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GENERALIZATION OF THE NOTION OF COMPLETENESS OF A RIEMANNIAN ANALYTIC MANIFOLD

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Abstract. In this paper, we discuss the concept of an analytic prolongation of a local Riemannian metric. We propose a generalization of the notion of completeness realized as an analytic prolongation of an arbitrary Riemannian metric. Various Riemannian metrics are studied, primarily those related to the structure of the Lie algebra $\mathfrak{g}$ of all Killing vector fields for a local metric. We introduce the notion of a quasi-complete manifold, which possesses the property of prolongability of all local isometries to isometries of the whole manifold. A classification of pseudo-complete manifolds of small dimensions is obtained. We present conditions for the Lie algebra of all Killing vector fields $\mathfrak{g}$ and its stationary subalgebra $\mathfrak{h}$ of a locally homogeneous pseudo-Riemannian manifold under which a locally homogeneous manifold can be analytically prolonged to a homogeneous manifold.

***Keywords and phrases*:** Riemannian manifold, pseudo-Riemannian manifold, Lie algebra, analytic continuation, vector field, Lie group, closed subgroup.

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1. Introduction. Consider a Riemannian analytic manifold M and a ball $U \subset M$ of small radius centered at some point $x_0 \in M$. By an analytic prolongation of a locally defined metric, we mean a Riemannian analytic manifold N for which there exists an analytic isometry $\varphi : U \to M$. We state the problem of constructing a natural analytic prolongation of a given metric. The natural condition of the nonprolongability of the required manifold was introduced by Helgason (see [1]) and Kobayashi and Nomizu (see [2]). However, the structure of nonprolongable manifolds may be quite extraordinary, for example, the simply connected covering of the right-hand half-plane with the punctured points $\bigl(1/n, k/n\bigr)$, $k, n \in \mathbb{N}$.

As a rule, in studies on the geometry of Riemannian spaces "in the whole," another essential requirement is the completeness of manifolds considered. For a complete, simply connected, analytic Riemannian manifold, any isometry $\varphi : U \to M$ between two connected open subsets $U \subset M$ and $V \subset M$ can be analytically prolonged to an isometry $\varphi : M \to M$ (see [1]).

However, in the general case, a ball U of a Riemannian analytic manifold cannot be isometrically embedded in a complete Riemannian analytic manifold, i.e., generally speaking, a locally given Riemannian metric cannot be extended analytically to the metric of a complete Riemannian manifold. The question arises about generalizing the concept of completeness. A natural generalization is the nonprolongability of a Riemannian analytic manifold.

We pose the following question: By given local properties of a Riemannian analytic metric, i.e., a metric defined on a small ball U, is it possible to construct a Riemannian analytic manifold M containing U as an open subset and admitting analytical prolongation of local isometries to isometries of the whole manifold? In other words, can any isometry $\varphi : U \to V$ between two connected open subsets $U \subset M$ and $V \subset M$ be analytically prolonged to an isometry $\varphi : M \to M$? An obstruction of such a prolongation is the following fact. Let $\mathfrak{g}$ be the Lie algebra of all Killing vector fields on the Riemannian analytic manifold M and $\mathfrak{h} \subset \mathfrak{g}$ be its stationary subalgebra: for a fixed point $p \in M$, we assume that $X \in \mathfrak{h}$ if and only if $X(p) = 0$. Let G be a simply connected subgroup generated by the algebra $\mathfrak{g}$ and H be its subgroup generated by the subalgebra $\mathfrak{h}$. Let G act on the simply connected

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manifold M; then the orbit of a fixed point $p \in M$ is a submanifold isometric to the factor-group G/H. However, the factor-group G/H is a manifold only in the case where the subgroup H is closed in G.

The goal of this work is to define a pseudo-complete manifold, which is the "most complete" analytical prolongation of an arbitrary locally given Riemannian analytic metric. The analytical prolongation of a locally given Riemannian metric is studied. We consider the cases of a completely inhomogeneous metric and a metric for which the Lie algebra of all Killing vector fields has no center. In these cases, we give the definition of a quasi-complete manifold M, which possesses the property of unique prolongability of all local orientation-preserving isometries $f : U \to V$, where U and V are connected open subsets of the manifold M, to an isometry $f : M \to M$. An oriented Riemannian analytic manifold whose algebra of vector fields has a zero center is said to be quasi-complete if it is nonprolongable and does not admit nontrivial local isometries preserving the orientation and all Killing vector fields.

Let us give a definition of a pseudo-complete manifold that leads to the "most complete" prolongation of a locally given metric and is applicable to an arbitrary locally given metric. An analytic, simply connected, oriented Riemannian manifold M is said to be pseudo-complete if it possesses the following properties:

(i) M is nonprolongable;
(ii) there are no local isometric orientation-preserving mappings $f : M \to N$, where N is a simply connected Riemannian analytic manifold and $f(M)$ is an open proper subset in N.

Among pseudo-complete manifolds, we distinguish the "most symmetric" regular pseudo-complete manifolds. Next, we study pseudo-complete manifolds of small dimensions and classify them.

The second goal is to study locally homogeneous manifolds, not only Riemannian, but also pseudo-Riemannian. We give conditions under which H is closed in G. The structure of nonclosed subgroups is well known. However, the nature of the corresponding studies are purely algebraic and do not take into account local properties of the Riemannian metric. A description of properties of nonclosed subgroups $U \subset M$ is contained in the classical work of A. I. Maltsev [3]. If a Lie subgroup H of a simply connected Lie group G is not closed in G, then the group G contains a torus T such that the intersection $H \cap T$ is an everywhere dense winding of this torus. However, this fact is difficult to establish based on local properties of a given Riemannian analytical metric, i.e., on properties of the Lie algebra $\mathfrak{g}$ and the stationary subalgebra $\mathfrak{h}$. Is it possible to find properties of the Lie algebra of all Killing vector fields such that the subgroup H defined by the stationary subalgebra $\mathfrak{h}$ is closed in the simply connected group G generated by the algebra $\mathfrak{g}$? We note Mostow's result, according to which H is closed in G if $\mathfrak{h}$ is semisimple. Moreover, Mostow proved that H is closed in G if $\dim \mathfrak{g} - \dim \mathfrak{h} < 5$ (see [4]).

We find necessary and sufficient properties of the Lie algebra $\mathfrak{g}$ of all Killing vector fields on a Riemannian analytic manifold M and its stationary subalgebra $\mathfrak{h}$ such that H is closed in G. Purely algebraic means are not enough here. To study the problem of the closedness of a stationary subgroup H in a simply connected group G, we use the study of the analytic continuation of a locally given Riemannian analytic metric. Manifolds that are analytic proongations of an arbitrary locally given Riemannian analytic metric have the same Lie algebra of all Killing vector fields. Therefore, the question of the closedness of the group H in G is equivalent to the question of the analytic prolongability of a given locally given Riemannian analytic metric on a locally homogeneous space to the metric of a complete manifold. The concept of analytic prolongation of a Riemannian analytic metric was used in the classical monographs of Helgason [1] and Kobayashi and Nomizu [2], but was not developed.

The case where $\mathfrak{g}$ has a zero center was studies in [5–8] not only for Riemannian manifolds, but also for pseudo-Riemannian spaces and affinely connected spaces. It is proved that in this case the subgroup H defined by the stationary subalgebra $\mathfrak{h}$ is closed in the simply connected group G generated by the algebra $\mathfrak{g}$. In addition to the algebraic approach, an analytical approach is being developed to study the analytic continuation of Riemannian analytic manifolds. One of the topics of this work is the study of locally homogeneous manifolds whose Lie algebra $\mathfrak{g}$ of all Killing vector fields has a nontrivial center $\mathfrak{z}$. We examine properties of the algebra $\mathfrak{g}$, its stationary subalgebra $\mathfrak{h}$, and the center $\mathfrak{z}$ that provide the closedness of the subgroup H defined by the stationary subalgebra $\mathfrak{h}$ in the simply connected group G generated by the algebra $\mathfrak{g}$. Let $\mathfrak{z}$ be the center of the algebra $\mathfrak{g}$, $\mathfrak{r}$ be its redical, and $[\mathfrak{g}, \mathfrak{g}]$ be its commutator subgroup. If

$$\dim\left(\mathfrak{h} \cap (\mathfrak{z} + [\mathfrak{g}, \mathfrak{g}])\right) = \dim\left(\mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]\right),$$

then H is closed in G. If for any semisimple subalgebra $\mathfrak{p} \subset \mathfrak{g}$ satisfying the condition $\mathfrak{p} + \mathfrak{r} = \mathfrak{g}$, where $\mathfrak{r}$ is the redical of $\mathfrak{g}$, the relation $(\mathfrak{p} + \mathfrak{z}) \cap \mathfrak{h} = \mathfrak{p} \cap \mathfrak{h}$ holds, then H is closed in G.

It is important to study the case of a completely inhomogeneous Riemannian metric, i.e., a metric that does not admit any motions (Killing fields). In this case, it is possible to define a so-called quasi-complete manifold, which possesses the property of nonprolongability and uniqueness for each locally defined, completely inhomogeneous metric (see [6]). The definition of a quasi-complete manifold can be generalized to the case where the Lie algebra of all Killing vector fields for a given locally defined Riemannian analytic metric has no center (see [5]). Such a manifold M possesses the property of maximal possible symmetry, i.e., any isometry $f : U \to V$ between connected open subsets of the manifold M can be analytically prolonged to an isometry $f : M \to M$. However, a quasi-complete manifold not only has the disadvantage that it is not defined for an arbitrary locally given metric, but in a certain sense it is not the "most complete" manifold. Therefore, below, for an arbitrary locally defined Riemannian metric, we present the concept of a pseudo-complete manifold, study its properties and relationships with quasi-complete manifolds, and also describe pseudo-complete manifolds in the case of small dimensions.

2. Analytic prolongation of Riemannian manifolds and generalization of the notion of completeness. The class of all locally isometric Riemannian analytic manifolds is called the class of manifolds generated by a given germ of a Riemannian analytic manifold, and each specific manifold from this class is called an analytic prolongation of this germ. A natural requirement for the analytic prolongation of a germ is the nonprolongability of the resulting manifold. Now we present precise definitions and formulations.

Definition 1. An analytic prolongation of a Riemannian analytic manifold M is a Riemannian analytic manifold N such that there is an analytic embedding of M into N as a proper open subset. A manifold that does not admit analytical prolongations is said to be nonprolongable.

Definition 2. A local isometry between two Riemannian analytic manifolds M and N is an isometry $\varphi : U \to V$ between open subsets $U \subset M$ and $V \subset M$. Manifolds that admit a local isometry between them are said to be locally isometric.

Any vector field $X \in \mathfrak{g}$ can be analytically prolonged along any curve on the manifold M; therefore, the Lie algebra $\mathfrak{g}$ determines the Lie algebra $\mathfrak{g}$ of Killing vector fields on any simply connected manifold N, which is localli isometric to M. This fact also holds for affinely connected manifolds.

Lemma 1. *Let M be an analytic affinely connected manifold, X be an infinitesimal affine transform defined in a domain $U \subset M$, and $\gamma(t)$, $0 \le t \le 1$, be a continuous curve in M such that $\gamma(0) \subset U$. Then the vector field is analytically prolongable along $\gamma(t)$. If curves $\gamma(t)$ and $\delta(t)$, where $0 \le t \le 1$, $\gamma(0) = \gamma(0)$, and $\gamma(1) = \gamma(1) = x_1$, are homotopic, then the prolongations of the vector fields to the point x_1 along these curves coincide.*

Proof. Assume that X can be analytically prolonged to a neighborhood of any point $\gamma(t)$ for $0 \le t \le t_1$. We prove that X can be prolonged to a neighborhood of the point $q = \gamma(t_1)$. Let V be a normal neighborhood of the point q, which is a normal neighborhood of each of its points (see [1]). Consider $t \le t_1$ such that $p = \gamma(t) \in V$.

The vector field X generates a local one-parameter group of isometries ϕ_s in a neighborhood of each point if $\gamma(t)$, $t \le t_1$. We prove that for all sufficiently small values of s, the local isometries ϕ_s can be analytically prolonged to a neighborhood of the point $q = \gamma(t_1)$. Then the vector field of velocities of this local group of isometries is an analytic prolongation of the vector field X to a neighborhood of the point q.

Consider a connected open set $V_0 \subset V$ containing the points p and q whose closure also leis in V, $\overline{V}_0 \subset V$, $p, q \in V_0$. Consider a small neighborhood $V' \subset V_0$ of the point q and join the point p with an arbitrary point $q' \in V'$ be a segment of a geodesic line $\alpha(t)$, $0 \le t \le 1$. Let

$$Y = \frac{d\alpha}{dt}(0) \in T_pM, \quad p_s = \varphi_s(p), \quad Y_s = \varphi_s(Y).$$

Draw a geodesic $\beta(t)$, $0 \le t \le 1$, from the point p_s such that $d\beta/dt = Y_s$. For sufficiently small values of s, we have $\beta(t) \in V_0$, $0 \le t \le 1$. We set $\varphi_s(q') = \beta(1)$. The mapping obtained is an analytic prolongation of the affine transform φ_s. □

It is important to study the case of a completely inhomogeneous Riemannian metric, i.e., a metric that does not admit any motions (Killing fields). In this case, it is possible to define a so-called quasi-complete manifold, which possesses the properties of nonprolongability and uniqueness for each locally defined, completely inhomogeneous metric (see [6]).

Definition 3. A Riemannian analytic manifold is called a completely inhomogeneous manifold if there are no Killing vector fields on it. A Riemannian metric of a completely inhomogeneous manifold is called an inhomogeneous metric.

By Lemma 1, all manifolds that are locally isometric to a completely inhomogeneous manifold are completely inhomogeneous.

Definition 4. A completely inhomogeneous, oriented, analytic Riemannian manifold is said to be quasi-complete if it is nonprolongable and does not admit nontrivial orientation-preserving local isometries onto itself.

We present basic properties of completely inhomogeneous quasi-complete manifolds (see [5]). For an arbitrary completely inhomogeneous manifold M, consider the set $S \subset M$ of all fixed points of various orientation-preserving local isometries of the manifold M onto itself.

Theorem 1. *For an arbitrary completely inhomogeneous, analytic Riemannian manifold M', theset $S \subset M'$ is an analytic subset of codimension ≥ 2. Therefore, $M' \setminus S$ is a connected manifold.*

Theorem 2. *For any completely inhomogeneous, analytic Riemannian manifold M', there exists a quasi-complete manifold M locally isometric to it and a locally isometric covering mapping $f : M' \setminus S \to M$. Thus, a quasi-complete manifold possesses the uniqueness property for each completely inhomogeneous, locally defined analytic Riemannian metric.*

The definition of a quasi-complete manifold can be generalized to the case where the Lie algebra of all Killing vector fields for a given locally defined analytic Riemannian metric has no center (see [5]). Many locally homogeneous manifolds, in particular, all locally symmetric spaces possess this property.

Definition 5. An analytic Riemannian manifold M is said to be locally homogeneous if at each point $p \in M$, Killing vector fields form a basis of the tangent space T_pM.

An equivalent definition of a locally homogeneous manifold M is as follows: for any points $p, q \in M$, there exists a local isometry φ of the manifold M such that $\varphi(p) = q$.

Definition 6. An oriented analytic Riemannian manifold whose algebra of vector fields has zero center is said to be quasi-complete if it is nonprolongable and does not admit nontrivial orientation-preserving Killing vector fields of local isometries onto itself.

We examine oriented analytic Riemannian manifolds whose Lie algebra of Killing vector fields has no center. We prove that each such manifold is locally isometric to a quasi-complete manifold, whereas locally homogeneous quasi-complete manifolds are complete homogeneous manifolds.

We denote by $Z(M)$ the pseudogroup of all local isometries of an analytic Riemannian manifold M that preserve the orientation and Killing vector fields: $\varphi \in Z(M)$ if and only if $\varphi X = X$ for any $X \in \mathfrak{g}$.

Lemma 2. *Let M be an analytic Riemannian manifold satisfying the property of unique prolongability of Killing vector fields and let the Lie algebra of all Killing vector fields on M have no center. Then the set $S \subset M$ consisting of fixed points of various isometries $\varphi \in Z(M)$ is an analytic subset of codimension ≥ 2.*

Proof. We prove that for any open set $U \subset M$ with compact closure, there is a finite number of local isometries from U onto itself belonging to the pseudogroup $Z(M)$. Assume the contrary and consider an infinite sequence of local isometries $\varphi_i \in Z(M)$ whose domains and sets of values lie in U. In the proof of [6, Lemm 3], by an infinite sequence of local isometries φ_i on some open set $V \subset U$, a Killing vector field X was constructed, which satisfies the following condition after passing to a subsequence: for any $t \in [-1, 1]$, there exists $k(i) \in \mathbb{N}$ such that

$$\lim_{i \to \infty} \varphi_i^{k(i)} = \operatorname{Exp} tX,$$

where $\operatorname{Exp} tX$ is the local one-parameter group of isometries generated by the vector field X. Therefore, for any vector field Y on V, there exists $i \in \mathbb{N}$ such that the following inequality holds;

$$\left| \left(\operatorname{Exp} tx \right)_* Y - Y \right| \leq \left| \varphi_i^{k(i)} Y - Y \right| + \left| \left(\operatorname{Exp} tX \right)_* Y - \varphi_i^{k(i)} Y \right|$$
$$\leq 0 + \left| Y - \left(\operatorname{Exp} t(-X) \right)_* \varphi_i^{k(i)} Y \right| \leq \frac{1}{2} \left| \left(\operatorname{Exp} tx \right)_* Y - Y \right|.$$

Therefore, for any $Y \in \mathfrak{g}$ we have $(\operatorname{Exp} tX)_* Y = Y$, i.e., $[X, Y] = 0$. However, this contradicts the absence of the center in the algebra $\mathfrak{g}$. This contradiction proved the existence of only a finite number of local isometries from U into U that belong to the pseudogroup $Z(M)$. As was proved in [6], this easily implies the fact that the set S is an analytic subset of codimension ≥ 2. By Lemma 2, the manifold $M \setminus S$ is connected. □

Lemma 3. *Let M be an analytic Riemannian manifold satisfying the property of unique prolongability of Killing vector fields and such that the Lie algebra of all Killing vector fields on M has no center. Then there exists a locally isometric covering mapping from $M \setminus S$ into an analytic Riemannian manifold M_1, which also satisfies the property of unique prolongability of Killing vector fields and such that the pseudogroup $Z(M_1)$ consists only of the identity transform.*

Proof. We take the factor-manifold of the manifold $M' \setminus S$ with respect to the pseudogroup $Z(M)$. The proof of Lemma 2 implies that for each point $x \subset M \setminus S$, there exists a neighborhood $U_{1x} \in M \setminus S$ of the point x, which does not admit nonidentity orientation-preserving local isometries onto itself that belong to the pseudogroup $Z(M)$. This proves that the factor-mapping π, which project the manifold $M \setminus S$ into set set $M_1 = M \setminus S/Z(M)$, is a covering mapping. Therefore, for each point $x \in M$, there exists a neighborhood $U_x \subset M_1$ of it and an open set $V_x \subset \pi^{-1}(U_x)$ such that the mapping π is a homeomorphism between the sets V_x and U_x. We introduce the Riemannian scalar product. Restricting the set $V_x \subset M \setminus S$ if necessary, we assume that V_x is a coordinate neighborhood of the point $y \in \pi^{-1}(U_x) \subset M \setminus S$. Then we take the set $U_x \subset M_1$ as a coordinate neighborhood of the point $x \in M_1$. Consider two such neighborhoods $U_1, U_2 \subset M_1$, $U_1 \cap U_2 \neq 0$. Note that the sets $V_1, V_2 \subset M \setminus S$ corresponding to the sets U_1 and U_2 may be nonintersecting. We set $\pi^{-1}(U_1 \cap U_2) \cap V_1 = V_{10}$ and $\pi^{-1}(U_1 \cap U_2) \cap V_2 = V_{12}$. Then there exists an isometry $\alpha : V_{10} \simeq V_{20}$. Let ψ_1 and ψ_2 be the coordinate mappings onto V_1 and V_2, respectively. Then $\psi_1 \pi^{-1}$ and $\psi_2 \pi^{-1}$ are the coordinate mappings onto U_1 and U_2.

Consider an arbitrary point $x \in M_1$ and arbitrary vectors $X, Y \in T_x M_1$. Also, consider a point $\pi^{-1}(x) \subset M \setminus S$ and vectors $X_1, Y_1 \in T_y M$ such that $\pi_* X_1 = X$ and $\pi_* Y_1 = Y$. We consider the Riemannian scalar product $\langle X, Y \rangle$, which is equal to the Riemannian scalar product $\langle X_1, Y_1 \rangle$ existing on T_xM. if we take another point $z \in \pi^{-1}(x)$ and vectors $X_2, Y_2 \in T_zM$ such that $\pi_*X_2 = X$ and $\pi_*Y_2 = Y$, then there exists a local isometry $\varphi \in Z(M)$ such that $\varphi(z) = y$, $\varphi_*X_2 = X_1$, and $\varphi_*Y_2 = Y_1$. Therefore, $\langle X_1, Y_1 \rangle = \langle X_2, Y_2 \rangle$. This proved the well posedness of the definition of the Riemannian metric on M_1.

The Riemannian manifold M_1 constructed above does not admit nonidentity orientation-preserving local isometries that induce the identity transform on the algebra of Killing vector fields $\mathfrak{g}$. The projection $\pi : M \setminus S \to M_1$ is a locally isometric covering mapping. It remains to prove the property of unique prolongability of Killing vector fields on M_1. Consider a Killing vector field X defined on an open set $U \subset M_1$ and open sets $U_0 \subset U$ and $V_0 \subset M \setminus S$ such that the covering mapping π is an isometry between the sets V_0 and U_0. Then the vector field $\pi_*^{-1}X$ can be uniquely prolonged from the set $V_0 \subset M$ on the whole manifold M and defines a vector field Y on M. Let points $y, z \in M \setminus S$ be such that $\pi(x) = \pi(y)$ and $\pi_*Y(z) = \pi_*\varphi_*Y(y)$. Since $\pi\varphi = \pi$ by the definition of π, we have $\pi_*\varphi_* = \pi_*$. Therefore, $\pi_*Y(z) = \pi_*\varphi_*Y(y) = \pi_*Y(y)$. This proved that the mapping π uniquely projects the vector field Y defined on M onto the vector field π_*Y defined on the manifold M_1. The vector field π_*Y is an analytic prolongation of the vector field X on the whole manifold M_1. □

Theorem 3. *An arbitrary analytic Riemannian manifold M whose Lie algebra of Killing vector fields has no center is locally isometric to a quasi-complete manifold.*

Proof. Consider an arbitrary analytic Riemannian manifold M' whose Lie algebra of Killing vector fields has no center. The manifold M_1 constructed in the proof of Lemma 3 does not admit local isometries onto itself that preserve the orientation and Killing vector fields. Then a certain maximal analytic prolongation of the manifold M_1 is a quasi-complete manifold M. We assume that all manifolds, which will be considered in the proof of Theorem 3, possess the property of unique analytic prolongability of Killing vector fields, i.e., the Lie algebra of all Killing vector fields is the same for all manifolds and coincides with $\mathfrak{g}$. If M' satisfies this condition, then the manifolds M_1 also satisfies it.

Consider the set Λ consisting of analytic prolongations M_α of the manifold M_1 that satisfy the property of unique prolongability of Killing vector fields and do not admit local isometries, which are identical on the algebra of all Killing vector fields. We consider a marked point on the manifold M_1, endow the manifold with a marked frame at the marked point, and consider the images of these point and frame in the manifolds $M_\alpha \in \Lambda$. On the set Λ, we introduce the following order relation: $M_\alpha \le M_\beta$ if there exists an isometric embedding $i_{\alpha\beta} : M_\alpha \to M_\beta$, which maps the marked point to the marked point and the marked frame to the marked frame. As a result, Λ becomes a partially ordered set. Consider an arbitrary linearly ordered subset Δ of the set Λ and construct the direct limit of the family of manifolds $M_\alpha \in \Delta$ and mappings $i_{\alpha\beta}$. We obtain a manifold M_0 possessing the following properties. For any manifold $M_\alpha \in \Delta$, there exists an isometric embedding $i_\alpha : M_\alpha \to M_0$ and, moreover, $i_\alpha(M_\alpha) \subset i_\beta(M_\beta)$ if $M_\alpha \le M_\beta$. Let $M_0 = \bigcup M_\alpha$. We prove that $M_\alpha \in \Lambda$. We can transfer an arbitrary vector field X from the manifold M_1 to the manifold $i_\alpha(M_\alpha) \subset M_0$ by using the embeddings $i_{1\alpha} : M_1 \to M_\alpha$ and $i_\alpha : M_\alpha \to M_0$; moreover, $(i_\alpha i_{1\alpha})_*X = (i_\beta i_{1\beta})_*X$ on $i_\alpha(M_\alpha) \cap i_\beta(M_\beta)$ and the Killing vector field $(i_\alpha i_{1\alpha})_*X$ is uniquely prolonged from the submanifold $i_\alpha(M_\alpha) \subset M_0$ to any submanifold $i_\beta(M_\beta) \subset M_0$, $M_\beta \ge M_\alpha$, and hence to the whole manifold M_0. Thus, a Killing vector field defined on a arbitrarily small open set $U \subset M_0$ can be uniquely prolonged to a Killing vector field on M_0.

Now we consider a local isometry $\varphi \in Z(M)$. Let $x_0 \in M_0$ be a point from the domain of the isometry φ. Then the points x_0 and $\varphi(x_0)$ line in some submanifold $i_\alpha(M_\alpha \subset M_0$. Therefore, $\varphi \in Z(i_\alpha(M_\alpha))$ and hence φ is the identity transform. We conclude that the pseudogroup $Z(M_0)$ consists only of the identity transform. Thus, for an arbitrary linearly ordered subset $\Delta \subset \Lambda$, we have constructed an upper boundary. By Zorn's lemma, the set Λ has a maximal element. We assert that the manifold M, which is the maximal element, is the required quasi-complete manifold. We prove that M is nonprolongable.

Assume the contrary and denote by N a nontrivial prolongation of the manifold M. Let, as above, $S \subset N$ be the set of fixed points of various local isometries from the pseudogroup $Z(N)$. As in the proof of Lemma 3, we consider the factor-manifold of the manifold $N \setminus S$. As a result, we obtain a manifold L satisfying the property of unique prolongability of Killing vector fields and admitting no local isometries that preserve the orientation and all Killing vector fields. We denote by i the embedding $i : M \to N$ and prove that $i(M) \cap S = \varnothing$. If $x \in i(M)$, then a certain normal ball B centered at x belongs to $i(M)$. Moreover, if $x \in S$, then there exists a local isometry $\varphi \in Z(M)$ satisfying the condition $\varphi(x) = X$. This isometry determines an isometry of the ball B into itself defined in the normal coordinates by the differential of the isometry φ, which is a linear mapping. However, the existence of such an isometry contradicts the triviality of the pseudogroup $Z(M)$. Thus, i yields the embedding $i : M \to N \setminus S$. The composite mapping $\pi i : M \to L$, where $\pi : N \setminus S \to L$ is the covering mapping constructed in the proof of Lemma 3, is also an embedding. If $\pi i(x) = \pi i(y)$, then there exists a local isometry $\varphi \in Z(M)$ such that $\varphi(x) = \varphi(y)$; therefore, $x = y$. Since M is a maximal element of the set Λ, πi is an isometry and $N \setminus S$ covers M.

We have the covering mapping $\pi : N \setminus S \to M$ and the embedding $i : M \to N \setminus S$; moreover, $i(M)$ is open in $N \setminus S$. Let $x_n \in i(M)$ be a sequence of points converging to $x \in N \setminus S$. Then the sequence $y_n = \pi(x_n)$ also converges to a point $y \in M$. Since $x_n = i(y_n)$, we have $x = i(y) \in M$. This proves the closedness of $i(M)$ in $N \setminus S$. Thus, $N \setminus S$ is not connected or $N \setminus S = N$. However, the nonconnectedness of $N \setminus S$ contradicts Lemma 2; therefore, $N \setminus S = M$. We prove that $S = \varnothing$. Assume the contrary and consider a normal ball B centered at some point $x \in S \subset N$. There exists an isometry of the ball B into itself; this isometry does not fix points of $B \setminus S$ and hence it is a nonidentical local isometry from the pseudogroup $Z(N \setminus S)$. Since $N \setminus S = M$, this contradicts the triviality of the pseudogroup $Z(M)$. This proves that $S \neq \varnothing$, $N = M$, and M is not prolongable. □

Theorem 4. *Let φ be a local isometry from a quasi-complete manifold M into a quasi-complete manifold N. Then φ can be prolonged to an isometry $\varphi : M \simeq N$.*

Proof. Consider an arbitrary point $X \in M$ and a smooth curve $\gamma(t)$, $0 \leq t \leq 1$, such that $\gamma(t) \in D(\varphi) \subset M$ and $\gamma(1) = x$. We prove that the isometry φ defined in the neighborhood $U = D(\varphi) \subset M$ of the point $x_0 = \gamma(0)$ can be prolonged along the curve γ. Assume that such a prolongation does not exist. Consider the minimal number $t_1 \in [0, 1]$ among all numbers t such that the isometry φ cannot be prolonged to a neighborhood of the point $\gamma(t)$ along the curve γ. However, we prove that, in contrast to the assumption, there exists a prolongation of φ to some neighborhood of the point $\gamma(t_1)$ along the curve γ.

Due to the assumption about t_1, the isometry φ is defined in some neighborhood of point $\gamma(t)$ for any $t \in [0, t_1)$; therefore, a curve $\delta(t) = \varphi(\gamma(t))$, $0 \leq t \leq t_1$, is defined on N. Let $x_1 = \gamma(t_1)$ and $\varepsilon > 0$ be such that the neighborhood $U_\varepsilon = \{x \in M,\ \rho(x, x_1) \leq \varepsilon\}$ is a normal neighborhood of each of its points. Since for any $y \in N$ and $\varepsilon_0 > 0$, there exists $\alpha > 0$ such that the inequalities

$$\Bigl|\rho\bigl(y, \delta(t')\bigr) - \rho\bigl(y, \delta(t'')\bigr)\Bigr| \leq \rho\bigl(\delta(t'), \delta(t'')\bigr) \leq \int\limits_{t'}^{t''} \sqrt{\langle \delta'(t), \delta'(t)\rangle} dt = \int\limits_{t'}^{t''} \langle \gamma'(t), \gamma''(t)\rangle dt < \varepsilon_0$$

are fulfilled for all $t', t'' \in [0, t_1)$ satisfying the conditions $|t_1 - t'| < \alpha$ and $|t_1 - t''| < \alpha$. Therefore, for any $y \in N$, there exists

$$\lim_{t \to t_1} \rho(y, \delta(t)) = \rho_1(y).$$

Consider the set

$$V_\varepsilon = \bigl\{y \in N \mid \rho_1(y) < \varepsilon\bigr\}.$$

There exists an isometry $\varphi = \psi^{-1}$ of some neighborhood $V_D \subset V_\varepsilon$ of the set

$$D = \bigl\{y \in N \mid y = \delta(t),\ t_2 \leq t < t_1\bigr\}$$

onto a neighborhood $U_D \subset U_\varepsilon$ of the set

$$B = \bigl\{x \in M \mid x = \gamma(t),\ t_2 \leq t < t_1\bigr\}.$$

We prove that ψ can be prolonged up to an isometry $\psi : V \simeq V$. First, we prove that ψ is prolongable along any curve $\nu(s)$, $0 \leq s \leq 1$, to V_ε, where $\nu(0) \in V_D$ and $\nu(1) = y$ is an arbitrary point in V_ε. Assume the contrary; then there exists a minimal number s_1 among all numbers $u \in [0,1]$ possessing the following property: ψ is not prolongable along the curve $\nu(s)$ to any neighborhood of the point $\nu(u)$. Let $\sigma > 0$ and $s_2 < s_1$ be such that the set

$$B_\sigma = \{y \in N \mid \rho(y, \nu(s_2)) < \sigma\}$$

is a normal neighborhood of the point $\nu(s_2)$ and $\rho(\nu(s_2; \nu(s_1)) < \sigma/2$. Therefore, $\nu(s_1) \in B_\sigma$. Using the linearity of the mapping ψ in the normal coordinates, one can prolong the isometry ψ defined in some neighborhood of the point $\nu(s_2)$ to an isometry ψ defined on the whole set B_σ, which is a neighborhood of the point $\nu(s)$. Therefore, the assumption on the nonprolongability of ψ along the curve $\nu(s)$ is invalid.

Now we prove that the prolongation of the isometry ψ to V_ε along various curves yields a single-valued mapping $\psi : V_\varepsilon \to U_\varepsilon$. Assume the contrary. Then there exists a closed Jordan curve $\nu(t)$, $0 \leq t \leq 1$, $\nu(0) = \nu(1)$, on V_ε such that the curve $\beta(t) = \psi(\nu(t))$ on U_ε is nonclosed, $\beta(0) \neq \beta(1)$. Since various analytic prolongations of the isometry ψ induce the same mapping on the algebra of Killing vector fields, then the isometry $\psi\psi^{-1}$, which maps $\beta(0)$ to $\beta(1)$, belongs to the pseudogroup $Z(M)$; this contradicts the fact that M is a quasi-complete manifold. One can similarly prove that a prolongation of a local isometry $\phi = \psi^{-1}$ from U_ε to V_ε defines a single-valued mapping on the set $\varphi(V_\varepsilon) \subset U_\varepsilon$. Thus, we have the isometric embedding $V_\varepsilon \to U_\varepsilon$. We prove that it is a surjective mapping. Assume the contrary. Gluing the manifolds N and U_ε along the mapping ψ, we obtain a nontrivial prolongation of the manifold N, which contradicts its nonprolongability. Therefore, we have an isometry $\psi : V_\varepsilon \to U_\varepsilon$. The inverse isometry $\psi^{-1} : U_\varepsilon \to V_\varepsilon$ yields a prolongation of the isometry ψ to a neighborhood U_ε of the point $\gamma(t_1)$ along the curve γ, in contrast to the initial assumption about t_1.

Thus, we have proved that a local isometry φ from M to N can be prolonged to any point $x \in M$ along an arbitrary curve on M. We have proved above that prolongations of an isometry ψ along various curves on V_ε yield a bijective mapping defined on the whole V_ε. In a similar way, one can prove that prolongations of φ along various curves on M yield an isometric embedding $\varphi : M \to N$. □

Corollary 1. *An arbitrary analytic Riemannian manifold whose Lie algebra of Killing vector fields has no center is locally isometric to a unique quasi-complete manifold. In other words, a locally defined analytic Riemannian metric whose Lie algebra of Killing vector fields has no center can be uniquely prolonged to a quasi-complete manifold.*

Proof. Let a quasi-complete manifold M be locally isometric to a manifold M' and let N be another quasi-complete manifold, which is locally isometric to the manifold M'. Then there exist a local isometry φ from N to M' and a local isometry ψ from M' to M. The superposition of the isometries φ and ψ is a local isometry from N to M. By Theorem 4, the local isometry $\psi\varphi$ can be prolonged to an isometry $M \simeq N$. □

Corollary 2. *Let $\mathfrak{g}$ be the Lie algebra of all Killing vector fields of an analytic Riemannian manifold M', which is diffeomorphic to a ball, and $\mathfrak{h}$ be its stationary subalgebra. Let G be a simply connected group generated by the algebra $\mathfrak{g}$ and H be its subgroup generated by the subalgebra $\mathfrak{h}$. If $\mathfrak{g}$ has no center, then H is closed in G.*

Proof. Since M' is diffeomorphic to a ball, its Killing vector fields can be analytically prolonged on it uniquely. By Theorem 3, the manifold M' is locally isomorphic to a quasi-complete manifold M, which possesses the same Lie algebra $\mathfrak{g}$ of all Killing vector fields and the same stationary subalgebra $\mathfrak{h}$. For an arbitrary vector field $X \in \mathfrak{g}$, for all $t < \delta$, elements of the one-parameter transformation group $\operatorname{Exp} tX$ are local isometries of the manifold M. By Theorem 4, they can be prolonged to isometries of the whole manifold M. Then the isometries $\operatorname{Exp} tX = (\operatorname{Exp} tX)^n$ are defined. Thus, the group G acts on M and H is the stationary subgroup of G. This means that the orbit of the group G on M can be covered by a homogeneous manifold $G \setminus H$. Therefore, H is closed in G. □

Note that a quasi-complete manifold is the most contracted, i.e., universally attracting, object of the category of all locally isometric manifolds. For any analytic Riemannian manifold M' whose algebra of Killing vector fields has no center, there exists a locally isometric mapping from M' into a quasi-complete manifold M defined on whole M'.

A quasi-complete manifold is unique in the class of all analytic prolongations of a given germ and possesses a series of remarkable properties (see [6]). First of them is the property of maximal symmetry: any local isometry $f : U \to V$ from a quasi-complete manifold M into itself can be analytically prolonged to an isometry $f : M \to M$. However, the notion of a quasi-complete manifold has not only the disadvantage that it is defined not for all locally defined analytic Riemannian metrics, but also it is not the "most complete" in a certain sense. Namely, there exists a germ of an analytic Riemannian manifold admitting a prolongation to a complete manifold whose prolongation to a quasi-complete is not a complete manifold.

Example 1. Consider the ellipsoid in the three-dimensional space defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

To obtain a quasi-complete manifold in the class of all analytic Riemannian manifolds locally isometric to an ellipsoid, one must remove six points of intersection with the coordinate axes from the ellipsoid and factorize the manifold obtained by the group of rotations by 180° about all coordinate axes.

It is possible to generalize the notion of completeness leading to the "most complete" manifold for an arbitrary germ of an analytic Riemannian manifold.

Definition 7. A simply connected analytic Riemannian manifold M is said to be pseudo-complete if it is possesses the following properties:

(1) M is nonprolongable;
(2) there are no locally isometric covering mappings $f : U \to M$, where N is a simply connected analytic Riemannian manifold and $f(M)$ is a proper open subset in N.

We examine analytic prolongations to a pseudo-complete manifold for various classes of germs of analytic Riemannian manifolds. First, we must prove the fact the an analytic prolongation to a pseudo-complete manifold exists for any germ of an analytic Riemannian manifold. In the general case, this prolongation is not unique, but various analytic prolongations of the same germ differ slightly.

Theorem 5. *Any locally defined analytic Riemannian manifold admits an analytic prolongation to a pseudo-complete manifold. If the class of locally isometric of analytic Riemannian manifolds contains a complete manifold, then this manifold is a unique pseudo-complete manifold of this class.*

Proof. On the set of all simply connected analytic prolongations of a given germ of analytic Riemannian manifold, we introduce the following order relation. We write $M \succeq N$ if there exists a locally isometric mapping $f : N \to M$. Thus, the set of simply connected, locally isometric analytic Riemannian manifolds becomes a partially ordered set. By Zorn's lemma, this set contains a maximal element. By definition, this element is a pseudo-complete manifold.

Consider a complete analytic Riemannian manifold M. Assume that M is not pseudo-complete; then there exists a locally isometric mapping $f : M \to N$ such that there exists a point $x \notin N$. Let $\gamma(t)$, $0 \leq t \leq 1$, be a geodesic curve joining the points $y \in f(M)$ and x. Then the preimage of this geodesic for $0 \leq t \leq \delta$ cannot be prolonged to a geodesic curve for all t on the manifold M, which contradicts the completeness of this manifold. Pseudo-complete manifolds are not unique in the class of all locally isometric analytic Riemannian manifolds. □

Example 2. Consider a germ of a two-dimensional analytic Riemannian manifold defined on the sphere with the metric

$$ds^2 = \frac{f(z,\overline{z})}{\sqrt{(1+|z|^6}}\, dz\, d\overline{z},$$

where $f(z,\overline{z})$ is an analytic function on the sphere satisfying the condition

$$f(z,\overline{z}) \neq |A'(z)|^2 f\big(A(z), A(\overline{z})\big)$$

for any linear fractional transform $A(z)$.

This metric has a singularity at the point $z = \infty$. The sphere with this metric is a pseudo-complete manifold. Using the transform $z = w^2 + a$, $a \in \mathbb{C}$, we remove the singularity at the point $z = \infty$. As a result, we obtain another sphere, which doubly covers the initial sphere and has the metric

$$ds^2 = \frac{4|w| f\big(w^2 + a, \overline{w}^2 + \overline{a}\big)}{\sqrt{1 + |w^2 + a^2|^6}}\, dw\, d\overline{w}.$$

This metric has a singularity at the point $w = 0$; this fact is natural since the sphere w is branched over sphere z at the point $z = a$ corresponding to the point $w = 0$. For various a, we obtain various pseudo-complete manifolds with the coordinate w.

Example 2 shows that there exist a large set on nonnatural pseudo-complete manifolds. To avoid branching over regular points, we restrict the notion of a pseudo-complete manifold.

Definition 8. An analytic simply connected Riemannian manifold M is called a regular pseudo-complete manifold if there is no a covering locally isometric mapping $f : M \setminus S \to N$ into another pseudo-complete manifold N, which is locally isometric to the manifold M.

Theorem 6. *A local isometry from a regular pseudo-complete manifold M into a pseudo-complete manifold N can be analytically prolonged along a continuous curve to any point of M except for an analytic subset S of codimension ≥ 2.*

Proof. We prove the theorem for the case where the Lie algebra of all Killing vector fields has no center. Consider the subsets $S \subset M$ and $S' \subset N$ consisting of all fixed points of local isometries that preserve the orientation of Killing vector fields. The sets S and S' are analytic subsets of the manifolds M and N of codimension ≥ 2 (see [5, 6]). Let M_0 be a quasi-complete manifold locally isometric to the manifolds M and N. Then there exist covering locally isometric mappings $f : M \setminus S \to M_0$ and $g : N \setminus S' \to M_0$ (see [5, 6]). Moreover, the definition of a regular pseudo-complete manifold implies that $f(M \setminus S) = M_0$ and $g(N \setminus S') = M_0$. Consider an arbitrary curve $\gamma(t) \subset M \setminus S$ such that the initial local isometry ϕ between the manifolds M and N contains the point $\gamma(0)$, its image $\delta(t) = f(\gamma(t)) \subset M_0$, and the connected component $\beta(t)$ of the preimage $g^-1((t)) \subset S'$ containing the point $\phi(\gamma(0))$. Then the initial local isometry ϕ can be analytically prolonged to an isometry of some neighborhood of the curve $\gamma(t)$, $0 \leq t \leq 1$, to some neighborhood of the curve $\beta(t)$, $0 \leq t \leq 1$, lying in $N \setminus S'$. □

For an arbitrary oriented analytic Riemannian manifold M, we denote by $Z(M)$ the pseudogroup consisting of all local isometries of the manifold M that preserve the orientation and all Killing vector fields. Consider the factor-manifold K_M of the manifold $M \setminus S$ by the pseudogroup $Z(M)$. We define the union of the manifolds K_M and K_N by gluing them along the set $K_{M\cup N}$. By the union of $M \cup N$ we mean the identification of maximal subsets on which the initial local isometry between the simply connected coverings $\widetilde{M}$ and $\widetilde{N}$ of the manifolds M and N can be prolonged. On the manifold $M \setminus S$, we consider the distribution $\mathfrak{z}^{\perp}$ consisting of vectors that are perpendicular to the center $\mathfrak{z}$ of the Lie algebra $\mathfrak{g}$ of all Killing vector fields.

Theorem 7. *A germ of an analytic Riemannian manifold such that the distribution $\mathfrak{z}^{\perp}$ of tangent vectors perpendicular to the center $\mathfrak{z}$ of the Lie algebra of all Killing vector fields is involutive admits* *an analytic prolongation to a regular pseudo-complete manifold. If one removes the set S of fixed points of local isometries preserving the orientation and all Killing vector fields from this pseudo-complete manifold M, then the simply connected covering $\widetilde{M \setminus S}$ of the manifold $M \setminus S$ is isometric to the direct product of the Euclidean space and the simply connected covering $\widetilde{K}$ of the quasi-complete manifold K, which is locally isometric to the completely geodesic submanifold tangent to $\mathfrak{z}^{\perp}$, i.e., $\widetilde{M \setminus S} \approx \mathbb{R}^k \times \widetilde{K}$.*

Proof. Since the distributions $\mathfrak{z}$ and $\mathfrak{z}^{\perp}$ are involutive, a certain neighborhood U of the marked point $p \in M$ has the form $U = V \times W$, where V is an open subset of the integral submanifold of the distribution $\mathfrak{z}$ and W is an open subset of the integral submanifold of the distribution $\mathfrak{z}^{\perp}$. Let $x^1, \dots, x^k$ be the coordinates on V and $y^1, \dots, y^m$ be the coordinates on W. Then in the coordinates $x^1, \dots, x^k, y^1, \dots, y^m$, the components g_{ij} are independent of $x^1, \dots, x^k$. Since the submanifolds V and W are perpendicular, the coefficients of $dx^i\, dy^j$ are equal to 0. Therefore, the metric on U has the form

$$ds^2 = ds_1^2(y) + f_{ij}(y)\, dx^i\, dx^j.$$

Due to the nonprolongability of pseudo-complete manifolds, $M \setminus S$ contains complete integral submanifold of the distribution $\mathfrak{z}$, i.e., the direct products of the Euclidean space and the torus $\mathbb{R}^s \times T^l$. Therefore, $M \setminus S$ is a bundles over $K' \subset K$ with the fibers $\mathbb{R}^s \times T^l$. Since the distribution $\mathfrak{z}^{\perp}$ is involutive, this bundle contains a section K' and hence is trivial, $M \setminus S = \mathbb{R}^s \times T^l \times K'$. Since M is nonprolongable, we conclude that $K' = K$. Therefore, the simply connected covering of the manifold $M \setminus S$ is isometric to the direct product of simply connected spaces: $\widetilde{M/S} \approx \mathbb{R}^k \times \widetilde{K}$. □

Corollary 3. *Consider an analytic Riemannian manifold M' of dimension n whose Lie algebra $\mathfrak{g}$ is commutative, i.e., coincides with its center $\mathfrak{z}$, and $\dim \mathfrak{g} = \dim \mathfrak{z} = n - 1$. Then there exists at most two pseudo-complete manifolds locally isometric to M'.*

Proof. Since $\operatorname{codim} \mathfrak{z} = 1$, we have $\dim \mathfrak{z}^{\perp} = 1$ and hence $\mathfrak{z}^{\perp}$ is involutive. By Theorem 5, for a pseudo-complete manifold M locally isometric to the manifold M', the following decomposition holds: $M \setminus S = \mathbb{R}^s \times T^l \times K$. The completely geodesic submanifold K is isometric either to the straight line $\mathbb{R}$, or to the circle S^1, or to the ray (a, ∞), or to the interval (a, b). Consider the factor-set $\overline{K} = M \setminus Z(M)$. If $K = \mathbb{R}$ or $K = S^l$, then $\overline{K} = K$. If $K = (a, \infty)$, then $\overline{K} = [a, \infty)$ or $\overline{K} = K = (a, \infty)$. If $K = (a, b)$, then $\overline{K} = [a, b)$, or $\overline{K} = (a, b]$, or $\overline{K} = [a, b]$, or $\overline{K} = K = (a, b)$.

In the case where $K = \mathbb{R}$ or $K = S^1$, then the corresponding germ of an analytic Riemannian manifold has a unique prolongation to a pseudo-complete manifold and this manifold is isometric to the Euclidean space. A prolongation of a germ to a pseudo-complete manifold is unique in the case $S = \infty$, i.e., $\overline{K} = K$.

Let $K = (a, \infty)$ and $\overline{K} = [a, \infty)$. Then points of the subset $S \subset M$ are mapped to the point $a \in \overline{K}$ under the factorization $\overline{K} = M/Z(M)$. The point $x \in S$ is a singular point of some field $X \in \mathfrak{z}$, $X(x) = 0$, and any isometry φ from M into itself for which $\varphi(x) = x$ has the form $\phi = \operatorname{Exp} tY$, $Y \in \mathfrak{z}$. Consider the subalgebra $\mathfrak{z}_0 \subset \mathfrak{z}$ consisting of Killing vector fields $X \in \mathfrak{z}$ that vanish at the point x, $X(x) = 0$. Then $\mathfrak{z}_0$ generates a group of isometries of some ball B, which can be analytically prolonged to a group of isometries of the manifold M and is isomorphic to the factor-group of the group $\mathfrak{z}_0 = \mathbb{R}^s$ by a certain lattice Γ acting on the manifold M. Then M is a complete manifold isometric to the space $\mathbb{R}^s \times T^l$. A similar construction can be used in the case where $K = (a, b)$ and $\overline{K} = [a, b)$ ot $\overline{K} = (a, b]$, i.e., where $\overline{K}$ is obtained from K by attaching a point a or b. In this case, the pseudo-complete manifold is also unique an is isometric to the manifold $\mathbb{R}^s \times T^l \times \overline{K}$; however, this manifold is not complete.

Finally, consider the case where $K = (a, b)$ and $\overline{K} = [a, b]$, i.e., where $\overline{K}$ is obtained from K by attaching two point a and b. Consider the pseudo-complete manifold M_1 and points of the set $S_1 \subset M_1$ projected to the point $a \in \overline{K}$. As in the previous cases, consider the manifold M_1' obtained by attaching the set S_1 to the factor-manifold of the manifold $M \setminus S$ by some lattice $\Gamma_1 \subset z = R^(n-1)$

such that $M_1' = \mathbb{R}^s \times T^l \times \overline{K_1}$, where $\overline{K_1} = [a, b)$. Similarly, consider the pseudo-complete manifold M_2 and point of the set $S_2 \subset M_2$ projected to the point $b \in K$. The manifold M_2' is obtained by attaching the set S_2 to the factor-manifold of the manifold $M \setminus S$ by some lattice $\Gamma_2 \subset \mathfrak{z} = R^(n-1)$ such that $M_2' = \mathbb{R}^s \times T^l \times \overline{K_2}$, where $\overline{K_2} = (a, b]$. If the lattices Γ_1 and Γ_2 do not coincide, then the manifolds $M_1 = M_1'$ and $M_2 = M_2'$ are two different pseudo-complete manifolds. If the lattices Γ_1 and Γ_2 coincide, then the manifolds M_1 and M_2 are isometric and determines a complete manifold $M = M_1 = M_2$. □

Now we describe pseudo-complete manifolds of low dimension.

Consider a germ $\mathcal{A}$ of a two-dimensional real-analytic Riemannian manifold. The dimension of the Lie algebra $\mathfrak{g}$ of Killing vector fields of a two-dimensional manifold is no greater than 3. If $\dim \mathfrak{g} = 3$, then the germ $\mathcal{A}$ is a germ of a manifold of constant curvature, which can be prolonged to a complete manifold: the sphere, the plane, or the hyperbolic plane. If $\dim \mathfrak{g} = 2$, then the germ $\mathcal{A}$ is a germ of a left-invariant Riemannian metric on a two-dimensional Lie group, which is the prolongation of the germ to a complete manifold. The case $\dim \mathfrak{g} = 1$ was described in the corollary of Theorem 5.

Consider completely inhomogeneous two-dimensional analytic Riemannian manifolds. The factor-manifold K constructed above as the union of all locally isometric factor-manifolds with respect to the pseudogroup of all local isometries preserving the orientation and all Killing vector fields is a quasi-complete manifold. Consider the set $\overline{K} = K \cup T$ obtained by attaching to the manifold K the images of the points $x \in S \subset M_\alpha$ under the factor-mappings $\pi : M_\alpha \to M_\alpha/Z(M_\alpha = \overline{K}_\alpha \subset \overline{K}$ defined on various analytic prolongations M_α of the germ $\mathcal{A}$. Then the subset $T \subset \overline{K}$ consists of isolated points and one can introduce on $\overline{K}$ the structure of an analytic manifold. Consider a point $z_0 \in T \subset \overline{K}$. Then there exists a sufficiently small ball U_0 centered at a point $x_0 \in U_0$ such that the factor-mapping $\pi : U_0 \to \overline{K}$ is the factorization of the ball U_0 by a finite group of rotations with center $x_0 \in U_0$, $\pi(x_0) = z_0$. If z is a complex coordinate on U_0, then the mapping π has the form $z \to w = z^m 4$ and the metric on the set $V_0 = \pi(U_0) \subset \overline{K}$ has the form $ds^2 = |w|^{(-2(m-1))/m} ds_1^2(w, \overline{w})$, where $ds_1^2(w, \overline{w})$ is an analytic Riemannian metric on the ball $V_0 \subset \overline{K}$.

We denote by $\widetilde{K}$ the simply connected covering of the set $\overline{K}$. Then the preimage $\widetilde{T} \subset \widetilde{K}$ of the set $T \subset \overline{K}$ is a discrete set of points $a_i \in \widetilde{K}$. On $\widetilde{K} \setminus \widetilde{T}$, an analytic Riemannian metric is uniquely defined such that the covering is locally isometric. Then the metric in neighborhoods of the points a_i has the form

$$ds^2 = |w|^{-2(m-1)/m} ds_1^2(w, \overline{w}).$$

A simply connected manifold $\widetilde{K}$ is diffeomorphic either to the complex plane, to the disk, or to the sphere.

Consider the case where $\widetilde{K}$ is identified with the complex plane $\mathbb{C}$. Then there exists a holomorphic function f on $\widetilde{K} \setminus \widetilde{T}$, which has branching of order m_i at the points a_i. This function is called the Weierstrass function with zeros of given order at given point:

$$f(z) = \prod_{i=1}^{\infty} \sqrt[m_i]{1 - \frac{z}{a_i}} \exp\left(\frac{1}{m_i}\left(\frac{z}{a_i} + \frac{1}{2}\left(\frac{z}{a_i}\right)^2 + \cdots + \frac{1}{p_i}\left(\frac{z}{a_i}\right)^{p_i}\right)\right),$$

where the numbers $p_i \in \mathbb{N}$ are such that for any $z \in \mathbb{C}$, the series

$$\sum_{i=1}^{\infty} \left(\frac{z}{a_i}\right)^{p_i}$$

converges. Consider the Riemannian surface M of the function $f(z)$. The surface M covers the complex plane $\mathbb{C}$ such that the covering mapping $\pi : M \to \mathbb{C}$ has branching of order m_i at the points $a_i \in \mathbb{C}$ and has no branching at other points. We define the following Riemannian metric on M: $g(X, Y) = g(\pi_* X, \pi_* Y)$, where $X, Y \in T_x M$ and $\pi_* X, \pi_* Y \in T_{\pi(x)} K$. This metric has no singularities at points $x_i \in M$ such that $\pi(x_i) = a_i$. It is easy to prove that for any simply connected manifold N locally isometric to M, each local isometry ϕ from N into M can be analytically prolonged to a locally isometric mapping $\phi : N \to M$. Thus, M is a unique analytic prolongation of the given germ to a pseudo-complete manifold.

In the case where $\widetilde{K}$ is a disk, similarly to the case of the complex plane, we can construct a unique analytic prolongation of the given germ to a pseudo-complete manifold. This manifold is also the Riemannian surface of a holomorphic function $f(z)$ on $\widetilde{K}$ with branching og order m_i at the points $a_i \in \widetilde{T} \subset \widetilde{K}$:

$$f(z) = \prod_{i=1}^{\infty} \sqrt[m_i]{\frac{z - a_i}{z - \alpha_i}} \exp \sum_{k=1}^{q_i} \frac{(a_i - \alpha_i)^k}{k(z - \alpha_i)^k},$$

where α_i is the point on the boundary of the disk closest to a_i and the numbers $q_i \in \mathbb{N}$ are such that

$$\left| \ln \frac{z - a_i}{z - \alpha_i} + \sum_{k=1}^{q_i} \frac{(a_i - \alpha_i)^k}{k(z - \alpha_i)^k} \right| \leq \frac{1}{2^i}.$$

Consider the case where $\widetilde{K}$ is the sphere. In this case, the set $\widetilde{T} \subset \widetilde{K}$ consists of a finite number of points $\alpha_0, \alpha_1, \ldots, \alpha_l$, at each of which the metric has a singularity of the form

$$ds^2 = |w|^{-2(m-1)/m} ds_1^2(w, \overline{w}).$$

The function

$$f(z) = \prod_{i=1}^{\infty} \sqrt[m_i]{\frac{z - a_i}{z - \alpha_0}}$$

on the sphere has branching of order m_i at the points a_i, $i = 1, 2, \ldots, l$, and branching of order m at the point α_0. As above, we consider the Riemannian surface M of the function $f(z)$. The covering mapping $\pi : M \to \widetilde{K} = S^2$ is a covering over $\widetilde{K} \setminus \widetilde{T}$ and has branching at the points $a_i \in \widetilde{T} \subset \widetilde{K}$, $i = 1, 2, \ldots, l$, and α_0 of orders indicated above. Then the metric on M induced by the metric on $\widetilde{K}$ and the covering mapping π has no singularities at the points $\pi^{-1}(a_i)$, but in the case where $m \neq m_0$, it has a singularity at the point $\pi^{-1}(a_0)$. The manifold obtained is a regular pseudo-complete manifold. Instead of the points $a_i \in \widetilde{T} \subset \widetilde{K}$, one can choose any other point $a_j \in \widetilde{T} \subset \widetilde{K}$ and construct another regular pseudo-complete manifold as was described above. Thus, we obtain all analytic prolongations of a given germ to a regular manifold.

Now we describe three-dimensional pseudo-complete manifolds. As above, we denote by $\mathfrak{z}$ the center of the Lie algebra $\mathfrak{g}$ of all Killing vector fields on the manifolds considered. If $\dim \mathfrak{z} = 3$, then the germ of a Riemannian manifold is homogeneous and, due to Mostow's result (see [4]), it can be prolonged to a homogeneous manifold. If $\dim \mathfrak{z} = 2$, then, by the corollary of Theorem 5, there exists at most two analytic prolongations of a given germ to a regular pseudo-complete manifold. The case where the algebra $\mathfrak{g}$ has no center, $\dim \mathfrak{z} = 0$, was analyzed in the proof of Theorem 4.

Consider the case where $\dim \mathfrak{z} = 2$. First, we examine the case where $\widetilde{K}$ is diffeomorphic to the plane. Consider the manifold $M_0 \approx \widetilde{K} \times \mathfrak{z}$. Let U_0 be a small ball endowed with a Riemannian metric and $V_0 = U_0/Z(U_0) \subset \widetilde{K}$. We extend the metric defined on U_0 to the manifold $V_0 \times \mathfrak{z}$. Let x^1, x^2, x^3 be coordinates on $V_0 \times \mathfrak{z}$ such that x^1 and x^2 are the coordinates on V_0 and x^3 is the coordinate on $\mathfrak{z}$. The components of the metric tensor $g_{ij}(x^1, x^2)$ are independent of x^3. The functions $g_{ij}(x^1, x^2)$ can be analytically prolonged along any curve $\widetilde{K}$; they define a metric on $M_0 \approx \widetilde{K} \times \mathfrak{z}$. Then $M_0 \setminus Z(M_0) = K$; therefore, $Z(M_0) = K \times \Gamma$, where Γ is the group of the covering $\widetilde{K} \to K$. Then for a regular pseudo-complete manifold M, we obtain the manifold $M/S = K \times \mathfrak{z}/\Gamma_0$, where Γ_0 is a discrete subgroup of the group $\mathfrak{z}$.

Now we consider the case where the factor-manifold K is diffeomorphic to the sphere. We split K into the union of two open disks $K = K_1 \cup K_2$. As above, we construct the Riemannian manifolds $M_1 = K_1 \times \mathbb{R}$ and $M_2 = K_2 \times \mathbb{R}$, which are analytic prolongations of the initial germ, whose submanifolds $\mathbb{R}$ are integral curves of the vector field $X \in \mathfrak{z}$. The local isometries f from M_1 into M_2 can be prolonged along any curve to $(K_1 \cap K_2) \times \mathbb{R}$. If such a prolongation is unique, then obtain a complete manifold $M \approx S^2 \times \mathbb{R}$, which is a prolongation of the given germ. Now we assume that there exists a closed curve $\gamma(t)$, $0 \leq t \leq 1$, on $(K_1 \Cap K_2) \times \mathbb{R}$, such that the prolongation of the isometry f along it is not single-valued, $f(\Gamma(0)) = y_1 \neq y_2 = f(\gamma(1))$. Let x^1, x^2, and x^3 be coordinates on M_1 such that x^1 and x^2 are the coordinates on K_1 and x^3 is the coordinate on$\mathbb{R}$. Let y^1, y^2, and y^3 be coordinates on M_2 such that y^1 and y^2 are the coordinates on K_2 and y^3 is the coordinate on $\mathbb{R}$. Since x^3 and y^3 are the coordinates on the Lie algebra $\mathfrak{z}$, the isometry f in the coordinates x^1, x^2, x^3, y^1, y^2, and y^3 has the form $y^1 = y^1(x^1, x^2)$, $y^2 = y^2(x^1, x^2)$, $y^3 = x^3 + f(x^1, x^2)$, where the functions y^1 and y^2 are the transition functions from the chart K_1 to the chart K_2 on the sphere and hence single-valued. The function $f(x^1, x^2)$ can be prolonged along the closed curve $\delta(t)$, $0 \leq t \leq 1$ to $K_1 \cap K_2$, but the prolongation is not single-valued. Let $f(\delta(1)) - f(\delta(0)) = a \in \mathbb{R}$. Consider the circle $S^1 = \mathbb{R}/\mathbb{Z}$. Then the prolongation of the function f along the curve δ is single-valued if we assume that f takes values not on the straight line $\mathbb{R}$ but on the circle S^1. Then the prolongation of the function f along the curves δ^n, $n \in \mathbb{Z}$, is also single-valued. Since any curve on $K_1 \cap K_2$ is homotopic to the curve δ^n, the prolongation of the function $f : U_0 \to S^1$ along various curves is single-valued on $K_1 \cap K_2$. In this case, the function f is the transition function of the bundle into circles over the sphere S^2, and we obtain compact lens spaces as analytic prolongations of the given germ.

3. Locally homogeneous manifolds whose Lie algebras of Killing vector fields have nontrivial centers. Now we examine the case where the Lie algebra $\mathfrak{g}$ has nonzero center $\mathfrak{z}$ and indicate the properties of the algebras $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{z}$ that provide the closedness of the subgroup H in G.

Define local group of local isometries. Consider an arbitrary analytic Riemannian manifold M, the Lie algebra $\mathfrak{g}$ of Killing vector fields on it, and the Lie group G with the Lie algebra $\mathfrak{g}$. A local group (chunk of a group) of local isometries of the manifold M is a small neighborhood of the neutral element of the group G. As a rule, the Lie algebra $\mathfrak{g}$ does not generate the group of isometries of the manifold M, but generates a pseudogroup of local isometries. We denote the local group by the same letter G as the group. The orbit of the local group of local isometries of the manifold M is a locally homogeneous manifold N. We also note that the local group H generated by the stationary subalgebra $\mathfrak{h}$ is the group of isometries of some ball centered at the marked point of the manifold M.

First, we examine some properties of the local group of local isometries from the point of view of abstract transformation groups. Consider the local group G as a subgroup of the group of local diffeomorphisms of the manifold M with a marked point, $G \subset \operatorname{Diff} M$. An element $\tilde{n} \in G \subset \operatorname{Diff} M$ is called a right multiplication if there exists an element $n \in G$ such that for all $x \in M$ such that $x = g(e)$, we have $\tilde{n}(x) = gn(e)$. A right multiplication by an element n is well defined if for any $h \in H$, there exists $h_1 \in H$ such that for any local isometry $g \in G$, the equality $ghn = gnh_1$ holds. In other words, n belongs to the normalizer $N(H)$ of the group H in G. We denote by N the local group consisting of elements $n \in G$ such that right multiplication by these elements in the group G generate local isometries of the manifold M and by $\mathfrak{n}$ its Lie algebra. Then $h \lhd \mathfrak{n} \subset G$. Note that right multiplications, i.e., elements $\tilde{n}$, and also elements of the center Z of the local group G belong to N. We denote by M_0 the orbit of the local group N on M. The adjoint action of elements $n \in N$, i.e., $g \mapsto n^{-1}gn$, determines local isometries on M_0.

We find a subgroup $G_0 \subset G$ consisting of "left multiplications." Consider the mapping f from the group G defined as the transformation group of the set G: $f(g) = g(e) = ge$, where e is the identity local isometry. Since $\tilde{n}(e) = en = n$, we assume that $f(\tilde{n}) = n$. On the set $f(G)$, we define multiplication $g_1 g_2 = g_1(e) g_2(e)$, which turns $f(G)$ into a subgroup $G_0 \subset G$. Left multiplications $g \in G_0$ can be complemented by right multiplications $\tilde{n}$, i.e., any element $g \in G \subset \operatorname{Diff} G$ such that $g(x) = gx$ for all $x \in G$, can be represented in the form $g = g_0 \tilde{n}$, $g_0 \tilde{n}(x) = gxn$ for all $x \in G$. Therefore, $G = G_0 N$ and $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{n}$.

Theorem 8. *Let $\mathfrak{g}$ be the Lie algebra of all Killing vector fields on a locally homogeneous analytic Riemannian manifold M and $\mathfrak{h}$ be its stationary subalgebra. Let G be a simply connected group generated by the algebra $\mathfrak{g}$ and H be its subgroup generated by the subalgebra $\mathfrak{h}$. Let $\mathfrak{z}$ be the center of the algebra $\mathfrak{g}$ and $[\mathfrak{g}, \mathfrak{g}]$ be its commutator subgroup. If $\mathfrak{h} \cap (\mathfrak{z} + [\mathfrak{g}, \mathfrak{g}]) = \mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$, then H is closed in G.*

Proof. Assume the contrary. Consider the closure $\bar{H}$ of the group H in G and the subalgebra $\bar{\mathfrak{h}} \subset \mathfrak{g}$ of the subgroup $\bar{H} \subset G$. The subalgebra $\mathfrak{h}$ is a normal subalgebra of the algebra $\bar{\mathfrak{h}}$ (see [3]). We assume that $X \in \mathfrak{h} \iff X(p) = 0$ for the marked point $p \in M$. Consider the one-parameter subgroup $\bar{h}_t \in \bar{H}$, $\bar{h}_t \notin H$, defined by the vector field $\bar{X} \in \bar{\mathfrak{h}}$, $\bar{X} \notin \mathfrak{h}$. It was proved in [3] that there exists a torus T in a simple compact subgroup $P \in G$ such that $H \cap T$ is an everywhere dense winding of the torus T. Therefore, we may assume that $\bar{\mathfrak{h}}_t \in T \subset P$. Then the Killing vector field $\bar{X}$ of tangent vectors of orbits of the local one-parameter group $\bar{h}_t$ belongs to the algebra $\mathfrak{t}$ of the group T and, therefore, $\bar{X} \in \mathfrak{t} \subset \mathfrak{p}$, where $\mathfrak{p}$ is the Lie algebra of the group P. There exist a neighborhood U of the neutral elements in the group G and the ball B_δ of radius δ centered at the marked point $p \in M$ such that all elements $g \in U$ of the group G define local isometries from the ball B_δ into the ball $B_{2\delta}$ of radius 2δ centered at $\in M$. Since elements $\bar{h}_t$ belong to the closure $\bar{H}$ of the group H in G, for any small t the inner automorphism $x \mapsto \bar{h}_t x \bar{h}^{-1}$ of the group G is the limit of the sequence of inner automorphisms $x \mapsto h_n x h_n^{-1}$, $h_n \in H$. For small t and large n, these automorphisms define local isometries from the ball B_δ into the ball $B_{2\delta}$.

All right multiplications commute with left multiplications, i.e., with elements of the group G_0, but they may note commute with each other. We prove that the local isometry $\bar{h}_t$ commutes with all right multiplications. For this we prove that the action of the element $\bar{h}_t$ in the group of inner automorphisms of the group G, $g \mapsto \bar{h}_t^{-1} g \bar{h}_t$, defines the identity mapping on M_0. Consider a sequence $h_n \in H$ converging to $\bar{h}_t$. Since H is a normal subgroup in N, we conclude that $nh_n = h_n n h_n'$, where h_n' $\in H$, and $nH = h_n^{-1} n h_n H$. Therefore, inner automorphisms $g \mapsto h_n^{-1} g h_n$ induce the identity mapping on M_0. Passing to the limit, we obtain that the inner automorphisms $g \mapsto \bar{h}_t^{-1} g \bar{h}_t$ induce the identity mapping on M_0.

Since the vector field X generating the local one-parameter group $\bar{h}_t$ belongs to a compact subalgebra of the algebra $\mathfrak{g}$, we conclude that X belongs to the commutator subgroup $[\mathfrak{g}, \mathfrak{g}]$ of the algebra $\mathfrak{g}$.

The vector field Z of tangent vectors of the orbits of the local one-parameter group z_t of right multiplications by $\bar{h}_t$ is a Killing vector field. We prove that Z commutes with all other Killing vector fields on M, i.e., $Z \in \mathfrak{z}$. Since for any $n \in \mathfrak{n}$, the elements $\bar{h}_t n$ and $n\bar{h}_t$ induce the same local isometry on M_0, we conclude that $\bar{h}_t$ induces the identity mapping on the algebra $\mathfrak{n}$. Therefore, the Killing vector fields Z and X belong to the center of the algebra $\mathfrak{n}$. For any elements $g_0 \in G_0$ and $\tilde{n} \in \tilde{\mathfrak{n}}$ considered as automorphisms of the group G, the equalities $g_0 \tilde{n}(x) = g_0(xn) = g_0 xn$ and $\tilde{n} g_0(x) = (g_0 x)n = g_0 xn$ are fulfilled. This means that the algebras $\tilde{\mathfrak{n}}$ and $\mathfrak{g}_0$ mutually commute. Therefore, the vector field Z commutes with the algebra $\mathfrak{g}_0$ and with the algebra $\mathfrak{g}_0 + \mathfrak{n}$, i.e., $Z \in \mathfrak{z}$.

Since $z_t \bar{h}_t^{-1} H = \bar{h}_t^{-1} H \bar{h}_t$, we have

$$\operatorname{Exp}(tZ) \operatorname{Exp}(-tX) = z_t \bar{h}_t^{-1} \in H.$$

Therefore, $Z - \bar{X} \in H$. Moreover, $\bar{X} \in \mathfrak{p} = [\mathfrak{p}, \mathfrak{p}] \subset [\mathfrak{g}, \mathfrak{g}]$. One can prove that $Z \notin [\mathfrak{g}, \mathfrak{g}]$. Then $Z - \bar{X} \notin [\mathfrak{g}, \mathfrak{g}]$, i.e.,

$$Z - \bar{X} \in \mathfrak{h} \cap (\mathfrak{z} + [\mathfrak{g}, \mathfrak{g}]).$$

On the other hand, since $Z - \bar{X} \in \mathfrak{h}$ and $Z - \bar{X} \notin [\mathfrak{g}, \mathfrak{g}]$, we have

$$Z - \bar{X} \in \mathfrak{h} \cap (\mathfrak{z} + [\mathfrak{g}, \mathfrak{g}]), \quad Z - \bar{X} \notin \mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}].$$

Then the vector field $Z - \bar{X}$ is stationary but does not belong to the commutator subgroup. Therefore, $\mathfrak{h} \cap (\mathfrak{z} + [\mathfrak{g}, \mathfrak{g}]) \neq \mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$. □

Theorem 9. *Let $\mathfrak{g}$ be the Lie algebra of all Killing vector fields on a locally homogeneous analytic pseudo-Riemannian manifold M, $\mathfrak{h}$ be its stationary subalgebra, $\mathfrak{z}$ be the center of the algebra $\mathfrak{g}$, and $\mathfrak{r}$*

be its redical. Let G be the simply connected subgroup generated by the algebra $\mathfrak{g}$ and H be its subgroup generated by the subalgebra $\mathfrak{h}$. If for any semisimple algebra $\mathfrak{p} \subset \mathfrak{g}$ satisfying the condition $\mathfrak{p} + \mathfrak{r} = \mathfrak{g}$, the equality $(\mathfrak{p} + \mathfrak{z}) \cap \mathfrak{h} = \mathfrak{p} \cap \mathfrak{h}$ holds, then H is closed in G.

Proof. Assume the contrary and consider the closure $\bar{H}$ of the group H in G. As in the proof of Theorem 5, we consider the one-parameter subgroup z_t generated by right multiplications by elements of the one-parameter group of local isometries $\bar{h}_t$ in the group G. Let $\bar{X}$ be the Killing vector field of tangent vectors of the orbits of the local one-parameter group of local isometries $\bar{h}_t^{-1}$ and Z be the Killing vector field of the local one-parameter group of local isometries z_t.

Let $\mathfrak{p}$ be a semisimple subalgebra of the algebra $\mathfrak{g}$ containing the vector field $\bar{X}$, $\bar{X} \in \mathfrak{p} \subset \mathfrak{g}$. We prove that $Z + \bar{X} \in \mathfrak{h}$ and $Z + \bar{X} \notin \mathfrak{p}$. In the simply connected Lie group G, we consider the radical R (the subgroup corresponding to the subalgebra $\mathfrak{r}$) and the semisimple subgroup P corresponding to the subalgebra $\mathfrak{p}$. Then R is a normal subgroup of the group G, $\mathfrak{r}$ is a normal subalgebra of the algebra $\mathfrak{g}$, $R \cap P = e$, $\mathfrak{r} \cap \mathfrak{p} = 0$, and the Levy–Malcev decomposition $G = RP$ holds.

The group G contains an open neighborhood of the identity element, which acts as a local group of local isometries in a neighborhood of the marked point $p \in M$. Since z_t belongs to the center R of the group G, we obtain that $z_t \in R$; since the subgroup H is a normal subgroup of the group $\bar{H}$ (see [1]), we have

$$\bar{h}_t^{-1} z_t H = \bar{h}_t^{-1} H \bar{h}_t = H.$$

Therefore, local isometries $\bar{h}_t^{-1} z_t$ fix the point p and hence they belong to the stationary subgroup H. Since $\bar{X} \in \mathfrak{p}$ and $Z \notin \mathfrak{p}$, we conclude that $(Z + \bar{X}) \notin \mathfrak{p}$. Since $(Z + \bar{X}) \in \mathfrak{h}$, this means that for a chosen maximal semisimple algebra $\mathfrak{p}$, the assertion $(\mathfrak{p} + \mathfrak{z}) \cap \mathfrak{h} \neq \mathfrak{p} \cap \mathfrak{h}$ holds. Theorem 9 is proved. □

4. Conclusion. In conclusion, we outline some unsolved problems associated with generalizing the completeness of analytic Riemannian manifolds.

(1) Obtain necessary and sufficient conditions for the closedness of a stationary subgroup of the group of local isometries on a Riemannian manifold. These conditions must be expressed in local terms, i.e., as properties of the Lie algebra of all Killing vector fields.

(2) Develop a theory of generalized complete manifolds for the case of a nontrivial center in the Lie algebra of all Killing vector fields. In particular, give a generalization of a quasi-complete manifold in the general case.

(3) Describe pseudo-complete manifolds in more detail in the general case and for specific Riemannian metrics.

(4) The conditions for the closedness of a stationary subgroup given in Theorems 8 and 9, are necessary and "almost sufficient." A problem is to find necessary and sufficient conditions for the closedness of a stationary subgroup in a simply connected group defined by the Lie algebra of all Killing vector fields.

REFERENCES

1. S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, Toronto (1978).
2. Sh. Kobayashi and K. Nomizu, *Foundations of Differential Geometry. Vol. 1*, Interscience, New York–London (1963).
3. A. I. Malcev, "On the theory of Lie groups in the large," *Mat. Sb.*, **16 (5)**, No. 2, 163–190 (1945).
4. G. D. Mostow, "Extensibility of Local Lie groups of transformations and groups on surfaces," *Ann. Math.*, **52**, 606–636 (1950).
5. V. A. Popov, "Analytic extension of locally defined Riemannian manifolds," *Mat. Zametki*, **36**, No. 4, 559–570 (1984).
6. V. A. Popov, "Extensibility of local isometry groups," *Mat. Sb.*, **135 (177)**, No. 1, 45–64 (1988).

7. V. A. Popov, “On the extendability of locally defined isometries of a pseudo-Riemannian manifolds,” *J. Math. Sci.*, **217**, No. 5, 624–627 (2016).
8. V. A. Popov, “On closeness of stationary subgroup of affine transformation groups,” *Lobachevskii J. Math.*, **38**, No. 4, 724–729 (2017).

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