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LOCALLY EUCLIDEAN METRICS AND THEIR ISOMETRIC REALIZATIONS

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Abstract. There are many works related to metrics and surfaces of positive and negative curvature. This paper is a survey of results related to locally Euclidean metrics and surfaces with such metrics. There are many problems included in the intersection of geometry, complex analysis, and differential equations that can become a source of new interesting research.

Keywords and phrases**:** locally Euclidean metric, natural representation, classification, isometric realization, developable surface, asymptotic coordinates, Monge—Ampère equation.

AMS Subject Classification**:** 53A05, 53C45

1. Definitions and statements. The work is based on the report read by the author at the International Conference "Classical and Modern Geometry" organized by the Moscow Pedagogical State University in connection with the 100th anniversary of the birth of Professor V. T. Bazylev.

The metric of an n-dimensional Riemannian manifold is said to be locally Euclidean if each of its points has a neighborhood isometric to a ball in the Euclidean space $\mathbb{R}^n$ with the standard metric $dx_1^2 + \ldots + dx_n^2$. There are quite a lot of such manifolds. For example, the metric of any n-dimensional polyhedral surface with punctured vertices is locally Euclidean. One of criteria of minimality of a two-dimensional surface with a known metric ds^2 is the Ricci test: A surface with a metric ds^2 is minimal if and only if the surface has nonpositive Gaussian curvature K and the metric $ds_e^2 = \sqrt{-K}ds^2$ is locally Euclidean outside the points where the curvature is zero. This means that on any minimal surface, there exists a locally Euclidean metric.

When studying locally Euclidean metrics, several questions immediately arise:

1. Let the metric of a Riemannian space be defined by a quadratic form
$$ds^2 = g_{ij}\, du^i\, du^j. \tag{1}$$
How can we know if it is locally Euclidean?
2. Let the metric (1) be locally Euclidean. How can we find an isometric mapping into $\mathbb{R}^n$, which exists by the definition of a locally Euclidean metric?
3. What are the properties of this mapping: smoothness class, domain of existence, behavior in general, etc.?
4. What theorems and problems of Euclidean geometry can be transferred to domains with a locally Euclidean metric?
5. Under which conditions can a given locally Euclidean metric be isometrically realized as the metric of a surface in a Euclidean space of an appropriate dimension?
6. What can be said about existence, properties, equations, etc. surfaces with locally Euclidean metric?

2. On the natural representation of locally Euclidean metrics. When specifying a locally Euclidean metric in the general form (1) in a certain domain D of the variables $u^1, \ldots, u^n$, we actually do not see the geometry; for example, geodesics in D are, in general, some curves, and we cannot see either their lengths or the angles between them even in the dimension $n = 2$ (all these quantities can only be found by calculations using the appropriate formulas). If we find an isometric mapping

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of the metric into $\mathbb{R}^n$, then geodesics will turn into straight lines, orthogonal lines will form right angles, etc. We can say that an isometric image in $\mathbb{R}^n$ of the domain D with a locally Euclidean metric represents, at least locally, a visual picture of the geometry in D. Therefore, it is natural to call this image a *natural* representation of a locally Euclidean geometry defined in D. Such a representation has so far been made only in the two-dimensional case. However, for any dimension, one can know in advance the smoothness class of transition from an arbitrary local Euclidean metric (1) to its natural representation and thus find the answer to one of the questions from Sec. 3. Namely, the following theorem on the smoothness of isometries holds.

Theorem 1 (see [2, 11, 13]). *Let two isometric n-dimensional Riemannian manifolds have the metrics*

$$ds^2 = g_{ij}\, du^i\, du^j, \quad d\sigma^2 = h_{ij}\, d\xi^i\, d\xi^j$$

of the smoothness class $C^{k,\alpha}$, $0 \le k < \infty$, $0 \le \alpha \le 1$, $k + \alpha > 0$. *Then an isometry between then has smoothness of class* $C^{k+1,\alpha}$. *In the case where these metrics are analytic or* C^∞*-smooth, isometries have the same smoothness class. If the metrics are defined by continuous forms, isometries in general are of the class* $C^{0,1}$.

Remark 2. The nontriviality of this theorem is that in the definition of an isometric map, there is a priori no requirement about its smoothness class, there is only a requirement that it preserve the distances between the corresponding points in the preimage and the image. This immediately implies that the isometry belongs to the Lipschitz class $C^{0,1}$.

Since the standard metric in $\mathbb{R}^n$ is analytic, the smooth of the isometry of a locally Euclidean metric in $\mathbb{R}^n$ is greater than its smoothness in the initial representation (1) by one.

Note that the same approach to the search for the natural representation can be used for metrics that are metrics of local constant negative or positive curvature, but no one has done this yet.

3. Case of two-dimensional locally Euclidean metrics. Since in this case, any metric can be reduced to isothermal coordinates, we can assume that the metric is given in the isothermal form:

$$ds^2 = \Lambda^2(\xi, \eta)(d\xi^2 + d\eta^2). \tag{2}$$

The following simple criterion holds: *The metric* (2) *with the continuous coefficient* Λ *is locally Euclidean if and only if the function* $\ln \Lambda$ *is harmonic.* One can easily verify that a function

$$z = x + iy = \Phi(\zeta), \quad \zeta = \xi + i\eta, \tag{3}$$

isometrically maps a domain D with the metric (2) onto the Euclidean plane (x, y) with the metric $dx^2 + dy^2$ if and only if it is holomorphic in D; moreover, it is related to the coefficient Λ by the equality

$$|\Phi'(\zeta)| = \Lambda(\zeta).$$

In the case of a simply connected domain D, the function $\Phi(\zeta)$ is defined by the modulus of its derivative (see [14, 16]), and the mapping constructed is simple-valued, i.e., the following theorem holds.

Theorem 3. *Any locally Euclidean metric defined in a simply connected domain can be isometrically immersed into the Euclidean plane.*

This means that one can visualize the geometry of this metric. However, the image under an immersion can have self-intersections, and it is important to identify cases where the metric can be *embedded* into the plane, i.e., where an abstract locally Euclidean metric is in fact a representation of the Euclidean geometry of a simply connected domain of the ordinary Euclidean plane. One can also indicate some interesting embedding criteria (i.e., criteria of univalent functions $\Phi(\zeta)$) expressed through the properties of the coefficient Λ (see [14, 16]).

Consider locally Euclidean metrics in multiply connected domains. In this case, the problem of the search for a natural representation is more complicated due to a larger number of solution options. Let the metric (2) be defined in an $(n+1)$-connected domain Ω with circular boundaries $\Gamma_j : |\zeta - \zeta_j| = R_j$, $1 \leq j \leq n$, located inside the circle $\Gamma_0 : |\zeta| = R_0$. As above, the holomorphic function $\Phi(\zeta)$, which locally conformally maps the domain Ω into the Euclidean plane, is defined by its derivative $\Phi'(\zeta)$ satisfying the condition $|\Phi'(\zeta)| = \Lambda(\zeta)$. The search for Φ' and Φ is a more difficult problem. For example, for a concentric annulus, we obtain the followingformulas :

$$\Phi'(\zeta) = \zeta^c A(\zeta), \quad \Phi(\zeta) = d_0 \ln \zeta + \zeta^{1+c} B(\zeta),$$

where $A(\zeta)$ and $B(\zeta)$ are single-valued holomorphic functions and c and d_0 are numerical (in general, complex-valued) coefficients, which can be explicitly expressed through the metric (2).

In the case of an integer exponent c, the function $\Phi'(\zeta)$ is single-valued; however, $\Phi(\zeta)$ is a single-valued function only under the additional condition $d_0 = 0$. For $d_0 \neq 0$, the mapping Φ has a logarithmic singularity and does not provide an isometric immersion into $\mathbb{R}^2$; since the image $\Phi(\Omega)$ is similar to the envelope of a cylinder, this singularity is said to be *cylindrical.* If c is noninteger, the mapping Φ is also not single-valued; since the image $\Phi(\Omega)$ is similar to the envelope of a cone, this singularity is said to be *conical.* Apparently, natural representations of locally Euclidean metrics do not correspond to *torsos.* This may be due to the fact that the spherical image of each torsional surface can be represented as the spherical image of some conical surface (see [18]), but, in general, this phenomenon—the absence of torso features in the internal geometry of domains with locally Euclidean metrics—requires a separate analysis.

Example 4. In the annulus $\Omega : R_0 \leq |\zeta| \leq R_1$, consider the locally Euclidean metric

$$ds^2 = \rho^{2a}(d\xi^2 + d\eta^2), \quad \rho^2 = \xi^2 + \eta^2. \tag{4}$$

We have

$$\Phi'(\zeta) = \zeta^a, \ \Phi(\zeta) = \frac{\zeta^{a+1}}{a+1}, \ a \neq -1; \quad \Phi(\zeta) = \ln \zeta, \ a = -1.$$

Therefore, for integer $a \neq -1$, the metric (4) can be isometrically immersed into $\mathbb{R}^2$ as an $|a+1|$-fold annulus, but for $a = -1$, an immersion does not occur. Also, immersions are absent for noninteger a: the image of the annulus Ω is a circular sector similar to the envelope of a cone.

4. Isometric immersions of locally Euclidean metrics in $\mathbb{R}^3$. We start with isometric immersions of locally Euclidean metrics in $\mathbb{R}^2$, which can actually be considered as a special case of searching for an isometric implementation of a given metric as a surface metric in the space or, in this case, in the form of a domain on the plane, which is possible only for metrics of constant curvature, including metrics of negative or positive curvature with their mapping to the Lobachevsky plane or sphere, respectively (this formulation of the problem has not yet been used anywhere).

The image of the domain D in $\mathbb{R}^2$ under the mapping (3) provides a visual representation of the locally Euclidean geometry. The mapping (3) may be univalent; then we obtain an *isometric embedding* of the metric (2) into $\mathbb{R}^3$ as a flat domain; the mapping may be single-valued but multivalent, i.e., with self-overlays, and then we obtain an *isometric immersion* of the metric into $\mathbb{R}^3$ as a domain of a plane. In particular, we recall that a simply connected domain with a locally Euclidean metrics can always be immersed into the plane $\mathbb{R}^2$.

Finally, the mapping $z = \Phi(\zeta)$ can be multivalued similarly to the logarithmic or power function with a noninteger exponent. Recall that the latter occurs only if the domain D is multiply connected, and cases of logarithmic or power singularities correspond to the cylindrical or conical structure of the corresponding surface in $\mathbb{R}^3$.

Now we consider isometric embeddings into $\mathbb{R}^3$. We saw above that a locally Euclidean metric can sometimes, but sometimes cannot, be embedded in $\mathbb{R}^2$. It turns out that this property is essential for the possibility of its isometric embedding in $\mathbb{R}^3$.

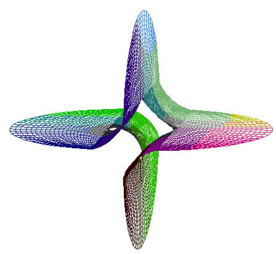

Fig. 1. Cylindrical embedding.

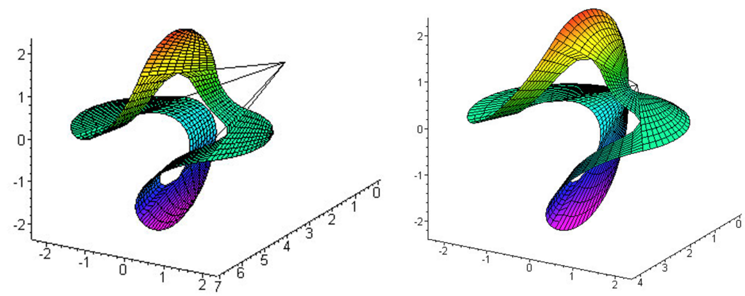

Fig. 2. Conical embeddings.

Theorem 5 (see [8, 9, 12]). *Assume that in a compact domain D with boundary of smoothness class C^2 or higher, a locally Euclidean metric is given, whcih admits an isometric* immersion *into E^2. Then it can be isometrically* embedded *into E^3 as a cylindrical or conical surface of the class C^∞ with boundary of the same smoothness as the boundary of D.*

Corollary 6. *Each two-dimensional compact simple connected domain with a locally Euclidean metric admits an isometric embedding into $\mathbb{R}^3$.*

Note that there are no analogs of this assertion for n-dimensional domains with locally Euclidean metrics (i.e., the minimal dimension of the Euclidean space into which an arbitrary locally Euclidean metric defined in an n-dimensional ball can be isometrically immersed or embedded is not yet known).

Example 7. The proof of Theorem 5 is constructive. An embedding is comstructed as a cylindrical or conical surface (see Figs. 1 and 2). Consider examples of embeddings of the metric (4) with $a = 1$:

$$ds^2 = 4(\xi^2 + \eta^2)(d\xi^2 + d\eta^2).$$

For this metric, we have

$$\Phi(\zeta) = \zeta^2;$$

therefore, the immersed image of the annulus Ω in $\mathbb{R}^2$ is the twofold annulus.

It was said above that any locally Euclidean metric defined in a simply connected domain can be isometrically embedded into $\mathbb{R}^3$. At the same time, we know that in a doubly connected domain, there are metrics of cylindrical and conical types that cannot be immersed into $\mathbb{R}^2$. Nevertheless, the following theorem holds.

Theorem 8 (see [12]). *Each locally Euclidean metric defined in a doubly connected domain can be isometrically immersed into $\mathbb{R}^3$.*

If a metric has a cylindrical singularity, its isometric immersion into $\mathbb{R}^3$ can be easily constructed. However, the construction of an immersion of a metric with a conical singularity requires some ingenuity.

For triply connected domains, the situation is completely different.

Theorem 9 (see [12]). *In triply connected domains, there exist locally Euclidean metrics that do not admit isometric immersions into $\mathbb{R}^3$ even in the class of C^1-smooth ruled surfaces.*

The corresponding example of a triply connected domain with a locally Euclidean metric is presented in Fig. 3. The locally Euclidean metric in this domain is obtained by identifying the segments A_1B_1 and A_2B_2 (we thus obtain the closed boundary line A_1A_2); another boundary curve K_1K_2 is

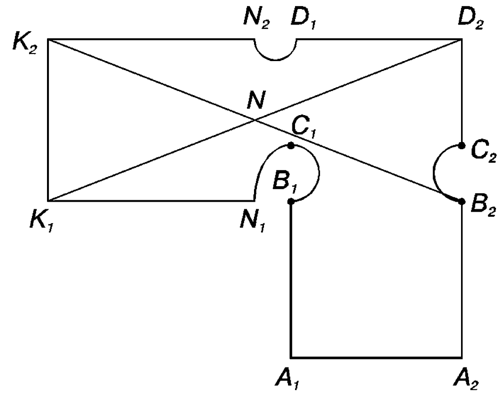

Fig. 3. Example of a triply connected domain from Theorem 9.

obtained by identifying the segments K_1N_1 and K_2N_2; the third boundary curve consists of the arcs $B_1C_1N_1N_2D_1D_2C_2B_2B_1$.

The nonexistence of an isometric immersion of this domain into $\mathbb{R}^3$ follows from the fact that there exist generatrices emanating from points of the line A_1A_2, which intersect with generatrices emanating from points of the line K_1K_2; this is possible only if all points of these generatrices are planar, but this is not so since the lines A_1A_2 and K_1K_2 are not sehments of straight lines.

Note that in [5, Sec. 3.2.1(F)], a statement about the existence of triply connected domains with locally Euclidean metrics that cannot be isometrically immersed into $\mathbb{R}^3$ in the class of classically regular developable surfaces was presented (without a proof).

5. Isometric immersions into $\mathbb{R}^3$ of the Möbius strip. The first explicit isometric embedding of a flat Möbius strip was constructed in 1989 (see [20]), although the first work [3] on this topic known to us was published back in 1898.[1] A smooth nonorientable surface with a locally Euclidean metric, diffeomorphic to a rectangular Möbius strip, is called a *Möbius strip*; if it is also isometric to some rectangular Möbius strip, then it is called a *standard Möbius strip*. In [15] (see also [16]), the following general statement was obtained.

Theorem 10. *An analytical closed curve Γ is the midline of a standard Möbius strip if and only if its curvature is antiperiodic and at the points where the curvature vanishes, the torsion of the curve has a zero of order no less than the curvature. Moreover, the Möbius strip, even in the class of C^2-smooth surfaces, is uniquely determined by its given analytical midline.*

One of the interesting consequences obtained in the proof of this theorem is the fact that closed analytic curves in $\mathbb{R}^3$ are divided into two disjoint classes, characterized by the periodicity or antiperiodicity of their principal normals and binormals.

The general Möbius strip is isometric to a flat strip with curvilinear lateral edges and identifiable parallel transverse edges. An equation of the general Möbius strip was obtained in [15] from a theorem connecting its generatrices with the midline.

Some time ago, the literature discussed the question of finding the simplest equation of the Möbius strip. This question can be posed in various ways. First, one searches for the simplest equation in parametric form, but there are no exact criteria for simplicity: whether the simplest equation of the middle line is needed or the simplest (in some sense) equation of the surface itself, including the representation of generatrices. The question can also include the requirement that the surface is isometric to a flat rectangle, i.e., is a standard M̈bius strip. The simplest representation of the midline

[1]However, it requires rethinking since it was stated in the same work that the surface determined by the equation $x^2(1 - z) = zy^2$ is one-sided, which obviously presupposes some special understanding of one-sidedness.

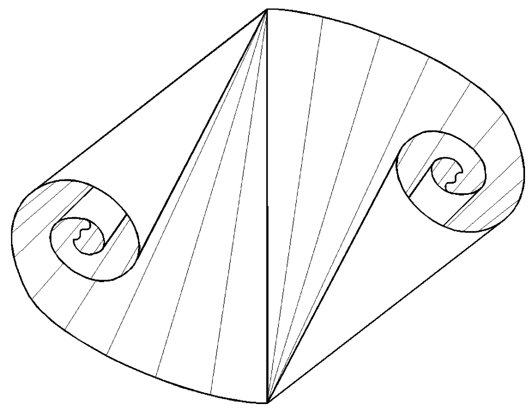

Fig. 4. Example from [7].

of the general Möbius strip was given in [15] in the form $(\cos\varphi, \sin\varphi, \cos\varphi\sin\varphi)$ as the intersection of two surfaces of second order (in [20], the midline was obtained as the intersection of two algebraic surfaces of orders 4 and 6; we also note that an analytic Möbius strip with a flat midline does not exist). Second, the question can be posed in a more specific way, namely, for the whole Möbius strip, find an algebraic equation of the smallest degree. In the 1930s, Wunderlich proposed an algebraic equation of order 40; the Schwartz equation in the algebraic form leads to an equation of degree 20. In [15], an equation of degree 7 for a general Möbius strip was found. However, we note that this equation does not describe the standard Möbius strip, so the question remains open: Which algebraic equation of the smallest degree describes the standard Mb̈ius strip? Note also that we do not claim that for a general Möbius strip, there are no equations of degree less than in [15].

6. Surfaces with locally Euclidean metrics. We are interesting in the structure of surfaces with locally Euclidean metrics. It is well known that such surfaces are called developable surfaces; they consist of rectilinear generatrices, which allow to parametrize these surfaces in the so-called *asymptotic coordinates* as follows:

$$\boldsymbol{r}(u,v) = \boldsymbol{a}(u) + v\boldsymbol{b}(u), \tag{5}$$

where $\boldsymbol{a} = \boldsymbol{a}(u)$ is the position vector of the directrix and $\boldsymbol{b}(u)$ is the unit vector of the generatrix. They satisfy the relations

$$\boldsymbol{b}(u) \nparallel \boldsymbol{a}'(u), \quad (\boldsymbol{a}'(u), \boldsymbol{b}(u), \boldsymbol{b}'(u)) = 0. \tag{6}$$

However, nothing is usually said about the connection between the smoothness classes of the surface and the vectors $\boldsymbol{a}(u)$ and $\boldsymbol{b}(u)$. Unexpectedly, it turned out that the surface can belong to the smoothness class C^∞, whereas the direction vector $\boldsymbol{b}(u)$ of the generatrix does not belong even to the smoothness class C^1, so is not possible to establish the real smoothness class of the surface based on the smoothness of its equation (5) in the asymptotic coordinates. An example of this situation is shown in Fig. 4 from [21] (the example itself was found in [7]).

Moreover, there exist surfaces with locally Euclidean metrics that do not contain any straight-line segments (see [1]). This situation is possible in the smoothness classes $C^{1,\alpha}$, $\alpha < 1/7$. Therefore, it is interesting to find conditions under which a surface with a locally Euclidean metric has the ordinary classical structure with generatrices. Such conditions were obtained by A. V. Pogorelov in 1956 (see [10, Chap. IX]). Pogorelov introduced the class of C^1-smooth surfaces of bounded external curvature. On such surfaces with locally Euclidean metrics, a rectilinear segment passes through each point and the tangent plane is constant along this segment. There are points (called *planar*), which have flat neighborhoods on the surface. Through any nonplanar point, a *unique* generatrix passes in both directions up to the edge of the surface. Sich surfaces are called *torsos*. Their structure guaranteed in the class of surfaces of bounded external curvature can serve a definition in the general case; then they are called *normal developable* surfaces (the term was proposed by Burago and Shefel). Our immediate goal is to find the smoothness properties of the direction curves and the field of generating vectors involved in the parameterization (5).

First, we consider the "good" cases where the surface has no flattening points, i.e., points at which all coefficients of the second quadratic form vanish. In this case, the smoothness of the surface and the vectors defining its equation in the asymptotic coordinates are in the natural relationship.

Theorem 11 (see [6, 16, 21, 22])**.** *A developable surface without flattening points has smoothness C^n, $n \geq 2$, if and only if it has the asymptotic parameterization* (5)*, where $\boldsymbol{a}(u) \in C^n$ and $\boldsymbol{b}(u) \in C^{n-1}$. However, in the general case, the relation $\boldsymbol{b}(u) \in C^n$ may be violated.*

The problem on an analytical description of normal developable surfaces in the general case was posed for a long time, back in the 1970s, but the solution was obtained quite recently. Namely, the following theorem was proved.

Theorem 12 (see [18])**.** *If a surface of the class C^1 is a normal developable surface of smoothness C^n, $n \geq 1$, then it belongs to Pogorelov's class of surfaces of bounded external curvature and possesses an asymptotic parameterization of the form* (5)*, where the direction vectors $\boldsymbol{b}(u)$ of generatrices belong to the Lipschitz class $C^{0,1}$, the directrix has smoothness C^n, and Eqs.* (6) *hold at points where the derivatives $\boldsymbol{b}'(u)$ exist.*

Of course, for all smoothness classes in this theorem, only the statement about the $C^{0,1}$-smoothness of the vectors $\boldsymbol{b}(u)$ is of interest; for the class C^1, it can be formulated as follows: if a C^1-smooth surface with locally Euclidean metric is a torso, then it is a surface of bounded external curvature (recall that the sufficiency was proved by A. V. Pogorelov).

Unfortunately, Theorem 12 does not provide any information about the actual smoothness of a normal developable surface based on differential properties of the vectors $\boldsymbol{a}(u)$ and $\boldsymbol{b}(u)$.

7. Monge–Ampère equation. The actual regularity of any surface can be verified by using its representation in the Cartesian coordinates. If a surface is defined by an equation $z = z(x, y)$, then its metric is locally Euclidean if the following equation is fulfilled

$$z_{xx}z_{yy} - z_{xy}^2 = 0. \tag{7}$$

This equation is called the *trivial* (or *degenerate*) Monge–Ampère equation.

Let us dwell on local properties of solutions of this equation, which have some analogs with solutions of elliptic equations: the phenomena of removability of singular points and an increase in the actual smoothness of solutions.

It is well known that the formally required smoothness of solutions to elliptic equations can actually be increased depending on the smoothness of the coefficients of the equation. A similar property holds for solutions of Eq. (7).

Theorem 13 (see [16, 21, 22])**.** *Let a solution $z(x, y)$ of Eq.* (7) *be such that $z(x, y) \in C^n$, $n \geq 2$, and $z_{xx} \neq 0$. Then*

$$\frac{z_{xy}}{z_{xx}} \in C^{n-1}, \qquad \frac{z_{yy}}{z_{xx}} \in C^{n-1}.$$

This property is essentially used in the proof of local properties associated with the behavior of solutions in neighborhoods of isolated singular points. For example, there is an analog of theorems on removable singular points, as in the case of harmonic functions, i.e., solutions of the Laplace equation. Let D and D_0 denote the domains $x^2 + y^2 < r^2$ and $0 < x^2 + y^2 < r^2$, respectively. Then the following statement holds.

Theorem 14. *Let a function $z = z(x, y)$ belong to the class $C^1(D) \cap C^2(D_0)$ and satisfy Eq.* (7) *in D_0. Then the second derivatives of the function $z(x, y)$ can be continuously prolonged to the point $(0, 0)$ such that the function $z(x, y)$ will be a function of the class $C^2(D)$.*

Now we discuss global properties of solutions of Eq. (7). In [10], it was proved that a complete C^1-smooth surface of bounded external curvature with a locally Euclidean metric is a cylinder. Therefore, any solution of Eq. (7) defined over the whole plane is a cylinder. However, one can raise the question of what will happen if we assume the existence of singular points at which the solution is not smooth. An example of such a solution is the equation of a right circular cone $z = \sqrt{x^2 + y^2}$. For the first time, such a question was posed by the author in 2009. In the case of a sole singular point, the answer is as follows: such a surface is an arbitrary conical surface (i.e., a developable surface in which all generatrices intersect at one point). Later, the author constructed other examples with a countable number of singular points. Yu. A. Aminov showed how to construct a solution whose singular points are located at the vertices of an arbitrary convex polygon. Finally, the following theorem was proved.

Theorem 15 (see [4, 19])**.** *Let M be an arbitrary finite set of points on the plane (x, y). Then Eq.* (7) *has solutions defined on the whole plane that belong to the class C^∞ everywhere except for points of the set M, at which they are continuous and look like conical surfaces with vertices at these points.*

8. Locally Euclidean metrics and conformal mappings. We conclude this paper with a partial answer to the 4th question formulated in Sec. 1. In [17], the problem of the search for a locally Euclidean metric in a circle according to its geodesic curvature given on the boundary was posed and solved in isothermal coordinates.

Under an isometric immersion into $\mathbb{R}^2$ of a required metric, the geodesic curvature of a curve transforms into the ordinary curvature of its image on the plane. Knowing the curvature of a curve as a function of the arc length allows one to find the curve itself by using natural equations, i.e., as a result, we can find the domain D bounded by this curve, and the isometric mapping of the circle with a locally Euclidean metric into $\mathbb{R}^2$ turns out to be a conformal mapping of the circle onto this domain D.

Thus, the search for a locally Euclidean metric in a circle with a given geodesic curvature of the boundary is similar to the problem of finding a domain on an ordinary plane by the curvature of its edge. The reduction of this problem to the problem of finding a locally Euclidean metric in a circle by the geodesic curvature of the boundary allows us to reduce the search for a conformal mapping of the circle onto the domain D with a known curvature of the boundary to the solution of some strongly nonlinear integro-differential equation.

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