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DIFFERENTIAL EQUATIONS WITH FRACTIONAL DERIVATIVES AND CHANGING DIRECTION OF EVOLUTION

Aleksandr Artyushin
Sobolev Institute of Mathematics SB RAS
4, Koptyuga Pr., Novosibirsk 630090, Russia
alexsp3@yandex.ru

Sirojiddin Dzhamalov*
Romanovsky Institute of Mathematics
of the Academy of Sciences of the Republic of Uzbekistan
9, University St., Tashkent 100174, Uzbekistan
Central Asian University
264, Milliy bog St., Tashkent 111221, Uzbekistan
siroj63@mail.ru

We study the fractional diffusion equation with changing direction of evolution. We consider the boundary value problem for this equation and prove the existence of a generalized solution. Bibliography: 10 *titles.*

1 Introduction

Introduce the notation: $T > 0$, $\Omega \subset R^m$ is a bounded domain with smooth boundary $\Gamma = \partial\Omega$, $Q = (0, T) \times \Omega$, $S = (0, T) \times \Gamma$, $0 < \nu < 1$. In the cylinder Q, we consider the following mixed problem for the model equation with the Gerasimov–Caputo fractional derivative:

$$\begin{aligned} &\partial_t^\nu (k(t,x)u(t,x)) - \Delta u(t,x) + \gamma u(t,x) = f(t,x), \\ &u(t,x)|_S = 0, \quad u(0,x) = u_0(x). \end{aligned} \tag{1.1}$$

The case $k(t,x) \geqslant k_0 > 0$ has been well studied.

In the case of a constant coefficient $k(x,t) \equiv k_0$, the study of this problem is based on the separation of variables or the use of the Laplace transform.

* To whom the correspondence should be addressed.

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In the case of variable coefficients, integral inequalities of the form

$$\int_0^T \psi(t,u)D^\nu(ku)\,dt \geqslant C(\|u\|_B),$$

where B is a Banach space, are usually used. Based on these inequalities, it is possible to obtain necessary a priori estimates. Some general inequalities of this kind can be found in [1]. We also mention the related works [2]–[6].

In this paper, we are interested in setting a well-posed problem in the case of an arbitrary behavior of the coefficient $k(t,x)$. It is clear that the Cauchy initial condition in (1.1) should be replaced in this case. For example, if $k(t,x) \geqslant 0$, a part of the set Ω is free from the initial data (see [6]). In the case of ordinary derivatives, the problem

$$\begin{aligned}
&k(t,x)u_t(t,x) - \Delta u(t,x) + \gamma u(t,x) = f(t,x),\\
&u(t,x)|_S = 0,\\
&u(0,x) = u_0(x), \quad x \in \Omega_0^+ = \{x|k(0,x) > 0\},\\
&u(T,x) = u_1(x), \quad x \in \Omega_T^- = \{x|k(T,x) < 0\}
\end{aligned}$$

is well posed under certain conditions (for more details we refer to [7], where the uniqueness of a generalized solution was studied). In the case of fractional derivatives, a similar problem is also well posed.

We reduce the problem to solving an equation of the form

$$(k(t,x)u(t,x))_t - D^\mu \Delta u(t,x) + \gamma D^\mu u(t,x) = D^\mu f(t,x).$$

Such equations (without degeneracy and sign change) were considered, for example, in [8] (see also the references therein).

2 Notation. Auxiliary Estimates

For $0 < \nu < 1$ and $t > 0$ we introduce the fractional integral of order ν with origin at a point a

$$J_a^\nu y(t) = \frac{\operatorname{sgn}(t-a)}{\Gamma(\nu)} \int_a^t \frac{y(s)}{|t-s|^{1-\nu}}\,ds,$$

the Riemann–Liouville fractional derivative of order ν

$$D_a^\nu y(t) = \frac{1}{\Gamma(1-\nu)} \frac{d}{dt} \int_a^t \frac{y(s)}{|t-s|^{\nu}}\,ds,$$

and the Gerasimov–Caputo fractional derivative

$$\partial_a^\nu y(t) = \frac{1}{\Gamma(1-\nu)} \int_a^t \frac{y'(s)}{|t-s|^{\nu}}\,ds.$$

In the case $a = 0$, we write $J^\nu y(t)$, $D^\nu y(t)$, $\partial^\nu y(t)$.

Let $y(t) \in C^1(0,T)$. Then for some constant $C > 0$ we have (see [1, 6])

$$\int_0^T y(t) D^\nu y(t)\,dt \geqslant C \int_0^T y^2(t)\Big(\frac{1}{t^\nu} + \frac{1}{(T-t)^\nu}\Big)\,dt + C\|y(t)\|^2_{W_2^{\nu/2}(0,T)}. \tag{2.1}$$

Using the equality

$$D^\nu y(t) = \frac{y(t)}{t^\nu \Gamma(1-\nu)} + \frac{\nu}{\Gamma(1-\nu)} \int_0^t \frac{y(t)-y(s)}{(t-s)^{1+\nu}}\,ds,$$

it is easy to get

$$2yD^\nu y(t) = D^\nu y^2(t) + \frac{y^2(t)}{t^\nu \Gamma(1-\nu)} + \frac{\nu}{\Gamma(1-\nu)} \int_0^t \frac{(y(t)-y(s))^2}{(t-s)^{1+\nu}}\,ds. \tag{2.2}$$

Then the estimate (2.1) immediately follows from (2.2).

Further, a direct calculation shows that $D^\nu J^\nu y(t) = J^\nu D^\nu y(t) = y(t)$. Moreover, for $0 < \mu \leqslant \nu < 1$

$$D^\mu J^\nu y(t) = J^{\nu-\mu} y(t) = J^{\nu-\mu} J^\mu D^\mu y(t) = J^\nu D^\mu y(t),$$

$$J^\mu D^\nu y(t) = D^{\nu-\mu} J^{\nu-\mu} J^\mu D^\nu y(t) = D^{\nu-\mu} y(t) = D^\nu J^\mu y(t).$$

It is obvious that the same equalities hold for operators with the origin at the point T.

For any smooth functions $f(t)$ and $g(t)$ we denote

$$J(f,g) = \int_0^T f(t) g(t)\,dt.$$

Then

$$J(f,g) = \int_0^T f(t) J_T^\nu D_T^\nu g(t)\,dt = \int_0^T J^\nu f(t) D_T^\nu g(t)\,dt.$$

For $0 < \mu < 1/2$ we have the estimates (see [9])

$$C_1\|D^\mu g(t)\|_{L_2(0,T)} \leqslant \|D^\mu g(t)\|_{L_2(0,T)} \leqslant C_2\|D^\mu g(t)\|_{L_2(0,T)}$$

which imply

$$|J(f,g)| \leqslant C_3\|J^\mu f(t)\|_{L_2(0,T)}\|D^\mu g(t)\|_{L_2(0,T)}, \tag{2.3}$$

where the constant C_3 depends on μ and T.

In what follows, we need to regularize fractional differentiation operators. For $0 < \mu < 1$ and $0 < \theta < 1$ we put

$$J_{\mu,\theta} y(t) = \frac{1}{\Gamma(\mu)} \int_0^t \frac{y(s)}{(t-s+\theta)^{1-\mu}}\,ds,$$

$$K_{\mu,\theta} y(t) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dt} \int_0^t \frac{y(s)}{(t-s+\theta)^{\mu}}\,ds.$$

It is clear that $K_{\mu,\theta}v(t,x) = \dfrac{d}{dt}J_{1-\mu,\theta}$. As in the case (2.1), for some constant $C_K = C_K(\mu,T) > 0$ independent of θ the following inequality holds:

$$\int_0^T y(t)K_{\mu,\theta}y(t)\,dt \geqslant C_K\|y(t)\|_{L_2(0,T)}. \tag{2.4}$$

Moreover,

$$\int_0^T y(t)K_{\mu,\theta}y(t)\,dt \geqslant C_1\|y(t)\|^2_{W_2^{\mu/2}(0,T)} - C_2\theta^{(1-\mu)/2}\|y(t)\|^2_{W_2^1(0,T)}. \tag{2.5}$$

Indeed, to prove this we need to estimate the term $D^\mu y(t) - K_{\mu,\theta}y(t)$. Denote

$$D(y(t),z(t)) = \int_0^T (D^\mu y(t) - K_{\mu,\theta}y(t))z(t)\,dt.$$

Lemma 2.1. *There exists a constant $C(\mu,T) > 0$ independent of θ and such that for any smooth functions $y(t)$, $z(t)$*

$$|D(y(t),z(t))| \leqslant C(\mu,T)\theta^{(1-\mu)/2}\|y(t)\|_{W_2^1(0,T)}\|z(t)\|_{W_2^1(0,T)}. \tag{2.6}$$

Proof. Denote

$$\rho_\theta(t) = \frac{1}{\Gamma(1-\mu)}\Big(\frac{1}{t^\mu} - \frac{1}{(t+\theta)^\mu}\Big).$$

Then

$$D^\mu y(t) - K_{\mu,\theta}y(t) = y(0)\rho_\theta(t) + \int_0^t y'(s)\rho_\theta(t-s)\,ds.$$

It is easy to see that

$$\int_0^T \rho_\theta(t)\,dt \leqslant C(T^{1-\mu} + \theta^{1-\mu} - (T+\theta)^{1-\mu}) \leqslant C\theta^{1-\mu}, \tag{2.7}$$

$$0 < \rho_\theta(t) = C\frac{t+\theta - t^\mu(t+\theta)^{1-\mu}}{t^\mu(t+\theta)} \leqslant C\frac{\theta}{t^\mu(t+\theta)}. \tag{2.8}$$

Denote

$$w(t) = \int_0^t y'(s)\rho_\theta(t-s)\,ds.$$

Using (2.7) and the Hausdorff–Young inequality, we get

$$\|w(t)\|_{L_2(0,T)} \leqslant C\theta^{1-\mu}\|y(t)\|_{W_2^1(0,T)}.$$

Thus, with the help of (2.8), we obtain the inequality

$$|D(y(t),z(t))| \leqslant C\theta^{1-\mu}\|y(t)\|_{W_2^1(0,T)}\|z(t)\|_{L_2(0,T)} + C|y(0)|I_z, \tag{2.9}$$

where

$$I_z = \int_0^T \frac{|z(t)|\theta}{t^\mu (t+\theta)}\,dt.$$

It is obvious that

$$I_z \leqslant \left(\int_0^T \frac{z^2(t)}{t^\mu}\,dt\right)^{1/2} \left(\int_0^T \frac{\theta^2}{t^\mu (t+\theta)^2}\,dt\right)^{1/2}.$$

To estimate the last integral, we divide the interval $(0, T)$ into two subintervals and write

$$\int_0^T \frac{\theta^2}{t^\mu (t+\theta)^2}\,dt \leqslant \int_0^\theta \frac{dt}{t^\mu} + \int_\theta^T \frac{\theta^2\,dt}{t^{2+\mu}} \leqslant C\theta^{1-\mu}. \tag{2.10}$$

Combining (2.9) with (2.10) and using the embedding theorems, we obtain (2.6). □

Corollary 2.1. *For any smooth function $y(t)$ the estimate* (2.5) *holds with some constants $C_1(\mu, T) > 0$ and $C_2(\mu, T) > 0$.*

Proof. It suffices to note that

$$\int_0^T K_{\mu,\theta} y(t) y(t)\,dt = \int_0^T D^\mu y(t) y(t)\,dt - \int_0^T (D^\mu y(t) - K_{\mu,\theta} y(t)) y(t)\,dt$$

and use the inequality (2.1) and Lemma 2.1. □

3 Statement of the Problem. Existence Theorem

Consider the problem

$$\partial^\nu (k(t,x)u(t,x)) - \Delta u(t,x) + \gamma u(t,x) = f(t,x), \tag{3.1}$$

$$u(t,x)|_S = 0, \tag{3.2}$$

$$u(0,x) = 0, \quad x \in \Omega_0^+ = \{x | k(0,x) > 0\}, \tag{3.3}$$

$$u(T,x) = 0, \quad x \in \Omega_T^- = \{x | k(T,x) < 0\}. \tag{3.4}$$

We set $\chi_0(x) = k(0,x)u(0,x)$ and $\chi_T(x) = k(T,x)u(T,x)$. Formally applying the operator J^ν to Equation (3.1), we arrive at the equality

$$k(t,x)u(t,x) - J^\nu \Delta u(t,x) + \gamma J^\nu u(t,x) = J^\nu f(t,x) + \chi_0(x), \quad t \in [0,T]. \tag{3.5}$$

In particular, for $t = T$ we have

$$\chi_T(x) - \chi_0(x) - J^\nu \Delta u(T,x) + \gamma J^\nu u(T,x) = J^\nu f(T,x). \tag{3.6}$$

Taking into account (3.3) and (3.4), we get

$$\operatorname{supp} \chi_0(x) \subseteq \Omega_0^-, \quad \operatorname{supp} \chi_T(x) \subseteq \Omega_T^+, \tag{3.7}$$

where Ω_0^- and Ω_T^+ are introduced in the same way as (3.3) and (3.4). Respectively, a function $u(t,x) \in L_2(0,T;\mathring{W}_2^1(\Omega))$ is referred to as a *generalized solution* to the problem (3.1)–(3.4) if

$$J^\nu u(t,x) \in C([0,T];\mathring{W}_2^1(\Omega)), \quad k(t,x)u(t,x) \in C([0,T],W_2^{-1}(\Omega)),$$

and (3.5)–(3.7) hold for some functions $\chi_0(x), \chi_T(x) \in L_1(\Omega)$.

Everywhere below, we set $\mu = 1-\nu$.

Theorem 3.1. *Let $\gamma > 0$, $k(t,x), k_t(t,x) \in L_\infty(Q)$, $D^{\mu/2} f(t,x) \in L_2(0,T;W_2^{-1}(\Omega))$, and for some $\gamma_0 > 0$*

$$2C_K\gamma + k_t(t,x) \geqslant \gamma_0, \quad (t,x) \in Q, \tag{3.8}$$

where the constant C_K is taken from (2.4). *Then the problem* (3.1)–(3.4) *has a generalized solution such that*

$$\|\chi_0\|^2_{L_2(\Omega)} + \|\chi_T\|^2_{L_2(\Omega)} + \|u\|^2_{W_2^{\mu/2}(0,T;W_2^1(\Omega))} \leqslant C\|f\|^2_{W_2^{\mu/2}(0,T;W_2^{-1}(\Omega))}.$$

Proof. As in [6], we use the regularization method proposed in [10]. However, we first formally apply the operator D^μ to Equation (3.1)

$$(k(t,x)u(t,x))_t - D^\mu \Delta u(t,x) + \gamma D^\mu u(t,x) = D^\mu f(t,x).$$

Let $0 < \varepsilon < 1$. We introduce a family of smooth functions $f_\varepsilon(t,x)$ such that

$$f_\varepsilon(0,x) = 0, \tag{3.9}$$

$$\lim_{\varepsilon\to 0} \|f_\varepsilon(t,x) - f(t,x)\|_{W_2^{\mu/2}(0,T;W_2^{-1}(\Omega))} = 0, \tag{3.10}$$

$$\lim_{\varepsilon\to 0} \varepsilon\|f_\varepsilon(t,x)\|^2_{W_2^1(0,T;L_2(\Omega))} = 0. \tag{3.11}$$

We note that the condition (3.9) is compatible with (3.10) since $\mu < 1$. Then we consider the problem (see [10])

$$-\varepsilon u_{tt} + (k(t,x)u(t,x))_t - D^\mu\Delta u(t,x) + \gamma D^\mu u(t,x) = D^\mu f_\varepsilon(t,x), \tag{3.12}$$

$$u(t,x)|_S = 0,$$

$$-\varepsilon u_t(0,x) + k^+(0,x)u(0,x) = 0, \tag{3.13}$$

$$-\varepsilon u_t(T,x) + k^-(T,x)u(T,x) = 0, \tag{3.14}$$

where we use the conventional notation

$$\eta^+ = \begin{cases} \eta, & \eta > 0, \\ 0, & \eta \leqslant 0, \end{cases} \quad \eta^- = \begin{cases} \eta, & \eta < 0, \\ 0, & \eta \geqslant 0. \end{cases}$$

In turn, the solvability of the problem (3.12)-(3.14) is proved by a modified Galerkin method. Let $\{w_k(x)\}_{k\in\mathbb{N}}$ be the system of $L_2(\Omega)$-orthonormal eigenfunctions of the problem

$$-\Delta w_k = \lambda_k w_k, \quad w_k(x)|_\Gamma = 0.$$

For any $n > 0$ we denote by $E_n \subset L_2(\Omega)$ the subspace of functions spanned by the vectors w_k, $k = 1, \dots, n$. Denote by P_n the orthogonal projection in $L_2(\Omega)$ onto the space E_n. Assuming that $\theta = \theta(\varepsilon, n)$ (we specify this value below), we consider the following problem in E_n:

$$-\varepsilon v_{ntt}(t,x) + P_n(k(t,x)v_n(t,x))_t - K_{\mu,\theta}\Delta v_n(t,x) + \gamma K_{\mu,\theta} v_n(t,x) = K_{\mu,\theta} f_{n\varepsilon}(t,x), \tag{3.15}$$

$$v_n(t,x)|_S = 0, \tag{3.16}$$

$$-\varepsilon v_{nt}(x,0) + P_n(k^+(0,x)v_n(0,x)) = 0, \tag{3.17}$$

$$-\varepsilon v_{nt}(T,x) + P_n(k^-(T,x)v_n(T,x)) = 0. \tag{3.18}$$

Here,

$$v_n(t,x) = \sum_{k=1}^{n} V_k(t) w_k(x), \quad f_{n\varepsilon}(t,x) = P_n f_\varepsilon(t,x) = \sum_{k=1}^{n} F_{k\varepsilon}(t) w_k(x).$$

We will omit the superscript n if this does not cause confusion. We note that the kernel $K_{\mu,\theta}$ has no singularity and can be integrated by parts. Therefore, we obtain a system of ordinary integro-differential equations with respect to the coefficients $V_k(t)$ the solvability of which follows from the uniqueness theorem. Hence it suffices to derive a suitable a priori estimate for the solution.

We multiply Equation (3.15) by $2v(t,x)$ and integrate over the cylinder Q

$$\begin{gathered}\int\limits_{\Omega} (k(0,x)v^2(0,x) + k(T,x)v(T,x)) + 2\varepsilon \int\limits_{Q} v_t^2 \, dQ + \int\limits_{Q} k_t v^2 \, dQ \\ + 2\int\limits_{Q} ((K_{\mu,\theta}\nabla v, \nabla v) + \gamma v K_{\mu,\theta} v) \, dQ = 2\int\limits_{Q} v K_{\mu,\theta} f_\varepsilon \, dQ.\end{gathered} \tag{3.19}$$

By (2.4), the operator $K_{\mu,\theta}$ is positive and the inequality (3.8) is satisfied. By (3.19), for some sufficiently large constant C (independent of θ) the following estimate holds (see also (3.9)):

$$\varepsilon\|v_t\|^2_{L_2(Q)} + \gamma_0\|v\|^2_{L_2(Q)} \leqslant C\|K_{\mu,\theta} f\|^2_{L_2(Q)} \leqslant C\|f\|^2_{W_2^1(0,T;L_2(\Omega))}. \tag{3.20}$$

Using (2.5), we obtain the lower estimate

$$2\int\limits_{Q} ((K_{\mu,\theta}\nabla v, \nabla v) + \gamma v K_{\mu,\theta} v) \, dQ \geqslant C\|v\|^2_{W_2^{\mu/2}(0,T;W_2^1(\Omega))} - C\theta^{(1-\mu)/2}\|v\|^2_{W_2^1(0,T;W_2^1(\Omega))}.$$

Since the space E_n is finite-dimensional, for some constant $C(n)$ we have

$$\|h\|^2_{W_2^1(\Omega)} \leqslant C(n)\|h\|^2_{L_2(\Omega)} \quad \forall h \in E_n.$$

By the estimate (3.20), for a sufficiently small θ

$$2\int\limits_{Q} ((K_{\mu,\theta}\nabla v, \nabla v) + \gamma v K_{\mu,\theta} v) \, dQ \geqslant C\|v\|^2_{W_2^{\mu/2}(0,T;W_2^1(\Omega))} - \varepsilon\|f\|^2_{W_2^1(0,T;L_2(\Omega))}. \tag{3.21}$$

The right-hand side of (3.19) can be estimated with the help of (2.3) and Lemma 2.1 as

$$\int\limits_Q vK_{\mu,\theta}f_\varepsilon\,dQ = \int\limits_Q vD^\mu f_\varepsilon\,dQ + \int\limits_Q v(K_{\mu,\theta}f_\varepsilon - D^\mu f_\varepsilon)\,dQ$$

$$= \int\limits_0^T \sum_k V_k(t)D^\mu F_{k\varepsilon}(t)\,dt + \int\limits_0^T \sum_k V_k(t)(K_{\mu,\theta}F_{k\varepsilon}(t) - D^\mu F_{k\varepsilon}(t))\,dt$$

$$\leqslant C\|f\|_{W_2^{\mu/2}(0,T;W_2^{-1}(\Omega))}\|v\|_{W_2^{\mu/2}(0,T;W_2^1(\Omega))} + C\theta^{(1-\mu)/2}\|f\|_{W_2^1(0,T;W_2^{-1}(\Omega))}\|v\|_{W_2^1(0,T;W_2^1(\Omega))}.$$

As above, choosing θ sufficiently small and taking into account the condition (3.11), we obtain the estimate

$$\int\limits_Q vK_{\mu,\theta}f_\varepsilon\,dQ \leqslant C\|f\|_{W_2^{\mu/2}(0,T;W_2^{-1}(\Omega))}\|v\|_{W_2^{\mu/2}(0,T;W_2^1(\Omega))} + \varepsilon\|f\|^2_{W_2^1(0,T;L_2(\Omega))}. \tag{3.22}$$

Combining the estimates (3.19), (3.21), (3.22), we finally obtain the estimate

$$\int\limits_\Omega (|k(0,x)|v^2(0,x) + |k(T,x)|v^2(T,x))\,dx + \varepsilon\int\limits_Q v_t^2\,dQ + \int\limits_Q v^2\,dQ$$

$$+ \int\limits_Q (|D^{\mu/2}\nabla v|^2 + |D^{\mu/2}v|^2)\,dQ \leqslant C\|D^{\mu/2}f_\varepsilon\|^2_{L_2(0,T;W_2^{-1}(\Omega))} + C\varepsilon\|f\|^2_{W_2^1(0,T;L_2(\Omega))}. \tag{3.23}$$

Thus, we have established the desired estimate, which means that the system (3.15)–(3.18) is uniquely solvable. We note that for all $(t,x)\in Q$

$$(-\varepsilon v_{nt} + P_n(kv_n))\big|_0^t = J_{\nu,\theta}(P_nf_\varepsilon + \Delta v_n - \gamma v_n).$$

By (3.17) and (3.18),

$$P_n(k^+v_n(T,x)) - P_n(k^-v_n(0,x)) = J_{\nu,\theta}(P_nf_\varepsilon(T,x) + \Delta v_n(T,x) - \gamma v_n(T,x)).$$

Now, we can pass to the limit as $n\to\infty$. Taking, if necessary, a subsequence, we can assume that for some functions $z_{\varepsilon 0}(x)$, $z_{\varepsilon T}(x)$, $u_\varepsilon(t,x)$ the following convergences hold:

$$\sqrt{|k(0,x)|}v_n(0,x) \rightharpoonup z_{\varepsilon 0}(x) \quad \text{weakly in } L_2(\Omega), \tag{3.24}$$

$$\sqrt{|k(T,x)|}v_n(T,x) \rightharpoonup z_{\varepsilon T}(x) \quad \text{weakly in } L_2(\Omega), \tag{3.25}$$

$$v_n(t,x) \rightharpoonup u_\varepsilon(t,x) \quad \text{weakly in } W_2^{\mu/2}(0,T;W_2^1(\Omega)), \tag{3.26}$$

$$v_{nt}(t,x) \rightharpoonup u_{\varepsilon t}(t,x) \quad \text{weakly in } L_2(Q) \tag{3.27}$$

as $n\to\infty$. By (3.23),

$$\varepsilon\|u_{\varepsilon t}\|^2_{L_2(Q)} + \|J^\nu u_\varepsilon\|_{W_2^{\frac{1+\nu}{2}}(0,T;W_2^1(\Omega))} \leqslant C. \tag{3.28}$$

In particular, $J^\nu u_\varepsilon(t,x)\in C([0,T];W_2^1(\Omega))$ and $J^\nu u_\varepsilon(0,x)=0$. The functions

$$\chi_{0\varepsilon}(x) = -\sqrt{|k^-(0,x)|}z_{\varepsilon 0}(x), \quad \chi_{T\varepsilon}(x) = \sqrt{|k^+(T,x)|}z_{\varepsilon T}(x)$$

satisfy (3.7), and, by the conditions (3.17) and (3.18), the following equalities hold:

$$-\varepsilon u_{\varepsilon t} + k u_\varepsilon - \chi_{0\varepsilon} = J_\nu(f_\varepsilon + \Delta u_\varepsilon - \gamma u_\varepsilon),$$

$$\chi_{T\varepsilon}(x) - \chi_{0\varepsilon}(x) = J_\nu(f_\varepsilon(T,x) + \Delta u_\varepsilon(T,x) - \gamma u_\varepsilon(T,x)).$$

Now, we can pass to the limit as $\varepsilon \to 0$ and then argue in a standard way. We only emphasize that, by the estimate (3.28), we can assume that $\varepsilon u_{\varepsilon t}(t,x) \to 0$ as $\varepsilon \to 0$ in $L_2(Q)$. □

Declarations

Data availability This manuscript has no associated data.

Ethical Conduct Not applicable.

Conflicts of interest The authors declare that there is no conflict of interest.

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