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A NEW FORMULA FOR THE NUMBER OF LABELED SERIES-PARALLEL GRAPHS

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Abstract. A series-parallel graph is a graph that does not contain a complete graph with four vertices as a minor. A new explicit simpler formula for the number of labeled series-parallel biconnected graphs with a given number of vertices is obtained.

***Keywords and phrases*:** enumeration, labeled graph, series-parallel graph, 2-connected graph, explicit formula.

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Definition 1. A graph is said to be *series-parallel* if it does not contain a subdivision of the complete graph K_4 (see [1]).

Series-parallel graphs are used for constructing robust communication networks (see [3]).

In [1], an asymptotic formula for the number of labeled connected and double connected series-parallel graphs with a large number of vertices was found. In [4], labeled series-parallel connected and biconnected graphs were listed in correspondence with the number if vertices. The numbers of labeled series-parallel tricyclic and tetracyclic blocks with a given number of vertices were found in [7] and [5], respectively. In [6], an explicit formula for the number of labeled series-parallel k-cyclic blocks with a given number of vertices was obtained.

Let B_n be the number of labeled biconnected series-parallel graphs with n vertices. In [4], the following formula was obtained:

$$B_n = (n-1)! \sum_{i=0}^{n-2} \sum_{j=0}^{n+i+1} \frac{(-2)^j}{i!} \Bigg[(j+1)^i \left(\binom{n-1}{i+2}\binom{n+i}{j} - 2\binom{n}{i+2}\binom{n+i-1}{j} \right) + j^i \left(\frac{1}{2}\binom{n-3}{i}\binom{n+i+1}{j} - \binom{n-2}{i}\binom{n+i}{j} \right) \Bigg]. \quad (1)$$

In this paper, we obtain a simpler explicit formula for the number of labeled series-parallel biconnected graphs with a given number of vertices.

Theorem 1. *The number B_n of labeled series-parallel biconnected graphs with n vertices is equal to*

$$B_n = (n-1)! \sum_{i=0}^{n-3} \sum_{j=0}^{n+i} \frac{(-2)^{j-1}}{(i+1)!} \binom{n-3}{i} \left[\binom{n+i}{j} j^i + 2i \binom{n+i-1}{j} (j+1)^{i-1} \right], \quad n \geq 3;$$

here we assume that $\binom{n}{m} = 0$ *for* $0 \leq n < m$ *and* $0^0 = 1$.

Proof. Let $b_{n,m}$ be the number of labeled series-parallel biconnected graphs with n vertices and m edges and

$$B(x,y) = \sum_{n=2}^{\infty} \sum_{m=1}^{\infty} b_{n,m} y^m \frac{x^n}{n!}, \quad B(x) = \sum_{n=2}^{\infty} B_n \frac{x^n}{n!}, \quad B'(x) = \sum_{n=2}^{\infty} B_n \frac{x^{n-1}}{(n-1)!}$$

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be the corresponding generating function. The following relations are well known (see [1]):

$$\frac{\partial B(x,y)}{\partial y} = \frac{x^2(1+D(x,y))}{2(1+y)}, \quad \ln\Big(\frac{1+D(x,y)}{1+y}\Big) = \frac{xD^2(x,y)}{1+xD(x,y)}.$$

Let $t = \ln((1+D)/(1+y))$; then we obtain

$$D = (1+y)e^t - 1, \quad x = \frac{t}{D(D-t)} = \frac{t}{((1+y)e^t-1)((1+y)e^t-t-1)}.$$

In [4], the following relation was found:

$$\frac{\partial B(x,y)}{\partial x} = \frac{xD(2-xD^2)}{2(1+xD)}.$$

Obviously, we have

$$\begin{aligned} D_0 = D(x,1) = 2e^t - 1, &\quad x = \frac{t}{D_0(D_0-t)} = \frac{t}{(2e^t-1)(2e^t-t-1)}, \\ B'(x) = \left[\frac{\partial B(x,y)}{\partial x}\right]_{y=1} &= \frac{xD_0(2-xD_0^2)}{2(1+xD_0)} = \frac{x}{2}(2D_0 - 2t - tD_0), \\ B'(x) = \frac{x}{2}f(t), &\quad f(t) = 4e^t - 2te^t - t - 2, \quad f'(t) = 2e^t - 1 - 2te^t. \end{aligned}$$

The following formula for the coefficients of expansion of an analytic function $f(t)$ into the series in powers of another analytic function $w(t)$ (the Bürmann–Lagrange formula; see [2]) is well known:

$$f(t) = d_0 + \sum_{n=1}^{\infty} d_n w^n(t), \quad d_n = \frac{1}{n}[t^{-1}]\frac{f'(t)}{w^n(t)}, \quad n \geq 1.$$

In the case considered, we have

$$\begin{aligned} f(t) = 4e^t - 2te^t - t - 2, &\quad x = \frac{t}{(2e^t-1)(2e^t-t-1)} = w(t), \\ d_n = \frac{1}{n}[t^{-1}](2e^t - 1 - 2te^t)(2e^t-1)^n(2e^t-t-1)^n t^{-n}, &\quad B'(x) = x + \frac{x}{2}\sum_{p=1}^{\infty} d_p x^p, \\ B_n = (n-1)![x^{-1}]B'(x)x^{-n} = \frac{(n-1)!}{2}[x^{-1}]\sum_{p=1}^{\infty} d_p x^{p+1-n} &= \frac{(n-1)!}{2}d_{n-2}, \quad n \geq 3, \\ B_n = \frac{(n-1)!}{2(n-2)}[t^{-1}](2e^t - 1 - 2te^t)(2e^t-1)^{n-2}(2e^t-t-1)^{n-2}t^{2-n}. \end{aligned}$$

Introduce the notation $N = \dfrac{(n-1)!}{2(n-2)}$. Using Newton's binomial formula and the Taylor expansion for the exponential function, we obtain

$$B_n = N[t^{-1}](2e^t-1)^{n-1}(2e^t-t-1)^{n-2}t^{2-n} - N[t^{-1}]2te^t(2e^t-1)^{n-2}(2e^t-t-1)^{n-2}t^{2-n}$$

$$= N[t^{-1}]\sum_{i=0}^{n-1}\binom{n-2}{i}(2e^t-1)^{n+i-1}(-t)^{n-2-i}t^{2-n}$$

$$- N[t^{-1}]2te^t\sum_{i=0}^{n-2}\binom{n-2}{i}(2e^t-1)^{n+i-2}(-t)^{n-2-i}t^{2-n}$$

$$= N[t^{-1}]\sum_{i=0}^{n-2}\binom{n-2}{i}(-1)^{n-i}\sum_{j=0}^{n+i-1}\binom{n+i-1}{j}2^j e^{jt}(-1)^{n+i-1-j}t^{-i}$$

$$- N[t^{-1}]\sum_{i=0}^{n-2}\binom{n-2}{i}(-1)^{n-i}\sum_{j=0}^{n+i-2}\binom{n+i-2}{j}2^{j+1}e^{(j+1)t}(-1)^{n+i-2-j}t^{1-i}$$

$$= N[t^{-1}]\sum_{i=0}^{n-2}\sum_{j=0}^{n+i-1}\binom{n-2}{i}(-1)^{j+1}\binom{n+i-1}{j}2^j\sum_{p=0}^{\infty}\frac{j^p t^{p-i}}{p!}$$

$$- N[t^{-1}]\sum_{i=0}^{n-2}\sum_{j=0}^{n+i-2}\binom{n-2}{i}(-1)^j\binom{n+i-2}{j}2^{j+1}\sum_{p=0}^{\infty}\frac{(j+1)^p t^{p+1-i}}{p!}$$

$$= N\sum_{i=1}^{n-2}\sum_{j=0}^{n+i-1}\binom{n-2}{i}(-1)^{j+1}\binom{n+i-1}{j}2^j\frac{j^{i-1}}{(i-1)!}$$

$$- N\sum_{i=2}^{n-2}\sum_{j=0}^{n+i-2}\binom{n-2}{i}(-1)^j\binom{n+i-2}{j}2^{j+1}\frac{(j+1)^{i-2}}{(i-2)!}. \qquad (2)$$

In the second term, the factorial in the denominator turns the term into zero for $i = 1$, whereas the second binomial coefficient vanishes for $j = n + i - 1$; therefore,

$$B_n = \frac{(n-1)!}{2(n-2)}\sum_{i=1}^{n-2}\sum_{j=0}^{n+i-1}\binom{n-2}{i}(-1)^{j+1}2^j\left[\frac{j^{i-1}}{(i-1)!}\binom{n+i-1}{j} + 2\binom{n+i-2}{j}2^{j+1}\frac{(j+1)^{i-2}}{(i-2)!}\right].$$

Using the identity

$$\frac{1}{n-2}\binom{n-2}{i} = \frac{1}{i}\binom{n-3}{i-1},$$

we find

$$B_n = (n-1)!\sum_{i=1}^{n-2}\sum_{j=0}^{n+i-1}\binom{n-3}{i-1}\frac{(-2)^{j-1}}{i!}\left[j^{i-1}\binom{n+i-1}{j} + 2(i-1)\binom{n+i-2}{j}2^{j+1}(j+1)^{i-2}\right].$$

Performing the change of variables $i' = i - 1$ and $i' \to i$, we arrive at the required assertion. □

Table 1.

n	3	4	5	6	7	8	9	10	11
B_n	1	9	152	3810	126402	5210576	256469544	14666168250	955097348870

In Table 1, the numbers B_n calculated by Theorem 1 and the Maple software are presented. They coincide with the numbers calculated by the formula (1) in [4].

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