g^0 = \overline{g}$*. Then for* $\tau > 0$ *and* $\varepsilon > 0$ *we have*

$$\left\| \mathbf{D}^p \left(f^\varepsilon e^{-\mathcal{A}_\varepsilon \tau} (f^\varepsilon)^* - f_0 e^{-\mathcal{A}^0 \tau} f_0 \right) \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \frac{\mathcal{C}_7 \varepsilon}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2p}} + \varepsilon)},$$

$$\left\| f^\varepsilon e^{-\mathcal{A}_\varepsilon \tau} (f^\varepsilon)^* - f_0 e^{-\mathcal{A}^0 \tau} f_0 \right\|_{L_2(\mathbb{R}^d)\to H^p(\mathbb{R}^d)} \le \mathcal{C}_9 \left(\frac{\varepsilon}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2p}} + \varepsilon)} + \frac{\varepsilon^2}{\tau^{\frac{1}{p}} + \varepsilon^2} \right).$$

Next, we consider the case where relations (7.17) are satisfied, i.e., $g^0 = \underline{g}$. In this case we have $\widetilde{g}(\mathbf{x}) = g^0$. Theorem 11.3 implies the following statement; its proof is completely similar to the proof of Corollary 13.3.

Corollary 14.3. *Suppose that relations* (7.17) *hold, i.e.,* $g^0 = \underline{g}$. *Then for* $\tau > 0$ *and* $\varepsilon > 0$ *we have*

$$\left\| g^\varepsilon b(\mathbf{D}) f^\varepsilon e^{-\mathcal{A}_\varepsilon \tau} (f^\varepsilon)^* - g^0 b(\mathbf{D}) f_0 e^{-\mathcal{A}^0 \tau} f_0 \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \frac{\mathcal{C}_6' \varepsilon}{\tau^{\frac{1}{2}} (\tau^{\frac{1}{2p}} + \varepsilon)}.$$

The constant $\mathcal{C}_6'$ *depends only on the problem data* (9.7).

15. Homogenization of the Parabolic Cauchy Problem

15.1. The problem with the operator $\widehat{\mathcal{A}}_\varepsilon$. Approximation of the solutions in $L_2(\mathbb{R}^d; \mathbb{C}^n)$. The results of Secs. 13 and 14 can be applied to homogenization of solutions of parabolic equations with periodic rapidly oscillating coefficients. Let $\widehat{\mathcal{A}}_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})$, $0 < T \leq \infty$. We study the behavior of the solution $\mathbf{u}_\varepsilon(\mathbf{x}, \tau)$ of the following problem:

$$\begin{aligned} &\frac{\partial \mathbf{u}_\varepsilon(\mathbf{x}, \tau)}{\partial \tau} = -b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) \mathbf{u}_\varepsilon(\mathbf{x}, \tau) + \mathbf{F}(\mathbf{x}, \tau), \quad \mathbf{x} \in \mathbb{R}^d, \ \tau \in (0, T), \\ &\mathbf{u}_\varepsilon(\mathbf{x}, 0) = \boldsymbol{\phi}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d, \end{aligned} \tag{15.1}$$

where $\boldsymbol{\phi} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$ and $\mathbf{F} \in L_q((0, T); L_2(\mathbb{R}^d; \mathbb{C}^n))$ with some $1 < q \leq \infty$. We have

$$\mathbf{u}_\varepsilon(\cdot, \tau) = e^{-\widehat{\mathcal{A}}_\varepsilon \tau} \boldsymbol{\phi} + \int_0^\tau e^{-\widehat{\mathcal{A}}_\varepsilon (\tau - \widetilde{\tau})} \mathbf{F}(\cdot, \widetilde{\tau}) \, d\widetilde{\tau}. \tag{15.2}$$

Let $\widehat{\mathcal{A}}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$ be the effective operator and let $\mathbf{u}_0(\mathbf{x}, \tau)$ be the solution of the "homogenized" problem

$$\begin{aligned} &\frac{\partial \mathbf{u}_0(\mathbf{x}, \tau)}{\partial \tau} = -b(\mathbf{D})^* g^0 b(\mathbf{D}) \mathbf{u}_0(\mathbf{x}, \tau) + \mathbf{F}(\mathbf{x}, \tau), \quad \mathbf{x} \in \mathbb{R}^d, \ \tau \in (0, T), \\ &\mathbf{u}_0(\mathbf{x}, 0) = \boldsymbol{\phi}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d. \end{aligned} \tag{15.3}$$

Similarly to (15.2),

$$\mathbf{u}_0(\cdot, \tau) = e^{-\widehat{\mathcal{A}}^0 \tau} \boldsymbol{\phi} + \int_0^\tau e^{-\widehat{\mathcal{A}}^0 (\tau - \widetilde{\tau})} \mathbf{F}(\cdot, \widetilde{\tau}) \, d\widetilde{\tau}. \tag{15.4}$$

Theorem 13.1 and representations (15.2), (15.4) imply that

$$\begin{aligned} &\|\mathbf{u}_\varepsilon(\cdot, \tau) - \mathbf{u}_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leq \widehat{\mathcal{C}}_1 \varepsilon (\tau^{\frac{1}{2p}} + \varepsilon)^{-1} \|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} \\ &+ \widehat{\mathcal{C}}_1 \varepsilon \int_0^\tau ((\tau - \widetilde{\tau})^{\frac{1}{2p}} + \varepsilon)^{-1} \|\mathbf{F}(\cdot, \widetilde{\tau})\|_{L_2(\mathbb{R}^d)} \, d\widetilde{\tau} \leq \widehat{\mathcal{C}}_1 \varepsilon (\tau^{\frac{1}{2p}} + \varepsilon)^{-1} \|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} \\ &+ \widehat{\mathcal{C}}_1 \varepsilon \|\mathbf{F}\|_{L_q((0,T); L_2(\mathbb{R}^d))} \left(\int_0^\tau ((\tau - \widetilde{\tau})^{\frac{1}{2p}} + \varepsilon)^{-q'} \, d\widetilde{\tau} \right)^{\frac{1}{q'}}, \end{aligned}$$

where $1 < q \leq \infty$ and $\dfrac{1}{q} + \dfrac{1}{q'} = 1$. Estimating the integral on the right, we arrive at the following result.

Theorem 15.1. *Let* $0 < T \leq \infty$ *and* $\mathbf{F} \in L_q((0, T); L_2(\mathbb{R}^d; \mathbb{C}^n))$ *for some* $1 < q \leq \infty$. *Let* $\boldsymbol{\phi} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$. *Suppose that* $\mathbf{u}_\varepsilon$ *is the solution of problem* (15.1) *and* $\mathbf{u}_0$ *is the solution of the homogenized problem* (15.3). *Then for any* $\tau \in (0, T)$ *and* $\varepsilon \to 0$ *the solution* $\mathbf{u}_\varepsilon(\cdot, \tau)$ *converges to* $\mathbf{u}_0(\cdot, \tau)$ *in the* L_2*-norm. For* $\tau \in (0, T)$ *and* $0 < \varepsilon \leq 1$ *we have*

$$\|\mathbf{u}_\varepsilon(\cdot, \tau) - \mathbf{u}_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leq \widehat{\mathcal{C}}_1 \varepsilon \left((\tau^{\frac{1}{2p}} + \varepsilon)^{-1} \|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} + \theta(q, \varepsilon, \tau) \|\mathbf{F}\|_{L_q((0,T); L_2(\mathbb{R}^d))} \right), \tag{15.5}$$

where

$$\theta(q,\varepsilon,\tau)=\begin{cases}c(p,q)\tau^{1-\frac{1}{2p}-\frac{1}{q}}, & \dfrac{2p}{2p-1}<q\le\infty,\\ (\log|\tau+1|+2p|\log\varepsilon|)^{\frac{1}{2p}}, & q=\dfrac{2p}{2p-1},\\ c(p,q)\varepsilon^{2p-1-\frac{2p}{q}}, & 1<q<\dfrac{2p}{2p-1}.\end{cases}\tag{15.6}$$

Here $c(p,q)$ is a constant depending only on p and q.

For a fixed $\tau\in(0,T)$, the coefficient of $\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)}$ in (15.5) is of order $O(\varepsilon)$. The coefficient of $\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}$ is of order $O(\varepsilon)$ for $\dfrac{2p}{2p-1}<q\le\infty$; $O(\varepsilon|\log\varepsilon|^{\frac{1}{2p}})$ for $q=\dfrac{2p}{2p-1}$; for $1<q<\dfrac{2p}{2p-1}$ its order $O(\varepsilon^{2p-\frac{2p}{q}})$ depends on q.

Similarly, under the additional assumption that $\widehat{G}(\boldsymbol{\theta})\equiv 0$, Theorem 13.2 implies the following statement.

Theorem 15.2. *Suppose that the assumptions of Theorem* 15.1 *are satisfied. Let $\widehat{G}(\boldsymbol{\theta})$ be the operator defined according to* (7.25), (7.26). *Assume that $\widehat{G}(\boldsymbol{\theta})=0$ for any $\boldsymbol{\theta}\in\mathbb{S}^{d-1}$. Then for $\tau\in(0,T)$ and $0<\varepsilon\le 1$ we have*

$$\|\mathbf{u}_\varepsilon(\cdot,\tau)-\mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)}\le\widehat{\mathcal{C}}_2\varepsilon^2\left((\tau^{\frac{1}{p}}+\varepsilon^2)^{-1}\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)}+\widetilde{\theta}(q,\varepsilon,\tau)\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}\right),\tag{15.7}$$

where

$$\widetilde{\theta}(q,\varepsilon,\tau)=\begin{cases}c(p,q)\tau^{1-\frac{1}{p}-\frac{1}{q}}, & \dfrac{p}{p-1}<q\le\infty,\\ (\log|\tau+1|+2p|\log\varepsilon|)^{\frac{1}{p}}, & q=\dfrac{p}{p-1},\\ c(p,q)\varepsilon^{2p-2-\frac{2p}{q}}, & 1<q<\dfrac{p}{p-1}.\end{cases}\tag{15.8}$$

Under the assumptions of Theorem 15.2, for a fixed $\tau\in(0,T)$ the coefficient of $\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)}$ in (15.7) is of order $O(\varepsilon^2)$. The coefficient of $\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}$ is of order $O(\varepsilon^2)$ for $\dfrac{p}{p-1}<q\le\infty$; $O(\varepsilon^2|\log\varepsilon|^{\frac{1}{p}})$ for $q=\dfrac{p}{p-1}$; for $1<q<\dfrac{p}{p-1}$ its order $O(\varepsilon^{2p-\frac{2p}{q}})$ depends on q.

15.2. The problem with the operator $\widehat{\mathcal{A}}_\varepsilon$. Approximation of the solutions in $L_q((0,T);L_2(\mathbb{R}^d;\mathbb{C}^n))$. Now, suppose that $0<T<\infty$ and $1\le q<\infty$. Applying Theorem 13.1, we can estimate the norm of the difference $\mathbf{u}_\varepsilon-\mathbf{u}_0$ in the class $L_q((0,T);L_2(\mathbb{R}^d;\mathbb{C}^n))$.

Theorem 15.3. *Let $0<T<\infty$ and let $\mathbf{F}\in L_q((0,T);L_2(\mathbb{R}^d;\mathbb{C}^n))$ with some $1\le q<\infty$. Let $\boldsymbol{\phi}\in L_2(\mathbb{R}^d;\mathbb{C}^n)$. Suppose that $\mathbf{u}_\varepsilon$ is the solution of problem* (15.1) *and $\mathbf{u}_0$ is the solution of the homogenized problem* (15.3). *Then for $0<\varepsilon\le 1$ we have*

$$\|\mathbf{u}_\varepsilon-\mathbf{u}_0\|_{L_q((0,T);L_2(\mathbb{R}^d))}\le\widehat{\mathcal{C}}_1\varepsilon\left(\rho(q,\varepsilon,T)\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)}+c(p,q)T^{1-\frac{1}{2p}}\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}\right),\tag{15.9}$$

where

$$\rho(q,\varepsilon,T)=\theta(q',\varepsilon,T)=\begin{cases}c(p,q)T^{\frac{1}{q}-\frac{1}{2p}}, & 1\le q<2p,\\ (\log(T+1)+2p|\log\varepsilon|)^{\frac{1}{2p}}, & q=2p,\\ c(p,q)\varepsilon^{\frac{2p}{q}-1}, & 2p<q<\infty.\end{cases}\tag{15.10}$$

Proof. By (13.2), (15.2), and (15.4),

$$\|\mathbf{u}_\varepsilon-\mathbf{u}_0\|_{L_q((0,T);L_2(\mathbb{R}^d))}\le\widehat{\mathcal{C}}_1\varepsilon\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)}\mathcal{I}_q(\varepsilon;T)^{\frac{1}{q}}+\widehat{\mathcal{C}}_1\varepsilon\Big(\int_0^T\mathcal{L}(\varepsilon,\tau;\mathbf{F})^q\,d\tau\Big)^{\frac{1}{q}},\tag{15.11}$$

where

$$\mathcal{I}_q(\varepsilon;T) := \int\limits_0^T \frac{d\tau}{(\tau^{\frac{1}{2p}}+\varepsilon)^q}, \quad \mathcal{L}(\varepsilon,\tau;\mathbf{F}) := \int\limits_0^\tau ((\tau-\widetilde{\tau})^{\frac{1}{2p}}+\varepsilon)^{-1}\|\mathbf{F}(\cdot,\widetilde{\tau})\|_{L_2(\mathbb{R}^d)}\,d\widetilde{\tau}.$$

Estimating the integral $\mathcal{I}_q(\varepsilon;T)$, we obtain

$$\mathcal{I}_q(\varepsilon;T)^{\frac{1}{q}} \le \rho(q,\varepsilon,T). \tag{15.12}$$

Now we estimate the second term in the right-hand side of (15.11). If $q=1$, changing the order of integration, we have

$$\begin{aligned}\int\limits_0^T \mathcal{L}(\varepsilon,\tau;\mathbf{F})\,d\tau &= \int\limits_0^T d\tau \int\limits_0^\tau ((\tau-\widetilde{\tau})^{\frac{1}{2p}}+\varepsilon)^{-1}\|\mathbf{F}(\cdot,\widetilde{\tau})\|_{L_2(\mathbb{R}^d)}\,d\widetilde{\tau}\\ &= \int\limits_0^T d\widetilde{\tau}\|\mathbf{F}(\cdot,\widetilde{\tau})\|_{L_2(\mathbb{R}^d)} \int\limits_{\widetilde{\tau}}^T ((\tau-\widetilde{\tau})^{\frac{1}{2p}}+\varepsilon)^{-1}\,d\tau.\end{aligned}$$

Estimating the internal integral by $\left(1-\dfrac{1}{2p}\right)^{-1} T^{1-\frac{1}{2p}}$, we obtain

$$\int\limits_0^T \mathcal{L}(\varepsilon,\tau;\mathbf{F})\,d\tau \le c(p)T^{1-\frac{1}{2p}}\|\mathbf{F}\|_{L_1((0,T);L_2(\mathbb{R}^d))}.$$

In the case where $1<q<\infty$, we apply the Hölder inequality

$$\mathcal{L}(\varepsilon,\tau;\mathbf{F}) \le \Big(\int\limits_0^\tau ((\tau-\widetilde{\tau})^{\frac{1}{2p}}+\varepsilon)^{-1}\,d\widetilde{\tau}\Big)^{\frac{1}{q'}} \Big(\int\limits_0^\tau ((\tau-\widetilde{\tau})^{\frac{1}{2p}}+\varepsilon)^{-1}\|\mathbf{F}(\cdot,\widetilde{\tau})\|^q_{L_2(\mathbb{R}^d)}\,d\widetilde{\tau}\Big)^{\frac{1}{q}}. \tag{15.13}$$

The integral in the first parenthesis does not exceed $\left(1-\dfrac{1}{2p}\right)^{-1} T^{1-\frac{1}{2p}}$. Using (15.13) and changing the order of integration, we arrive at the inequality

$$\begin{aligned}\int\limits_0^T \mathcal{L}(\varepsilon,\tau;\mathbf{F})^q\,d\tau &\le c(p,q)T^{(1-\frac{1}{2p})\frac{q}{q'}} \int\limits_0^T d\tau \int\limits_0^\tau ((\tau-\widetilde{\tau})^{\frac{1}{2p}}+\varepsilon)^{-1}\|\mathbf{F}(\cdot,\widetilde{\tau})\|^q_{L_2(\mathbb{R}^d)}\,d\widetilde{\tau}\\ &= c(p,q)T^{(1-\frac{1}{2p})\frac{q}{q'}} \int\limits_0^T d\widetilde{\tau}\|\mathbf{F}(\cdot,\widetilde{\tau})\|^q_{L_2(\mathbb{R}^d)} \int\limits_{\widetilde{\tau}}^T ((\tau-\widetilde{\tau})^{\frac{1}{2p}}+\varepsilon)^{-1}\,d\tau\\ &\le \tilde{c}(p,q)T^{(1-\frac{1}{2p})(\frac{q}{q'}+1)}\|\mathbf{F}\|^q_{L_q((0,T);L_2(\mathbb{R}^d))}.\end{aligned}$$

As a result, for $1\le q<\infty$ we have

$$\Big(\int\limits_0^T \mathcal{L}(\varepsilon,\tau;\mathbf{F})^q\,d\tau\Big)^{\frac{1}{q}} \le c(p,q)T^{1-\frac{1}{2p}}\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}. \tag{15.14}$$

Now, relations (15.11), (15.12), and (15.14) imply the required estimate (15.9). □

In estimate (15.9) the coefficient of $\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}$ is $O(\varepsilon)$, and the coefficient of $\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)}$ is of order $O(\varepsilon)$ for $1\le q<2p$; $O(\varepsilon|\log\varepsilon|^{\frac{1}{2p}})$ for $q=2p$; and for $2p<q<\infty$ its order $O(\varepsilon^{\frac{2p}{q}})$ depends on q.

Similarly, under the additional assumption that $\widehat{G}(\boldsymbol{\theta}) \equiv 0$, Theorem 13.2 implies the following statement.

Theorem 15.4. *Suppose that the assumptions of Theorem* 15.3 *are satisfied. Let* $\widehat{G}(\boldsymbol{\theta})$ *be the operator defined according to* (7.25), (7.26). *Assume that* $\widehat{G}(\boldsymbol{\theta}) = 0$ *for any* $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. *Then for* $0 < \varepsilon \leq 1$ *we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_q((0,T);L_2(\mathbb{R}^d))} \leq \widehat{\mathcal{C}}_2 \varepsilon^2 \left(\widetilde{\rho}(q,\varepsilon,T) \|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} + c(p,q) T^{1-\frac{1}{p}} \|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))} \right), \tag{15.15}$$

where

$$\widetilde{\rho}(q,\varepsilon,T) = \widetilde{\theta}(q',\varepsilon,T) = \begin{cases} c(p,q) T^{\frac{1}{q}-\frac{1}{p}}, & 1 \leq q < p, \\ (\log(T+1) + 2p|\log \varepsilon|)^{\frac{1}{p}}, & q = p, \\ c(p,q) \varepsilon^{\frac{2p}{q}-2}, & p < q < \infty. \end{cases} \tag{15.16}$$

In estimate (15.15) the coefficient of $\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}$ is $O(\varepsilon^2)$, and the coefficient of $\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)}$ is of order $O(\varepsilon^2)$ for $1 \leq q < p$; $O(\varepsilon^2 |\log \varepsilon|^{\frac{1}{p}})$ for $q = p$; and for $p < q < \infty$ its order $O(\varepsilon^{\frac{2p}{q}})$ depends on q.

15.3. The problem with the operator $\widehat{\mathcal{A}}_\varepsilon$. Approximation of the solutions in $H^p(\mathbb{R}^d;\mathbb{C}^n)$. Next, we obtain approximation of the solution $\mathbf{u}_\varepsilon$ of problem (15.1) in the norm of $H^p(\mathbb{R}^d;\mathbb{C}^n)$ and also approximation of the "flux" $\mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$ in the norm of $L_2(\mathbb{R}^d;\mathbb{C}^m)$. In the case where $\mathbf{F} = 0$, Theorem 13.5 directly implies the following result.

Theorem 15.5. *Let* $\mathbf{F} = 0$ *and let* $\boldsymbol{\phi} \in L_2(\mathbb{R}^d;\mathbb{C}^n)$. *Suppose that* $\mathbf{u}_\varepsilon$ *is the solution of problem* (15.1) *and* $\mathbf{u}_0$ *is the solution of the homogenized problem* (15.3). *Let* $\mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$. *Suppose that* Λ *is the* Γ*-periodic solution of problem* (7.9) *and* $\widetilde{g}$ *is the matrix-valued function defined by* (7.13). *Then for* $\tau \in (0,T)$ *and* $0 < \varepsilon \leq \tau^{\frac{1}{2p}}$ *we have*

$$\|\mathbf{D}^p \left(\mathbf{u}_\varepsilon(\cdot,\tau) - \mathbf{u}_0(\cdot,\tau) - \varepsilon^p \Lambda^\varepsilon b(\mathbf{D})\mathbf{u}_0(\cdot,\tau) \right)\|_{L_2(\mathbb{R}^d)} \leq \widehat{\mathcal{C}}_7^\circ \varepsilon \tau^{-\frac{1}{2}-\frac{1}{2p}} \|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)},$$

$$\|\mathbf{p}_\varepsilon(\cdot,\tau) - \widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)} \leq \widehat{\mathcal{C}}_6^\circ \varepsilon \tau^{-\frac{1}{2}-\frac{1}{2p}} \|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)}.$$

In turn, Theorem 13.6 implies the following statement.

Theorem 15.6. *Suppose that the assumptions of Theorem* 15.5 *are satisfied.*

1°. *For* $\tau \in (0,T)$ *and* $0 < \varepsilon \leq \tau^{\frac{1}{2p}}$ *we have*

$$\|\mathbf{u}_\varepsilon(\cdot,\tau) - \mathbf{u}_0(\cdot,\tau) - \varepsilon^p \Lambda^\varepsilon b(\mathbf{D})\mathbf{u}_0(\cdot,\tau)\|_{H^p(\mathbb{R}^d)} \leq \widehat{\mathcal{C}}_8^\circ \varepsilon \tau^{-\frac{1}{2p}} \|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)}.$$

2°. *In addition, suppose that* $\widehat{G}(\boldsymbol{\theta}) \equiv 0$. *Then for* $\tau \in (0,T)$ *and* $0 < \varepsilon \leq \tau^{\frac{1}{2p}}$ *we have*

$$\|\mathbf{u}_\varepsilon(\cdot,\tau) - \mathbf{u}_0(\cdot,\tau) - \varepsilon^p \Lambda^\varepsilon b(\mathbf{D})\mathbf{u}_0(\cdot,\tau)\|_{H^p(\mathbb{R}^d)} \leq \widehat{\mathcal{C}}_9^\circ \left(\varepsilon \tau^{-\frac{1}{2}-\frac{1}{2p}} + \varepsilon^2 \tau^{-\frac{1}{p}} \right) \|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)}.$$

In the case where $\mathbf{F} \neq 0$, we need approximations of the exponential $e^{-\widehat{\mathcal{A}}_\varepsilon \tau}$ for all values of $0 < \tau < T$ (not only for $\tau \geq \varepsilon^{2p}$), therefore, we rely on Theorem 13.3.

Theorem 15.7. *Let* $\mathbf{F} \in L_q((0,T);L_2(\mathbb{R}^d;\mathbb{C}^n))$ *for some* $2 < q \leq \infty$ *and let* $\boldsymbol{\phi} \in L_2(\mathbb{R}^d;\mathbb{C}^n)$. *Suppose that* $\mathbf{u}_\varepsilon$ *is the solution of problem* (15.1) *and* $\mathbf{u}_0$ *is the solution of the homogenized problem* (15.3). *Let* $\mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$. *Suppose that* Λ *is the* Γ*-periodic solution of problem* (7.9) *and* $\widetilde{g}$ *is the matrix-valued function defined by* (7.13). *Suppose that* Π_ε *is the operator defined in* (13.5). *Then for* $\tau \in (0,T)$ *and*

$\varepsilon > 0$ *we have*

$$\|\mathbf{D}^p\left(\mathbf{u}_\varepsilon(\cdot,\tau) - \mathbf{u}_0(\cdot,\tau) - \varepsilon^p\Lambda^\varepsilon b(\mathbf{D})(\Pi_\varepsilon\mathbf{u}_0)(\cdot,\tau)\right)\|_{L_2(\mathbb{R}^d)}$$
$$\leq \frac{\widehat{\mathcal{C}}_7\varepsilon}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2p}}+\varepsilon)}\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} + \widehat{\mathcal{C}}_7\,\varepsilon\,\Theta(q,\varepsilon,\tau)\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}, \tag{15.17}$$

$$\|\mathbf{p}_\varepsilon(\cdot,\tau) - \widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)}$$
$$\leq \frac{\widehat{\mathcal{C}}_6\varepsilon}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2p}}+\varepsilon)}\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} + \widehat{\mathcal{C}}_6\,\varepsilon\,\Theta(q,\varepsilon,\tau)\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}. \tag{15.18}$$

Here

$$\Theta(q,\varepsilon,\tau) = \begin{cases} c(p,q)\tau^{\frac{1}{2}-\frac{1}{q}-\frac{1}{2p}}, & \dfrac{2p}{p-1} < q \leq \infty, \\ c(p)\left(\log|\tau+1| + |\log\varepsilon|\right)^{\frac{1}{2}+\frac{1}{2p}}, & q = \dfrac{2p}{p-1}, \\ c(p,q)\varepsilon^{p-1-\frac{2p}{q}}, & 2 < q < \dfrac{2p}{p-1}. \end{cases} \tag{15.19}$$

Proof. Theorem 13.3 and representations (15.2), (15.4) imply the estimate

$$\|\mathbf{D}^p\left(\mathbf{u}_\varepsilon(\cdot,\tau) - \mathbf{u}_0(\cdot,\tau) - \varepsilon^p\Lambda^\varepsilon b(\mathbf{D})(\Pi_\varepsilon\mathbf{u}_0)(\cdot,\tau)\right)\|_{L_2(\mathbb{R}^d)}$$
$$\leq \frac{\widehat{\mathcal{C}}_7\varepsilon}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2p}}+\varepsilon)}\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} + \widehat{\mathcal{C}}_7\varepsilon\int_0^\tau \frac{\|\mathbf{F}(\cdot,\widetilde{\tau})\|_{L_2(\mathbb{R}^d)}}{(\tau-\widetilde{\tau})^{\frac{1}{2}}((\tau-\widetilde{\tau})^{\frac{1}{2p}}+\varepsilon)}\,d\widetilde{\tau}$$
$$\leq \frac{\widehat{\mathcal{C}}_7\varepsilon}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2p}}+\varepsilon)}\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} + \widehat{\mathcal{C}}_7\varepsilon\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}\left(\int_0^\tau \frac{d\widetilde{\tau}}{(\tau-\widetilde{\tau})^{\frac{q'}{2}}((\tau-\widetilde{\tau})^{\frac{1}{2p}}+\varepsilon)^{q'}}\right)^{\frac{1}{q'}},$$

where $\dfrac{1}{q}+\dfrac{1}{q'}=1$. The integral on the right converges if $q'<2$, i.e., $q>2$. Estimating the integral on the right, we arrive at the inequality (15.17).

Similarly, combining (13.9) and representations (15.2), (15.4), we deduce estimate (15.18). □

Next, Theorem 13.4 implies the following statement.

Theorem 15.8. *Suppose that the assumptions of Theorem* 15.7 *are satisfied.*

1°. *For* $0<\tau<T$ *and* $\varepsilon>0$ *we have*

$$\|\mathbf{u}_\varepsilon(\cdot,\tau) - \mathbf{u}_0(\cdot,\tau) - \varepsilon^p\Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon\mathbf{u}_0(\cdot,\tau)\|_{H^p(\mathbb{R}^d)}$$
$$\leq \frac{\widehat{\mathcal{C}}_8(1+\tau^{-\frac{1}{2}})\varepsilon}{\tau^{\frac{1}{2p}}+\varepsilon}\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} + \widehat{\mathcal{C}}_8\varepsilon\left(c(p,q)\tau^{1-\frac{1}{2p}-\frac{1}{q}} + \Theta(q,\varepsilon,\tau)\right)\|\mathbf{F}\|_{L_q((0,T),L_2(\mathbb{R}^d))}, \tag{15.20}$$

where $\Theta(q,\varepsilon,\tau)$ *is given by* (15.19).

2°. *In addition, assume that* $\widehat{G}(\boldsymbol{\theta})\equiv 0$. *Then for* $\tau\in(0,T)$ *and* $\varepsilon>0$ *we have*

$$\|\mathbf{u}_\varepsilon(\cdot,\tau) - \mathbf{u}_0(\cdot,\tau) - \varepsilon^p\Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon\mathbf{u}_0(\cdot,\tau)\|_{H^p(\mathbb{R}^d)}$$
$$\leq \widehat{\mathcal{C}}_9\left(\frac{\varepsilon}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2p}}+\varepsilon)} + \frac{\varepsilon^2}{\tau^{\frac{1}{p}}+\varepsilon^2}\right)\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} + \widehat{\mathcal{C}}_9\left(\varepsilon\Theta(q,\varepsilon,\tau) + c(p,q)\varepsilon^2\tau^{1-\frac{1}{p}-\frac{1}{q}}\right)\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}. \tag{15.21}$$

Proof. To check inequality (15.20), we apply (13.13), (15.2), and (15.4):

$$\|\mathbf{u}_\varepsilon(\cdot,\tau)-\mathbf{u}_0(\cdot,\tau)-\varepsilon^p\Lambda^\varepsilon b(\mathbf{D})(\Pi_\varepsilon\mathbf{u}_0)(\cdot,\tau)\|_{H^p(\mathbb{R}^d)}$$
$$\leq \frac{\widehat{\mathcal{C}}_8(1+\tau^{-\frac{1}{2}})\varepsilon}{\tau^{\frac{1}{2p}}+\varepsilon}\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)}+\widehat{\mathcal{C}}_8\varepsilon\int\limits_0^\tau\frac{(1+(\tau-\widetilde{\tau})^{-\frac{1}{2}})\|\mathbf{F}(\cdot,\widetilde{\tau})\|_{L_2(\mathbb{R}^d)}}{(\tau-\widetilde{\tau})^{\frac{1}{2p}}+\varepsilon}\,d\widetilde{\tau}$$
$$\leq \frac{\widehat{\mathcal{C}}_8(1+\tau^{-\frac{1}{2}})\varepsilon}{\tau^{\frac{1}{2p}}+\varepsilon}\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)}+\widehat{\mathcal{C}}_8\varepsilon\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}\Bigg(\int\limits_0^\tau\frac{d\widetilde{\tau}}{((\tau-\widetilde{\tau})^{\frac{1}{2p}}+\varepsilon)^{q'}}\Bigg)^{\frac{1}{q'}}$$
$$+\widehat{\mathcal{C}}_8\varepsilon\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}\Bigg(\int\limits_0^\tau\frac{d\widetilde{\tau}}{(\tau-\widetilde{\tau})^{\frac{q'}{2}}((\tau-\widetilde{\tau})^{\frac{1}{2p}}+\varepsilon)^{q'}}\Bigg)^{\frac{1}{q'}}.$$

Estimating the integrals on the right, we arrive at the inequality

$$\|\mathbf{u}_\varepsilon(\cdot,\tau)-\mathbf{u}_0(\cdot,\tau)-\varepsilon^p\Lambda^\varepsilon b(\mathbf{D})(\Pi_\varepsilon\mathbf{u}_0)(\cdot,\tau)\|_{H^p(\mathbb{R}^d)}$$
$$\leq \frac{\widehat{\mathcal{C}}_8(1+\tau^{-\frac{1}{2}})\varepsilon}{\tau^{\frac{1}{2p}}+\varepsilon}\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)}+\widehat{\mathcal{C}}_8\varepsilon\left(\theta(q,\varepsilon,\tau)+\Theta(q,\varepsilon,\tau)\right)\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}.$$

Since $\theta(q,\varepsilon,\tau)=c(p,q)\tau^{1-\frac{1}{2p}-\frac{1}{q}}$ for $q>2$, this implies (15.20).

Similarly, applying inequality (13.14) (which holds under the condition $\widehat{G}(\boldsymbol{\theta})\equiv 0$), we obtain

$$\|\mathbf{u}_\varepsilon(\cdot,\tau)-\mathbf{u}_0(\cdot,\tau)-\varepsilon^p\Lambda^\varepsilon b(\mathbf{D})(\Pi_\varepsilon\mathbf{u}_0)(\cdot,\tau)\|_{H^p(\mathbb{R}^d)}$$
$$\leq\widehat{\mathcal{C}}_9\left(\frac{\varepsilon}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2p}}+\varepsilon)}+\frac{\varepsilon^2}{\tau^{\frac{1}{p}}+\varepsilon^2}\right)\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)}+\widehat{\mathcal{C}}_9\varepsilon^2\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}\Bigg(\int\limits_0^\tau\frac{d\widetilde{\tau}}{((\tau-\widetilde{\tau})^{\frac{1}{p}}+\varepsilon^2)^{q'}}\Bigg)^{\frac{1}{q'}}$$
$$+\widehat{\mathcal{C}}_9\varepsilon\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}\Bigg(\int\limits_0^\tau\frac{d\widetilde{\tau}}{(\tau-\widetilde{\tau})^{\frac{q'}{2}}((\tau-\widetilde{\tau})^{\frac{1}{2p}}+\varepsilon)^{q'}}\Bigg)^{\frac{1}{q'}}$$
$$\leq\widehat{\mathcal{C}}_9\left(\frac{\varepsilon}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2p}}+\varepsilon)}+\frac{\varepsilon^2}{\tau^{\frac{1}{p}}+\varepsilon^2}\right)\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)}+\widehat{\mathcal{C}}_9\left(\varepsilon^2\widetilde{\theta}(q,\varepsilon,\tau)+\varepsilon\Theta(q,\varepsilon,\tau)\right)\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}.$$

Since $\widetilde{\theta}(q,\varepsilon,\tau)=c(p,q)\tau^{1-\frac{1}{p}-\frac{1}{q}}$ for $q>2$, this implies (15.21). □

Under the assumptions of Theorems 15.7 and 15.8, in estimates (15.17), (15.18), (15.20), (15.21) for a fixed $\tau\in(0,T)$ the coefficient of $\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)}$ is or order $O(\varepsilon)$. The coefficient of $\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}$ is of order $O(\varepsilon)$ for $\dfrac{2p}{p-1}<q\leq\infty$; $O(\varepsilon|\log\varepsilon|^{\frac{1}{2}+\frac{1}{2p}})$ for $q=\dfrac{2p}{p-1}$; and for $2<q<\dfrac{2p}{p-1}$ its order $O(\varepsilon^{p-\frac{2p}{q}})$ depends on q.

15.4. The problem with the operator $\widehat{\mathcal{A}}_\varepsilon$. Approximation of the solutions in the space $L_q((0,T);\ H^p(\mathbb{R}^d;\mathbb{C}^n))$. Applying Theorem 13.3, it is possible to approximate the solution $\mathbf{u}_\varepsilon$ in the class $L_q((0,T);H^p(\mathbb{R}^d;\mathbb{C}^n))$ in the case where $0<T<\infty$ and $1\leq q<2$.

Theorem 15.9. *Let $0<T<\infty$ and let $\mathbf{F}\in L_q((0,T);L_2(\mathbb{R}^d;\mathbb{C}^n))$ with some $1\leq q<2$. Let $\boldsymbol{\phi}\in L_2(\mathbb{R}^d;\mathbb{C}^n)$. Suppose that $\mathbf{u}_\varepsilon$ is the solution of problem (15.1) and $\mathbf{u}_0$ is the solution of the homogenized problem (15.3). Let $\mathbf{p}_\varepsilon=g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$. Suppose that Λ is the Γ-periodic solution of the problem (7.9) and $\widetilde{g}$ is the matrix-valued function defined by (7.13). Let Π_ε be the operator given* *by* (13.5)*. Then for* $0 < \varepsilon \leq 1$ *we have*

$$\begin{aligned} \|\mathbf{D}^p\left(\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon^p \Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon \mathbf{u}_0\right)\|_{L_q((0,T);L_2(\mathbb{R}^d))} &\leq \widehat{\mathcal{C}}_7 \varepsilon \sigma(q,\varepsilon,T)\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} \\ &+ \widehat{\mathcal{C}}_7 \varepsilon c(p,q) T^{\frac{1}{2}-\frac{1}{2p}} \|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}, \end{aligned} \tag{15.22}$$

$$\begin{aligned} \|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon b(\mathbf{D})\Pi_\varepsilon \mathbf{u}_0\|_{L_q((0,T);L_2(\mathbb{R}^d))} &\leq \widehat{\mathcal{C}}_6 \varepsilon \sigma(q,\varepsilon,T)\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} \\ &+ \widehat{\mathcal{C}}_6 \varepsilon c(p,q) T^{\frac{1}{2}-\frac{1}{2p}} \|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}, \end{aligned} \tag{15.23}$$

where

$$\sigma(q,\varepsilon,T) = \Theta(q',\varepsilon,T) = \begin{cases} c(p,q) T^{\frac{1}{q}-\frac{1}{2}-\frac{1}{2p}}, & 1 \leq q < \dfrac{2p}{p+1}, \\ c(p)\left(\log(T+1) + |\log\varepsilon|\right)^{\frac{1}{2}+\frac{1}{2p}}, & q = \dfrac{2p}{p+1}, \\ c(p,q)\varepsilon^{\frac{2p}{q}-p-1}, & \dfrac{2p}{p+1} < q < 2. \end{cases} \tag{15.24}$$

Proof. By (13.8), (15.2), and (15.4),

$$\|\mathbf{D}^p\left(\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon^p \Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon \mathbf{u}_0\right)\|_{L_q((0,T);L_2(\mathbb{R}^d))} \leq \widehat{\mathcal{C}}_7 \varepsilon \|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} \mathcal{J}_q(\varepsilon;T)^{\frac{1}{q}} + \widehat{\mathcal{C}}_7 \varepsilon \Big(\int_0^T \mathcal{M}(\varepsilon,\tau;\mathbf{F})^q \, d\tau\Big)^{\frac{1}{q}}, \tag{15.25}$$

where

$$\mathcal{J}_q(\varepsilon;T) := \int_0^T \frac{d\tau}{\tau^{\frac{q}{2}}(\tau^{\frac{1}{2p}} + \varepsilon)^q}, \quad \mathcal{M}(\varepsilon,\tau;\mathbf{F}) := \int_0^\tau \frac{\|\mathbf{F}(\cdot,\widetilde{\tau})\|_{L_2(\mathbb{R}^d)} \, d\widetilde{\tau}}{(\tau-\widetilde{\tau})^{\frac{1}{2}}((\tau-\widetilde{\tau})^{\frac{1}{2p}} + \varepsilon)}.$$

Estimating the integral $\mathcal{J}_q(\varepsilon;T)$, we obtain

$$\mathcal{J}_q(\varepsilon;T)^{\frac{1}{q}} \leq \sigma(q,\varepsilon,T). \tag{15.26}$$

Now we estimate the second term in the right-hand side of (15.25). If $q = 1$, changing the order of integration, we have

$$\int_0^T \mathcal{M}(\varepsilon,\tau;\mathbf{F}) \, d\tau = \int_0^T d\tau \int_0^\tau \frac{\|\mathbf{F}(\cdot,\widetilde{\tau})\|_{L_2(\mathbb{R}^d)} \, d\widetilde{\tau}}{(\tau-\widetilde{\tau})^{\frac{1}{2}}((\tau-\widetilde{\tau})^{\frac{1}{2p}} + \varepsilon)} = \int_0^T d\widetilde{\tau} \|\mathbf{F}(\cdot,\widetilde{\tau})\|_{L_2(\mathbb{R}^d)} \int_{\widetilde{\tau}}^T \frac{d\tau}{(\tau-\widetilde{\tau})^{\frac{1}{2}}((\tau-\widetilde{\tau})^{\frac{1}{2p}} + \varepsilon)}.$$

Estimating the internal integral by $\left(\dfrac{1}{2} - \dfrac{1}{2p}\right)^{-1} T^{\frac{1}{2}-\frac{1}{2p}}$, we obtain

$$\int_0^T \mathcal{M}(\varepsilon,\tau;\mathbf{F}) \, d\tau \leq c(p) T^{\frac{1}{2}-\frac{1}{2p}} \|\mathbf{F}\|_{L_1((0,T);L_2(\mathbb{R}^d))}.$$

In the case where $1 < q < \infty$, we apply the Hölder inequality:

$$\mathcal{M}(\varepsilon,\tau;\mathbf{F}) \leq \left(\int_0^\tau \frac{d\widetilde{\tau}}{(\tau-\widetilde{\tau})^{\frac{1}{2}}((\tau-\widetilde{\tau})^{\frac{1}{2p}} + \varepsilon)}\right)^{\frac{1}{q'}} \left(\int_0^\tau \frac{\|\mathbf{F}(\cdot,\widetilde{\tau})\|^q_{L_2(\mathbb{R}^d)} \, d\widetilde{\tau}}{(\tau-\widetilde{\tau})^{\frac{1}{2}}((\tau-\widetilde{\tau})^{\frac{1}{2p}} + \varepsilon)}\right)^{\frac{1}{q}}. \tag{15.27}$$

The integral in the first parenthesis on the right does not exceed $\left(\frac{1}{2}-\frac{1}{2p}\right)^{-1} T^{\frac{1}{2}-\frac{1}{2p}}$. Using (15.27) and changing the order of integration, we arrive at the inequality

$$\begin{aligned}\int_0^T \mathcal{M}(\varepsilon,\tau;\mathbf{F})^q\,d\tau &\le c(p,q)T^{(\frac{1}{2}-\frac{1}{2p})\frac{q}{q'}} \int_0^T d\tau \int_0^\tau \frac{\|\mathbf{F}(\cdot,\widetilde{\tau})\|^q_{L_2(\mathbb{R}^d)}\,d\widetilde{\tau}}{(\tau-\widetilde{\tau})^{\frac{1}{2}}((\tau-\widetilde{\tau})^{\frac{1}{2p}}+\varepsilon)} \\ &= c(p,q)T^{(\frac{1}{2}-\frac{1}{2p})\frac{q}{q'}} \int_0^T d\widetilde{\tau}\|\mathbf{F}(\cdot,\widetilde{\tau})\|^q_{L_2(\mathbb{R}^d)} \int_{\widetilde{\tau}}^T \frac{d\widetilde{\tau}}{(\tau-\widetilde{\tau})^{\frac{1}{2}}((\tau-\widetilde{\tau})^{\frac{1}{2p}}+\varepsilon)} \\ &\le \tilde{c}(p,q)T^{(\frac{1}{2}-\frac{1}{2p})(\frac{q}{q'}+1)}\|\mathbf{F}\|^q_{L_q((0,T);L_2(\mathbb{R}^d))}.\end{aligned}$$

As a result, for all $1 \le q < \infty$ we obtain

$$\Big(\int_0^T \mathcal{M}(\varepsilon,\tau;\mathbf{F})^q\,d\tau\Big)^{\frac{1}{q}} \le c(p,q)T^{\frac{1}{2}-\frac{1}{2p}}\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}. \tag{15.28}$$

Now, relations (15.25), (15.26), and (15.28) imply the required estimate (15.22).

Inequality (15.23) is checked similarly with the help of (13.9). □

Similarly, Theorem 13.4 implies the following statement.

Theorem 15.10. *Suppose that the assumptions of Theorem* 15.9 *are satisfied.*

1°. *For* $0 < \varepsilon \le 1$ *we have*

$$\begin{aligned}\|\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon^p\Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon \mathbf{u}_0\|_{L_q((0,T);H^p(\mathbb{R}^d))} &\le \widehat{\mathcal{C}}_8\varepsilon\left(c(p,q)T^{\frac{1}{q}-\frac{1}{2p}}+\sigma(q,\varepsilon,T)\right)\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} \\ &+\widehat{\mathcal{C}}_8\varepsilon\left(T^{1-\frac{1}{2p}}+T^{\frac{1}{2}-\frac{1}{2p}}\right)\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}.\end{aligned} \tag{15.29}$$

Here $\sigma(q,\varepsilon,T)$ *is defined by* (15.24).

2°. *In addition, suppose that* $\widehat{G}(\boldsymbol{\theta}) \equiv 0$. *Then for* $0 < \varepsilon \le 1$ *we have*

$$\begin{aligned}\|\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon^p\Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon \mathbf{u}_0\|_{L_q((0,T);H^p(\mathbb{R}^d))} &\le \widehat{\mathcal{C}}_9\left(c(p,q)\varepsilon^2T^{\frac{1}{q}-\frac{1}{p}}+\varepsilon\sigma(q,\varepsilon,T)\right)\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} \\ &+\widehat{\mathcal{C}}_9 c(p,q)\left(\varepsilon T^{\frac{1}{2}-\frac{1}{2p}}+\varepsilon^2T^{1-\frac{1}{p}}\right)\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}.\end{aligned} \tag{15.30}$$

Proof. By (13.13), (15.2), and (15.4),

$$\begin{aligned}&\|\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon^p\Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon \mathbf{u}_0\|_{L_q((0,T);H^p(\mathbb{R}^d))} \le \widehat{\mathcal{C}}_8\varepsilon\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)}\left(\mathcal{I}_q(\varepsilon;T)^{\frac{1}{q}}+\mathcal{J}_q(\varepsilon;T)^{\frac{1}{q}}\right) \\ &+\widehat{\mathcal{C}}_8\varepsilon\Big(\int_0^T \mathcal{L}(\varepsilon,\tau;\mathbf{F})^q\,d\tau\Big)^{\frac{1}{q}}+\widehat{\mathcal{C}}_8\varepsilon\Big(\int_0^T \mathcal{M}(\varepsilon,\tau;\mathbf{F})^q\,d\tau\Big)^{\frac{1}{q}} \\ &\le \widehat{\mathcal{C}}_8\varepsilon\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)}\left(\rho(q,\varepsilon,T)+\sigma(q,\varepsilon,T)\right)+\widehat{\mathcal{C}}_8\varepsilon\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}c(p,q)\left(T^{1-\frac{1}{2p}}+T^{\frac{1}{2}-\frac{1}{2p}}\right).\end{aligned}$$

In the last passage, we have used (15.12), (15.14), (15.26), and (15.28). Taking into account that $\rho(q,\varepsilon,T) = c(p,q)T^{\frac{1}{q}-\frac{1}{2p}}$ for $1 \le q < 2$, we arrive at inequality (15.29).

Similarly, under condition that $\widehat{G}(\boldsymbol{\theta}) \equiv 0$, we check estimate (15.30) with the help of (13.14). □

In estimates (15.22), (15.23), (15.29), and (15.30) the coefficient of $\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}$ is $O(\varepsilon)$, and the coefficient of $\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)}$ is of order $O(\varepsilon)$ for $1 \le q < \dfrac{2p}{p+1}$; $O(\varepsilon|\log\varepsilon|^{\frac{1}{2}+\frac{1}{2p}})$ for $q = \dfrac{2p}{p+1}$; and for $\dfrac{2p}{p+1} < q < 2$ its order $O(\varepsilon^{\frac{2p}{q}-p})$ depends on q.

15.5. More general Cauchy problem. Approximation of the solutions in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. Now, we consider a more general Cauchy problem. Let $Q(\mathbf{x})$ be a measurable Γ-periodic matrix-valued function in $\mathbb{R}^d$ of size $n \times n$. It is assumed that $Q(\mathbf{x})$ is uniformly positive definite and bounded:

$$\mathfrak{c}'\mathbf{1}_n \le Q(\mathbf{x}) \le \mathfrak{c}''\mathbf{1}_n, \quad \mathbf{x} \in \mathbb{R}^d, \quad 0 < \mathfrak{c}' \le \mathfrak{c}'' < \infty.$$

Let $0 < T \le \infty$. We study the behavior of the solution $\mathbf{v}_\varepsilon(\mathbf{x},\tau)$ of the following problem:

$$\begin{aligned} Q^\varepsilon(\mathbf{x})\frac{\partial \mathbf{v}_\varepsilon(\mathbf{x},\tau)}{\partial\tau} &= -b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})\mathbf{v}_\varepsilon(\mathbf{x},\tau) + \mathbf{F}(\mathbf{x},\tau), \quad \mathbf{x} \in \mathbb{R}^d,\ \tau \in (0,T), \\ Q^\varepsilon(\mathbf{x})\mathbf{v}_\varepsilon(\mathbf{x},0) &= \boldsymbol{\phi}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d, \end{aligned} \tag{15.31}$$

where $\boldsymbol{\phi} \in L_2(\mathbb{R}^d;\mathbb{C}^n)$ and $\mathbf{F} \in L_q((0,T);L_2(\mathbb{R}^d;\mathbb{C}^n))$ with some $1 < q \le \infty$.

We represent the matrix-valued function $Q(\mathbf{x})^{-1}$ in a factorized form: $Q(\mathbf{x})^{-1} = f(\mathbf{x})f^*(\mathbf{x})$, where $f(\mathbf{x})$ is a Γ-periodic matrix-valued function of size $n \times n$ such that $f, f^{-1} \in L_\infty(\mathbb{R}^d)$. (For instance, one can take $f(\mathbf{x}) = Q(\mathbf{x})^{-1/2}$.)

We substitute $\mathbf{w}_\varepsilon = (f^\varepsilon)^{-1}\mathbf{v}_\varepsilon$. Then $\mathbf{w}_\varepsilon$ is the solution of the following problem:

$$\begin{aligned} \frac{\partial \mathbf{w}_\varepsilon(\mathbf{x},\tau)}{\partial\tau} &= -(f^\varepsilon(\mathbf{x}))^* b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) f^\varepsilon(\mathbf{x})\mathbf{w}_\varepsilon(\mathbf{x},\tau) + (f^\varepsilon(\mathbf{x}))^*\mathbf{F}(\mathbf{x},\tau), \quad \mathbf{x} \in \mathbb{R}^d,\ \tau \in (0,T), \\ \mathbf{w}_\varepsilon(\mathbf{x},0) &= (f^\varepsilon(\mathbf{x}))^*\boldsymbol{\phi}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d. \end{aligned}$$

Let $\mathcal{A}_\varepsilon = (f^\varepsilon(\mathbf{x}))^* b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) f^\varepsilon(\mathbf{x})$. Then

$$\mathbf{w}_\varepsilon(\cdot,\tau) = e^{-\mathcal{A}_\varepsilon\tau}(f^\varepsilon)^*\boldsymbol{\phi} + \int_0^\tau e^{-\mathcal{A}_\varepsilon(\tau-\widetilde{\tau})}(f^\varepsilon)^*\mathbf{F}(\cdot,\widetilde{\tau})\,d\widetilde{\tau}.$$

Hence,

$$\mathbf{v}_\varepsilon(\cdot,\tau) = f^\varepsilon e^{-\mathcal{A}_\varepsilon\tau}(f^\varepsilon)^*\boldsymbol{\phi} + \int_0^\tau f^\varepsilon e^{-\mathcal{A}_\varepsilon(\tau-\widetilde{\tau})}(f^\varepsilon)^*\mathbf{F}(\cdot,\widetilde{\tau})\,d\widetilde{\tau}. \tag{15.32}$$

Let f_0 be given by (9.1) and let $\mathcal{A}^0 = f_0 b(\mathbf{D})^* g^0 b(\mathbf{D}) f_0$. Suppose that $\mathbf{v}_0$ is the solution of the "homogenized" problem

$$\begin{aligned} \overline{Q}\frac{\partial \mathbf{v}_0(\mathbf{x},\tau)}{\partial\tau} &= -b(\mathbf{D})^* g^0 b(\mathbf{D})\mathbf{v}_0(\mathbf{x},\tau) + \mathbf{F}(\mathbf{x},\tau), \quad \mathbf{x} \in \mathbb{R}^d,\ \tau \in (0,T), \\ \overline{Q}\mathbf{v}_0(\mathbf{x},0) &= \boldsymbol{\phi}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d. \end{aligned} \tag{15.33}$$

Similarly to (15.32),

$$\mathbf{v}_0(\cdot,\tau) = f_0 e^{-\mathcal{A}^0\tau} f_0\boldsymbol{\phi} + \int_0^\tau f_0 e^{-\mathcal{A}^0(\tau-\widetilde{\tau})} f_0\mathbf{F}(\cdot,\widetilde{\tau})\,d\widetilde{\tau}. \tag{15.34}$$

Applying Theorems 14.1, 14.2 and using representations (15.32), (15.34), we arrive at the following results.

Theorem 15.11. *Let $0 < T \le \infty$ and let $\mathbf{F} \in L_q((0,T);L_2(\mathbb{R}^d;\mathbb{C}^n))$ with some $1 < q \le \infty$. Let $\boldsymbol{\phi} \in L_2(\mathbb{R}^d;\mathbb{C}^n)$. Suppose that $\mathbf{v}_\varepsilon$ is the solution of problem* (15.31) *and $\mathbf{v}_0$ is the solution of the* *homogenized problem* (15.33). *Then for any* $\tau \in (0,T)$ *and* $\varepsilon \to 0$ *the solution* $\mathbf{v}_\varepsilon(\cdot,\tau)$ *converges to* $\mathbf{v}_0(\cdot,\tau)$ *in the* L_2*-norm. For* $\tau \in (0,T)$ *and* $0 < \varepsilon \le 1$ *we have*

$$\|\mathbf{v}_\varepsilon(\cdot,\tau) - \mathbf{v}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)} \le \mathcal{C}_1\varepsilon\left((\tau^{\frac{1}{2p}} + \varepsilon)^{-1}\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} + \theta(q,\varepsilon,\tau)\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}\right),$$

where $\theta(q,\varepsilon,\tau)$ *is defined by* (15.6).

Theorem 15.12. *Suppose that the assumptions of Theorem* 15.11 *are satisfied. Let* $\widehat{G}(\boldsymbol{\theta})$ *be the operator defined according to* (7.25), (7.26). *Suppose that* $\widehat{G}(\boldsymbol{\theta}) = 0$ *for all* $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. *Then for* $\tau \in (0,T)$ *and* $0 < \varepsilon \le 1$ *we have*

$$\|\mathbf{v}_\varepsilon(\cdot,\tau) - \mathbf{v}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)} \le \mathcal{C}_2\varepsilon^2\left((\tau^{\frac{1}{p}} + \varepsilon^2)^{-1}\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} + \widetilde{\theta}(q,\varepsilon,\tau)\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}\right),$$

where $\widetilde{\theta}(q,\varepsilon,\tau)$ *is defined by* (15.8).

15.6. More general Cauchy problem. Approximation of solutions in $L_q((0,T);L_2(\mathbb{R}^d;\mathbb{C}^n))$. By analogy with the proof of Theorem 15.3, we deduce the following statement from Theorem 14.1.

Theorem 15.13. *Let* $0 < T < \infty$ *and let* $\mathbf{F} \in L_q((0,T);L_2(\mathbb{R}^d;\mathbb{C}^n))$ *with some* $1 \le q < \infty$. *Let* $\boldsymbol{\phi} \in L_2(\mathbb{R}^d;\mathbb{C}^n)$. *Suppose that* $\mathbf{v}_\varepsilon$ *is the solution of problem* (15.31) *and* $\mathbf{v}_0$ *is the solution of the homogenized problem* (15.33). *Then for* $0 < \varepsilon \le 1$ *we have*

$$\|\mathbf{v}_\varepsilon - \mathbf{v}_0\|_{L_q((0,T);L_2(\mathbb{R}^d))} \le \mathcal{C}_1\varepsilon\left(\rho(q,\varepsilon,T)\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} + c(p,q)T^{1-\frac{1}{2p}}\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}\right),$$

where $\rho(q,\varepsilon,T)$ *is defined by* (15.10).

Similarly, applying Theorem 14.2, we obtain the following result.

Theorem 15.14. *Suppose that the assumptions of Theorem* 15.13 *are satisfied. Let* $\widehat{G}(\boldsymbol{\theta})$ *be the operator defined according to* (7.25), (7.26). *Assume that* $\widehat{G}(\boldsymbol{\theta}) = 0$ *for any* $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. *Then for* $0 < \varepsilon \le 1$ *we have*

$$\|\mathbf{v}_\varepsilon - \mathbf{v}_0\|_{L_q((0,T);L_2(\mathbb{R}^d))} \le \mathcal{C}_2\varepsilon^2\left(\widetilde{\rho}(q,\varepsilon,T)\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} + c(p,q)T^{1-\frac{1}{p}}\|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}\right),$$

where $\widetilde{\rho}(q,\varepsilon,T)$ *is defined by* (15.16).

15.7. More general Cauchy problem. Approximation of the solutions in $H^p(\mathbb{R}^d;\mathbb{C}^n)$. Now, we obtain approximation of the solution $\mathbf{v}_\varepsilon$ of problem (15.31) in the norm of $H^p(\mathbb{R}^d;\mathbb{C}^n)$ and also approximation of the "flux" $\mathbf{q}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon$ in the norm of $L_2(\mathbb{R}^d;\mathbb{C}^m)$. In the case where $\mathbf{F} = 0$, Theorem 14.5 directly implies the following result.

Theorem 15.15. *Let* $\mathbf{F} = 0$ *and let* $\boldsymbol{\phi} \in L_2(\mathbb{R}^d;\mathbb{C}^n)$. *Suppose that* $\mathbf{v}_\varepsilon$ *is the solution of problem* (15.31) *and* $\mathbf{v}_0$ *is the solution of the homogenized problem* (15.33). *Let* $\mathbf{q}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon$. *Suppose that* Λ *is the* Γ*-periodic solution of problem* (7.9) *and* $\widetilde{g}$ *is the matrix-valued function defined by* (7.13). *Then for* $\tau \in (0,T)$ *and* $0 < \varepsilon \le \tau^{\frac{1}{2p}}$ *we have*

$$\|\mathbf{D}^p\left(\mathbf{v}_\varepsilon(\cdot,\tau) - \mathbf{v}_0(\cdot,\tau) - \varepsilon^p\Lambda^\varepsilon b(\mathbf{D})\mathbf{v}_0(\cdot,\tau)\right)\|_{L_2(\mathbb{R}^d)} \le \mathcal{C}_7^\circ\varepsilon\tau^{-\frac{1}{2}-\frac{1}{2p}}\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)},$$

$$\|\mathbf{q}_\varepsilon(\cdot,\tau) - \widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{v}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)} \le \mathcal{C}_6^\circ\varepsilon\tau^{-\frac{1}{2}-\frac{1}{2p}}\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)}.$$

Applying Theorem 14.6, we obtain the following statement.

Theorem 15.16. *Suppose that the assumptions of Theorems* 15.15 *are satisfied.*

1°. *For* $\tau \in (0,T)$ *and* $0 < \varepsilon \le \tau^{\frac{1}{2p}}$ *we have*

$$\|\mathbf{v}_\varepsilon(\cdot,\tau) - \mathbf{v}_0(\cdot,\tau) - \varepsilon^p\Lambda^\varepsilon b(\mathbf{D})\mathbf{v}_0(\cdot,\tau)\|_{H^p(\mathbb{R}^d)} \le \mathcal{C}_8^\circ\varepsilon\tau^{-\frac{1}{2p}}\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)}.$$

2°. *In addition, assume that* $\widehat{G}(\boldsymbol{\theta}) \equiv 0$. *Then for* $\tau \in (0,T)$ *and* $0 < \varepsilon \le \tau^{\frac{1}{2p}}$ *we have*

$$\|\mathbf{v}_\varepsilon(\cdot,\tau) - \mathbf{v}_0(\cdot,\tau) - \varepsilon^p \Lambda^\varepsilon b(\mathbf{D})\mathbf{v}_0(\cdot,\tau)\|_{H^p(\mathbb{R}^d)} \le \mathcal{C}_9^\circ \left(\varepsilon \tau^{-\frac{1}{2}-\frac{1}{2p}} + \varepsilon^2 \tau^{-\frac{1}{p}}\right) \|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)}.$$

If $\mathbf{F} \neq 0$, then we need approximations of the sandwiched exponential $f^\varepsilon e^{-\mathcal{A}_\varepsilon \tau}(f^\varepsilon)^*$ for all values of $0 < \tau < T$ (not only for $\tau \ge \varepsilon^{2p}$), and therefore we rely on Theorem 14.3.

Theorem 15.17. *Let* $\mathbf{F} \in L_q((0,T); L_2(\mathbb{R}^d;\mathbb{C}^n))$ *with some* $2 < q \le \infty$. *Let* $\boldsymbol{\phi} \in L_2(\mathbb{R}^d;\mathbb{C}^n)$. *Suppose that* $\mathbf{v}_\varepsilon$ *is the solution of problem* (15.31) *and* $\mathbf{v}_0$ *is the solution of the homogenized problem* (15.33). *Let* $\mathbf{q}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon$. *Suppose that* Λ *is the* Γ*-periodic solution of problem* (7.9) *and* $\widetilde{g}$ *is the matrix-valued function defined by* (7.13). *Let* Π_ε *be the operator defined by* (13.5). *Then for* $\tau \in (0,T)$ *and* $\varepsilon > 0$ *we have*

$$\begin{aligned}
&\|\mathbf{D}^p(\mathbf{v}_\varepsilon(\cdot,\tau) - \mathbf{v}_0(\cdot,\tau) - \varepsilon^p \Lambda^\varepsilon b(\mathbf{D})(\Pi_\varepsilon \mathbf{v}_0)(\cdot,\tau))\|_{L_2(\mathbb{R}^d)} \\
&\le \frac{\mathcal{C}_7 \varepsilon}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2p}} + \varepsilon)} \|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} + \mathcal{C}_7\, \varepsilon\, \Theta(q,\varepsilon,\tau) \|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}, \\
&\|\mathbf{q}_\varepsilon(\cdot,\tau) - \widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)} \\
&\le \frac{\mathcal{C}_6 \varepsilon}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2p}} + \varepsilon)} \|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} + \mathcal{C}_6\, \varepsilon\, \Theta(q,\varepsilon,\tau) \|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}.
\end{aligned}$$

Here $\Theta(q,\varepsilon,\tau)$ *is given by* (15.19).

Finally, applying Theorem 14.4, we obtain the following result.

Theorem 15.18. *Suppose that the assumptions of Theorem* 15.17 *are satisfied.*

1°. *For* $\tau \in (0,T)$ *and* $\varepsilon > 0$ *we have*

$$\begin{aligned}
&\|\mathbf{v}_\varepsilon(\cdot,\tau) - \mathbf{v}_0(\cdot,\tau) - \varepsilon^p \Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon \mathbf{u}_0(\cdot,\tau)\|_{H^p(\mathbb{R}^d)} \\
&\le \frac{\mathcal{C}_8(1+\tau^{-\frac{1}{2}})\varepsilon}{\tau^{\frac{1}{2p}} + \varepsilon} \|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} + \mathcal{C}_8 \varepsilon \left(c(p,q)\tau^{1-\frac{1}{2p}-\frac{1}{q}} + \Theta(q,\varepsilon,\tau)\right) \|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}.
\end{aligned}$$

2°. *In addition, assume that* $\widehat{G}(\boldsymbol{\theta}) \equiv 0$. *Then for* $\tau \in (0,T)$ *and* $\varepsilon > 0$ *we have*

$$\begin{aligned}
&\|\mathbf{v}_\varepsilon(\cdot,\tau) - \mathbf{v}_0(\cdot,\tau) - \varepsilon^p \Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon \mathbf{u}_0(\cdot,\tau)\|_{H^p(\mathbb{R}^d)} \\
&\le \mathcal{C}_9 \left(\frac{\varepsilon}{\tau^{\frac{1}{2}}(\tau^{\frac{1}{2p}} + \varepsilon)} + \frac{\varepsilon^2}{\tau^{\frac{1}{p}} + \varepsilon^2}\right) \|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} + \mathcal{C}_9 \left(\varepsilon \Theta(q,\varepsilon,\tau) + c(p,q)\varepsilon^2 \tau^{1-\frac{1}{p}-\frac{1}{q}}\right) \|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}.
\end{aligned}$$

15.8. More general Cauchy problem. Approximation of solutions in $L_q((0,T); H^p(\mathbb{R}^d;\mathbb{C}^n))$. By analogy with the proof of Theorem 15.9, applying Theorem 14.3, we deduce the following result.

Theorem 15.19. *Let* $0 < T < \infty$ *and let* $\mathbf{F} \in L_q((0,T); L_2(\mathbb{R}^d;\mathbb{C}^n))$ *with some* $1 \le q < 2$. *Let* $\boldsymbol{\phi} \in L_2(\mathbb{R}^d;\mathbb{C}^n)$. *Suppose that* $\mathbf{v}_\varepsilon$ *is the solution of problem* (15.31) *and* $\mathbf{v}_0$ *is the solution of the homogenized problem* (15.33). *Let* $\mathbf{q}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon$. *Suppose that* Λ *is the* Γ*-periodic solution of problem* (7.9) *and* $\widetilde{g}$ *is the matrix-valued function defined by* (7.13). *Let* Π_ε *be the operator defined by* (13.5). *Then for* $0 < \varepsilon \le 1$ *we have*

$$\begin{aligned}
\|\mathbf{D}^p(\mathbf{v}_\varepsilon - \mathbf{v}_0 - \varepsilon^p \Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon \mathbf{v}_0)\|_{L_q((0,T);L_2(\mathbb{R}^d))} &\le \mathcal{C}_7 \varepsilon \sigma(q,\varepsilon,T) \|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} \\
&\quad + \mathcal{C}_7 \varepsilon c(p,q) T^{\frac{1}{2}-\frac{1}{2p}} \|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}, \\
\|\mathbf{q}_\varepsilon - \widetilde{g}^\varepsilon b(\mathbf{D})\Pi_\varepsilon \mathbf{v}_0\|_{L_q((0,T);L_2(\mathbb{R}^d))} &\le \mathcal{C}_6 \varepsilon \sigma(q,\varepsilon,T) \|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} \\
&\quad + \mathcal{C}_6 \varepsilon c(p,q) T^{\frac{1}{2}-\frac{1}{2p}} \|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))},
\end{aligned}$$

where $\sigma(q,\varepsilon,T)$ *is given by* (15.24).

Similarly, Theorem 14.4 implies the following statement.

Theorem 15.20. *Suppose that the assumptions of Theorem* 15.19 *are satisfied.*

1°. *For* $0 < \varepsilon \le 1$ *we have*

$$\|\mathbf{v}_\varepsilon - \mathbf{v}_0 - \varepsilon^p \Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon \mathbf{v}_0\|_{L_q((0,T);H^p(\mathbb{R}^d))} \le \mathcal{C}_8 \varepsilon \left(c(p,q) T^{\frac{1}{q}-\frac{1}{2p}} + \sigma(q,\varepsilon,T) \right) \|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} + \mathcal{C}_8 \varepsilon \left(T^{1-\frac{1}{2p}} + T^{\frac{1}{2}-\frac{1}{2p}} \right) \|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}.$$

Here $\sigma(q,\varepsilon,T)$ *is defined by* (15.24).

2°. *In addition, assume that* $\widehat{G}(\boldsymbol{\theta}) \equiv 0$. *Then for* $0 < \varepsilon \le 1$ *we have*

$$\|\mathbf{v}_\varepsilon - \mathbf{v}_0 - \varepsilon^p \Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon \mathbf{v}_0\|_{L_q((0,T);H^p(\mathbb{R}^d))} \le \mathcal{C}_9 \left(c(p,q) \varepsilon^2 T^{\frac{1}{q}-\frac{1}{p}} + \varepsilon \sigma(q,\varepsilon,T) \right) \|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} + \mathcal{C}_9 c(p,q) \left(\varepsilon T^{\frac{1}{2}-\frac{1}{2p}} + \varepsilon^2 T^{1-\frac{1}{p}} \right) \|\mathbf{F}\|_{L_q((0,T);L_2(\mathbb{R}^d))}.$$

16. Appendix. Another Way to Get Results on Homogenization of the Operator Exponential

The results on approximation of the operator exponential can be derived by the method of integrating the resolvent along a suitable contour in the complex plane. This method was applied by Meshkova in [9], where second-order operators A_ε were considered. To implement such an approach, one needs to have "ready" results on approximation of the resolvent $(A_\varepsilon - \zeta I)^{-1}$ at an arbitrary regular point $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$ with two-parametric error estimates (depending on ε and ζ). For the second-order operators, such estimates were obtained in [21].

For higher-order operators of form $\widehat{\mathcal{A}}_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})$, the required approximations for the resolvent with two-parametric error estimates were obtained in [8]. On their basis, another proof of Theorems 13.1 and 13.3 can be given.

Let $\zeta = |\zeta| e^{i\varphi} \in \mathbb{C} \setminus \mathbb{R}_+$. We put $c(\varphi) := \dfrac{1}{\sin \varphi}$ for $\varphi \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$ and $c(\varphi) := 1$ for $\varphi \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. In [8, Sec. 8], it was proved that

$$\left\| (\widehat{\mathcal{A}}_\varepsilon - \zeta I)^{-1} - (\widehat{\mathcal{A}}^0 - \zeta I)^{-1} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \le C' c(\varphi)^2 \varepsilon |\zeta|^{\frac{1}{2p}-1}, \tag{16.1}$$

$$\left\| \mathbf{D}^p \left((\widehat{\mathcal{A}}_\varepsilon - \zeta I)^{-1} - (\widehat{\mathcal{A}}^0 - \zeta I)^{-1} - \varepsilon^p \Lambda^\varepsilon b(\mathbf{D}) (\widehat{\mathcal{A}}^0 - \zeta I)^{-1} \Pi_\varepsilon \right) \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \le C'' c(\varphi)^2 \varepsilon |\zeta|^{\frac{1}{2p}-\frac{1}{2}}. \tag{16.2}$$

We use representation for the operator exponential $e^{-\tau \widehat{\mathcal{A}}_\varepsilon}$, $\tau > 0$, via the integral of the resolvent along the contour enclosing the spectrum of the operator $\widehat{\mathcal{A}}_\varepsilon$ (see [7]):

$$e^{-\tau \widehat{\mathcal{A}}_\varepsilon} = -\frac{1}{2\pi i} \int_{\gamma_\tau} e^{-\zeta \tau} (\widehat{\mathcal{A}}_\varepsilon - \zeta I)^{-1} \, d\zeta. \tag{16.3}$$

We choose the contor γ_τ depending on the parameter τ in the same way as in [9]: $\gamma_\tau = \widetilde{\gamma}_\tau \cup \widehat{\gamma}_\tau$, where

$$\widetilde{\gamma}_\tau := \left\{ \zeta \in \mathbb{C} : \zeta = \rho e^{i\frac{\pi}{4}} : \rho \in [\tau, \infty) \right\} \cup \left\{ \zeta \in \mathbb{C} : \zeta = \rho e^{i\frac{7\pi}{4}} : \rho \in [\tau, \infty) \right\},$$

$$\widehat{\gamma}_\tau := \left\{ \zeta \in \mathbb{C} : \zeta = \tau e^{i\varphi} : \frac{\pi}{4} \le \varphi \le \frac{7\pi}{4} \right\}.$$

Using representation (16.3) for $e^{-\tau\widehat{\mathcal{A}}_\varepsilon}$ and a similar representation for $e^{-\tau\widehat{\mathcal{A}}^0}$, we have

$$\begin{aligned} e^{-\tau\widehat{\mathcal{A}}_\varepsilon} - e^{-\tau\widehat{\mathcal{A}}^0} &= -\frac{1}{2\pi i}\int\limits_{\gamma_\tau} e^{-\zeta\tau}\Big((\widehat{\mathcal{A}}_\varepsilon - \zeta I)^{-1} - (\widehat{\mathcal{A}}^0 - \zeta I)^{-1}\Big)\,d\zeta \\ &= -\frac{1}{2\pi i\tau}\int\limits_{\gamma_1} e^{-\eta}\left(\Big(\widehat{\mathcal{A}}_\varepsilon - \frac{\eta}{\tau}I\Big)^{-1} - \Big(\widehat{\mathcal{A}}^0 - \frac{\eta}{\tau}I\Big)^{-1}\right)d\eta. \end{aligned} \tag{16.4}$$

The second equality is obtained by changing the variable $\eta = \zeta\tau$.

Applying representation (16.4) and estimate (16.1), we arrive at the inequality

$$\left\| e^{-\tau\widehat{\mathcal{A}}_\varepsilon} - e^{-\tau\widehat{\mathcal{A}}^0}\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \frac{C'\varepsilon}{\pi\tau}\int\limits_{\gamma_1} e^{-\operatorname{Re}\eta}\big(\tau^{-1}|\eta|\big)^{\frac{1}{2p}-1}\,|d\eta|.$$

We took into account that $c(\varphi) \le \sqrt{2}$ for the points of the contour γ_τ. The integral is understood here as an integral over the arc length. Estimating this integral, we arrive at the inequality

$$\left\| e^{-\tau\widehat{\mathcal{A}}_\varepsilon} - e^{-\tau\widehat{\mathcal{A}}^0}\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \frac{\widetilde{C}'\varepsilon}{\tau^{\frac{1}{2p}}}. \tag{16.5}$$

Applying (16.5) for $\tau \ge 1$ and estimating the left-hand side by 2 for $0 < \tau < 1$, we obtain an inequality of form (13.2). Thus, we have proved Theorem 13.1 in a different way.

Similarly, using (16.2) and representation (16.3), it is easy to get the estimate

$$\left\| \mathbf{D}^p\left(e^{-\tau\widehat{\mathcal{A}}_\varepsilon} - e^{-\tau\widehat{\mathcal{A}}^0} - \varepsilon^p\widehat{\mathcal{K}}(\varepsilon,\tau)\right)\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le \frac{\widetilde{C}''\varepsilon}{\tau^{\frac{1}{2}+\frac{1}{2p}}}.$$

From here it is easy to pass to inequality (13.8).

Thus, general results on the behavior of the exponential $e^{-\tau\widehat{\mathcal{A}}_\varepsilon}$ can be deduced from the results of [8] about approximations of the resolvent. However, we could not use the same way to get other results of the paper, since we have no "ready" results on the behavior of the resolvent in the case of improved results under the additional condition (that $\widehat{G}(\boldsymbol{\theta}) \equiv 0$), nor in the case of a more general operator of the form $\mathcal{A}_\varepsilon = (f^\varepsilon)^* b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) f^\varepsilon$.

Therefore, we preferred to carry out independent considerations for the operator exponential in the spirit of the operator-theoretic approach.

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