DOI 10.1007/s10958-023-06786-3
Journal of Mathematical Sciences, Vol. 276, No. 5, November, 2023

Locally finite ultrametric spaces and labeled trees

Oleksiy Dovgoshey, Alexander Kostikov

Abstract. It is shown that a locally finite ultrametric space (X, d) is generated by a labeled tree if and only if for every open ball $B \subseteq X$ there is a point $c \in B$ such that $d(x, c) = \operatorname{diam} B$ whenever $x \in B$ and $x \neq c$. For every finite ultrametric space Y, we construct an ultrametric space Z having the smallest possible number of points such that Z is generated by a labeled tree and Y is isometric to a subspace of Z. It is proved that for a given Y such a space Z is unique up to isometry.

Keywords. Complete multipartite graph, diameter of ultrametric space, labeled tree, locally finite ultrametric space.

1. Introduction

In what follows, we denote by $\mathbb{R}^+$ the half-open interval $[0, \infty)$.

The *metric* on a set X is a function $d\colon X \times X \to \mathbb{R}^+$ such that for all x, y, $z \in X$

1. $d(x, y) = d(y, x)$,
2. $(d(x, y) = 0) \Leftrightarrow (x = y)$,
3. $d(x, y) \leq d(x, z) + d(z, y)$.

A metric space (X, d) is *ultrametric* if the *strong triangle inequality*

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}.$$

holds for all x, y, $z \in X$. In this case, the function d is called *an ultrametric* on X.

Definition 1.1. Let (X, d) and (Y, ρ) be metric spaces. A mapping $\Phi\colon X \to Y$ is an *isometric embedding* if

$$d(x, y) = \rho(\Phi(x), \Phi(y))$$

holds for all x, $y \in X$. A bijective isometric embedding is said to be an *isometry.* Metric spaces are *isometric* if there is an isometry of these spaces.

Let (X, d) be a metric space. An *open ball* with a *radius* $r > 0$ and a *center* $c \in X$ is the set

$$B_r(c) = \{x \in X : d(c, x) < r\}.$$

Definition 1.2. A metric space (X, d) is called *locally finite* if $\operatorname{card} B$ is finite for every open ball $B \subseteq X$.

In addition to open balls, we also need some other subsets of (X, d), which we will call the centered spheres.

Translated from Ukrains'kiĭ Matematychnyĭ Visnyk, Vol. 20, No. 3, pp. 350–380, July–September, 2023.
Original article submitted August 08, 2023

Definition 1.3. Let (X, d) be a metric space. A set $C \subseteq X$ is a *centered sphere* in (X, d) if there are $c \in C$, the center of C, and $r \in \mathbb{R}^+$, the radius of C, such that

$$C = \{x \in X : d(x, c) = r\} \cup \{c\}. \tag{1.1}$$

Equality (1.1) means that C is the sphere $\{x \in X : d(x, c) = r\}$ with the added center c.

We denote by $\mathbf{B}_X = \mathbf{B}_{X,d}$ and $\mathbf{Cs}_X = \mathbf{Cs}_{X,d}$ the sets of all open balls of the metric space (X, d) and, respectively, the sets of all centered spheres of this space.

Definition 1.4. The *labeled tree* is a pair (T, l), where T is a tree and l is a mapping defined on the vertex set $V(T)$.

In what follows, we consider only the nonnegative real-valued labelings $l : V(T) \to \mathbb{R}^+$.

Following [6], we define a mapping $d_l : V(T) \times V(T) \to \mathbb{R}^+$ as

$$d_l(u, v) = \begin{cases} 0 & \text{if } u = v, \\ \max\limits_{w \in V(P)} l(w) & \text{if } u \neq v, \end{cases} \tag{1.2}$$

where P is the path joining u and v in T.

Theorem 1.5 ([7])**.** Let $T = T(l)$ be a labeled tree. Then the function d_l is an ultrametric on $V(T)$ if and only if the inequality

$$\max\{l(u), l(v)\} > 0$$

holds for every edge $\{u, v\}$ of T.

Let us introduce a class **UGVL** (Ultrametrics Generating by Vertex Labelings) by the rule: An ultrametric space (X, d) belongs to **UGVL** if and only if there is a labeled tree $T = T(l)$ satisfying $X = V(T)$ and $d(x, y) = d_l(x, y)$ for all x, $y \in X$. If $(X, d) \in$ **UGVL**, then we say that (X, d) is generated by a labeled tree or that (X, d) is a **UGVL**-space.

The following conjecture was formulated in [7].

Conjecture 1.6. Let (X, d) be a nonempty, totally bounded ultrametric space. If all points of X are isolated, then the following statements are equivalent:

1. $(X, d) \in$ **UGVL**.
2. $\mathbf{B}_{X,d} \subseteq \mathbf{Cs}_{X,d}$.

E. Petrov proved in [10] the validity of the conjecture for finite ultrametric spaces using some other terms and the technique of Gurvich–Vyalyi representing trees. We repeat this result in Theorem 4.4 in Section 4 of the paper.

In Theorem 4.6, it is shown that the equivalence

$$((X, d) \in \mathbf{UGVL}) \Leftrightarrow (\mathbf{B}_{X,d} \subseteq \mathbf{Cs}_{X,d})$$

is valid for all nonempty locally finite ultrametric spaces (X, d).

Theorem 4.9 shows that $\mathbf{Cs}_{X,d} \subseteq \mathbf{B}_{X,d}$ holds if and only if d is a discrete metric on X.

In Theorem 5.9, we construct the "minimal" **UGVL**-extensions of an arbitrary finite ultrametric space and prove that all such minimal extensions are isometric.

2. Preliminaries. Trees and complete multipartite graphs

The *simple graph* is a pair (V, E) consisting of a nonempty set V and a set E whose elements are unordered pairs $\{u, v\}$ of different points u, $v \in V$. For a graph $G = (V, E)$, the sets $V = V(G)$ and $E = E(G)$ are called *the set of vertices* and *the set of edges*, respectively. We say that G is *empty* if $E(G) = \varnothing$. A graph G is *finite* if $V(G)$ is a finite set. A graph H is, by definition, a *subgraph* of a graph G if the inclusions $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ are valid. In this case, we simply write $H \subseteq G$.

The *path* is a finite nonempty graph P whose vertices can be numbered so that

$$V(P) = \{x_0, x_1, \ldots, x_k\},\ k \geqslant 1,$$
$$E(P) = \bigl\{\{x_0, x_1\}, \ldots, \{x_{k-1}, x_k\}\bigr\}.$$

In this case, we say that P is a path joining x_0 and x_k.

A graph G is *connected* if for every two distinct u, $v \in V(G)$ there is a path in G joining u and v.

A finite graph C with $\operatorname{card} V(G) \geqslant 3$ is a *cycle* if there is an enumeration of its vertices without repetitions such that

$$V(C) = \{v_1, \ldots, v_n\},$$
$$E(C) = \{\{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}.$$

Definition 2.1. A connected graph without cycles is called a *tree*.

A tree T may have a distinguished vertex r called the *root*; in this case, $T = T(r)$ is called a *rooted tree*.

Definition 2.2. If u and v are vertices of a rooted tree $T = T(r)$, then u is a *successor* of v if the path $P \subseteq T$ joining u and r contains the node v. A successor u of a node v is said to be a *direct successor* of the node v if $\{u, v\} \in E(T)$ holds.

Let $T = T(r)$ be a rooted tree and let v be a node of T. Denote by $\delta^+(v)$ the *out-degree* of v, i.e., $\delta^+(v)$ is the number of direct successors of v. The root r is a leaf of T if and only if $\delta^+(r) \leqslant 1$. Moreover, for a vertex v different from the root r, the equality $\delta^+(v) = 0$ holds if and only if v is a leaf of T.

Recall the definition of the isomorphic rooted trees.

Definition 2.3. Let $T_1 = T_1(r_1)$ and $T_2 = T_2(r_2)$ be rooted trees. A bijection $f\colon V(T_1) \to V(T_2)$ is an *isomorphism of the rooted trees* of T_1 and T_2 if $f(r_1) = r_2$ and

$$(\{u, v\} \in E(T_1)) \Leftrightarrow (\{f(u), f(v)\} \in E(T_2)).$$

The rooted trees T_1 and T_2 are isomorphic if there exists an isomorphism $f\colon V(T_1) \to V(T_2)$.

Definition 2.4. Let $T_i = T_i(r_i, l_i)$ be labeled rooted trees with the roots r_i and the labeling $l_i\colon V(T_i) \to \mathbb{R}^+$, $i = 1, 2$. An isomorphism $f\colon V(T_1) \to V(T_2)$ of the rooted trees $T_1(r_1)$ and $T_2(r_2)$ is an *isomorphism of the labeled rooted trees* $T_1(r_1, l_1)$ and $T_2(r_2, l_2)$ if the equality

$$l_2(f(v)) = l_1(v)$$

holds for every $v \in V(T_1)$. The labeled rooted trees $T_1(r_1, l_1)$ and $T_2(r_2, l_2)$ are isomorphic if there is an isomorphism of these trees.

We will say that a tree T is a *star* if there is a vertex $c \in V(T)$, the center of T, such that c and v are adjacent for every $v \in V(T) \setminus \{c\}$.

Proposition 2.5. A finite connected graph G with $\operatorname{card} V(G) = n$ is a tree if and only if $\operatorname{card} E(G) = n - 1$.

For the proof see, for example, Corollary 1.5.3 in [3].

The next simple proposition directly follows from Definition 2.1 and the definition of subgraphs of a graph.

Proposition 2.6. Let T be a tree and let G be a connected subgraph of T. Then G is a subtree of T.

Let $\{G_i : i \in I\}$ be a nonempty family of graphs such that

$$\left(\bigcup_{i\in I} V(G_i)\right) \cap \left(\bigcup_{i\in I} E(G_i)\right) = \varnothing.$$

Then the union $\bigcup_{i\in I} G_i$ is a graph H with

$$V(H) = \bigcup_{i\in I} V(G_i), \quad E(H) = \bigcup_{i\in I} E(G_i).$$

The definition of connectedness of graphs implies the following.

Proposition 2.7. Let $\{G_i : i \in I\}$ be a nonvoid family of connected subgraphs of a graph G. If the set $\bigcap_{i\in I} V(G_i)$ is nonempty, then $\bigcup_{i\in I} G_i$ is a connected subgraph of G.

In the proof of Theorem 4.6, we will also use the following simple fact.

Proposition 2.8. Let T_1, T_2, $T_3 \ldots$ be a sequence of trees satisfying the inclusion

$$T_i \subseteq T_{i+1} \tag{2.1}$$

for every integer $i \geqslant 1$. Then the graph

$$T\colon = \bigcup_{i=1}^{\infty} T_i \tag{2.2}$$

is a tree.

Proof. Indeed, T is a connected graph by Proposition 2.7. Suppose that we can find a cycle $C \subseteq T$. Since C is a finite graph, inclusion (2.1) and equality (2.2) imply that there is an integer $i_0 \geqslant 1$ such that

$$T_{i_0} \supseteq C.$$

The last inclusion is impossible, since T_{i_0} is a tree. Thus T also is a tree. □

Definition 2.9. Let G be a graph and let $k \geqslant 2$ be a cardinal number. The graph G is *complete k-partite* if the vertex set $V(G)$ can be partitioned into k nonempty, disjoint subsets, or *parts*, in such a way that no edge has both ends in the same part and any two vertices in different parts are adjacent.

We will say that G is a *complete multipartite graph* if there is a cardinal number k such that G is complete k-partite.

Lemma 2.10. Let G be a complete multipartite graph. Then the following conditions statements are equivalent:

1. There is a star $S \subseteq G$ such that $V(S) = V(G)$.
2. At least one part of G contains exactly one point.

Proof. $1 \Rightarrow 2$. Let $S \subseteq G$ be a star with the center c and let $V(S) = V(G)$. Then there is a part A of G such that $c \in A$. If u is a point of A and $u \neq c$, then, by Definition 2.9, the points u and c are nonadjacent in G. Now $S \subseteq G$ implies that these points are also nonadjacent in S, contrary to the definition of stars. Thus, the part A contains the point c only.

$2 \Rightarrow 1$. Let A be a part of G and let $\operatorname{card} A = 1$ hold. Write c for the unique point of A and consider the star S with the center c and $V(S) = V(G)$. Then $S \subseteq G$ follows from Definition 2.9. □

3. Preliminaries. Balls and centered spheres in ultrametric spaces

Let (X, ρ) be a metric space and let A be a subset of X. Recall that the *diameter* of A is the quantity

$$\operatorname{diam} A = \sup\{\rho(x, y) \colon x, y \in A\}. \tag{3.1}$$

Definition 3.1. If (X, ρ) is a metric space with $\operatorname{card} X \geqslant 2$, then the *diametrical graph* of (X, ρ) is a graph $G = G_{X,\rho}$ such that $V(G) = X$ holds and

$$\big(\{u, v\} \in E(G)\big) \Leftrightarrow \big(\rho(u, v) = \operatorname{diam} X\big)$$

is valid for all u, $v \in V(G)$.

The following theorem directly follows from Theorem 3.1 in [4].

Theorem 3.2. Let (X, ρ) be an ultrametric space with $\operatorname{card} X \geqslant 2$. If a diametrical graph $G_{X,\rho}$ is nonempty, then it is a complete multipartite graph.

The next lemma shows that the radius of any centered ultrametric sphere is equal to its diameter.

Lemma 3.3. Let C be a centered sphere in an ultrametric space (X, d) and let $\operatorname{card} C \geqslant 2$. If $c \in C$ and $r \in \mathbb{R}^+$ satisfy the condition

$$C = \{x \in X \colon d(x, c) = r\} \cup \{c\}, \tag{3.2}$$

then the equality

$$r = \operatorname{diam} C \tag{3.3}$$

holds.

Proof. The inequality $\operatorname{card} C \geqslant 2$ implies that there is a point $x \in C$ such that $d(x, c) = r$. Consequently,

$$r \leqslant \operatorname{diam} C \tag{3.4}$$

holds. Now using (3.2) and the strong triangle inequality, we obtain

$$d(u, v) \leqslant \max\{d(u, c), d(v, c)\} \leqslant r \tag{3.5}$$

for all u, $v \in C$. Equality (3.3) follows from (3.4) and (3.5). □

Lemma 2.10, Lemma 3.3 and Theorem 3.2 give us the following.

Corollary 3.4. Let (Y, ρ) be an ultrametric space with a nonempty diametrical graph $G_{Y,\rho}$. Then the following statements are equivalent:

1. $Y \in \mathbf{Cs}_{Y,\rho}$.

2. At least one part of the complete multipartite graph $G_{Y,\rho}$ contains exactly one point.

3. There is a star $S \subseteq G_{Y,\rho}$ such that $V(S) = V(G_{Y,\rho})$.

The next result is a special case of Proposition 3.3 from [1].

Lemma 3.5. Let (X, ρ) be a metric space with $\operatorname{card} X \geqslant 2$. If a diametrical graph $G_{X,\rho}$ is complete multipartite, then every part of $G_{X,\rho}$ is an open ball with a center $c \in X$ and the radius $r = \operatorname{diam} X$ and, conversely, every $B_r(c) \in \mathbf{B}_X$ with $r = \operatorname{diam} X$ is a part of $G_{X,\rho}$.

Using the last lemma, we obtain a refinement of Theorem 3.2.

Theorem 3.6. Let (X, ρ) be an ultrametric space with $\operatorname{card} X \geqslant 2$. If a diametrical graph $G_{X,\rho}$ is nonempty, then $G_{X,\rho}$ is complete multipartite and, moreover, the set of all parts of $G_{X,\rho}$ is the same as the set of all open balls of radius $r = \operatorname{diam} X$.

The following proposition claims that every point of an arbitrary ultrametric ball is the center of this ball.

Proposition 3.7. Let (X, d) be an ultrametric space. Then for every ball $B_r(c)$ and every $a \in B_r(c)$, we have $B_r(c) = B_r(a)$.

This directly follows from Proposition 18.4 in [12], so we omit the proof here.

Corollary 3.8. Let (X, d) be an ultrametric space. Then the inclusion

$$\mathbf{B}_{B,d|_{B\times B}} \subseteq \mathbf{B}_{X,d}$$

holds for every $B \in \mathbf{B}_X$.

As in the case of Corollary 3.8, Proposition 3.7 implies the following.

Corollary 3.9. Let (X, d) be an ultrametric space and let $B \in \mathbf{B}_{X,d}$. Then the inclusion

$$\mathbf{Cs}_{B,d|_{B\times B}} \subseteq \mathbf{Cs}_{X,d}$$

holds.

The following proposition describes some useful properties of locally finite ultrametric spaces.

Proposition 3.10. Let (X, d) be a locally finite ultrametric space, $c \in X$, and let $\mathbf{B}^c_{X,d}$ be the set of all open balls containing the point c,

$$\mathbf{B}^c_{X,d} = \{B \in \mathbf{B}_{X,d} : c \in B\}.$$

The following statements hold:

1. The mapping
$$\mathbf{B}^c_{X,d} \ni B \mapsto \operatorname{diam} B \in \mathbb{R}^+ \tag{3.6}$$
is injective.

2. If X is infinite, then there is a sequence $(B_1, B_2, \ldots, B_n, \ldots)$ of balls such that
$$\mathbf{B}^c_{X,d} = \{B_1, B_2, \ldots, B_n, \ldots\}, \tag{3.7}$$
with
$$\lim_{n\to\infty} \operatorname{diam} B_n = \infty \tag{3.8}$$
and
$$\operatorname{diam} B_n < \operatorname{diam} B_{n+1} \tag{3.9}$$
for every positive integer n.

Proof. (i). Since (X, d) is locally finite, every $B \in \mathbf{B}^c_{X,d}$ can be represented as
$$B = \{x \in X : d(x, c) \leqslant \operatorname{diam} B\},$$
which implies the injectivity of mapping (3.6).

(ii). Let X be infinite. Since (X, d) is locally finite, the set
$$\{d(c, x) : x \in X\} \cap [0, t]$$
is finite for every $t \in \mathbb{R}^+$. Moreover, the set
$$D^c_1 = \{d(c, x) : x \in X\}$$
is unbounded because every bounded locally finite metric space is finite. Using the last two assertions, it is easy to check that the sets D^c_1 and
$$\mathbb{N} = \{1, 2, \ldots, n, \ldots\}$$
are order-isomorphic as subsets of the order set $(\mathbb{R}^+, \leqslant)$. Let $\Phi : \mathbb{N} \to D^c_1$ be an order-isomorphism of $\mathbb{N}$ and D^c_1. Write
$$t_n := \Phi(n)$$
for every $n \in \mathbb{N}$. Then, by definition of the order-isomorphisms, we have
$$t_n < t_{n+1} \tag{3.10}$$
for each $n \in \mathbb{N}$. Furthermore, the limit relation
$$\lim_{n\to\infty} t_n = \infty \tag{3.11}$$
holds, since D^c_1 is an unbounded subset of $\mathbb{R}^+$.

Let us now denote by D^c_2 the set $\{\operatorname{diam} B : B \in \mathbf{B}^c_{X,d}\}$. We claim that the equality
$$D^c_1 = D^c_2 \tag{3.12}$$
holds.

Indeed, since (X,d) is locally finite, in each $B \in \mathbf{B}^c_{X,d}$ we can find $p \in B$ satisfying the equality

$$d(c,p) = \operatorname{diam} B.$$

Consequently, the inclusion

$$D^c_1 \subseteq D^c_2 \tag{3.13}$$

holds. Now, again using the local finiteness of (X,d) for each $a \in X$, we can find $\varepsilon > 0$ such that the set

$$\{x \in X \colon d(c,a) < d(c,x) < d(c,a) + \varepsilon\}$$

is empty, which implies the equality

$$\operatorname{diam} B_r(c) = d(c,a),$$

whenever $r \in (d(c,a), d(c,a)+\varepsilon)$. Thus the inclusion

$$D^c_2 \supseteq D^c_1 \tag{3.14}$$

holds. Equality(3.12) follows from (3.13) and (3.14).

Statement (i) implies that there is a bijection $F \colon D^c_2 \to \mathbf{B}^c_{X,d}$ satisfying the equality

$$\operatorname{diam} F(t) = t$$

for every $t \in D^c_2$.

Let us consider now the bijective mapping

$$\mathbb{N} \xrightarrow{\Phi} D^c_1 \xrightarrow{\mathrm{Id}} D^c_2 \xrightarrow{F} \mathbf{B}^c_{X,d},$$

where $\mathrm{Id} \colon D^c_1 \to D^c_2$ is the identical mapping, and define $B_n \in \mathbf{B}^c_{X,d}$ as the value of this mapping at point $n \in \mathbb{N}$. Then (3.8) and (3.9) follow from (3.10) and (3.11), respectively. $\square$

4. Characterization of locally finite UGVL-spaces

First of all, we note that the class **UGVL** contains all nonempty ultrametric spaces with at most 3 points.

Proposition 4.1. Let (X,d) be a nonempty ultrametric space. If the inequality $\operatorname{card} X \leqslant 3$ holds, then $(X,d) \in \mathbf{UGVL}$ and every $B \in \mathbf{B}_X$ is a centered sphere in (X,d).

Proof. If $\operatorname{card} X = 1$ or $\operatorname{card} X = 2$, then the proposition is trivially valid. Let us consider the case when $\operatorname{card} X = 3$, $X = \{x,y,z\}$.

Every triangle in any ultrametric space is isosceles, and its base has a length less than or equal to that of its legs. Thus, we may suppose that

$$d(x,y) = d(y,z) = a \text{ and } d(z,x) = b,$$

with $a \geqslant b > 0$. Let us consider now a labeled path $P_1 = P_1(l)$ with

$$V(P_1) = \{y,x,z\} \text{ and } E(P_1) = \{\{y,x\},\{x,z\}\}$$

and th labeling $l\colon V(P_1) \to \mathbb{R}^+$ such that

$$l(y) = a, \quad l(x) = 0 \quad \text{and} \quad l(z) = b$$

(see Figure 1). Then P_1 is a labeled tree. A simple calculation shows that $d = d_l$ holds, where d_l is defined by (1.2) with $T = P_1$. Thus, (X, d) belongs to the class **UGVL** by definition.

Let us prove that every $B \in \mathbf{B}_X$ is a centered sphere in (X, d). The last statement holds if $\operatorname{card} B = 1$, which follows from (1.1) with $S = B$ and $r = 0$.

If $\operatorname{card} B = 2$ or $\operatorname{card} B = 3$, then to see that B is a centered sphere, one can use Corollary 3.4 with $(Y, \rho) = (B, d|_{B\times B})$. □

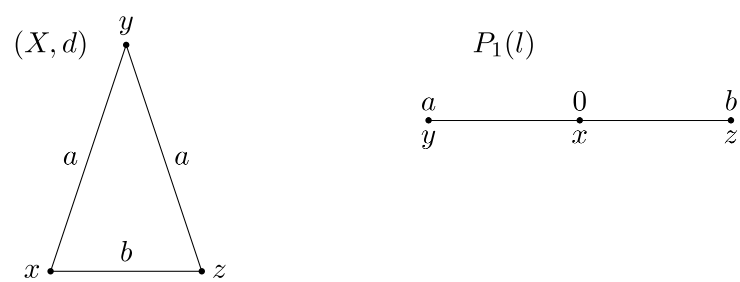

Figure 1: The ultrametric triangle (X, d) is generated by the labeled path $P_1(l)$.

The following example shows that 3 is the best possible constant in Proposition 4.1.

Example 4.2. Let us consider the four-point ultrametric space (X, d) depicted by Figure 2. To see that there is no labeled tree for which

$$d_l = d \tag{4.1}$$

holds, suppose that, for some tree T with $V(T) = \{x, y, z, t\}$ and $l\colon V(T) \to \mathbb{R}^+$, (4.1) holds. Then, using (1.2), we obtain

$$d_l(x, z) = 1 = \max\{l(x), l(z)\},$$
$$d_l(y, t) = 1 = \max\{l(y), l(t)\},$$

which implies

$$\operatorname{diam} X = \max\{l(x), l(y), l(z), l(t)\} = 1$$

contrary to $\operatorname{diam} X \geqslant d(x, y) = 2$.

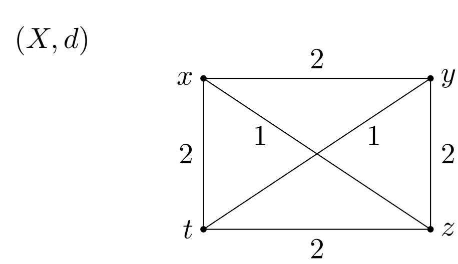

Figure 2: The four-point ultrametric space (X, d) does not belong to **UGVL**.

Let us show that every open ball in a **UGVL**-space is also a **UGVL**-space.

Lemma 4.3. Let $(X, d) \in \mathbf{UGVL}$ and let $T(l)$ be a labeled tree generating (X, d). Then, for every $B^1 \in \mathbf{B}_X$ there is a subtree T^1 of T such that

$$V(T^1) = B^1 \tag{4.2}$$

and

$$d|_{B^1 \times B^1} = d_{l^1} \tag{4.3}$$

holds, where l^1 is the restriction of the labeling $l\colon V(T) \to \mathbb{R}^+$ on the set $V(T^1)$.

Proof. Let $B^1 = B_{r_1}(c_1)$ be an arbitrary open ball in (X, d). If $\operatorname{card} B^1 = 1$ holds, then the empty tree T^1 with $V(T^1) = \{c_1\}$ satisfy also (4.3).

Suppose that $\operatorname{card} B^1 \geqslant 2$ and consider the family

$$\mathcal{F}_{B^1} = \{P^x \colon x \in B^1, x \neq c_1\},$$

where P^x is a unique path joining c_1 and x in T. Then, by Proposition 2.7, the union

$$T^1 := \bigcup_{P^x \in \mathcal{F}_{B^1}} P^x \tag{4.4}$$

is a connected subgraph of T and, consequently, T^1 is a subtree of T by Proposition 2.6. It follows directly from (4.4) that the inclusion $V(T^1) \supseteq B^1$ holds. Thus, to prove equality (4.2), it suffices to show that the inclusion

$$V(P^x) \subseteq B^1 \tag{4.5}$$

is valid for every $P^x \in \mathcal{F}_{B^1}$.

Let us consider an arbitrary $P^x \in \mathcal{F}_{B^1}$,

$$\begin{aligned} &V(P^x) = \{x_0, x_1, \ldots, x_k\},\\ &E(P^x) = \big\{\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{k-1}, x_k\}\big\}, \quad k \geqslant 1, \end{aligned}$$

$x_0 = c_1$ and $x_k = x$. Then, using (1.2), we obtain

$$\begin{aligned} d(x_0, x_j) = d_l(x_0, x_j) &= \max_{1 \leqslant i \leqslant j} l(v_i)\\ &\leqslant \max_{1 \leqslant i \leqslant k} l(v_i) = d(x_0, x) < r_1 \end{aligned}$$

for every $j \in \{1, \ldots, k\}$. Thus,

$$x_j \in B^1 \tag{4.6}$$

holds for every $j \in \{1, \ldots, k\}$. Now $x_0 = c_1$, $c_1 \in B$ and (4.6) imply (4.5).

To complete the proof, it suffices to note that (4.3) follows from (1.2), since we have $d = d_l$ and, for every pair of distinct u, $v \in V(T^1)$, there is a unique path P joining u and v in T, and that $P \subseteq T^1$ (because T^1 is a subtree of T). ☐

The next theorem can be proved using the Gurvich–Vyalyi representing tree technique (see Theorem 4.1 in [10]) but we will give an independent proof, which allows us to obtain a similar result for locally finite spaces.

Theorem 4.4. The statements

1. $(X, d) \in \mathbf{UGVL}$

   and

2. $\mathbf{B}_{X,d} \subseteq \mathbf{Cs}_{X,d}$

are equivalent for every finite nonempty ultrametric space (X, d).

Proof. $1 \Rightarrow 2$. By Proposition 4.1, the logical equivalence $1 \Leftrightarrow 2$ is valid if $\operatorname{card} X \leqslant 3$ holds. Thus, without loss of generality, we can assume that

$$\operatorname{card} X \geqslant 4. \tag{4.7}$$

Let (X, d) belong to the class $\mathbf{UGVL}$. Then there is a labeled tree $T = T(l)$ such that $V(T) = X$ and $d_l = d$ hold. We must show that the inclusion

$$\mathbf{Cs}_{X,d} \supseteq \mathbf{B}_{X,d} \tag{4.8}$$

is valid, i.e., every open ball B in (X, d) is a centered sphere in (X, d). Let us make sure that the last statement is true for the case $B = X$.

The finiteness of X and inequality (4.7) imply that the diametrical graph $G_{X,d}$ is nonempty. Using Corollary 3.4, we obtain that $X \in \mathbf{Cs}_{X,d}$ holds if and only if at least one part of the complete multipartite graph $G_{X,d}$ contains exactly one point. Let $\{A_1, \ldots, A_k\}$ be the set of all parts of $G_{X,d}$. Suppose on the contrary that the inequality

$$\operatorname{card} A_i \geqslant 2 \tag{4.9}$$

holds for every $i \in \{1, \ldots, k\}$. Let us consider a subset $\{c_1, \ldots, c_k\}$ of the set X such that $c_i \in A_i$ for every $i \in \{1, \ldots, k\}$. Then, by Theorem 3.6 and Proposition 3.7, for every $i \in \{1, \ldots, k\}$ we have

$$A_i = B_r(c_i) \tag{4.10}$$

with $r = \operatorname{diam} X$. Lemma 4.3 implies now that all ultrametric spaces $(A_1, d|_{A_1 \times A_1}), \ldots, (A_k, d|_{A_k \times A_k})$ belong to the class $\mathbf{UGVL}$. In particular, by Lemma 4.3, there are labeled subtrees $T^1(l_1), \ldots, T^k(l_k)$ of the labeled tree $T(l)$ such that

$$V(T^i) = A_i \quad \text{and} \quad d|_{A_i \times A_i} = d_{l_i} \tag{4.11}$$

hold with $l_i = l|_{A_i}$ for every $i \in \{1, \ldots, k\}$. Now using formula (4.10) with $r = \operatorname{diam} X$ and (4.11), we obtain the strict inequality

$$\max_{u \in A_i} l(u) < \operatorname{diam} X \tag{4.12}$$

for every $i \in \{1, \ldots, k\}$. Since the number k of the parts of $G_{X,d}$ is finite and $\{A_1, \ldots, A_k\}$ is a partition of X, inequality (4.12) gives us

$$\max_{u \in X} l(u) = \max_{1 \leqslant i \leqslant k} \max_{u \in A_i} l(u) < \operatorname{diam} X. \tag{4.13}$$

Now to complete the proof of the validity of $1 \Rightarrow 2$, it suffices to note that the finiteness of X and the definition of the ultrametric d_l imply the equality

$$\max_{u \in X} l(u) = \operatorname{diam} X,$$

contrary to (4.13).

$2 \Rightarrow 1$. We must show that

$$(X, d) \in \mathbf{UGVL} \tag{4.14}$$

whenever (X, d) is a finite nonempty ultrametric space satisfying the inclusion

$$\mathbf{B}_{X,d} \subseteq \mathbf{Cs}_{X,d}. \tag{4.15}$$

To prove the above statement, we will use the induction on $\operatorname{card} X$.

By Proposition 4.1, we obtain that (4.15) $\Rightarrow$ (4.14) is valid for every ultrametric space (X, d) with $1 \leqslant \operatorname{card} X \leqslant 3$.

Let $n \geqslant 3$ be a given integer number. Suppose that (4.15) $\Rightarrow$ (4.14) is valid if

$$1 \leqslant \operatorname{card} X \leqslant n. \tag{4.16}$$

Let us consider an arbitrary fixed ultrametric space (X, d) such that $\operatorname{card} X = n + 1$ and (4.15) holds.

Let $\{A_1, \ldots, A_k\}$ be the set of all parts of the diametrical graph $G_{X,d}$. By Theorem 3.6, every A_i, $i \in \{1, \ldots, k\}$, is an open ball in (X, d). Now, using Corollaries 3.8 and 3.9, we see that (4.15) implies the inclusion

$$\mathbf{B}_{A_i, d|_{A_i \times A_i}} \subseteq \mathbf{Cs}_{A_i, d|_{A_i \times A_i}} \tag{4.17}$$

for every $i \in \{1, \ldots, k\}$. Since each A_i is a proper subset of X, the induction hypothesis gives us the membership

$$\left(A_i, d|_{A_i \times A_i}\right) \in \mathbf{UGVL}$$

for every $i \in \{1, \ldots, k\}$. Consequently, for every $i \in \{1, \ldots, k\}$, we can find a labeled tree $T^i(l_i)$ such that

$$V(T^i) = A_i \quad \text{and} \quad d|_{A_i \times A_i} = d_{l_i}. \tag{4.18}$$

Let $\{c_1, \ldots, c_k\}$ be a subset of the set X such that $c_i \in A_i$ holds for every $i \in \{1, \ldots, k\}$. By Corollary 3.4, equality (4.15) implies that there is $i \in \{1, \ldots, k\}$ such that $\operatorname{card} A_i = 1$. Without loss of generality, we may assume that the set A_1 is a singleton, $A_1 = \{c_1\}$.

Let us expand the labeled tree $T^i = T^i(l_i)$ to a labeled tree $T_1^i = T_1^i(l_{i,1})$ by the rule: $V(T_1^i) = \{c_1\} \cup V(T^i)$, $E(T_1^i) = \{c_1, c_i\} \cup E(T^i)$ and

$$l_{i,1} = \begin{cases} l_i(u) & \text{if } u \in V(T^i) \\ \operatorname{diam} X & \text{if } u = c_1 \end{cases} \tag{4.19}$$

for every $i \in \{2, \ldots, k\}$.

By Proposition 2.7, the graph

$$T = \bigcup_{i=2}^{k} T_1^i$$

is connected. Now, using Proposition 2.5, we can prove that T is a tree. Indeed, by Proposition 2.5, T is a tree iff earlier–"if and only if"

$$\operatorname{card} V(T) - \operatorname{card} E(T) = 1. \tag{4.20}$$

To prove the last equality we note that

$$\begin{aligned}\operatorname{card} V(T) &= \sum_{i=1}^{k} \operatorname{card} A_i = 1 + \sum_{i=2}^{k} \operatorname{card} V(T^i) \\ &= 1 + \sum_{i=2}^{k} \big(\operatorname{card} V(T_1^i) - 1\big) = 2 - k + \sum_{i=2}^{k} \operatorname{card} V(T_1^i).\end{aligned}$$

and

$$\operatorname{card} E(T) = \sum_{i=2}^{k} \operatorname{card} E(T_1^i).$$

Consequently, we have

$$\operatorname{card} V(T) - \operatorname{card} E(T) = 2 - k + \sum_{i=2}^{k} \big(\operatorname{card} V(T_1^i) - \operatorname{card} E(T_1^i)\big). \tag{4.21}$$

Since every T_1^i is a tree, $\operatorname{card} V(T_1^i) - \operatorname{card} E(T_1^i) = 1$ holds for each $i \in \{2, \dots, k\}$. Thus, the right half of formula (4.21) can be written as

$$2 - k + \sum_{i=2}^{k} \big(\operatorname{card} V(T_1^i) - \operatorname{card} E(T_1^i)\big) = 2 - k + (k-1) = 1,$$

which implies (4.20).

Using (4.19), we can find a labeling $l\colon V(T) \to \mathbb{R}^+$ such that

$$l|_{V(T_1^i)} = l_{i,1} \tag{4.22}$$

holds for every $i \in \{2, \dots, k\}$. Then we have $V(T) = X$; in addition, equalities (4.18), (4.19), and (4.22) imply the equality $d_l = d$. Thus, (4.14) is valid. □

The second part of the proof of Theorem 4.4 (see, in particular, formula (4.22)) gives us the following.

Corollary 4.5. Let $(Y, \rho) \in \mathbf{UGVL}$ be finite, let the diametrical graph $G_{Y,\rho}$ be complete multipartite with parts $B_1^Y, \dots, B_n^Y$, and let $T_1 = T_1(l), \dots, T_n = T_n(l)$ be labeled trees generating, respectively, the ultrametric spaces $\left(B_1^Y, \rho_{|B_1^Y \times B_1^Y}\right), \dots, \left(B_n^Y, \rho_{|B_n^Y \times B_n^Y}\right)$. Then there exists a labeled tree $T = T(l)$ generating (Y, ρ) such that $T_i \subseteq T$ and $l_{|V(T_i)} = l_i$ for every $i = 1, 2 \dots, n$.

Let us now turn to the case of locally finite ultrametric spaces.

The following theorem is the first main result of the paper.

Theorem 4.6. Let (X, d) be a locally finite nonempty ultrametric space. Then the following statements are equivalent:

1. $(X, d) \in \mathbf{UGVL}$.

2. $\mathbf{B}_{X,d} \subseteq \mathbf{Cs}_{X,d}$.

3. For every $B \in \mathbf{B}_{X,d}$ with $\operatorname{card} B \geqslant 2$, there is a star S such that $V(S) = B$ and $S \subseteq G_{B,d_{|B\times B}}$, where $G_{B,d_{|B\times B}}$ is the diametrical graph of the space $(B, d_{|B\times B})$.

Proof. Corollaries 3.7, 3.8, and 3.4 show that the logical equivalence $(ii) \Leftrightarrow (iii)$ is valid.

Moreover, if $(X,d) \in \mathbf{UGVL}$ holds, then, for every $B \in \mathbf{B}_{X,d}$ we have $(B, d_{|B\times B}) \in \mathbf{UGVL}$ by Lemma 4.3. Consequently, using Corollary 3.7, Corollary 3.8, and the finiteness of balls in locally finite metric spaces, we see that the validity of $(i) \Rightarrow (ii)$ follows from Theorem 4.4.

To complete the proof, it suffices to show that $(ii) \Rightarrow (i)$ is also valid.

Let us consider the case when (X,d) is infinite. In the case when (X,d) is finite, the validity of $(ii) \Rightarrow (i)$ was proved in Theorem 4.4.

Suppose that (ii) holds. Let c be a point of X and let

$$\mathbf{B}^c_{X,d} \colon = \{B \in \mathbf{B}_{X,d} \colon c \in B\}.$$

Then, by Proposition 3.10, there exists an infinite sequence $(B_n)_{n\in\mathbb{N}}$ of open balls satisfying the conditions:

(s_1) $\operatorname{diam} B_n < \operatorname{diam} B_{n+1}$ for every $n \in \mathbb{N}$;

(s_2) $\mathbf{B}^c_{X,d} = \{B_n \colon n \in \mathbb{N}\}$.

Every open ball is a finite subset of X because (X,d) is locally finite. Consequently, by Theorem 4.4, the ultrametric space $\left(B_n, d_{|B_n\times B_n}\right)$ belongs to the class **UGVL** for every $n \in \mathbb{N}$.

Now using Corollary 4.5 and statements (s_1), we can find a sequence $(T_n)_{n\in\mathbb{N}}$ of labeled trees $T_n = T_n(l_n)$ such that:

(s_3) $\left(B_n, d_{|B_n\times B_n}\right)$ is generated by $T_n(l_n)$;

(s_4) $T_n \subseteq T_{n+1}$ and $l_{n+1|V(T_n)} = l_n$ for every $n \in \mathbb{N}$.

Write

$$T \colon = \bigcup_{n=1}^{\infty} T_n.$$

Then T is a tree by Proposition 2.8. From statements (s_4) and the equality

$$V(T) = \bigcup_{n=1}^{\infty} V(T_n), \tag{4.23}$$

it follows that there is a labeling $l \colon V(T) \to \mathbb{R}^+$ such that

$$l_{|V(T)} = l_n \tag{4.24}$$

for every $n \in \mathbb{N}$. Statements (s_2) and (s_3) give us

$$X = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} V(T_n),$$

which together with (4.23) implies the equality

$$V(T) = X.$$

Now using the last equality, equality (4.24), and (s_3)–(s_4), it is easy to show that (X,d) is generated by the labeled tree $T = T(l)$. □

Theorem 4.6 claims, in particular, that the inclusion $\mathbf{B}_{X,d} \subseteq \mathbf{Cs}_{X,d}$ implies $(X, d) \in \mathbf{UGVL}$ for locally finite ultrametric spaces (X, d). In the rest of the Section, we want to show that the reserve inclusion $\mathbf{Cs}_{X,d} \subseteq \mathbf{B}_{X,d}$ holds iff the metric d is discrete.

We say that a metric $d\colon X \times X \to \mathbb{R}^+$ is *discrete* if there is a constant $k > 0$ such that

$$d(x, y) = k \tag{4.25}$$

whenever x and y are distinct points of X.

Remark 4.7. The standard definition of *discrete metric* can be formulated as follows: "A metric on X is discrete if the distance from each point of X to every other point of X is one." (See, for example, [13, p. 4].)

Lemma 4.8. The following conditions statements are equivalent for every metric space (X, d):

1. The metric d is discrete.
2. For each $x \in X$ there is $k > 0$ such that (4.25) holds whenever $y \in X \setminus \{x\}$.

Proof. $1 \Rightarrow 2$. This implication is trivially valid.

$2 \Rightarrow 1$. Let 2 hold but d not be a discrete metric. Then there are some points $x, y, u, v \in X$ such that

$$d(x, y) \neq d(u, v), \tag{4.26}$$

and

$$\min\{d(x, y), d(u, v)\} > 0. \tag{4.27}$$

If the sets $\{x, y\}$ and $\{u, v\}$ have a common point, then, without loss of generality, we suppose $x = u$. From (4.26) and (4.27), it follows that

$$x \neq y, \ u \neq v, \quad \text{and } d(x, y) \neq d(u, v),$$

contrary to 2. Consequently, the sets $\{x, y\}$ and $\{u, v\}$ are disjoint.

Now using condition 2 again, we obtain

$$d(x, y) = d(x, u) \neq 0$$

and

$$d(u, x) = d(u, v) \neq 0,$$

which implies $d(x, y) = d(u, v)$. The last equality contradicts (4.26). The validity of $2 \Rightarrow 1$ follows. □

Theorem 4.9. Let (X, d) be a nonempty ultrametric space. Then the following statements are equivalent:

1. $\mathbf{B}_{X,d} \supseteq \mathbf{Cs}_{X,d}$.
2. The metric d is discrete.
3. $\mathbf{B}_{X,d} = \mathbf{Cs}_{X,d}$.

Proof. The implications $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$ are evidently valid. Let us prove the validity of $(i) \Rightarrow (ii)$.

Let (i) hold but d not be a discrete metric. Then, by Lemma 4.8, there are distinct points $a, b, c \in X$ such that

$$d(c,a) > d(c,b) > 0. \tag{4.28}$$

Write

$$C\colon = \{x \in X\colon d(x,c) = r\} \cup \{c\}, \tag{4.29}$$

where

$$r = d(c,a). \tag{4.30}$$

Then C is a centered sphere in $(X,d), a \in C$, and

$$b \notin C, \tag{4.31}$$

by (4.28)–(4.30). By condition (i), there is $B_1 \in \mathbf{B}_{X,d}$,

$$B_1\colon = \{x \in X\colon d(x,c_1) < r_1\}$$

such that $B_1 = C$. Since $c \in C$, Proposition 3.7 implies that a center c of the centered sphere C also is the center of the ball B_1,

$$B_1 = \{x \in X\colon d(x,c) < r_1\}. \tag{4.32}$$

Now using (4.30) and $a \in C$, we obtain the inequality $r < r_1$. Consequently, $b \in B_1$ holds by (4.28) and (4.32). To complete the proof, it suffices to note that $b \in B_1$ and $B_1 = C$ give us $b \in C$, contrary to (4.31). □

5. Isometric embedding of finite ultrametric spaces in UGVL-spaces

Now we want to show that any finite ultrametric space can be extended to some minimal **UGVL**-space, and that such an extension is unique up to isometry.

Definition 5.1. Let (X,d) be an ultrametric space. An **UGVL**-space (Y,ρ) is a UGVL-*extension* of (X,d) if there is $Y_1 \subseteq Y$ such that $(Y_1, \rho|_{Y_1 \times Y_1})$ is isometric to (X,d).

In what follows, we will say that a UGVL-extension (Y,ρ) of (X,d) is *minimal* if, for every proper subset Y_0 of Y, the ultrametric space $(Y_0, \rho|_{Y_0 \times Y_0})$ is not a UGVL-extension of (X,d).

Example 5.2. Let (X,d), $X = \{x,y,z,t\}$, be a four-point ultrametric space depicted in Figure 2. It was shown in Example 4.2 that $(X,d) \notin$ **UGVL**. Let us consider the five-point set $\overline{X} = \{x,y,z,t,w\}$ and define an ultrametric $\overline{d}$ on $\overline{X}$ such that $\overline{d}|_{X\times X} = d$ and $\overline{d}(w,p) = 2$ whenever $p \in X$ (see Figure 3). It is easy to see that only $\overline{X}$, $\{x,z\}$, and $\{t,y\}$ are non-singleton open balls in $(\overline{X},\overline{d})$. Since each of these sets is a centered sphere, $(\overline{X},\overline{d}) \in$ **UGVL** by Theorem 4.4.

Example 5.3. Let (X,d) be infinite and let $d\colon X \times X \to \mathbb{R}^+$ be discrete. Then (X,d) is a UGVL-extension of itself, but there is no minimal **UGVL**-extension of (X,d).

To construct minimal **UGVL**-extensions of a finite ultrametric space, we will use the Gurvich–Vyalyi representing trees. Recall the procedure for constructing such trees.

With every finite nonempty ultrametric space (X,d), we can associate a labeled rooted tree $T(X,l)$ by the following rule (see [9,11]). The root of $T(X,l)$ is the set X. If X is a one-point set, then $T(X,l)$

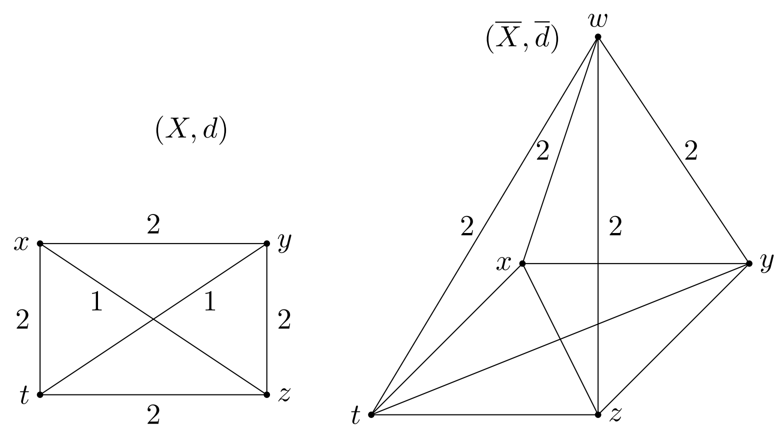

Figure 3: The pyramid $(\overline{X},\overline{d})$ is a minimal **UGVL**-extension of the quadruple (X,d).

is a rooted tree consisting of one node X with the label 0. Let $|X| \geqslant 2$. According to Theorem 3.6, the diametral graph $G_{X,d}$ is complete multipartite with the parts $X_1,\dots,X_k$, where all $X_1,\dots,X_k$ are open balls in (X,d). In this case, the root of the tree $T(X,l)$ is labeled by $\operatorname{diam} X > 0$ and, moreover, $T(X,l)$ has the nodes $X_1, \dots, X_k$, $k \geqslant 2$, of the first level with the labels

$$l(X_i) = \operatorname{diam} X_i, \quad i = 1,\dots,k. \tag{5.1}$$

The nodes of the first level labeled by 0 are leaves, and those indicated by $\operatorname{diam} X_i > 0$ are internal nodes of the tree $T(X,l)$. If the first level has no internal nodes, then the tree $T(X,l)$ is constructed. Otherwise, by repeating the above-described procedure with X_i corresponding to the internal nodes of the first level, we obtain nodes of the second level, etc. Since X is a finite set, all vertices at some level will be leaves, and the construction of $T(X,l)$ is completed.

We will say that the labeled rooted tree $T(X,l)$ is a *representing tree* of (X,d).

It can be shown that for any finite ultrametric space (X,d) the vertex set of the representing tree $T = T(X,l)$ coincides with the set of all open balls of (X,d),

$$V(T) = \mathbf{B}_X \tag{5.2}$$

(see, for example, Theorem 1.6 in [9]). Using Theorem 1.5 and equality (5.2), we see that the defined-above labeling $l\colon V(T) \to \mathbb{R}^+$ generates an ultrametric on the set $\mathbf{B}_X$. It is interesting to note that this ultrametric is the Hausdorff metric d_H on $\mathbf{B}_X$,

$$d_H(B_1,B_2) = \max\{\sup_{x\in B_1} d(x,B_2), \sup_{x\in B_2} d(x,B_1)\},$$

where

$$d(x,B) = \inf_{b\in B} d(x,b).$$

Proposition 5.4. Let (X,d) be a finite nonempty ultrametric space with the Gurvich–Vyalyi representing tree $T(X,l)$. Then $(\mathbf{B}_X, d_H)$ is a UGVL-ultrametric space generated by the labeled tree $T(l) = T(X,l)$.

For the proof, see Theorem 2.5 in [5].

The next characterization of finite **UGVL**-spaces can be considered as a reformulation of Theorem 4.1 from [10].

Theorem 5.5. Let (X,d) be a finite nonempty ultrametric space. Then $(X,d) \in \mathbf{UGVL}$ if and only if every internal node of the representing tree $T(X,l)$ has at least one direct successor which is a leaf of $T(X,l)$.

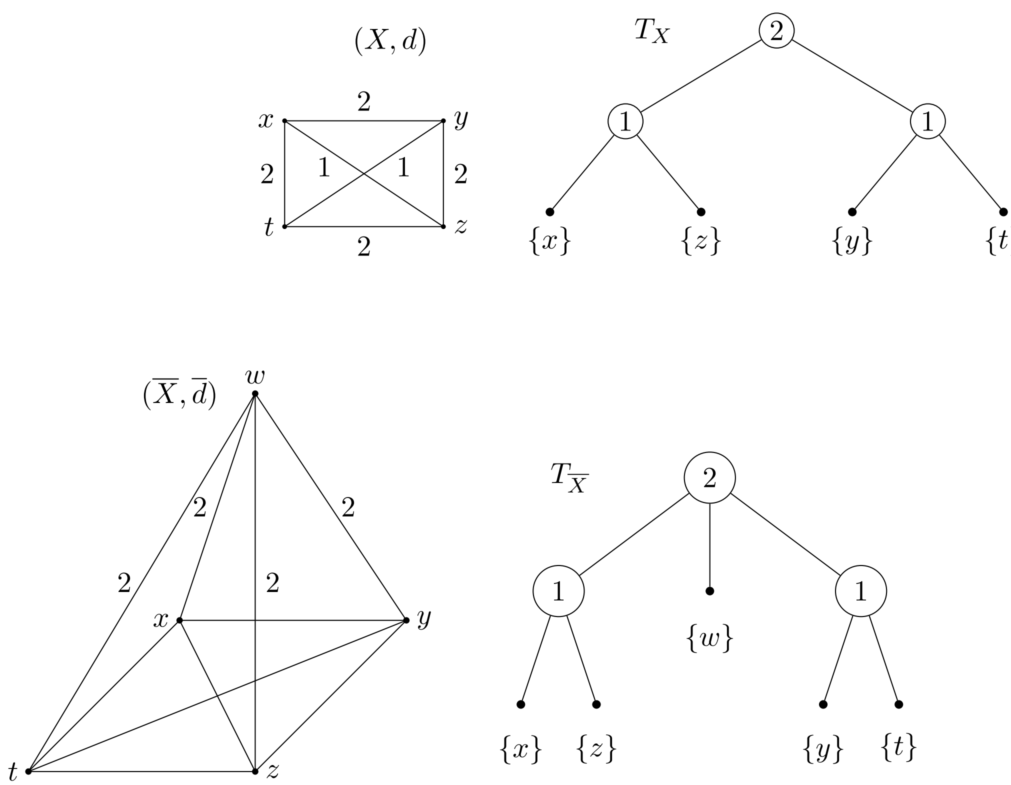

Figure 4: The representing trees of the quadruple (X,d) and the pyramid $(\overline{X},\overline{d})$. The singleton $\{w\}$ is a leaf of the root of $T_{\overline{X}}$.

The next proposition and equality (5.2) show that Theorems 4.4 and 5.5 are really equivalent.

Proposition 5.6. Let (X,d) be a finite ultrametric space, $B \in \mathbf{B}_X$ be an internal node of the representing tree $T(X,l)$, and let c be a point of B. Then the ball B is a centered sphere with the center c if and only if the singleton $\{c\}$ is a leaf of $T(X,l)$ and, simultaneously, a direct successor of the internal node B.

Proof. The validity of this assertion is checked using the above procedure for constructing $T(X,l)$, and Corollary 3.4 with $Y = B$ and $\rho = d_{|B\times B}$. □

The proofs of the following two theorems can be found in [8].

Theorem 5.7. Let (X,d) and (Y,ρ) be nonempty finite ultrametric spaces. Then the representing trees of these spaces are isomorphic as labeled rooted trees if and only if (X,d) and (Y,ρ) are isometric.

Theorem 5.8. Let $T = T(r,l)$ be a finite labeled rooted tree with the root r and the labeling $l\colon V(T) \to \mathbb{R}^+$. Then the following two conditions are equivalent.

1. For every $u \in V(T)$, we have $\delta^+(u) \neq 1$ and

$$(\delta^+(u) = 0) \Leftrightarrow (l(u) = 0);$$

in addition, the inequality

$$l(v) < l(u) \tag{5.3}$$

holds whenever v is a direct successor of u.

2. There is a finite ultrametric space (X, d) such that the representing tree of (X, d) and T are isomorphic as labeled rooted trees.

For every finite nonempty ultrametric space, we denote by $\Delta(X, d)$ the number of open balls that are not centered spheres in (X, d),

$$\Delta(X, d) := \operatorname{card}(\mathbf{B}_{X,d} \setminus \mathbf{Cs}_{X,d}). \tag{5.4}$$

The following theorem is the second main result of the paper.

Theorem 5.9. Let (X, d) be a finite nonempty ultrametric space. Then the following statements hold.

1. Every **UGVL**-extension (Y, ρ) of (X, d) satisfies the inequality

$$\operatorname{card} Y \geqslant \Delta(X, d) + \operatorname{card} X. \tag{5.5}$$

2. A **UGVL**-extension (Y, ρ) of (X, d) is minimal if and only if the equality

$$\operatorname{card} Y = \Delta(X, d) + \operatorname{card} X \tag{5.6}$$

holds.

3. All minimal **UGVL**-extensions of (X, d) are isometric.

Proof. Let (Y, ρ) be a UGVL-extension of (X, d), and let $\Phi\colon X \to Y$ be an isometric embedding of (X, d) in (Y, ρ). Write

$$Y_1 \colon = \Phi(X) \quad \text{and} \quad \rho_1 \colon = \rho_{|Y_1 \times Y_1}.$$

Then (Y, ρ) is a UGVL-extension of (Y_1, ρ_1), the equalities $\Delta(X, d) = \Delta(Y_1, \rho_1)$ and $\operatorname{card} X = \operatorname{card} Y_1$ hold, and, moreover, (Y, ρ) is minimal for (X, d) iff it is minimal for (Y_1, ρ_1). Thus, without loss of generality, we may consider only those **UGVL**-extensions of the space (X, d) that are superspaces of this space.

(i). Let us consider an arbitrary **UGVL**-extension (Y, ρ) of (X, d). Since X is finite, inequality (5.5) evidently holds if Y is infinite.

Suppose that (Y, ρ) is a finite ultrametric space, and define a mapping $F\colon \mathbf{B}_{X,d} \to \mathbf{B}_{Y,\rho}$ as

$$F(B) \colon = \{y \in Y \colon \rho(y, b) \leq \operatorname{diam} B\}, \tag{5.7}$$

where b is an arbitrary given point of B.

We claim that F is an injective mapping. Let us prove it. As in the proof of statement (i) of Proposition 3.10, we have the equality

$$B = \{x \in X \colon d(x, b) \leq \operatorname{diam} B\} \tag{5.8}$$

for all $B \in \mathbf{B}_{X,d}$ and $b \in B$. Since (X, d) is a subspace of (Y, ρ), (5.7) and (5.8) imply

$$B \subseteq F(B) \tag{5.9}$$

for every $B \in \mathbf{B}_{X,d}$.

Let B_1 and B_2 be open balls in (X, d) such that

$$F(B_1) = F(B_2). \tag{5.10}$$

We must show that

$$B_1 = B_2. \tag{5.11}$$

Inclusion (5.9) and equality (5.10) give us

$$B_2 \subseteq F(B_1) \quad \text{and} \quad B_1 \subseteq F(B_2). \tag{5.12}$$

Let b_i be an arbitrary point of B_i, $i = 1$, 2. The equality $d = \rho_{|X \times X}$, (5.7), and (5.12) imply

$$d(b_1, b_2) \leqslant \operatorname{diam} B_1$$

and

$$d(b_1, b_2) \leqslant \operatorname{diam} B_2.$$

Hence the membership relations $b_1 \in B_2$ and $b_2 \in B_1$ are valid. Since b_i is an arbitrary point of B_i, $i = 1, 2$, it implies

$$B_1 \subseteq B_2 \quad \text{and} \quad B_2 \subseteq B_1.$$

Equality (5.10) follows. Thus $F\colon \mathbf{B}_{X,d} \to \mathbf{B}_{Y,\rho}$ is injective.

Let us prove the validity of inequality (5.5).

Since (Y, ρ) belongs to the class **UGVL**, Theorem 4.4 implies that the ball $F(B)$ is a centered sphere in (Y, ρ) for every $B \in \mathbf{B}_{X,d}$. If c is a center of the centered sphere $F(B) \in \mathbf{Cs}_{Y,\rho}$ and

$$B \in \mathbf{B}_{X,d} \setminus \mathbf{Cs}_{X,d}, \tag{5.13}$$

then c is not a point of the ball B,

$$c \notin B. \tag{5.14}$$

Indeed, since (X, d) is a subspace (Y, ρ), Definition 1.3 and the membership $c \in B$ imply that B is a centered sphere in (X, d)

$$B \in \mathbf{Cs}_{X,d},$$

contrary to (5.13). Thus, (5.14) holds. By Proposition 5.6, the set $\{c\}$ is a leaf of T_Y and, at the same time, a direct successor of the node $F(B)$ of the representing tree T_Y. Since different nodes of T_Y do not have common direct successors, inequality(5.5) follows from the injectivity of the mapping $F\colon \mathbf{B}_{X,d} \to \mathbf{B}_{Y,\rho}$.

(ii). Let (Y, ρ) be a UGVL-extension of (X, d), and let equality (5.6) hold. Then Y is a finite set. Since every proper subset Y_0 of the finite set Y satisfies the inequality

$$\operatorname{card} Y_0 < \operatorname{card} Y,$$

equality(5.6) implies

$$\operatorname{card} Y_0 < \Delta(X, d) + \operatorname{card} X.$$

Consequently, $(Y_0, \rho_{|Y_0\times Y_0})$ is not a UGVL-extension of (X, d) by statement (i). Thus, if equality (5.6) holds, then (Y, ρ) is a minimal **UGVL**-extension of (X, d).

Let (Y, ρ) be a minimal **UGVL**-extension of (X, d). We will prove that (Y, ρ) satisfies equality (5.6).

First of all, prove that any minimal **UGVL**-extension $W(\lambda)$ of a given finite nonempty ultrametric space $S(\delta)$ is also a finite ultrametric space. By Proposition 4.1, it suffices to consider the case

$$\operatorname{card} S \geqslant 4. \tag{5.15}$$

Let $T = T(l)$ be a labeled tree generating the space (W, λ). As was noted in the first part of the proof, we may also suppose that $W \supseteq S$. Hence S is a subset of the vertex set $V(T)$. The union of all paths connecting in T different points of S is a finite subtree T^S of T. It should be noted that this union is nonempty due to (5.15). Let us define a labeling $l^S\colon V(T^S) \to \mathbb{R}^+$ as $l^S = l_{|V(T^S)}$. Then the ultrametric space $(V(T^S), \delta^S)$ with

$$\delta^S = \lambda_{|V(T^S)\times V(T^S)}$$

is a superspace for (S, δ) and a finite subspace of (W, λ); moreover, $(V(T^S), \delta^S)$ is generated by the labeled tree $T^S(l^S)$. Consequently, $(V(T^S), \delta^S) \in \mathbf{UGVL}$ holds, which implies

$$(V(T^S), \delta^S) = (W, \lambda)$$

due to the minimality of (W, λ).

Thus (Y, ρ) is a finite ultrametric space.

Let $\mathbf{B}^I_{X,d}$ and $\mathbf{B}^I_{Y,\rho}$ be defined as

$$\mathbf{B}^I_{X,d}\colon = \{B \in \mathbf{B}_{X,d}\colon \operatorname{diam} B > 0\}, \tag{5.16}$$

and

$$\mathbf{B}^I_{Y,\rho}\colon = \{B \in \mathbf{B}_{Y,\rho}\colon \operatorname{diam} B > 0\}. \tag{5.17}$$

We claim that the equality

$$\operatorname{diam}(B \cap X) = \operatorname{diam} B \tag{5.18}$$

holds for every $B \in \mathbf{B}^I_{Y,\rho}$. For the case $B = Y$, equality (5.18) can be written as

$$\operatorname{diam}(Y \cap X) = \operatorname{diam} Y. \tag{5.19}$$

Suppose, on the contrary, that

$$\operatorname{diam}(Y \cap X) < \operatorname{diam} Y.$$

Then, by Theorem 3.6, there is an open ball $B_r(y_0) \in \mathbf{B}_{Y,\rho}$ with $r = \operatorname{diam} Y$ such that

$$X \subseteq B_r(y_0). \tag{5.20}$$

The diametrical graph $G_{Y,\rho}$ is complete multipartite; hence the set $Y \setminus B_r(y_0)$ is nonvoid,

$$\operatorname{card}(Y \setminus B_r(y_0)) > 0. \tag{5.21}$$

The Gurvich–Vyalyi representing tree $T_{B_r(y_0)}$ is a subtree of the representing tree T_Y with a vertex set $V(T_{B_r(y_0)})$ consisting of all successor of the vertex $B_r(y_0)$ in T_Y. By Theorem 5.5, the ultrametric space $\big(B_{r(y_0)}, \rho_{|B_r(y_0)\times B_r(y_0)}\big)$ belongs to the class **UGVL**. Hence it is a UGVL-extension of (X, d) by (5.20). Now inequality (5.21) shows that (Y, ρ) is not a minimal **UGVL**-extension of (X, d) contrary to the supposition. Thus (5.19) holds.

Let $B_1^0, B_2^0, \dots, B_n^0$ be the paths of diametrical graph $G_{Y,\rho}$. Suppose that $B_1^0 \in \mathbf{B}_{Y,\rho}^I$ and $\operatorname{diam}(B_1^0 \cap X) < \operatorname{diam} B_2^0$. Then using Theorem 3.6 and Corollary 3.8, we can find $B_1^1 \in \mathbf{B}_{Y,\rho}^I$ such that B_1^1 is a path of the complete multipartite graph $G_{B_1^0, \rho_{|B_1^0 \times B_1^0}}$ satisfying

$$B_1^1 \supseteq B_1^0 \cap X. \tag{5.22}$$

Write

$$Y_1 \colon = (Y \setminus B_1^0) \cup B_1^1. \tag{5.23}$$

Then $B_1^0, B_2^0, \dots, B_n^0$ are the paths of the diametrical graph $G_{Y_1, \rho_{|Y_1 \times Y_1}}$ and, consequently,

$$\mathbf{B}_{Y_1, \rho_{|Y_1 \times Y_1}} \subseteq \mathbf{B}_{Y,\rho}$$

holds by Corollary 3.8. Now using Theorem 5.5, we see that $(Y_1, \rho_{|Y_1 \times Y_1}) \in \mathbf{UGVL}$ and $X \subseteq Y_1$ by (5.22)–(5.23). Consequently, $(Y_1, \rho_{|Y_1 \times Y_1})$ is a UGVL-extension of (X, d). Since B_1^1 is a proper subset of B_2^0, equality (5.23) implies

$$\operatorname{card} Y_1 < \operatorname{card} Y.$$

Hence (Y, ρ) is not a minimal **UGVL**-extension of (X, d), contrary to the supposition. Thus the equality

$$\operatorname{diam}(B_1^0 \cap X) = \operatorname{diam} B_1^0$$

holds.

Similarly, we obtain

$$\operatorname{diam}(B_i^0 \cap X) = \operatorname{diam} B_i^0 \tag{5.24}$$

whenever $i = 2, \dots, n$ and $B_i^0 \in \mathbf{B}_{Y,\rho}^I$. Thus (5.18) holds for all internal nodes of T_X having the first level. Now considering the diametrical graphs $G_{B_1^0, \rho_{|B_1^0 \times B_1^0}}, \dots, G_{B_n^0, \rho_{|B_n^0 \times B_n^0}}$ instead of the graph $G_{Y,\rho}$, we obtain (5.18) for the nodes of the second level, and so on. Consequently, (5.18) holds for every $B \in \mathbf{B}_{Y,\rho}^I$ due to the finiteness of (Y, ρ) and the equality

$$V(T_Y) = \mathbf{B}_{Y,\rho}.$$

Let T_X and T_Y be the representing trees of (X, d) and, respectively, (Y, ρ). Using Proposition 5.6, we can find a subset Y_0 of Y such that for every $y_0 \in Y_0$, the singleton $\{y_0\}$ is a leaf of an internal node $F(B)$ with

$$B \in \mathbf{B}_{X,d} \setminus \mathbf{Cs}_{X,d};$$

in addition, for different points $y_1, y_2 \in Y_0$, the singletons $\{y_1\}$ and $\{y_2\}$ are leaves of different internal nodes of T_Y. Removing from T_Y all leaves of type $\{y\}$ for $y \in Y \setminus (X \cup Y_0)$, we obtain a labeled rooted subtree T^1 of T_Y. Let us define a subset Y_1 of Y as

$$Y_1 \colon = X \cup Y_0.$$

Using Theorems 5.7 and 5.8, we can prove that T^1 and the representing tree T_{Y_1} of the ultrametric space $(Y_1, \rho_{|Y_1 \times Y_1})$ are isomorphic as labeled rooted trees. We only note that

$$\operatorname{diam} B = \operatorname{diam}(B \cap X)$$

holds for every $B \in \mathbf{B}_{Y_1, \rho_{|Y_1 \times Y_1}}^I$ because (5.18) and

$$B \cap X \subseteq B \cap Y_1 \subseteq B$$

are valid for every $B \in \mathbf{B}^I_{Y,\rho}$. It follows directly from the definition of T^1 that every internal node of T^1 has a leaf. Since T^1 and T_{Y_1} are isomorphic as labeled rooted trees, every internal node of T_{Y_1} also has a leaf. Consequently, $(Y_1, \rho_{|Y_1\times Y_1})$ is a UGVL-extension of (X, d). The sets X and Y_0 are disjoint and the equality

$$\operatorname{card} Y_0 = \operatorname{card}(\mathbf{B}_{X,d} \setminus \mathbf{Cs}_{X,d})$$

holds. Consequently, we have

$$\operatorname{card} Y_1 = \operatorname{card} X + \operatorname{card} Y_0 = \operatorname{card} X + \Delta(X, d).$$

To complete the proof of equality (5.6), it suffices to note that

$$(Y, \rho) = (Y_1, \rho_{|Y_1\times Y_1})$$

because (Y, ρ) is a minimal **UGVL**-extension of (X, d), $(Y_1, \rho_{|Y_1\times Y_1})$ is a UGVL-extension of (X, d), and $Y_1 \subseteq Y$.

(iii). Let (Y_1, ρ_1) and (Y_2, ρ_2) be minimal (**UGVL**)-extensions of (X, d). We must prove that (Y_1, ρ_1) and (Y_2, ρ_2) are isometric as metric spaces. It was noted above that we can consider only the case when

$$Y_1 \supseteq X \quad \text{and} \quad Y_2 \supseteq X.$$

It was noted in the final part of the proof of statement *(ii)* that, for every minimal **UGVL**-extension (Y, ρ) of (X, d), the representing tree T_Y is isomorphic to the labeled rooted tree that can be obtained from T_X by gluing a leaf to each vertex $B \in \mathbf{B}_{X,d} \setminus \mathbf{Cs}_{X,d}$.

Hence the representing trees T_{Y_1} and T_{Y_2} are isomorphic. Consequently, (Y_1, ρ_1) and (Y_2, ρ_2) are isometric by Theorem 5.7. □

Let $\mathfrak{M}$ be a set of finite nonempty ultrametric spaces. A **UGVL**-space (Y, ρ) is said to be *$\mathfrak{M}$-universal* if (Y, ρ) is a UGVL-extension for every $(X, d) \in \mathfrak{M}$.

We say that (Y, ρ) is *minimal $\mathfrak{M}$-universal* if, for every proper **UGVL**-subspace $(Y_1, \rho_{|Y_1\times Y_1})$ of (Y, ρ), there is $(X, d) \in \mathfrak{M}$ such that $(Y_1, \rho_{|Y_1\times Y_1})$ is not a UGVL-extension of (X, d).

Problem 5.10. Find a condition under which $\mathfrak{M}$ admits a minimal universal **UGVL**-extension.

Some results connected to minimal universal metric spaces can be found in [2].

Declarations

Declaration of competing interest
The authors declare no conflict of interest.

Data availability
The manuscript has no associated data.

Acknowledgement
Oleksiy Dovgoshey was supported by the Finnish Society of Sciences and Letters (Project "Intrinsic Metrics of Domains in Geometric Function Theory and Graphs").

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Oleksiy Dovgoshey
Institute of Applied Mathematics and Mechanics of the NAS of Ukraine, Slovyans'k, Ukraine,
Department of Mathematics and Statistics, University of Turku, Turku, Finland
E-Mail: oleksiy.dovgoshey@gmail.com

Alexander Kostikov
LLC Technical University "Metinvest Politechnic", Zaporizhzhia, Ukraine
E-Mail: alexkst63@gmail.com