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THE VOLTERRA THEORY OF INTEGRO-DIFFERENTIAL EQUATIONS

Alexandre Soldatov *
National Research University "Moscow Power Engineering Institute"
14, Krasnokazarmennaya St., Moscow 111250, Russia
Academy of Sciences of the Republic of Sakha (Yakutia)
33, Lenin Av., Yakutsk 677007, Russia
soldatov48@gmail.com

Sarvar Zaripov
Tajik National University
17, Rudaki St., Dushanbe 734025, Tajikistan
sarvar8383@list.ru

We establish the solvability of a Volterra integro-differential equation with logarithmic kernel in a class of weighted spaces on a finite interval with power singularities at the endpoints of the interval. Bibliography: 10 *titles.*

Much attention was paid to integro-differential equations with different properties of the kernel were considered, for example, in [1]–[4]. The case of logarithmic kernels was studied [5]–[7] in detail. We also mention the works [1, 8] devoted to the study of integro-differential equations and the conjugates. Our work continues the study of equations of this type in this direction

1 Volterra Integral Operators in Weighted Spaces

Following [9], we introduce the weighted space $C_\lambda = C_{\lambda_0,\lambda_1}([0,1],0,1)$, $\lambda = (\lambda_0,\lambda_1) \in \mathbb{R}^2$, of continuous complex-valued functions $\varphi(t)$, $0 < t < 1$, such that $\varphi_0(t) = t^{-\lambda_0}(1-t)^{-\lambda_1}\varphi(t)$ is bounded. This space is Banach with respect to the norm $|\varphi| = \sup_t |\varphi_0(t)|$. The space C^n_λ, $n = 1, 2, \dots$ of differentiable functions is defined inductively by the condition $\varphi \in C^{n-1}_\lambda$, $\varphi' \in C^{n-1}_{\lambda-1}$ or, in terms of the weight differentiation operator

$$(D\varphi)(t) = t(1-t)\varphi'(t) \qquad (1.1),$$

$\varphi, D\varphi \in C^{n-1}_{\lambda-1}$. In particular, the operator D is bounded as an operator from C^n_λ to C^{n-1}_λ. We note that if λ_0 or λ_1 is positive, then the constant functions do not belong to the space C^1_λ

* To whom the correspondence should be addressed.

and, consequently, the kernel of D is zero, i.e., the operator D is injective in C^1_λ. Thus, we can introduce the right inverse operator $D^{(-1)}$ which is the simplest Volterra operator and acts by the formula

$$(D_0^{(-1)}\varphi)(x) = \int_0^x \frac{\varphi(t)dt}{t(1-t)}, \tag{1.2$_0$}$$

for $\lambda_0 > 0$ or

$$(D_1^{(-1)}\varphi)(x) = -\int_x^1 \frac{\varphi(t)dt}{t(1-t)}, \tag{1.2$_1$}$$

for $\lambda_1 > 0$. For operators in the space C^n_λ were well studied [10]. We formulate the corresponding result [10, Theorem 2.10.1] adapted to the case under consideration.

Lemma 1.1. (a) *Assume that $\lambda_0 > 0$ and $\lambda_1 \neq 0$. Then for $\lambda_1 < 0$ the operator $D_0^{(-1)}$ is bounded from C^n_λ to C^{n+1}_λ, and for $\lambda_1 > 0$ from the subspace of C^n_λ defined by*

$$\int_0^1 \frac{\varphi(t)dt}{t(1-t)} = 0. \tag{1.3}$$

(b) *Assume that $\lambda_1 > 0$ and $\lambda_0 \neq 0$. Then for $\lambda_0 < 0$ the operator $D_0^{(-1)}$ is bounded from C^n_λ to C^{n+1}_λ and for $\lambda_0 > 0$ from the subspace of C^n_λ defined by* (1.3).

We note that the operator

$$(T\varphi)(t) = \varphi(1-t), \tag{1.4}$$

realizes an isomorphism between C^n_λ and $C^n_{\widetilde{\lambda}}$, where $\widetilde{\lambda}_0 = \lambda_1$ and $\widetilde{\lambda}_1 = \lambda_0$. The operators (1.2) are associated with T by the relation

$$TD_1^{(-1)}T = D_0^{(-1)}. \tag{1.5}$$

Therefore, statements (a) and (b) are equivalent.

We consider the Volterra integral operator in the space C_λ

$$(I\varphi)(x) = \int_0^x K(x,t)\frac{\varphi(t)dt}{t(1-t)}, \quad 0 < x < 1, \tag{1.6}$$

with the kernel

$$K = A_0 + A_1 L + + \ldots + A_{n-1}L^{n-1}, \quad L(x,t) = \ln\Big(\frac{x}{1-x}\frac{1-t}{t}\Big),$$

where $A_j \in \mathbb{C}$.

Lemma 1.2. *The operator I is bounded in the space C_λ for $\lambda_0 > 0 > \lambda_1$.*

Proof. As above, we introduce the space $C_\delta([0,a],0)$, $\delta \in \mathbb{R}$, of continuous functions $\varphi(x)$, $0 < x \leqslant a$, such that $\varphi_0(t) = t^{-\delta}\varphi(t)$ is bounded. Then for $1/2 < a < 1$ the space $C_\lambda = C_{\lambda_0,\lambda_1}([0,1],0,1)$ can be described by the conditions

$$\varphi(t) \in C_{\lambda_0}([0,a],0), \quad \varphi(1-t) \in C_{\lambda_1}([0,a],0). \tag{1.7}$$

We consider the Volterra integral operators

$$(I^0\varphi)(x) = \int_0^x c(x,t) \ln^r \left(\frac{t}{x}\right) \frac{\varphi(t)dt}{t}, \quad 0 < x \leqslant a,$$

$$(I^1\varphi)(x) = \int_x^a c(x,t) \ln^r \left(\frac{t}{x}\right) \frac{\varphi(t)dt}{t}, \quad 0 < x \leqslant a,$$

where r is an integer and the function $c(x,t)$, $0 < x, t \leqslant a$, is continues and bounded.

The operators I^0 and I^1 are bounded in the space $C_\delta([0,a],0)$ for $\delta > 0$ and $\delta < 0$, which becomes clear if we write

$$(I^0\varphi)(x) = x^\delta \int_0^1 c(x,xs)(\ln s)^r s^{\delta-1}\varphi_0(xs)ds, \quad \delta > 0,$$

$$(I^1\varphi)(x) = x^\delta \int_1^{a/x} c(x,xs)(\ln s)^r s^{\delta-1}\varphi_0(xs)ds, \quad \delta < 0,$$

where $\varphi_0(t) = t^{-\delta}\varphi(t)$ is bounded.

Now, we consider I as an operator on $[0,a]$. It is obvious that

$$\frac{K(x,t)}{1-t} = \sum_{j=0}^{n-1} c_j(x,t) \ln^j \left(\frac{x}{t}\right), \quad 0 < x, t < a,$$

with some bounded continuous functions c_j. This operator can be represented as the sum of operators of the form I^0. Therefore, taking into account the inequality $\lambda_0 > 0$, we conclude that this operator is bounded in $C_{\lambda_0}([0,a],0)$.

Using (1.4), we write

$$(I\varphi)(1-x) = \int_x^a K(1-x,1-t)\frac{\varphi(1-t)dt}{t(1-t)} + \psi(x), \quad 0 < x, t < a,$$

where

$$\psi(x) = \int_a^1 K(1-x,1-t)\frac{\varphi(1-t)dt}{t(1-t)} = \int_0^{1-a} K(1-x,t)\frac{\varphi(t)dt}{t(1-t)}.$$

As above, the operator defined by the first term is the finite sum of operators of the form I^1 and, consequently, by the inequality $\lambda_1 < 0$, is limited to $C_{\lambda_1}([0,a],0)$. It is obvious that the second term, can be represented as an operator with limited action $C_{\lambda_0}([0,1-a],0) \to C_{\lambda_1}([0,a],0)$. Together with description (3) of the space C_λ, we obtain the required assertion. $\square$

Let us consider the connection between I and the weighted differentiation operator $D = D_x$. It is obvious that $I\varphi$ is continuously differentiable on $(0,1)$ and

$$D(I\varphi) = A_0\varphi + I_1\varphi, \tag{1.8}$$

where I_1 is defined in a similar way as (1.6) via $K_1 = D_x K$. Applying the operation D_x to the function $L(x,t)$ in (1.6), we obtain the relations $D_x L = 1$ and $D_x L^s = sL^{s-1}$. Hence $K_1 = A_1 + 2A_2 L + \ldots + (n-1)A_{n-1}L^{n-2}$. By Lemma 1.2 applied to I_1, the operator I is bounded from C_λ to C_λ^1. From (1.8) it follows that

$$DI = A_0 + I_1, \quad D^2 I = A_0 D + A_1 + I_2, \ldots,$$
$$D^s I = A_0 D^{s-1} + 1!A_1 D^{s-2} + \ldots + (s-1)!A_{s-1} + I_s, \quad s \leqslant n,$$

where I_s is defined via

$$D^s K = s!A_s + 2\cdot 3\cdots(s+1)A_{s+1} + \ldots,$$

Thus, the operator I is bounded from C_λ^{n-1} to C_λ^n. Since $D^n K = 0$, we have

$$D^n I = A_0 D^{n-1} + 1!A_1 D^{n-2} + \ldots + (n-1)!A_{n-1}. \tag{1.9}$$

2 Volterra Integro-Differential Equation

In the space C_λ^{n+1}, $\lambda_0 > 0$, we consider the equation

$$D\varphi + I\varphi = f \tag{2.1}$$

with $f \in C_\lambda^n$. By (1.9), the function φ satisfies the ordinary differential equation

$$P(D)\varphi = D^n f, \tag{2.2}$$

where

$$P(\zeta) = \zeta^{n+1} + A_0\zeta^{n-1} + \ldots + (n-1)!A_{n-1}.$$

In what follows, we assume that all the zeros $\zeta_0, \zeta_1, \ldots, \zeta_n$ of the polynomial P are simple.

We first consider the homogeneous equation

$$P(D)\varphi = 0. \tag{2.3}$$

For the sake of brevity we set

$$q(t) = \frac{t}{1-t}, \quad 0 < t < 1 \tag{2.4}$$

and associate with a complex number ζ the function $q^\zeta(t) = [q(t)]^\zeta$.

The straight lines $\operatorname{Re}\zeta = \lambda_0$ and $\operatorname{Re}\zeta = -\lambda_1$ divide the complex plane into open right G^+ and left G^- half-planes and the strip G^0 between these planes ($\lambda_0 \neq -\lambda_1$). Thus,

$$\begin{aligned} &G^- = \{\operatorname{Re}\zeta < -\lambda_1\}, \quad G^+ = \{\operatorname{Re}\zeta > \lambda_0\}, \quad \lambda_0 + \lambda_1 \geqslant 0,\\ &G^- = \{\operatorname{Re}\zeta < \lambda_0\}, \quad G^+ = \{\operatorname{Re}\zeta > -\lambda_1\}, \quad \lambda_0 + \lambda_1 \leqslant 0. \end{aligned} \tag{2.5}$$

Lemma 2.1. *Let all the zeros $\zeta_0, \zeta_1, \ldots, \zeta_n$ of the polynomial P lie outside the lines* $\operatorname{Re}\zeta = \lambda_0$ *and* $\operatorname{Re}\zeta = -\lambda_1$. *Then for $\lambda_0 + \lambda_1 \geqslant 0$ the homogeneous problem* (2.3) *in the class C_λ^{n+1} has only the zero solution, and for $\lambda_0 + \lambda_1 < 0$ the functions q^{ζ_i}, $\zeta_i \in G^0$ form the basis for the space of solutions to this problem.*

Proof. The obvious equalities

$$Dq = q, \quad Dq^\zeta = \zeta q^\zeta \tag{2.6}$$

for the function (2.4) show that $P(D)q^\zeta = P(\zeta)q^\zeta$. Hence the linearly independent functions $\varphi_j = q^{\zeta_j}$, $0 \leqslant j \leqslant n$, form a basis for the space of solutions to the homogeneous equation (2.3) in the class $C^{n+1}(0,1)$ of $(n+1)$ times continuously differentiable functions on $(0,1)$. Therefore, it suffices to show that these functions belong to C_λ. By assumption, all points ζ_i belong to $G^- \cup G^0 \cup G^+$. According to (2.4),

$$|q^\zeta(t)| = t^{\lambda_0+\nu_0}(1-t)^{\lambda_1+\nu_1}, \quad \nu_0 = \operatorname{Re}\zeta - \lambda_0, \quad \nu_1 = -\operatorname{Re}\zeta - \lambda_1.$$

If $\zeta \in G^\pm$, then in both cases (2.5) one of the numbers ν_j is negative and, consequently, the function q^ζ does not belong to the space C_λ. If $\zeta \in G^0$, then both numbers ν_j are positive for $\lambda_0 + \lambda_1 < 0$ or negative for $\lambda_0 + \lambda_1 < 0$. Hence $q^\zeta \in C_\lambda$ only in the first case, which completes the proof of the lemma. □

By Lemma 2.1, it suffices to construct a particular solution to the inhomogeneous equation (2.2) which, in the case $\lambda_0 > 0$, is also a particular solution to the original equation (2.1). Let $f \in C^n_\lambda$. Then

$$fq^{-\zeta} \in C_\delta, \quad \delta_0 = \lambda_0 - \operatorname{Re}\zeta, \quad \delta_1 = \lambda_1 + \operatorname{Re}\zeta.$$

According to (2.5),

$$\begin{aligned} &\delta_0 > 0, \quad \delta_1 < 0, \quad \zeta \in G^-, \\ &\delta_0 < 0, \quad \delta_1 > 0, \quad \zeta \in G^+, \\ &\delta_0 > 0, \quad \delta_1 > 0, \quad \zeta \in G^0, \quad \lambda_0 + \lambda_1 < 0, \\ &\delta_0 < 0, \quad \delta_1 < 0, \quad \zeta \in G^0, \quad \lambda_0 + \lambda_1 > 0. \end{aligned}$$

By Lemma 1.1, each of the functions

$$u^-_\zeta(x) = \int_0^x \frac{q^\zeta(x)}{q^\zeta(t)} \frac{f(t)dt}{t(1-t)}; \quad \zeta \in G^-, \tag{2.7$^-$}$$

and

$$u^+_\zeta(x) = -\int_x^1 \frac{q^\zeta(x)}{q^\zeta(t)} \frac{f(t)dt}{t(1-t)} \in C^{n+1}_\lambda, \quad \zeta \in G^+. \tag{2.7$^+$}$$

belongs to C^{n+1}_λ.

Assume that $\zeta \in G^0$, $\lambda_0 + \lambda_1 < 0$, and

$$\int_0^1 \frac{f(t)}{q^\zeta(t)} \frac{dt}{t(1-t)} = 0. \tag{2.8}$$

Then $u^+_\zeta = u^-_\zeta$, and we can set

$$u^0_\zeta = u^\pm_\zeta, \quad \zeta \in G^0, \quad \lambda_0 + \lambda_1 < 0. \tag{2.9}$$

By Lemma 1.1, this function also belongs to the class C_λ^{n+1}. In the latter case, $\zeta \in G^0$, $\lambda_0 + \lambda_1 < 0$, the integrals in (2.9) do not necessarily have sense, so, this case should be excluded from our consideration.

We formulate the main result of the paper about the solvability of Equation (2.1) in two cases depending on the sign of $\lambda_0 + \lambda_1$.

Theorem 2.1. *Let $\lambda_0 + \lambda_1 \geqslant 0$, and let all the zeros $\zeta_0, \zeta_1, \dots, \zeta_n$ of the polynomial P lie outside the strip $\overline{G}^0 = \{\lambda_0 \leqslant \operatorname{Re} \zeta \leqslant -\lambda_1\}$, i.e., in $G^+ \cup G^-$. Then Equation (2.1) with $f \in C_\lambda^n$, $\lambda_0 > 0$, is uniquely solvable in the class C_λ^{n+1} and the solution is expressed as*

$$\varphi = \varphi^+ + \varphi^-, \quad \varphi^\pm = \sum_{\zeta_i \in G^\pm} c_i u_{\zeta_i}^\pm, \tag{2.10}$$

where $c_0, c_1, \dots, c_n$ satisfy the system

$$\sum_{i=0}^{n} P_{n-j}(\zeta_i) c_i = \begin{cases} 0, & 0 \leqslant j \leqslant n-1, \\ 1, & j = n, \end{cases}$$

with polynomials $P_0 = 1$, $P_1(\zeta) = \zeta$, and

$$P_s(\zeta) = A_{n-s+1} + A_{n-s+2}\zeta + \ldots + A_{n-1}\zeta^{s-2} + \zeta^s, \ 2 \leqslant s \leqslant n.$$

Proof. As was already mentioned, Equations (2.1) and (2.2) are equivalent in the class C_λ^{n+1}, $\lambda_0 > 0$. According to (2.6), for $\varphi = u_\zeta^\pm$, $\zeta \in G^\pm$,

$$D\varphi = f + \zeta\varphi, \quad D^2\varphi = f + \zeta Df + \zeta^2\varphi, \dots,$$
$$D^{s+1}\varphi = \zeta^s f + \zeta^{s-1} Df + \ldots + D^s f + \zeta^{s+1}\varphi.$$

Substituting these expressions into $P(D)$, taking into account that $P(\zeta) = 0$, and making elementary calculations, we obtain the equality

$$P(D)\varphi = P_n(\zeta) f + P_{n-1}(\zeta) Df + \ldots + P_1(\zeta) D^{n-1} f + P_0 D^n f + P(\zeta)\varphi$$

with the polynomials P_j of degree j defined above. Hence

$$P(D) u_{\zeta_i}^\pm = \sum_{j=0}^{n} P_{n-j}(\zeta_i) D^j f, \quad \zeta_i \in G^\pm.$$

Then for the linear combination (2.10) of $u_{\zeta_i}^\pm$ we can write

$$P(D)\varphi = \sum_{j=0}^{n} \left[\sum_{i=0}^{n} P_{n-j}(\zeta_i) c_i \right] D^j f, \tag{2.12}$$

where we took into account that all the points $\zeta_0, \dots, \zeta_n$ belong to $G^+ \cup G^-$ by assumption.

It is easy to see that the system (2.11) is uniquely solvable with respect to c_i. Indeed, let a matrix with entries $P_{n-j}(\zeta_i)$, $0 \leqslant i, j \leqslant n$, have zero determinant. Then some nontrivial linear combination of P_j, which is a polynomial of the nth degree, vanishes at $n+1$ points $\zeta_0, \dots \zeta_n$.

Therefore, the linear combination is equal to zero, which contradicts the linear independence of the polynomials $P_0, \dots, P_n$.

For the solution $c_0, \dots, c_n$ of this system, the equality (2.12) means that the function (2.10) is a partial solution to Equation (2.2). This fact, together with Lemma 1.2, completes the proof. ☐

We note that the solution to the system (2.11) is given by $c_i = \dfrac{\det M_i}{\det M}$, where M denotes the matrix with entries $P_{n-j}(\zeta_i)$ and the matrix M_i is obtained by replacing the ith column of M with the vector $(0, \dots, 0, 1)$.

It remains to consider the case $\lambda_0 + \lambda_1 < 0$. Arguing in the same way as in Theorem 2.1 we arrive at the following result.

Theorem 2.2. *Let* $\lambda_0 + \lambda_1 < 0$, *and let and all the zeros of the polynomial* P *lie outside the lines* $\operatorname{Re} \zeta = (-1)^k \lambda_k$, $k = 0.1$. *Let a function* $f \in C_\lambda^{n+1}$ *satisfy the orthogonality conditions*

$$\int_0^1 f(t)[q(t)]^{-\zeta_i} = 0, \quad \zeta_i \in G^0. \tag{2.13}$$

Then a solution to this equation in the class C_λ^{n+1}, $\lambda_0 > 0$, *is given by*

$$\varphi = \varphi^- + \varphi^0 + \varphi^+,$$

where $\varphi^\pm$ *are defined in* (2.10), φ^0 *has a similar meaning with respect to* G^0, *and the coefficients* $c_0, c_1, \dots, c_n$ *satisfy* (2.11).

By duality, it is easy to show that the orthogonality conditions (2.13) of Theorem 2.2 are not only sufficient, but also necessary for the solvability of Equation (2.1).

We consider the bilinear form

$$(\varphi, \psi) = \int_0^1 \varphi(t)\psi(t)\frac{dt}{t(1-t)}. \tag{2.14}$$

This form is continuous on $C_\lambda \times C_\delta$ for $\lambda + \delta > 0$. The operator $D' = -D$ is adjoint to D relative to this form in the sense that

$$(D\varphi, \psi) = -(\varphi, D\psi). \tag{2.15}$$

It remains to take into account that the product $\varphi\psi$ of $\varphi \in C_\lambda^1$ and $\psi \in C_\delta^1$ belongs to the space $C_{\lambda+\delta}$ and vanishes for $\lambda + \delta > 0$ at the endpoints of the segment.

We consider the Volterra integral operator

$$(I'\psi)(x) = \int_x^1 K(t, x)\frac{\psi(t)dt}{t(1-t)}$$

which is the adjoint to (1.6). The following counterpart of Lemma 1.2 holds.

Lemma 2.2. *The operator* I' *is bounded in* C_δ *for* $\delta_0 < 0 < \delta_1$.

Proof. We argue in the same as in Lemma 1.2. In this case, it is easier to use (1.4) sending C_λ to $C_{\lambda'}$ with the weight order $\lambda' = (\lambda_1, \lambda_0)$. Since $K(1-x, 1-t) = K(t, x)$, we have $I' = TIT$, which reduces Lemma 2.1 to Lemma 1.2.

We note that a similar relation is also valid for the operator D, i.e., $TDT = -D$. In particular, for $\lambda_0 + \lambda_1 < 0$, in the notation of Lemma 2.1, the functions $\psi_i = T\varphi_i$ are solutions to the homogeneous adjoint equation

$$-D\psi + I'\psi = 0. \tag{2.16}$$

It is clear that

$$\psi_i(x) = \varphi_i(1-x) = q^{-\zeta_i}(x). \tag{2.17}$$

Changing the integration order¡ we obtain the duality relation

$$(I\varphi, \psi) = (\varphi, I'\psi).$$

By (2.15), it follows that for any $\varphi \in C^1_\lambda$ the solution $\psi \in C^1_\delta$ to Equation (2.16) satisfies the following equality for $\lambda + \delta > 0$:

$$(D\varphi + I\varphi, \psi) = (\varphi, -D\psi + I'\psi) = 0.$$

In particular, we have the equalities $(f, \psi_i) = 0$, $\zeta_i \in G^0$, which coincide with the orthogonality conditions (2.13) In Theorem 2.2. Hence these conditions are necessary and sufficient for the solvability of Equation (2.1) in the case $\lambda_0 + \lambda_1 < 0$. □

Declarations

Data availability This manuscript has no associated data.

Ethical Conduct Not applicable.

Conflicts of interest The authors declare that there is no conflict of interest.

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