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APPLICATION OF THE VARIATIONAL METHOD OF HOMOGENEOUS SOLUTIONS IN THE AXISYMMETRIC PROBLEM OF THE THEORY OF ELASTICITY FOR A FINITE CYLINDER WITH REGARD FOR ITS OWN WEIGHT

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We use the variational method of homogeneous solutions to investigate the stress-strain state of a solid finite cylinder with regard for its own weight. The lateral surface of the cylinder is fixed and the end faces are free of loads. The general solution is represented in the form of superposition of the solutions of the problems for an inhomogeneous system of equations with homogeneous conditions imposed on the end faces of the cylinder (principal state) and a homogeneous system of equations with inhomogeneous conditions on the end faces of the cylinder (perturbed state). The problem of determination of the perturbed state is reduced to infinite systems of linear algebraic equations solved by the method of reduction. Examples of numerical realization of the solution are presented.

Keywords: variational method of homogeneous solutions, axisymmetric problem, finite cylinder, Love function, own weight.

Introduction

Despite the development of computational complexes aimed at software realization of the numerical methods, analytic approaches to the solution of boundary-value problems of the mechanics of deformable solids have not lost their urgency. The exact analytic solutions serve as an irreplaceable tool both for the verification of the accumulated numerical results and for getting approximate formulas used in engineering applications. Thus, in particular, for the solution of axisymmetric problems of the theory of elasticity for cylindrical bodies, it is customary to use the methods of singular integral equations [10], expansions in Fourier–Bessel series [14], cross superposition [2, 8], integral transformations [4], direct integration [1, 16], and homogeneous solutions [12, 13].

Note that the problems of determination of the stress-strain state of a finite cylinder with regard for its own weight form an important class of axisymmetric problems of the theory of elasticity [5, 6, 15]. In [6], by the method of integral transformations, the numerical solutions of axisymmetric problems of the theory of elasticity for a cylinder of finite length with free cylindrical surface were found by taking into account its own weight. In [5], this method was used to solve a similar problem for a cylinder with conditions of sliding fastening imposed on the lower end face, an axisymmetric normal load applied to the upper end face, and the rigidly restrained lateral surface. The exact analysis of the strain and stress fields in a finite circular elastic cylinder under the action of its own weight was presented in [15], where the influence of the end faces was analyzed. It was shown that the influence of the end faces on the stress-strain state is significant but restricted to a local region near the end face, where the stress and strain distributions noticeably differ from the distributions corresponding to the sim-

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plified solution for the uniaxial stressed state.

The aim of the present paper is to develop the variational method of homogeneous solutions [12] for solving an axisymmetric problem of the theory of elasticity for a finite solid cylinder with restrained lateral surface and and free end faces under the action of its own weight.

1. Statement of the Problem

Consider an axisymmetric problem of the theory of elasticity for a solid cylinder $\mathcal{V} = \{0 \le r \le 1, -b \le z < b\}$ under the action of its own weight. Here, r and z are the dimensionless radial and axial coordinates and $b = \text{const}$. Assume that the lateral surface $\mathcal{S}_0 = \{r = 1, -b \le z \le b\}$ of the cylinder is fixed, i.e.,

$$u_r\big|_{r=1} = 0 \quad \text{and} \quad u_z\big|_{r=1} = 0, \tag{1}$$

and the end faces $\mathcal{S}_1 = \{0 \le r \le 1, z = -b\}$ and $\mathcal{S}_2 = \{0 \le r \le 1, z = b\}$ are free of force loads:

$$\sigma_{zz}\big|_{z=\pm b} = 0 \quad \text{and} \quad \sigma_{rz}\big|_{z=\pm b} = 0. \tag{2}$$

Here, u_r and u_z are, respectively, the radial and axial components of the vector of displacements and σ_{zz} and σ_{rz} are, respectively, the axial and tangential stresses.

The elastic equilibrium of the cylinder $\mathcal{V}$ is described by the following equations [7]:

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{\partial}{\partial z}\sigma_{rz} - \frac{1}{r}\sigma_{\theta\theta} &= 0, \\ \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz}) + \frac{\partial}{\partial z}\sigma_{zz} &= \rho g, \end{aligned} \tag{3}$$

where ρg is the weight of the unit volume of the body. The relationship between the components of the strain tensor ε_{rr}, ε_{zz}, $\varepsilon_{\theta\theta}$, and ε_{rz} and the components of the vector of displacements is determined by the following Cauchy relations [7]:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \quad \varepsilon_{\theta\theta} = \frac{u_r}{r}, \quad \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right). \tag{4}$$

Moreover, the relationship between the components of the strain tensor and the components of the stress tensor σ_{rr}, σ_{zz}, $\sigma_{\theta\theta}$, and σ_{rz} is determined by the physical relations of Hooke's law [7], namely,

$$\sigma_{rr} = \frac{2\mu}{1-2\nu}\big((1-\nu)\varepsilon_{rr} + \nu(\varepsilon_{zz} + \varepsilon_{\theta\theta})\big),$$

$$\sigma_{zz} = \frac{2\mu}{1-2\nu}\big((1-\nu)\varepsilon_{zz} + \nu(\varepsilon_{rr} + \varepsilon_{\theta\theta})\big), \tag{5}$$

$$\sigma_{\theta\theta} = \frac{2\mu}{1-2\nu}\big((1-\nu)\varepsilon_{\theta\theta} + \nu(\varepsilon_{zz} + \varepsilon_{rr})\big), \quad \sigma_{rz} = 2\mu\varepsilon_{rz}.$$

Here, $\mu = E/(2(1+\nu))$ is the shear modulus, E is Young's modulus, and ν is Poisson's ratio.

By using relations (4) and (5), we represent the balance equations (3) in displacements in the following form [3]:

$$\nabla^2 u_r + \frac{1}{1-2\nu}\frac{\partial\varepsilon}{\partial r} - \frac{u_r}{r^2} = 0,$$

$$\nabla^2 u_z + \frac{1}{1-2\nu}\frac{\partial\varepsilon}{\partial z} = \rho g, \quad \varepsilon \equiv \varepsilon_{rr} + \varepsilon_{zz} + \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) - \frac{\partial u_z}{\partial z}, \tag{6}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

is the axisymmetric Laplace operator.

Our aim is to determine the components of the vector of displacements and stress tensor satisfying Eqs. (3) and (6) and guaranteeing the validity of conditions (1) and (2) together with relations (4) and (5).

2. Construction of the Solution

We construct the solution of problem (1), (2), (4)–(6) in the form of sums as follows:

$$u_r(r,z) = u_r^0(r,z) + \tilde{u}_r(r,z), \quad u_z(r,z) = u_z^0(r,z) + \tilde{u}_z(r,z),$$

where $u_r^0(r,z)$, $u_z^0(r,z)$ is a partial solution of the inhomogeneous system (6) and $\tilde{u}_r(r,z)$, $\tilde{u}_z(r,z)$ is the general solution of the homogeneous system

$$\nabla^2 \tilde{u}_r + \frac{1}{1-2\nu}\frac{\partial\tilde{\varepsilon}}{\partial r} - \frac{\tilde{u}_r}{r^2} = 0,$$

$$\nabla^2 \tilde{u}_z + \frac{1}{1-2\nu}\frac{\partial\tilde{\varepsilon}}{\partial z} = 0, \quad \tilde{\varepsilon} = \frac{1}{r}\frac{\partial}{\partial r}(r\tilde{u}_r) - \frac{\partial \tilde{u}_z}{\partial z} \tag{7}$$

corresponding to the inhomogeneous system (6).

2.1. Principal State. The problem of determination of the principal state is reduced to finding the solution of the inhomogeneous system of equations (6) with the following homogeneous boundary conditions:

$$\sigma_{rr}^{0}\Big|_{r=1} = 0, \qquad \sigma_{rz}^{0}\Big|_{r=1} = 0, \tag{8}$$

$$u_{z}^{0}\Big|_{z=-b} = 0 \qquad \sigma_{rz}^{0}\Big|_{z=-b} = 0, \qquad \sigma_{zz}^{0}\Big|_{z=b} = 0, \qquad \sigma_{rz}^{0}\Big|_{z=b} = 0. \tag{9}$$

The components of the stress tensor σ_{rr}^0, σ_{zz}^0, $\sigma_{\theta\theta}^0$, and σ_{rz}^0 satisfying the balance equations (3) and guaranteeing the validity of conditions (8) and (9) can be simply found in the following form:

$$\sigma_{rr}^{0} = 0, \qquad \sigma_{zz}^{0} = -\rho g(b-z), \qquad \sigma_{\theta\theta}^{0} = 0, \qquad \sigma_{rz}^{0} = 0. \tag{10}$$

The corresponding components of the strain tensor in the principal state are expressed in terms of the components of stresses by the formulas [7]

$$\varepsilon_{\theta\theta}^{0} = \frac{u_r^0}{r} = \frac{1}{E}\left(\sigma_{\theta\theta}^{0} - \nu(\sigma_{rr}^{0} + \sigma_{zz}^{0})\right),$$

$$\varepsilon_{zz}^{0} = \frac{\partial u_z^0}{\partial z} = \frac{1}{E}\left(\sigma_{zz}^{0} - \nu(\sigma_{rr}^{0} + \sigma_{\theta\theta}^{0})\right).$$

From these formulas, in view of Eq. (10) and the first condition in (9), we obtain the components of the vector of displacements in the principal state

$$u_r^0 = \frac{\nu\rho g r}{E}(b-z) \qquad \text{and} \qquad u_z^0 = \frac{\rho g}{2E}\left((b-z)^2 - 4b^2\right). \tag{11}$$

2.2. Perturbed State. The problem of determination of the perturbed state is reduced to finding the solution of the homogeneous system of equations (7). The indicated equation is reduced to the following homogeneous biharmonic equation [3, 7]:

$$\nabla^2\nabla^2\tilde{\chi} = 0 \tag{12}$$

satisfying the boundary conditions imposed on the surface $\mathcal{S}_0$

$$\tilde{u}_r\big|_{r=1} = -u_r^0(z) \qquad \text{and} \qquad \tilde{u}_z\big|_{r=1} = -u_z^0(z)$$

and the inhomogeneous conditions on the surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$:

$$\tilde{\sigma}_{zz}\big|_{z=-b} = -\sigma_{zz}^{0}\Big|_{z=-b}, \qquad \tilde{\sigma}_{rz}\big|_{z=-b} = 0, \qquad \tilde{\sigma}_{zz}\big|_{z=b} = 0, \qquad \tilde{\sigma}_{rz}\big|_{z=b} = 0.$$

Here, σ_{zz}^0 is given by the second formula in (10) and $\tilde{\chi}$ is the Love function introduced by the following formulas [3]:

$$\frac{1}{2\mu}\tilde{\sigma}_{rr} = \frac{\partial}{\partial z}\left(\nu\nabla^2\tilde{\chi} - \frac{\partial^2\tilde{\chi}}{\partial r^2}\right), \qquad \frac{1}{2\mu}\tilde{\sigma}_{\theta\theta} = \frac{\partial}{\partial z}\left(\nu\nabla^2\tilde{\chi} - \frac{1}{r}\frac{\partial\tilde{\chi}}{\partial r}\right),$$

$$\frac{1}{2\mu}\tilde{\sigma}_{zz} = \frac{\partial}{\partial z}\left((2-\nu)\nabla^2\tilde{\chi} - \frac{\partial^2\tilde{\chi}}{\partial z^2}\right), \qquad \frac{1}{2\mu}\tilde{\sigma}_{rz} = \frac{\partial}{\partial r}\left((1-\nu)\nabla^2\tilde{\chi} - \frac{\partial^2\tilde{\chi}}{\partial z^2}\right),$$

$$\tilde{u}_r = -\frac{\partial^2\tilde{\chi}}{\partial r\partial z}, \qquad \tilde{u}_z = 2(1-\nu)\nabla^2\tilde{\chi} + \frac{\partial^2\tilde{\chi}}{\partial z^2}.$$

Adding the corresponding components of stresses for the principal and perturbed states

$$\sigma^0_{rr}(r) + \tilde{\sigma}_{rr}(r,z) = \sigma_{rr}(r,z), \qquad \sigma^0_{\theta\theta}(r) + \tilde{\sigma}_{\theta\theta}(r,z) = \sigma_{\theta\theta}(r,z),$$

$$\sigma^0_{zz}(r) + \tilde{\sigma}_{zz}(r,z) = \sigma_{zz}(r,z), \qquad \tilde{\sigma}_{rz}(r,z) = \sigma_{rz}(r,z),$$

we get the stress-strain state of the cylinder $\mathcal{V}$.

The unknown Love function $\tilde{\chi}$ can be represented in the form of the sum of two components

$$\tilde{\chi} = \chi' + \chi'',$$

each of which is determined from the corresponding perturbed problem for the homogeneous equation (12).

The first disturbed problem is to find the solution of Eq. (12) with homogeneous conditions imposed on $\mathcal{S}_0$:

$$u'_r\big|_{r=1} = 0, \qquad u'_z\big|_{r=1} = 0, \tag{13}$$

and inhomogeneous conditions imposed on $\mathcal{S}_1$ and $\mathcal{S}_2$:

$$\sigma'_{zz}\big|_{z=-b} = \sigma_1, \qquad \sigma'_{zz}\big|_{z=b} = \sigma_2,$$
$$\sigma'_{rz}\big|_{z=-b} = \tau_1, \qquad \sigma'_{rz}\big|_{z=b} = \tau_2. \tag{14}$$

We find the solution of the first perturbed problem as the sum of the symmetric and antisymmetric components representing the functions on the right-hand sides of relations (14), respectively, in the form

$$\sigma_1 = -\frac{1}{2}\sigma^0_{zz}, \qquad \sigma_2 = -\frac{1}{2}\sigma^0_{zz}, \qquad \tau_1 = 0, \quad \tau_2 = 0, \tag{15}$$

$$\sigma_1 = -\frac{1}{2}\sigma^0_{zz}, \qquad \sigma_2 = \frac{1}{2}\sigma^0_{zz}, \qquad \tau_1 = 0, \quad \tau_2 = 0. \tag{16}$$

In (15) and (16), the function σ^0_{zz} is determined by the second formula in (10).

We now represent the Love function $\chi'(r,z)$ for the symmetric problem in the form

$$\chi'(r,z) = C\sinh(\gamma z) f(r). \tag{17}$$

At the same time, for the antisymmetric problem, we get

$$\chi'(r,z) = C\cosh(\gamma z) f(r). \tag{18}$$

Here, C is an unknown coefficient. Substituting representations (17) and (18) in Eq. (12), we find the function $f(r)$ in the following form [12]:

$$f(r) = ArJ_1(\gamma r) - \frac{2B}{\pi\gamma} J_0(\gamma r), \tag{19}$$

where A and B are unknown coefficients, $J_0(\gamma r)$ and $J_1(\gamma r)$ are, respectively, the zero- and first-order Bessel functions of the first kind and γ is an eigenvalue, which is a solution of the transcendental characteristic equation

$$\gamma\left(J_0^2(\gamma)+J_1^2(\gamma)\right) - 4(\nu-1)J_0(\gamma)J_1(\gamma) = 0. \tag{20}$$

In view of (19), conditions (13) are reduced to the solution of the following homogeneous system of equations for the constants A and B:

$$\gamma\pi J_0(\gamma)A + 2J_1(\gamma)B = 0, \qquad \pi\left(\gamma J_1(\gamma) - 4(\nu-1)J_0(\gamma)\right)A - 2J_0(\gamma)B = 0. \tag{21}$$

The compatibility of this system is guaranteed by the validity of Eq. (20).

The sole trivial real root $\gamma = 0$ of Eq. (20) is of no practical interest. Therefore, we focus our attention on the determination of the infinite sequence of complex roots γ_k. The first 20 roots $\gamma_k = \alpha_k + i\beta_k$ of Eq. (20) for $\nu = 0.25$ are presented in Table 1.

Note that the first equation in system (21) can be written as follows:

$$\kappa_k = \frac{A}{B} = -\frac{2J_1(\gamma_k)}{\pi\gamma_k J_0(\gamma_k)}. \tag{22}$$

Then the solution of the first perturbed problem is represented in the form

$$\chi'(r,z) = \frac{1}{2}\sum_{k=1}^{\infty}\left(C_k\chi'_k(r,z) + \bar{C}_k\bar{\chi}'_k(r,z)\right), \tag{23}$$

where $\chi'_k = \sinh(\gamma_k z) f_k(r)$, $\chi'_k = \cosh(\gamma_k z) f_k(r)$, $f_k(r) = rJ_1(\gamma_k r)\kappa_k - 2J_0(\gamma_k r)/(\pi\gamma_k)$, γ_k are the roots of Eq. (20), and κ_k is given by (22).

Table 1

k	α_k	β_k	k	α_k	β_k
1	3.0253	0.5565	11	34.5301	1.7716
2	6.1991	0.9173	12	37.6733	1.8151
3	9.3583	1.1217	13	40.8164	1.8551
4	12.5109	1.2659	14	43.9593	1.8921
5	15.6601	1.3775	15	47.1021	1.9266
6	18.8072	1.4687	16	50.2447	1.9588
7	21.9531	1.5457	17	53.3872	1.9891
8	25.0981	1.6124	18	56.5297	2.0177
9	28.2425	1.6713	19	59.6721	2.0447
10	31.3864	1.7239	20	62.8143	2.0703

The second perturbed problem is to find the solution of Eq. (12) with inhomogeneous conditions imposed on $\mathcal{S}_0$:

$$u_r''\big|_{r=1} = u_1(z), \quad u_z''\big|_{r=1} = u_2(z) \tag{24}$$

and homogenous conditions imposed on $\mathcal{S}_1$ and $\mathcal{S}_2$:

$$\sigma_{zz}''\big|_{z=\pm b} = 0, \quad \sigma_{rz}''\big|_{z=\pm b} = 0. \tag{25}$$

As in the first perturbed problem, we represent the solution in the form of the sum of symmetric and antisymmetric parts for which the right-hand sides of conditions (24) can be represented in the form:

$$u_1(z) = -\frac{1}{2}\big(u_r^0(1,z)+u_r^0(1,-z)\big), \quad u_2(z) = -\frac{1}{2}\big(u_z^0(1,z)-u_z^0(1,-z)\big),$$

$$u_1(z) = -\frac{1}{2}\big(u_r^0(1,z)-u_r^0(1,-z)\big), \quad u_2(z) = -\frac{1}{2}\big(u_z^0(1,z)+u_z^0(1,-z)\big).$$

Here, the functions u_r^0 and u_z^0 are given by relations (11).

We represent the solution of Eq. (12) in the form

$$\chi''(r,z) = CJ_0(\gamma r)\varphi(z), \tag{26}$$

where

$$\varphi(z) = (A + Bz)\cosh(\gamma z) + (C + Dz)\sinh(\gamma z)$$

is a solution of the differential equation

$$\varphi^{IV}(z) - 2\gamma^2\varphi''(z) + \gamma^4\varphi(z) = 0,$$

A, B, C, and D are arbitrary constants, and γ is an eigenvalue. We represent the function $\varphi(z)$ as the sum of the odd

$$\varphi(z) = L_1 \sinh(\gamma z) + L_2 z \cosh(\gamma z), \qquad L_1 = A, \quad L_2 = D, \tag{27}$$

and even

$$\varphi(z) = L_1 \cosh(\gamma z) + L_2 z \sinh(\gamma z), \qquad L_1 = C, \quad L_2 = B, \tag{28}$$

parts, each of which depends on two arbitrary constants L_1 and L_2.

In view of (27) and (28), the validity of conditions (25) is reduced to the homogeneous systems of linear algebraic equations:

$$\begin{cases} \gamma\cosh(\gamma b)L_1 + ((2\nu - 1)\cosh(\gamma b) + \gamma b \sinh(\gamma b))L_2 = 0, \\ \gamma\sinh(\gamma b)L_1 + (2\nu\sinh(\gamma b) + \gamma b\cosh(\gamma b))L_2 = 0, \end{cases} \tag{29}$$

$$\begin{cases} \gamma\sinh(\gamma b)L_1 + ((2\nu - 1)\sinh(\gamma b) + \gamma b \cosh(\gamma b))L_2 = 0, \\ \gamma\cosh(\gamma b)L_1 + (2\nu\cosh(\gamma b) + \gamma b\sinh(\gamma b))L_2 = 0, \end{cases} \tag{30}$$

whose compatibility conditions yield the following transcendental equations:

$$\sinh(2\gamma b) + 2\gamma b = 0, \tag{31}$$

$$\sinh(2\gamma b) - 2\gamma b = 0. \tag{32}$$

Each of Eqs. (31) and (32) has a unique trivial real root, which is of no practical interest for our subsequent analysis. This is why we restrict ourselves to the case of infinite sequences of complex roots γ_k of these two equations. Hence, the solutions of systems (29) and (30) are given by sequences of the pairs of the complex numbers L_{1k}, L_{2k}, $k = 1, 2, \ldots$, such that $L_{1k}/L_{2k} = \kappa_k$, and κ_k are complex constants expressed, for the cases of odd and even functions $\varphi(z)$, in terms of the roots of Eqs. (31) and (32):

$$\kappa_k = -\frac{2\nu}{\gamma_k} - \frac{b}{\tanh(\gamma_k b)} \quad \text{and} \quad \kappa_k = -\frac{2\nu}{\gamma_k} - b\tanh(\gamma_k b).$$

If γ_k is the root of Eq. (31) or Eq. (32), then $-\gamma_k$, $\bar{\gamma}_k$, and $-\bar{\gamma}_k$ are the roots of these equations (the overbar denotes the operation of complex conjugation). Real and imaginary parts of the roots $g_k \equiv \gamma_k b = \alpha_k + i\beta_k$ of Eqs. (31) and (32) satisfy, respectively, the following systems of transcendental equations:

$$\begin{cases} \sinh(2\alpha_k)\cos(2\beta_k) + 2\alpha_k = 0, \\ \cosh(2\alpha_k)\sin(2\beta_k) + 2\beta_k = 0 \end{cases}$$

and

$$\begin{cases} \sinh(2\alpha_k)\cos(2\beta_k) - 2\alpha_k = 0, \\ \cosh(2\alpha_k)\sin(2\beta_k) - 2\beta_k = 0. \end{cases}$$

The asymptotic values of these roots as $k \to \infty$ are given by the formulas

$$\alpha_k^a = \frac{1}{2}\ln(\pi + 4\pi k), \qquad \beta_k^a = -\frac{\pi}{4} + \pi k$$

and

$$\alpha_k^a = \frac{1}{2}\ln(\pi + 4\pi k), \qquad \beta_k^a = \frac{\pi}{4} + \pi k,$$

respectively.

Thus, we get two systems of homogeneous complex roots of the biharmonic equation (12) for the odd (27) and even (28) functions $\varphi(z)$. This means that, according to representation (26), the corresponding expressions for the Love functions take the form

$$\chi_k''(r,z) = J_0(\gamma_k r)\big(\kappa_k \sinh(\gamma_k z) + z\cosh(\gamma_k z)\big),$$

$$\chi_k''(r,z) = J_0(\gamma_k r)\big(\kappa_k \cosh(\gamma_k z) + z\sinh(\gamma_k z)\big).$$

Hence, we get the solution of the second perturbed problem in the form

$$\chi''(r,z) = \frac{1}{2}\sum_{k=1}^{\infty}\big(C_k\chi_k''(r,z) + \bar{C}_k\bar{\chi}_k''(r,z)\big), \tag{33}$$

where C_k are unknown complex constants.

2.3. Components of Stresses and Displacements. The sum of components of the stress tensor and the vector of displacements in the principal state and two perturbed states gives the complete solution of the original problem:

$$\sigma_{rr}(r,z) = \sigma^0_{rr}(r,z) + \sigma'_{rr}(r,z) + \sigma''_{rr}(r,z),$$

$$\sigma_{\theta\theta}(r,z) = \sigma^0_{\theta\theta}(r,z) + \sigma'_{\theta\theta}(r,z) + \sigma''_{\theta\theta}(r,z),$$

$$\sigma_{zz}(r,z) = \sigma^0_{zz}(r,z) + \sigma'_{zz}(r,z) + \sigma''_{zz}(r,z),$$

$$\sigma_{rz}(r,z) = \sigma^0_{rz}(r,z) + \sigma'_{rz}(r,z) + \sigma''_{rz}(r,z),$$

$$u_r(r,z) = u^0_r(r,z) + u'_r(r,z) + u''_r(r,z),$$

$$u_z(r,z) = u^0_z(r,z) + u'_z(r,z) + u''_z(r,z).$$

For the perturbed problem, the stresses and displacements can be represented in the form

$$\begin{aligned}
\tilde{\sigma}_{rr}(r,z) &= \frac{1}{2}\sum_{k=1}^{\infty}\left(C_k\sigma_{krr}(r,z)+\bar{C}_k\bar{\sigma}_{krr}(r,z)\right),\\
\tilde{\sigma}_{zz}(r,z) &= \frac{1}{2}\sum_{k=1}^{\infty}\left(C_k\sigma_{kzz}(r,z)+\bar{C}_k\bar{\sigma}_{kzz}(r,z)\right),\\
\tilde{\sigma}_{\theta\theta}(r,z) &= \frac{1}{2}\sum_{k=1}^{\infty}\left(C_k\sigma_{k\theta\theta}(r,z)+\bar{C}_k\bar{\sigma}_{k\theta\theta}(r,z)\right),\\
\tilde{\sigma}_{rz}(r,z) &= \frac{1}{2}\sum_{k=1}^{\infty}\left(C_k\sigma_{krz}(r,z)+\bar{C}_k\bar{\sigma}_{krz}(r,z)\right),\\
\tilde{u}_r(r,z) &= \frac{1}{2}\sum_{k=1}^{\infty}\left(C_k u_{kr}(r,z)+\bar{C}_k\bar{u}_{kr}(r,z)\right) + az + d,\\
\tilde{u}_z(r,z) &= \frac{1}{2}\sum_{k=1}^{\infty}\left(C_k u_{kz}(r,z)+\bar{C}_k\bar{u}_{kz}(r,z)\right) + c,
\end{aligned} \tag{34}$$

where a, c, and d are unknown constants. In relations (34), we have used the following notation:

– for *the first symmetric perturbed* problem:

$$\begin{aligned}
\sigma_{krr} &= -2\mu\gamma_k^2 \cosh(\gamma_k z)\Bigg(\kappa_k\big((1-2\nu)J_0(\gamma_k r)-\gamma_k r J_1(\gamma_k r)\big) \\
&\qquad + \frac{2}{\pi\gamma_k r}(\gamma_k r J_0(\gamma_k r)-J_1(\gamma_k r))\Bigg), \\
\sigma_{kzz} &= -2\mu\gamma_k^2 \cosh(\gamma_k z)\Bigg(-\frac{2}{\pi}J_0(\gamma_k r) \\
&\qquad + \kappa_k\big(2(\nu-2)J_0(\gamma_k r)+\gamma_k r J_1(\gamma_k r)\big)\Bigg), \\
\sigma_{k\theta\theta} &= -2\mu\gamma_k \cosh(\gamma_k z)\Bigg((1-2\nu)\gamma_k\kappa_k J_0(\gamma_k r) + \frac{2}{\pi r}J_1(\gamma_k r)\Bigg), \\
\sigma_{krz} &= 2\mu\gamma_k^2 \sinh(\gamma_k z)\Bigg(-\frac{2}{\pi}J_1(\gamma_k r) \\
&\qquad + \kappa_k\big((2\nu-2)J_1(\gamma_k r)-\gamma_k r J_0(\gamma_k r)\big)\Bigg), \\
u_{kr} &= -\frac{\gamma_k}{\pi}\cosh(\gamma_k z)\big(2J_1(\gamma_k r)+\pi\gamma_k\kappa_k r J_0(\gamma_k r)\big), \\
u_{kz} &= \gamma_k \sinh(\gamma_k z)\Bigg(\big(\gamma_k r J_1(\gamma_k r)+4(1-\nu)J_0(\gamma_k r)\big)\kappa_k - \frac{2}{\pi}J_0(\gamma_k r)\Bigg);
\end{aligned} \tag{35}$$

– *for the first antisymmetric perturbed* problem, it is necessary to replace $\cosh(\gamma_k z)$ with $\sinh(\gamma_k z)$ and $\sinh(\gamma_k z)$ with $\cosh(\gamma_k z)$ in relations (35);

– *for the second symmetric perturbed* problem:

$$\begin{aligned}
\sigma_{krr} &= 2\mu\frac{\gamma_k}{r}\big((\gamma_k r(2\nu+1+\gamma_k\kappa_k)J_0(\gamma_k r)-(1+\gamma_k\kappa_k)J_1(\gamma_k r)\big) \\
&\qquad \times \cosh(\gamma_k z)+\gamma_k z\big(\gamma_k r J_0(\gamma_k r)-J_1(\gamma_k r)\big)\sinh(\gamma_k z)\big), \\
\sigma_{kzz} &= -2\mu(\gamma_k)^2 J_0(\gamma_k r)\big(\gamma_k z\sinh(\gamma_k z)+(2\nu-1+\gamma_k\kappa_k)\cosh(\gamma_k z)\big), \\
\sigma_{k\theta\theta} &= 2\mu\frac{\gamma_k}{r}\big((2\nu\gamma_k r J_0(\gamma_k r)+(1+\gamma_k\kappa_k)J_1(\gamma_k r)\big)\cosh(\gamma_k z) \\
&\qquad + \gamma_k z J_1(\gamma_k r)\sinh(\gamma_k z)\big),
\end{aligned}$$

$$\sigma_{krz} = 2\mu\gamma_k^2 J_1(\gamma_k r)\big((2\nu+\gamma_k\kappa_k)\sinh(\gamma_k z)+\gamma_k z\cosh(\gamma_k z)\big),$$

$$u_{kr} = \gamma_k J_1(\gamma_k r)\big(\gamma_k z\sinh(\gamma_k z)+(1+\gamma_k\kappa_k)\cosh(\gamma_k z)\big),$$

$$u_{kz} = \gamma_k J_0(\gamma_k r)\big((\gamma_k\kappa_k-4\nu+6)\sinh(\gamma_k z)+\gamma_k z\cosh(\gamma_k z)\big),$$

– for *the second antisymmetric perturbed* problem:

$$\sigma_{krr} = 2\mu\frac{\gamma_k}{r}\big((\gamma_k r(2\nu+1+\gamma_k\kappa_k)J_0(\gamma_k r)-(1+\gamma_k\kappa_k)J_1(\gamma_k r))\sinh(\gamma_k z)$$

$$+\big(\gamma_k z(\gamma_k r J_0(\gamma_k r)-J_1(\gamma_k r)\big)\cosh(\gamma_k z)\big),$$

$$\sigma_{kzz} = -2\mu(\gamma_k)^2 J_0(\gamma_k r)\big(\gamma_k z\cosh(\gamma_k z)+(2\nu-1+\gamma_k\kappa_k)\sinh(\gamma_k z)\big),$$

$$\sigma_{k\theta\theta} = 2\mu\frac{\gamma_k}{r}\big((2\nu\gamma_k r J_0(\gamma_k r)+(1+\gamma_k\kappa_k)J_1(\gamma_k r))\sinh(\gamma_k z)$$

$$+\gamma_k z J_1(\gamma_k r)\cosh(\gamma_k z)\big),$$

$$\sigma_{krz} = 2\mu\gamma_k^2 J_1(\gamma_k r)\big((2\nu+\gamma_k\kappa_k)\cosh(\gamma_k z)+\gamma_k z\sinh(\gamma_k z)\big),$$

$$u_{kr} = \gamma_k J_1(\gamma_k r)\big(\gamma_k z\cosh(\gamma_k z)+(1+\gamma_k\kappa_k)\sinh(\gamma_k z)\big),$$

$$u_{kz} = \gamma_k J_0(\gamma_k r)\big((\gamma_k\kappa_k-4\nu+6)\cosh(\gamma_k z)+\gamma_k z\sinh(\gamma_k z)\big).$$

For the first symmetric and antisymmetric perturbed problems, the constants are equal to $a=c=d=0$, for the second symmetric perturbed problem, we get $a=c=0$, and for the antisymmetric problem, $d=0$.

2.4. Variational Method of Homogeneous Solutions for the Perturbed Problem. To find the unknown coefficients C_k, we assume that solutions (23) and (33) satisfy conditions (14) and (24), respectively, and apply the variational approach [9, 11–13]. For this purpose, we introduce quadratic functionals for the first and second perturbed problems, respectively,

$$F_1 = \int_0^1 \Big((\tilde{\sigma}_{zz}\big|_{z=-b}-\sigma_1(r))^2+(\tilde{\sigma}_{rz}\big|_{z=-b}-\tau_1(r))^2\Big)dr,$$

$$F_2 = \int_0^b \Big((\tilde{u}_r\big|_{r=1}-u_1(z))^2+(\tilde{u}_z\big|_{r=1}-u_2(z))^2\Big)dz. \qquad (36)$$

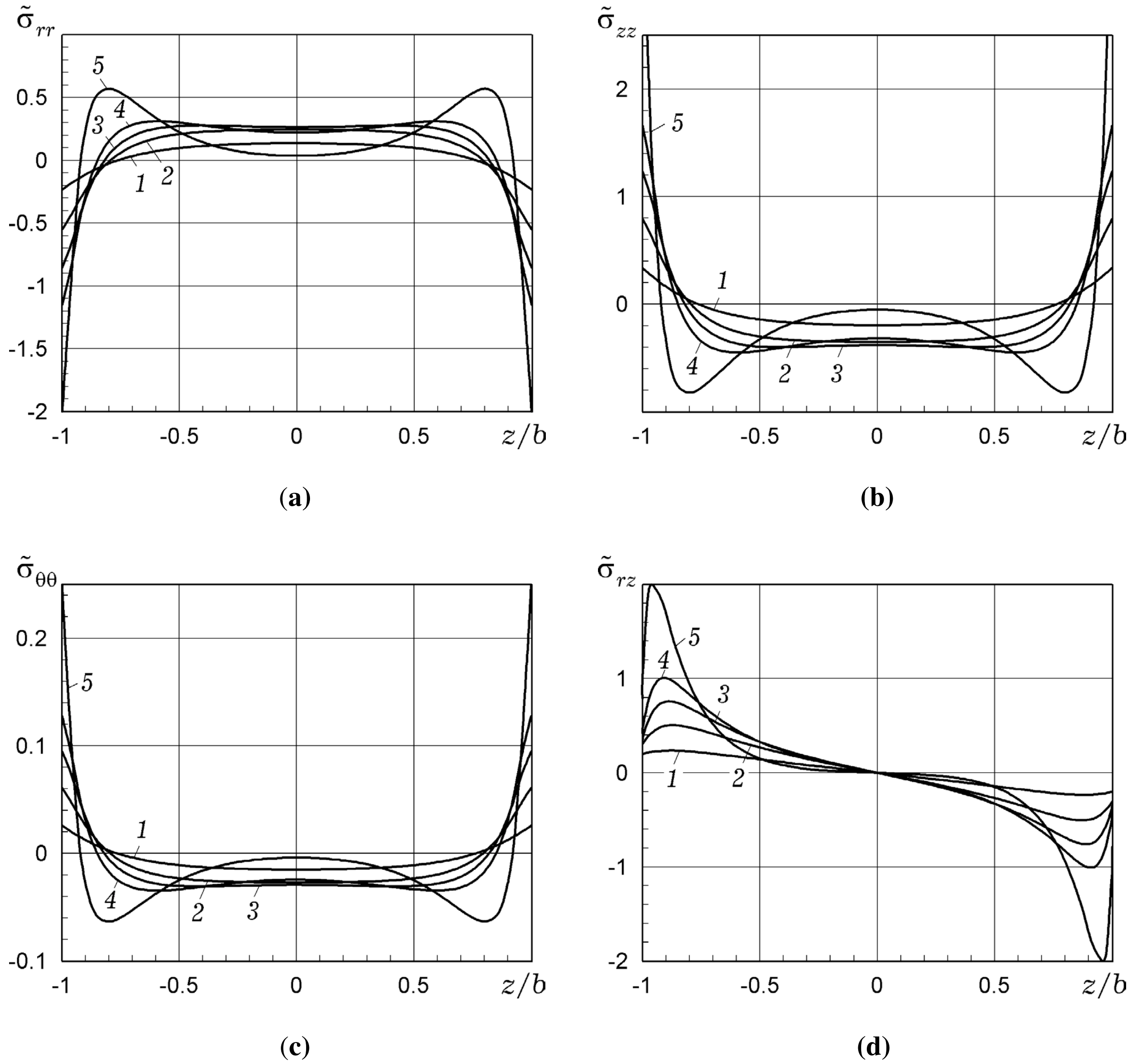


Fig. 1

Substituting representations (34) in functionals (36) and applying the necessary conditions of minimum, namely,

$$\frac{\partial F_i}{\partial C_m} = 0, \quad \frac{\partial F_i}{\partial \bar{C}_m} = 0, \quad \frac{\partial F_i}{\partial a} = 0, \quad \frac{\partial F_i}{\partial c} = 0, \quad \frac{\partial F_i}{\partial d} = 0,$$

where $m = 1, 2, \ldots,$ $i = 1, 2,$ we arrive at the infinite systems of linear algebraic equations for the complex constants $C_k^1 = C_k$ and $C_k^2 = \bar{C}_k$:

$$\sum_{k=1}^{\infty} \sum_{p=1}^{2} M_{mk}^{\ell p} C_k^p = K_m^{\ell}. \tag{37}$$

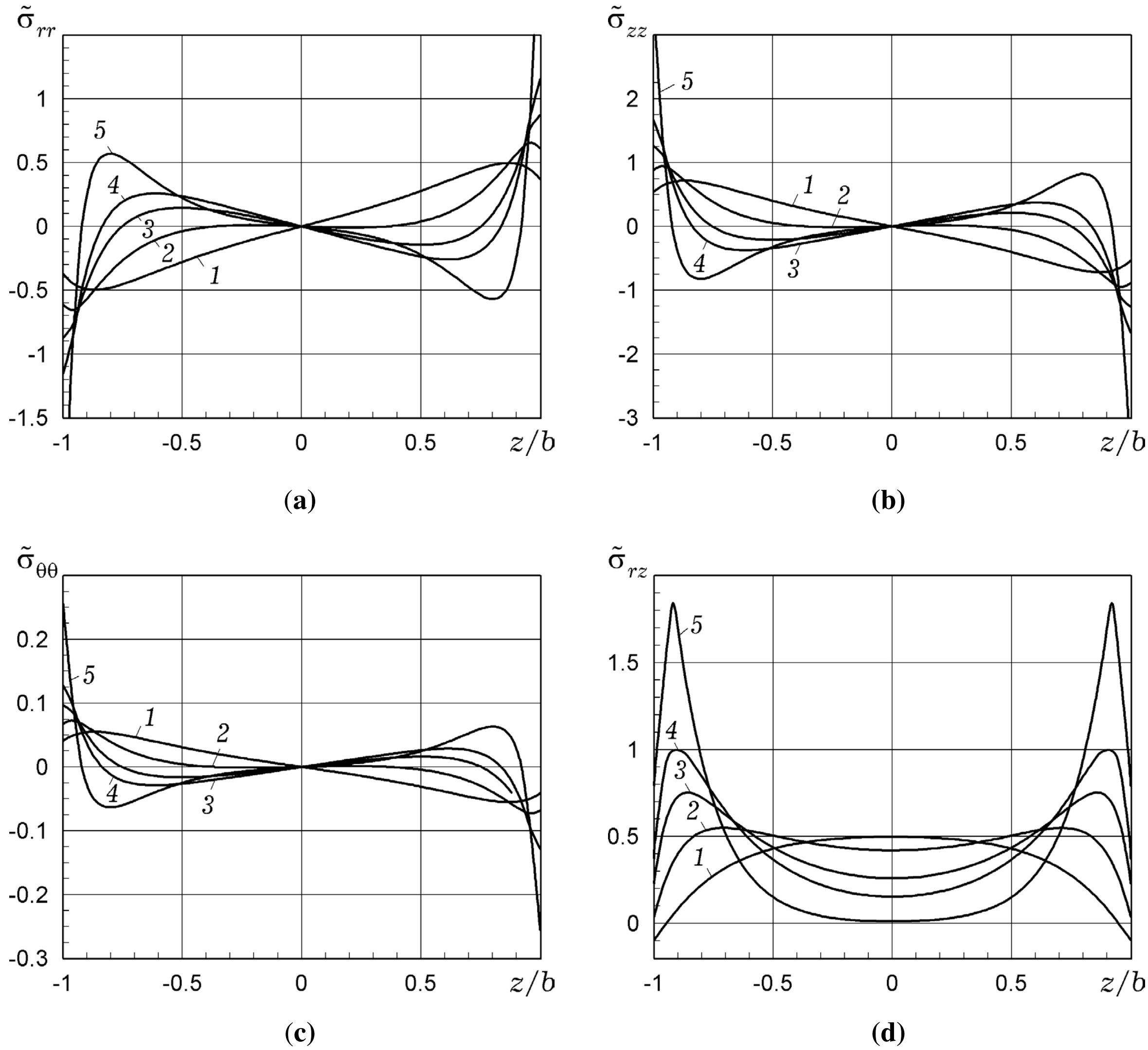


Fig. 2

The coefficients $M_{mk}^{\ell p}$, K_m^{ℓ}, $\ell = 1, 2$, $m = 1, 2, \ldots$, of system (37) are determined by the following formulas:

– for *the first symmetric and antisymmetric perturbed* problems:

$$M_{mk}^{\ell p} = \frac{1}{2}\int_0^1 \left(\sigma_{kzz}^{p}(r,-b)\sigma_{mzz}^{\ell}(r,-b) + \sigma_{krz}^{p}(r,-b)\sigma_{mrz}^{\ell}(r,-b)\right) dr\,,$$

$$K_m^{\ell} = \int_0^1 \left(\sigma_1(r)\sigma_{mzz}^{\ell}(r,-b) + \tau_1(r)\sigma_{mrz}^{\ell}(r,-b)\right) dr\,, \tag{38}$$

– for *the second symmetric perturbed* problem:

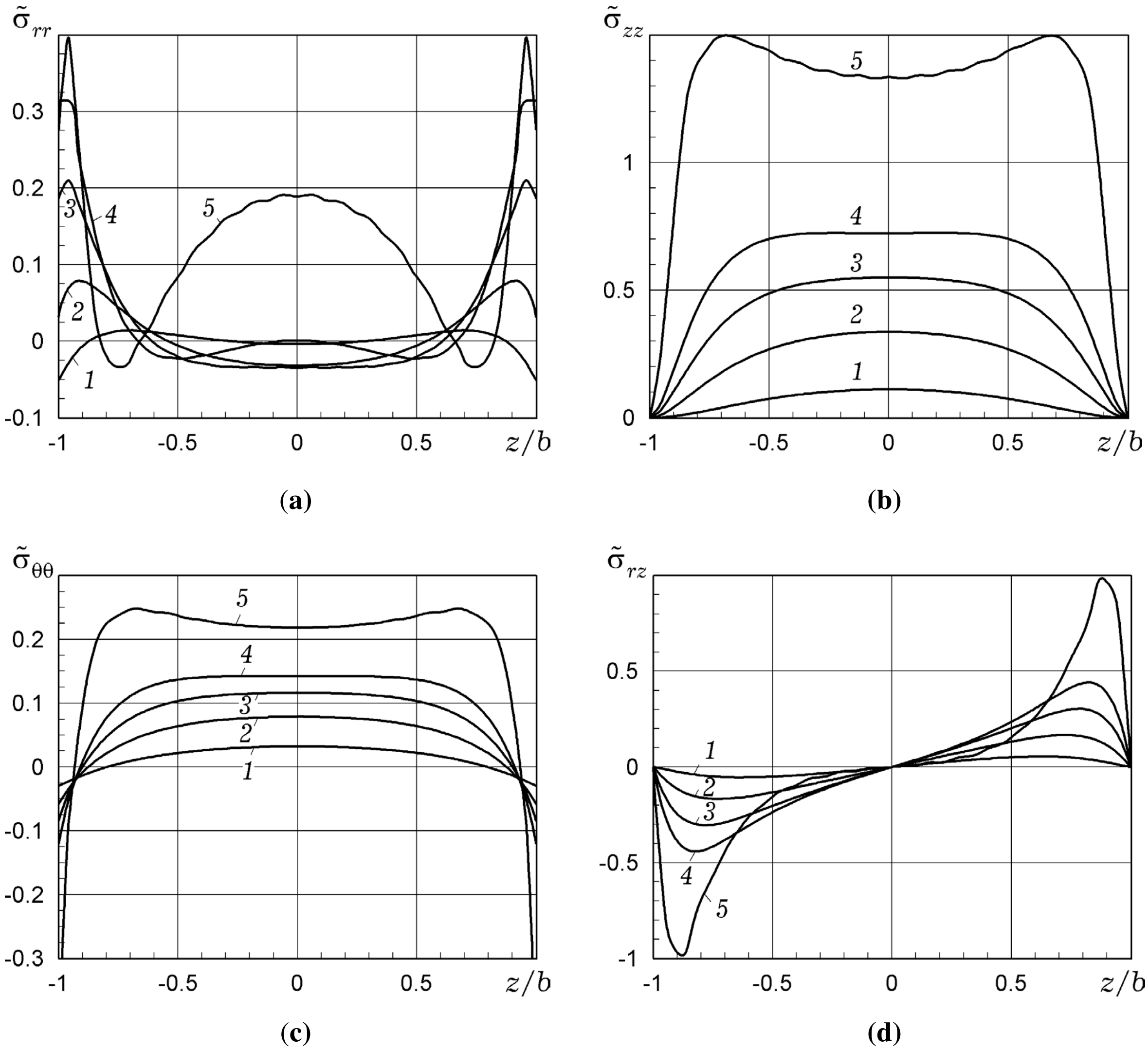


Fig. 3

$$M_{mk}^{\ell p} = \frac{1}{2}\int_0^b \left(u_{kr}^p(1,z)\,u_{mr}^{\ell}(1,z)+u_{kz}^p(1,z)\,u_{mz}^{\ell}(1,z)\right)dz$$

$$-\frac{1}{2b}\int_0^b u_{kr}^p(1,z)\,dz\int_0^b u_{mr}^{\ell}(1,z)\,dz,$$

(39)

$$K_m^{\ell} = \int_0^b \left(u_1(z)\,u_{mr}^{\ell}(1,z)+u_2(z)\,u_{mz}^{\ell}(1,z)\right)dz$$

$$-\frac{1}{b}\int_0^b u_1(z)\,dz\int_0^b u_{mr}^{\ell}(1,z)\,dz,$$

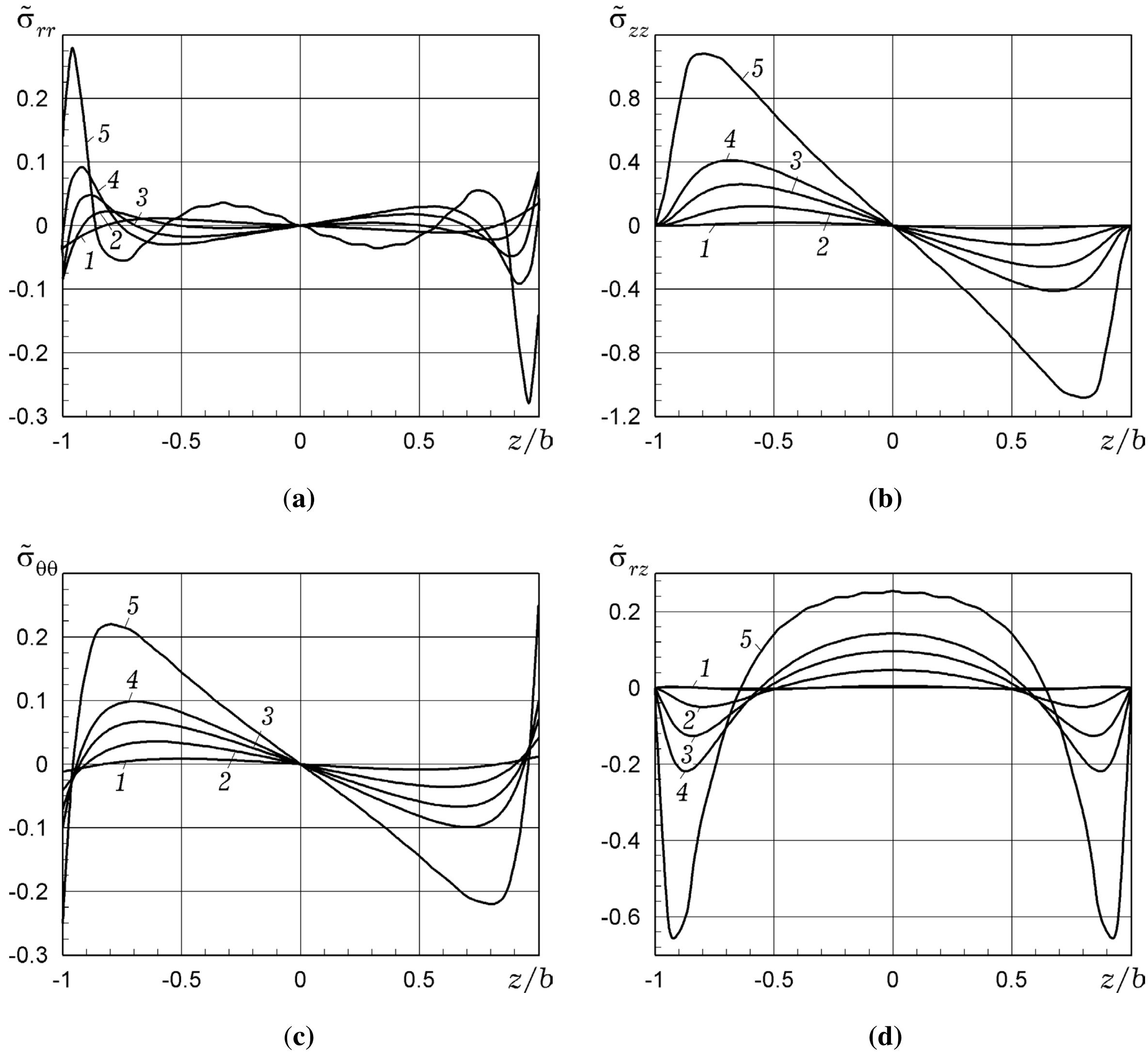


Fig. 4

– for *the second antisymmetric perturbed* problem:

$$M_{mk}^{\ell p} = \frac{1}{2}\int_0^b \left(u_{kr}^p(1,z)\,u_{mr}^\ell(1,z)+u_{kz}^p(1,z)\,u_{mz}^\ell(1,z)\right)dz$$

$$-\frac{3}{2b^3}\int_0^b u_{kr}^p(1,z)\,dz\int_0^b u_{mr}^\ell(1,z)\,dz-\frac{1}{2b}\int_0^b u_{kz}^p(1,z)\,dz\int_0^b u_{mz}^\ell(1,z)\,dz,$$

(40)

$$K_m^\ell = \int_0^b \left(u_1(z)u_{mr}^\ell(1,z)+u_2(z)u_{mz}^\ell(1,z)\right)dz$$

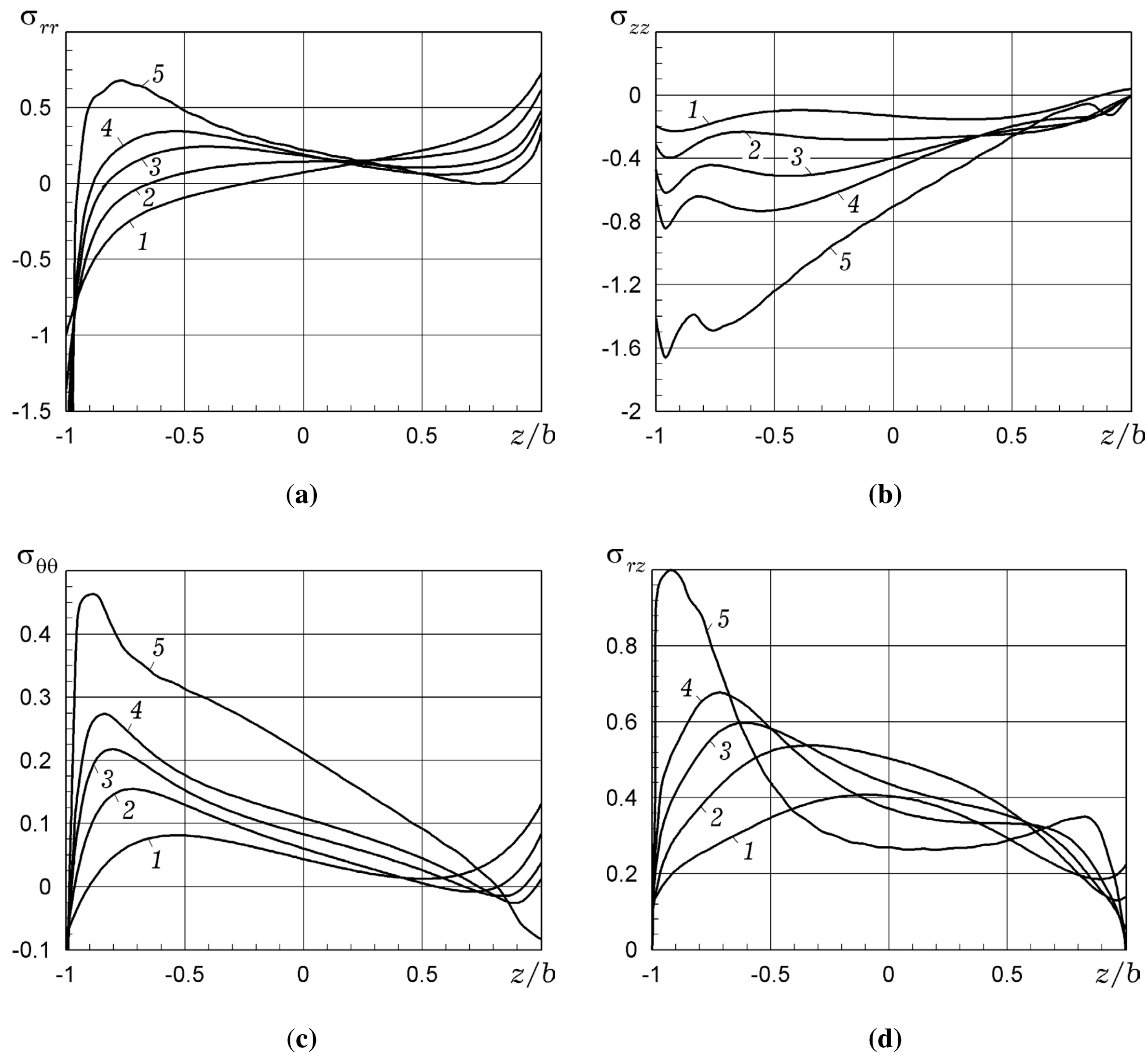


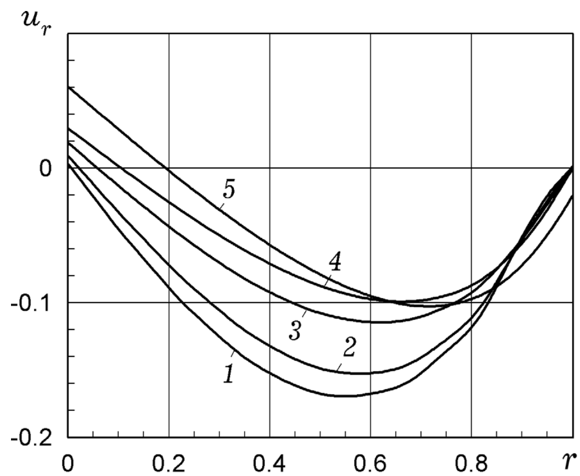
Fig. 5

$$-\frac{3}{b^3}\int_0^b u_1(z)\,dz\int_0^b u_{mr}^{\ell}(1,z)\,dz-\frac{1}{b}\int_0^b u_2(z)\,dz\int_0^b u_{mz}^{\ell}(1,z)\,dz.$$

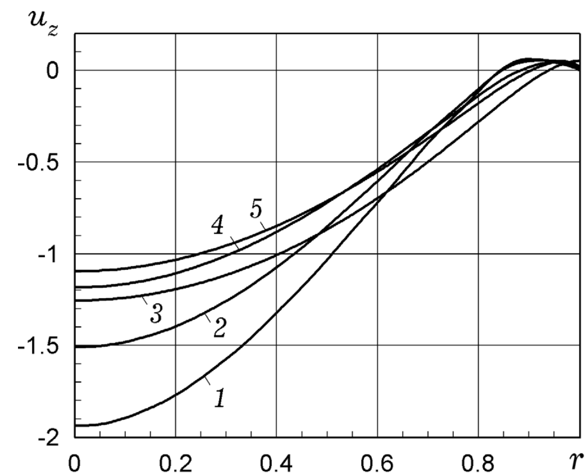
The constants a, c, and d are given by the following formulas:

$$a=-\frac{3}{b^3}\int_0^b \frac{1}{2}\sum_{k=1}^{\infty}(C_k u_{kr}(1,z)+\overline{C}_k\overline{u}_{kr}(1,z)-u_1(r))z\,dz,$$

$$c=-\frac{1}{b}\int_0^b \frac{1}{2}\sum_{k=1}^{\infty}\left(C_k u_{kz}(1,z)+\overline{C}_k\overline{u}_{kz}(1,z)-u_2(r)\right)dz,$$



(a)



(b)

Fig. 6

$$d = -\frac{1}{b}\int_0^b \frac{1}{2}\sum_{k=1}^{\infty}\left(C_k u_{kr}(1,z)+\bar{C}_k \bar{u}_{kr}(1,z)-u_1(r)\right)dz.$$

In relations (38)–(40), we have introduced the following notation: $\sigma^1_{kzz}=\sigma_{kzz}$, $\sigma^2_{kzz}=\bar{\sigma}_{kzz}$, $\sigma^1_{krz}=\sigma_{krz}$, $\sigma^2_{krz}=\bar{\sigma}_{krz}$, $u^1_{kr}=u_{kr}$, $u^2_{kr}=\bar{u}_{kr}$, $u^1_{kz}=u_{kz}$, and $u^2_{kz}=\bar{u}_{kz}$.

3. Examples of Numerical Investigations of the Solution

The system of equations (37) was solved by the method of reduction [11] according which finite systems of N equations for N unknown coefficients are considered instead of the corresponding infinite systems. For the results presented in what follows, we restricted ourselves to $N=15$. The calculations were performed for $\rho g=1$ and $\nu=0.25$, and the components of the stress tensor were normalized to Young's modulus.

For the first perturbed symmetric problem, we present the distributions of components of the stress tensor $\tilde{\sigma}_{rr}$, $\tilde{\sigma}_{zz}$, $\tilde{\sigma}_{\theta\theta}$, and $\tilde{\sigma}_{rz}$ along the axial coordinate z normalized by b in Fig. 1; for the first perturbed antisymmetric problem, the corresponding distributions are shown in Fig. 2; for the second perturbed symmetric problem, they are shown in Fig. 3, and for the second perturbed antisymmetric problem, in Fig. 4. Curves ***1–5*** in these figures correspond to the values $b=0.25$, 0.5, 0.75, 1.0, and 2.0, respectively. The components of the stress tensor were found on the surface S_0.

It is easy to see that the stresses in the perturbed state have a well-pronounced edge effect, i.e., the gradients of the components of the stress tensor noticeably increase as b increases in the vicinity of the end faces of the cylinder.

In Fig. 5, we show the distributions of components of the stress tensor in the original problem σ_{rr}, σ_{zz}, $\sigma_{\theta\theta}$, and σ_{rz} along the axial coordinate z referred to the parameter b for the following values of length of the cylinder: $b=0.25$, 0.5, 0.75, 1, 2 (curves ***1–5***, respectively) and $r=1$.

The dependences of the components of the vector of displacements u_r and u_z on the radial coordinate r for the end face of the cylinder S_2 are shown in Fig. 6 for the following lengths of the cylinder: $b = 0.25, 1.5, 1.75, 1, 2$ (curves ***1–5***, respectively).

CONCLUSIONS

By using the variational method of homogeneous solutions, we formulate and solve the axisymmetric problem of the theory of elasticity for a finite cylindrical body under the action of its own weight. The solution of the problem is represented in the form of superposition of the principal and perturbed states. The problem of determination of the perturbed state is split into two problems, for each of which we analyze the symmetric and antisymmetric cases about the middle cross-section of the cylinder. The solution of each of these four problems is represented in the form of expansion in the complete systems of complex axisymmetric homogeneous solutions of the Lamé equations. We introduce quadratic functionals for solving the first and second perturbed problems, which determine the deviations of solutions from the corresponding given inhomogeneous conditions in the quadratic norm. The conditions of minimum for the functionals give infinite systems of linear algebraic equations with respect to the coefficients of expansion of solutions, which are solved by the method of reduction preserving finitely many terms of expansions in the series. We also analyze the efficiency of application of the proposed variational approach to the solution of axisymmetric problems for different lengths of the cylinder.

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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