DOI 10.1007/s10958-023-06771-w
Journal of Mathematical Sciences, Vol. 276, No. 4, November, 2023

ON PRESCRIBED VALUES OF THE OPERATOR OF SECTIONAL CURVATURE ON THREE-DIMENSIONAL LOCALLY HOMOGENEOUS LORENTZIAN MANIFOLDS

S. V. Klepikova and O. P. Khromova

UDC 514.765

Abstract. In this paper, the problem of prescribed values of the operator of sectional curvature on a three-dimensional locally homogeneous Lorentzian manifolds is solved. Necessary and sufficient conditions for the operator of sectional curvature of such manifolds are obtained.

***Keywords and phrases*:** Lie algebra, Lie group, left-invariant Lorentzian metric, curvature operator, spectrum.

***AMS Subject Classification*:** 53B20, 53C30, 53C50

1. Introduction. Problems of restoring a (pseudo) Riemannian manifold from the prescribed spectrum of the curvature operator are an important direction in the study of curvature operators. Riemannian locally homogeneous spaces with prescribed values of the spectrum of the Ricci operator were defined by O. Kowalski and S. Nikcevic in [8]. Left-invariant Lorentzian metrics on three-dimensional Lie groups were studied by G. Calvaruso and O. Kowalski; in [4], they examined the existence of a Lie group with a left-invariant Lorentzian metric and given values of the spectrum of the Ricci operator. The inhomogeneous case was considered in [3, 7].

Similar results were obtained by D. N. Oskorbin, E. D. Rodionov, and O. P. Khromova for the operator of one-dimensional curvature and the operator of sectional curvature in the case of three-dimensional Lie groups with left-invariant Riemannian metrics (see [6, 12]). Some results on restoring (pseudo)Riemannian metrics and affine connections from a given curvature tensor are contained in [9, 10].

The main purpose of this work is to study the problem of prescribed values of the operator of sectional curvature $\mathcal{K}$ on three-dimensional locally homogeneous Lorentzian manifolds.

In contrast to the case of a Riemannian metric, where an orthonormal basis diagonalizing the operator matrix $\mathcal{K}$ always exists, in the Lorentzian case, there exist various situations known as *Segré types* (see [1]):

(i) the Segré type $\{111\}$: the operator $\mathcal{K}$ has three real eigenvalues (possibly coinciding) with the corresponding one-dimensional eigenspaces;
(ii) the Segré type $\{1z\overline{z}\}$: the operator $\mathcal{K}$ has one real and two complex conjugate eigenvalues;
(iii) the Segré type $\{21\}$: the operator $\mathcal{K}$ has two real eigenvalues (possibly coinciding), one of them has algebraic multiplicity 2; each of these eigenvalues has a one-dimensional eigenspace;
(iv) the Segré type $\{3\}$: the operator $\mathcal{K}$ has one real eigenvalue of algebraic multiplicity 3; the corresponding eigenspace has dimension 1.

2. Three-dimensional homogeneous Lorentzian manifolds. Let (M, g) be a three-dimensional homogeneous manifold with a Lorentzian metric g of signature $(+, +, -)$. We denote by ∇ the Levi-Civita connection and by R the curvature tensor defined as follows:

$$R(X, Y)Z = [\nabla_Y, \nabla_X]Z + \nabla_{[X,Y]}Z.$$

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematicheskie Obzory, Vol. 180, Proceedings of the International Conference "Classical and Modern Geometry" Dedicated to the 100th Anniversary of Professor V. T. Bazylev. Moscow, April 22-25, 2019. Part 2, 2020.

The Lorentzian metric g induces the scalar product $\langle \cdot, \cdot \rangle$ on the bundle $\Lambda^2 M$ by the rule

$$\langle X_1 \wedge X_2, Y_1 \wedge Y_2 \rangle = \det(\langle X_i, Y_i \rangle).$$

The curvature tensor R at any point can be treated as an operator $\mathcal{K}\colon \Lambda^2 M \to \Lambda^2 M$ called the *operator of sectional curvature* and defined by the equality

$$\langle X \wedge Y, \mathcal{K}(Z \wedge T) \rangle = R(X, Y, Z, T).$$

The study of curvature operators on three-dimensional locally homogeneous Lorentzian manifolds is based on the following theorem.

Theorem 1 (G. Calvaruso, see [2]). *Let (M, g) be a three-dimensional locally homogeneous Lorentzian manifold. Either (M, g) is locally symmetric or it is locally isometric to a three-dimensional Lie group with a left-invariant Lorentzian metric.*

The following classification of three-dimensional Lorentzian Lie groups was obtained in [13].

Theorem 2. *Let G be a three-dimensional Lie group with a left-invariant Lorentzian metric. The following assertions hold.*

1. *If G is unimodular, then there exists a a prseudo-orthonormal basis $\{e_1, e_2, e_3\}$ of the Lie algebra of the Lie group G such that the metric Lie algebra of the Lie group G is contained in the following list*:
   (a) *the case $\mathcal{A}_1$*:
   $$[e_1, e_2] = \lambda_3 e_3, \quad [e_1, e_3] = -\lambda_2 e_2, \quad [e_2, e_3] = \lambda_1 e_1,$$
   where e_1 is a time-like vector;
   (b) *the case $\mathcal{A}_2$*:
   $$[e_1, e_2] = (1 - \lambda_2) e_3 - e_2, \quad [e_1, e_3] = e_3 - (1 + \lambda_2) e_2, \quad [e_2, e_3] = \lambda_1 e_1,$$
   where e_3 is a time-like vector;
   (c) *the case $\mathcal{A}_3$*:
   $$[e_1, e_2] = e_1 - \lambda e_3, \quad [e_1, e_3] = -\lambda e_2 - e_1, \quad [e_2, e_3] = \lambda_1 e_1 + e_2 + e_3,$$
   where e_3 is a time-like vector;
   (d) *the case $\mathcal{A}_4$*:
   $$[e_1, e_2] = \lambda_3 e_2, \quad [e_1, e_3] = -\beta e_1 - \alpha e_2, \quad [e_2, e_3] = -\alpha e_1 + \beta e_2,$$
   where e_1 is a time-like vector and $\beta \neq 0$.
2. *If G is not unimodular, then there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ of the Lie algebra of the Lie group G such that the metric Lie algebra of the Lie group G is contained in the following list*:
   (a) *the case $\mathcal{A}$*:
   $$[e_1, e_2] = 0, \quad [e_1, e_3] = \lambda \sin\varphi e_1 - \mu \cos\varphi e_2, \quad [e_2, e_3] = \lambda \cos\varphi e_1 + \mu \sin\varphi e_2,$$
   where e_3 is a time-like vector and $\sin\varphi \neq 0$, $\lambda + \mu \neq 0$, $\lambda \geq 0$, $\mu \geq 0$;
   (b) *the case $\mathcal{B}$*:
   $$[e_1, e_2] = 0, \quad [e_1, e_3] = te_1 - se_2, \quad [e_2, e_3] = pe_1 + qe_2,$$
   with nonzero $\langle e_2, e_2 \rangle = -\langle e_1, e_3 \rangle = 1$ and $q \neq t$;
   (c) *the case $\mathcal{C}_1$*:
   $$[e_1, e_2] = 0, \quad [e_1, e_3] = se_1 + pe_2, \quad [e_2, e_3] = pe_1 + qe_2,$$
   where e_2 is a time-like vector and $q \neq s$;
   (d) *the case $\mathcal{C}_2$*:
   $$[e_1, e_2] = 0, \quad [e_1, e_3] = qe_1 - re_2, \quad [e_2, e_3] = pe_1 + qe_2,$$
   where e_2 is a time-like vector and $q \neq 0$, $p + r \neq 0$.

Table 1. Three-dimensional unimodular Lie algebras

Lie algebra	Conditions for structural constants			
	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$
su(2)	$(+,+,+)$	–	–	–
$\mathrm{sl}(2,\mathbb{R})$	$(+,+,-)$	$\lambda_1 \neq 0,\ \lambda_2 \neq 0$	$\lambda \neq 0$	$\lambda_3 \neq 0$
e(2)	$(+,+,0)$	–	–	–
e(1, 1)	$(+,-,0)$	$\lambda_1 = 0,\ \lambda_2 \neq 0$ or $\lambda_1 \neq 0,\ \lambda_2 = 0$	$\lambda = 0$	$\lambda_3 = 0$
h	$(+,0,0)$	$\lambda_1 = 0,\ \lambda_2 = 0$	–	–
$\mathbb{R}^3$	$(0,0,0)$	–	–	–

Remark 1. There exist exactly six nonisomorphic three-dimensional unimodular Lie algebras and the corresponding types of unimodular three-dimensional Lie groups (see [11]). They are listed in Table 1 together with the conditions for their structural constants. The sign "—" means that the corresponding type of basis for the corresponding Lie algebra is impossible. For the Lie algebrs $\mathcal{A}_1$, only the signs of the triple $(\lambda_1, \lambda_2, \lambda_3)$ are indicated (up to permutations and changes of sign). Note that similar bases were also constructed in [2, 5].

The following classification result for three-dimensional Lorentzian locally symmetric spaces was obtained in [2].

Theorem 3. *A three-dimensional locally symmetric manifold (M, g) is locally isometric to one of the following manifolds*:

(I) *one of the Lorentzian spatial form $\mathbb{R}^3_1$, $\mathbb{S}^3_1$, or $\mathbb{H}^3_1$ (with zero, positive, or negative sectional curvature, respectively)*;
(II) *one of the direct products $\mathbb{R} \times \mathbb{S}^2_1$, $\mathbb{R} \times \mathbb{H}^2_1$, $\mathbb{S}^2 \times \mathbb{R}_1$, or $\mathbb{H}^2 \times \mathbb{R}_1$*;
(III) *a manifold with a Lorentzian metric g, which admits a local coordinate system (u_1, u_2, u_3) in which the metric tensor has the form*

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f(u_2, u_3) \end{pmatrix},$$

where $\varepsilon = \pm 1$, $f(u_2, u_3) = u_2^2 \alpha + u_2 \beta(u_3) + \xi(u_3)$, $\alpha \in \mathbb{R}$, and β and ξ are arbitrary smooth functions.

3. Three-dimensional Lorentzian Lie groups.

In the sequel, by "three-dimensional Lorentzian Lie group $(G, \mathfrak{g})$" we mean a three-dimensional Lie group G with a left-invariant Lorentzian metric g and a metric Lie algebra $\mathfrak{g}$.

Theorem 4. *A three-dimensional unimodular Lorentz Lie group $(G, \mathcal{A}_1)$ with the operator of sectional curvature $\mathcal{K}$ exists if and only if $\mathcal{K}$ has the Segré type $\{111\}$ and the eigenvalues k_1, k_2, and k_3 satisfy at least one of the following conditions*:

(i) $k_1 = k_2 = k_3 = 0$;
(ii) *exactly two of the three sums $k_1 + k_2$, $k_1 + k_3$, and $k_2 + k_3$ are equal to zero*;
(iii) $(k_1 + k_2)(k_1 + k_3)(k_2 + k_3) < 0$.

Proof. In the case considered, the matrix of the operator of sectional curvature $\mathcal{K}$ is diagonal, the eigenvalues k_1, k_2, and k_3 are real and equal to

$$k_1 = -\frac{1}{4}(\lambda_3 + \lambda_1 - \lambda_2)^2 + \lambda_1(\lambda_1 + \lambda_3),$$
$$k_2 = -\frac{1}{4}(\lambda_1 + \lambda_2 + \lambda_3)^2 + \lambda_2(\lambda_1 + \lambda_2),$$
$$k_3 = -\frac{1}{4}(\lambda_1 + \lambda_2 - \lambda_3)^2 + \lambda_3(\lambda_3 - \lambda_2).$$

Let

$$\mu_i = \frac{1}{2}(-\lambda_1 + \lambda_2 + \lambda_3) + \delta_i\lambda_i,$$

where $\delta_1 = 1$, $\delta_2 = \delta_3 = -1$. Then

$$k_1 + k_2 = -2\mu_1\mu_2, \quad k_1 + k_3 = -2\mu_1\mu_3, \quad k_2 + k_3 = -2\mu_2\mu_3.$$

Assume that at least two of the numbers μ_i are equal to zero. Then all right-hand sides of the above equations vanish and the system has a unique solution $k_1 = k_2 = k_3 = 0$.

If one of μ_i is equal to zero, then exactly two of the numbers $k_1 + k_2$, $k_1 + k_3$, and $k_2 + k_3$ are equal to zero.

Assume that $\mu_i \neq 0$; then

$$\mu_i^2 = -\frac{1}{2}\frac{(k_1 + k_2)(k_1 + k_3)(k_2 + k_3)}{(k_1 + k_2 + k_3 - k_i)^2},$$

and the system is solvable if and only if

$$(k_1 + k_2)(k_1 + k_3)(k_2 + k_3) < 0. \qquad \square$$

Theorem 5. *A three-dimensional unimodular Lorentzian Lie group* $(G, \mathcal{A}_2)$ *with the operator of sectional curvature* $\mathcal{K}$ *exists if and only if one of the following conditions is fulfilled*:

(i) $\mathcal{K}$ *has the Segré type* $\{111\}$ *with the eigenvalues* $k_1 = -k_2 = -k_3 \geq 0$ (*up to a permutation*);

(ii) $\mathcal{K}$ *has the Segré type* $\{12\}$ *with the eigenvalues*

  (a) $k_1 = k_2 = 0$ *orr*

  (b) $k_2 < 0$.

Proof. In the case considered, the matrix of the operator of sectional curvature $\mathcal{K}$ has the form

$$\mathcal{K} = \begin{pmatrix} \frac{3}{4}\lambda_1^2 - \lambda_1\lambda_2 & 0 & 0 \\ 0 & 2\lambda_2 - \lambda_1 - \frac{1}{4}\lambda_1^2 & 2\lambda_2 - \lambda_1 \\ 0 & -2\lambda_2 + \lambda_1 & -2\lambda_2 + \lambda_1 - \frac{1}{4}\lambda_1^2 \end{pmatrix}.$$

If $\lambda_1 = 2\lambda_2$, then the matrix is diagonalizable with the eigenvalues $k_1 = -k_2 = -k_3 \geq 0$. In the opposite case, the matrix of the operator $\mathcal{K}$ has the following Jordan form:

$$\mathcal{K} = \begin{pmatrix} -\lambda_1\lambda_2 + \frac{3}{4}\lambda_1^2 & 0 & 0 \\ 0 & -\frac{1}{4}\lambda_1^2 & 1 \\ 0 & 0 & -\frac{1}{4}\lambda_1^2 \end{pmatrix};$$

the eigenvalues are

$$k_1 = -\lambda_1\lambda_2 + \frac{3}{4}\lambda_1^2, \quad k_2 = -\frac{1}{4}\lambda_1^2 \leq 0.$$

If $k_2 = 0$, then $\lambda_1 = 0$ and all eigenvalues are equal to zero.

Assume that $k_2 < 0$; then $\lambda_1 = \pm 2\sqrt{-k_2}$. Expressing λ_2, we obtain

$$\lambda_2 = \mp\frac{k_1 + 3k_2}{2\sqrt{-k_2}}. \qquad \square$$

Theorem 6. *A three-dimensional unimodular Lorentzian Lie group* $(G, \mathcal{A}_3)$ *with the operator of sectional curvature* $\mathcal{K}$ *exists if and only if one of the following conditions is fulfilled*:

(i) $\mathcal{K}$ *has the Segré type* $\{21\}$ *with eigenvalues* $k_1 = k_2 = 0$;

(ii) $\mathcal{K}$ *has the Segré type* $\{3\}$ *with eigenvalue* $k_1 < 0$.

Proof. In this case, the matrix of the operator of sectional curvature $\mathcal{K}$ has the form

$$\mathcal{K} = \begin{pmatrix} -\frac{1}{4}\lambda^2 & \lambda & -\lambda \\ \lambda & 2-\frac{1}{4}\lambda^2 & -2 \\ \lambda & 2 & -2-\frac{1}{4}\lambda^2 \end{pmatrix}.$$

Its coinciding eigenvalues are equal to $-\lambda^2/4 \leq 0$. If $\lambda = 0$, then $\mathcal{K}$ has the Segré type $\{21\}$ and $k_1 = k_2 = 0$. If $\lambda \neq 0$, then $\mathcal{K}$ has the Segré type $\{3\}$ and $k_1 < 0$. □

Theorem 7. *A three-dimensional unimodular Lorentzian Lie group* $(G, \mathcal{A}_4)$ *with the operator of sectional curvature* $\mathcal{K}$ *exists if and only if one of the following conditions is fulfilled*:

(i) *the operator* $\mathcal{K}$ *has the Segré type* $\{111\}$ *and* $k_1 = k_2 = -k_3 < 0$ (*up to a permutation*);

(ii) *the operator* $\mathcal{K}$ *has the Segré type* $\{z\bar{z}1\}$ *and* $k_1 + k_2 < 0$.

Proof. In this case, the operator $\mathcal{K}$ has the eigenvalues

$$k_1 = \overline{k_2} = -\beta^2 - \frac{1}{4}\lambda_3^2 \pm i\beta(2\alpha - \lambda_3), \quad k_3 = -\lambda_3\alpha + \frac{3}{4}\lambda_3^2 + \beta^2,$$

where i is the imaginary unit.

Let $x = -\beta^2 - \lambda_3^2/4 < 0$ and $y = \beta(2\alpha - \lambda_3)$; then $k_1 = \overline{k_2} = x + iy$. If $\lambda_3 = 2\alpha$, then

$$k_3 = \alpha^2 + \beta^2 > 0, \quad x = -\alpha^2 - \beta^2 < 0, \quad y = 0.$$

Therefore, $k_1 = k_2 = -k_3 < 0$ and the eigenvalues are real.

Assume that $\lambda_3 \neq 2\alpha$; then

$$\beta = \frac{y}{2\alpha - \lambda_3} \neq 0, \quad k_3 + x = -\lambda_3\alpha + \frac{1}{2}\lambda_3^2.$$

If $\lambda_3 = 0$, then

$$k_3 = \beta^2, \quad x = -\beta^2, \quad y = 2\alpha\beta;$$

otherwise, $\lambda_3 \neq 0$ and $k_3 + x \neq 0$. Then

$$\alpha = \frac{\lambda_3^2 - 2k_3 - 2x}{2\lambda_3}, \quad \lambda_3 = \pm 2|k_3 + x| \frac{\sqrt{-x}}{\sqrt{(k_3+x)^2 + y^2}}.$$ □

Theorem 8. *A three-dimensional nonunimodular Lorentzian Lie group* $(G, \mathcal{A})$ *with the operator of sectional curvature* $\mathcal{K}$ *exists if and only if* $\mathcal{K}$ *has the Segré type* $\{111\}$ *and the eigenvalues* k_1, k_2, *and* k_3 *satisfy at least one of the following conditions*:

(i) $k_1 = k_2 = k_3 > 0$;

(ii) $k_1k_3 \leq k_2^2 < \left(\frac{k_1 + k_3}{2}\right)^2$ *and* $k_2 < \frac{k_1 + k_3}{2}$ (*up to a permutation*).

Proof. In this case, the matrix of the operator of sectional curvature $\mathcal{K}$ is diagonalizable and has the real eigenvalues

$$k_1 = -\frac{1}{4}(\lambda+\mu)^2\cos^2\varphi + \lambda^2, \quad k_2 = -\frac{1}{4}(\lambda+\mu)^2\cos^2\varphi + \mu\lambda, \quad k_3 = -\frac{1}{4}(\lambda+\mu)^2\cos^2\varphi + \mu^2. \quad (1)$$

Note that $k_1 - 2k_2 + k_3 = (\lambda - \mu)^2 \geq 0$. If this expression is equal to zero, then $\lambda = \mu \neq 0$ and $k_1 = k_2 = k_3 = \lambda^2\sin^2\varphi > 0$.

Assume that $k_1 - 2k_2 + k_3 > 0$ and $\lambda \neq \mu$. Consider the differences

$$k_1 - k_2 = \lambda(\lambda - \mu), \quad k_1 - k_3 = (\lambda - \mu)(\lambda + \mu) \neq 0, \quad k_3 - k_2 = \mu(\mu - \lambda).$$

Since $\lambda \geq 0$ and $\mu \geq 0$, the system has a solution

$$\lambda = \frac{|k_1 - k_2|}{\sqrt{k_1 - 2k_2 + k_3}}, \quad \mu = \frac{|k_2 - k_3|}{\sqrt{k_1 - 2k_2 + k_3}}.$$

Substituting this solution into (1), we obtain

$$0 \leq \cos^2 \varphi = 4\frac{k_2^2 - k_1 k_3}{(k_1 - k_3)^2} < 1.$$

Therefore, $k_2^2 - k_1 k_3 \geq 0$ and $(k_1 + k_3)^2 - 4k_2^2 > 0$. □

Theorem 9. *A three-dimensional nonunimodular Lorentzian Lie group $(G, \mathcal{B})$ with the operator of sectional curvature $\mathcal{K}$ exists if and only if one of the following conditions is fulfilled:*

(i) *the operator $\mathcal{K}$ has the Segré type $\{111\}$ and $k_1 = -3k_2 = -3k_3 \geq 0$ (up to a permutation);*
(ii) *the operator $\mathcal{K}$ has the Segré type $\{12\}$ and $k_1 = -3k_2 \geq 0$.*

Proof. In this case, the matrix of the operator of sectional curvature $\mathcal{K}$ has the form

$$\mathcal{K} = \begin{pmatrix} -\frac{1}{4}s^2 & qs & ps - q^2 - qt \\ 0 & \frac{3}{4}s^2 & -qs \\ 0 & 0 & -\frac{1}{4}s^2 \end{pmatrix}.$$

If $s = q = 0$ or $s = 0$ and $q = -t$, then the matrix of the operator $\mathcal{K}$ is trivial.

If $ps - qt = 0$, then the matrix of the operator of sectional curvature $\mathcal{K}$ has the Segré type $\{111\}$ with the eigenvalues $k_1 = 3s^2/4$ and $k_2 = k_3 = -s^2/4$.

Otherwise, the matrix of the operator $\mathcal{K}$ has the Segré type $\{12\}$ with the eigenvalues $k_1 = 3s^2/4$ and $k_2 = -s^2/4$. □

Theorem 10. *A three-dimensional nonunimodulal Lorentzian Lie group $(G, \mathcal{C}_1)$ with the operator of sectional curvature $\mathcal{K}$ exists if and only if one of the following conditions is fulfilled:*

(i) *the operator $\mathcal{K}$ has the Segré type $\{111\}$ with the eigenvalues $k_2 \leq 0$ and $k_3 \leq 0$ such that $k_2^2 + k_3^2 > 0$ and $|k_1| \leq \sqrt{k_2 k_3}$ (up to a permutation);*
(ii) *the operator $\mathcal{K}$ has the Segré type $\{1z\overline{z}\}$ and $|k_1| \leq -\dfrac{k_2 + k_3}{2}$.*

Proof. In this case, the matrix of the operator of sectional curvature $\mathcal{K}$ is diagonalizable and its eigenvalues are

$$\begin{aligned} k_1 &= qs, \\ k_2 &= -\frac{1}{2}q^2 - \frac{1}{2}s^2 + \frac{1}{2}|q + s|\sqrt{(q - s)^2 - 4p^2}, \\ k_3 &= -\frac{1}{2}q^2 - \frac{1}{2}s^2 - \frac{1}{2}|q + s|\sqrt{(q - s)^2 - 4p^2}, \end{aligned} \tag{2}$$

where $k_2 k_3 - k_1^2 = (q + s)^2 p^2 \geq 0$. If this expression is equal to zero, then

$$p = 0, \ s = \pm\sqrt{-k_2}, \ q = \pm\sqrt{-k_3}, \ k_1 = qs \quad \text{orr} \quad q = -s, \ k_1 = k_2 = k_3 = -s^2 < 0.$$

Otherwise,

$$k_2 k_3 - k_1^2 > 0, \quad p^2 = \frac{k_2 k_3 - k_1^2}{(q + s)^2}.$$

Consider the system

$$k_1 = qs, \quad k_2 + k_3 = -q^2 - s^2 < 0;$$

it has a solution

$$s = \pm\frac{1}{2}\sqrt{2\sqrt{(k_2+k_3)^2-4k_1^2}-2(k_2+k_3)}, q = \frac{k_1}{s},$$

where $(k_2+k_3)^2 - 4k_1^2 \geq 0$.

Note that even in the case where k_2 and k_3 are complex conjugate eigenvalues, the numbers k_2k_3 and k_2+k_3 are real. □

Theorem 11. *A three-dimensional nonunimodular Lorentzian Lie group* $(G, \mathcal{C}_2)$ *with the operator of sectional curvature* $\mathcal{K}$ *exists if and only if one of the following conditions is fulfilled*:

(i) *the operator* $\mathcal{K}$ *the Segré type* $\{111\}$ *with the eigenvalues* $\left|\frac{k_2+k_3}{2}\right| < -k_1$ *(up to a permutation)*;

(ii) *the operator* $\mathcal{K}$ *has the Segré type* $\{1z\overline{z}\}$ *and* $\left|\frac{k_2+k_3}{2}\right| < -k_1$.

Proof. In this case, the matrix of the operator of sectional curvature $\mathcal{K}$ is diagonalizable and has the eigenvalues

$$k_1 = -q^2 - \frac{1}{4}(p+r)^2 < 0,$$
$$k_2 = \frac{1}{4}(p+r)^2 - q^2 + \frac{1}{2}|p+r|\sqrt{(p-r)^2-4q^2},$$
$$k_3 = \frac{1}{4}(p+r)^2 - q^2 - \frac{1}{2}|p+r|\sqrt{(p-r)^2-4q^2}.$$

We have

$$k_2+k_3+2k_1 = -4q^2 < 0 \quad \Rightarrow \quad q = \pm\frac{1}{2}\sqrt{-k_2-k_3-2k_1}.$$

Let $x = p+r \neq 0$, $y = p-r$. Then

$$x^2 = k_2+k_3-2k_1 > 0, \quad 4k_1^2 - 4k_2k_3 = y^2x^2 \geq 0.$$

This system has a solution

$$x = \pm\sqrt{k_2+k_3-2k_1}, \quad y = \pm 2\frac{\sqrt{k_1^2-k_2k_3}}{\sqrt{k_2+k_3-2k_1}}.$$

Note that even in the case where k_2 and k_3 are complex conjugate eigenvalues, the numbers k_2k_3 and k_2+k_3 are real. □

4. Three-dimensional locally symmetric Lorentzian manifolds. Theorem 3 allows one to divide the problem of studying the operator of sectional curvature of three-dimensional locally symmetric Lorentzian manifolds into three subproblems. At the same time, it is obvious that the operator of sectional curvature $\mathcal{K}$ is diagonalizable for Lorentzian manifolds of constant sectional curvature $\mathbb{R}^3_1$, $\mathbb{S}^3_1$, and $mathbbH^3_1$ (i.e., the operator $\mathcal{K}$ has the Segré type $\{111\}$) and $\mathcal{K}$ has three coinciding eigenvalues (zero, positive, or negative, respectively).

In the case of direct products (the case (II) of Theorem 3), the operator of sectional curvature $\mathcal{K}$ has the Segré type $\{111\}$ with two zero and one nonzero eigenvalues.

In the case (III) of Theorem 3, the matrix of the operator of sectional curvature has the form

$$\mathcal{K} = \begin{pmatrix} 0 & 0 & \alpha/\varepsilon \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then the operator of sectional curvature $\mathcal{K}$ has the Segré type either $\{111\}$ with the eigenvalues $k_1 = k_2 = k_3 = 0$ for $\alpha = 0$ or $\{12\}$ with $k_1 = k_2 = 0$ for $\alpha \neq 0$. Therefore, the following theorem holds.

Theorem 12. *A three-dimensional Lorentzian locally symmetric manifold with the operator of sectional curvature $\mathcal{K}$ exists if and only if one of the following conditions is fulfilled*:

(i) *the operator $\mathcal{K}$ has the Segré type $\{111\}$ with the same eigenvalues*;
(ii) *the operator $\mathcal{K}$ has the Segré type $\{111\}$ with two zero and one nonzero eigenvalues*;
(iii) *the operator $\mathcal{K}$ has the Segré type $\{12\}$ with zero eigenvalues.*

5. Operator of sectional curvature on three-dimensional locally homogeneous Lorentzian manifolds. In this section, using the results of the previous sections, we determine a possible form of the operator of sectional curvature on three-dimensional locally homogeneous Lorentzian manifolds. We start with the case of a nondiagonalizable operator of sectional curvature.

Theorem 13. *A three-dimensional locally homogeneous Lorentzian manifold (M, g) with the nondiagonalizable operator of sectional curvature $\mathcal{K}$ exists if and only if $\mathcal{K}$ satisfies one of the following conditions*:

(i) *the operator $\mathcal{K}$ has the Segré type $\{12\}$ and*
  (a) *both eigenvalues are equal to zero orr*
  (b) $k_2 < 0$;
(ii) *the operator $\mathcal{K}$ has the Segré type $\{3\}$ with a negative eigenvalue*;
(iii) *the operator $\mathcal{K}$ has the Segré type $\{1z\overline{z}\}$ and*
  (a) *complex conjugate eigenvalues have a negative real part or*
  (b) $0 \leq \dfrac{k_2 + k_3}{2} < -k_1$.

Proof. The Segré type $\{12\}$ is possible only for three-dimensional Lorentzian Lie groups $(G, \mathcal{A}_2)$, $(G, \mathcal{A}_3)$, or $(G, \mathcal{B})$ and for the Lorentzian locally symmetric space (see the case (III) of Theorem 3). Then the case (i) of Theorem 13 follows from Theorems 5, 6, 9, and 12.

Among all variants listed above, the Segré type $\{3\}$ is possible only for the three-dimensional Lie group $(G, \mathcal{A}_3)$. Then Theorem 6 implies the case (ii) of Theorem 13.

If $\mathcal{K}$ has the Segré type $\{1z\overline{z}\}$, then (M, g) is locally isometric to the three-dimensional Lorentzian Lie group $(G, \mathcal{A}_4)$, $(G, \mathcal{C}_1)$, or $(G, \mathcal{C}_2)$. Then the case (iii) of Theorem 13 follows from Theorems 7, 10, and 11. □

The following theorem is a consequence of the results obtained above.

Theorem 14. *A three-dimensional locally homogeneous Lorentzian manifold (M, g) with the nondiagonalizable operator of sectional curvature $\mathcal{K}$ with the Segré type $\{111\}$ exists if and only if the eigenvalues k_1, k_2, and k_3 satisfy at least one of the following conditions*:

(i) *all eigenvalues coincide*;
(ii) *two eigenvalues are equal to zero and the third eigenvalue is nonzero*;
(iii) *exactly two of the numbers $k_1 + k_2$, $k_1 + k_3$, and $k_2 + k_3$ are equal to zero*;
(iv) $(k_1 + k_2)(k_1 + k_3)(k_2 + k_3) < 0$;
(v) *up to a permutation,*

$$k_2 k_3 \leq k_1^2 < \left(\frac{k_2 + k_3}{2}\right)^2 \quad \text{and} \quad k_1 < \frac{k_2 + k_3}{2};$$

(vi) *up to a permutation,*

$$k_2 < 0, \quad k_3 < 0, \quad |k_1| \leq \sqrt{k_2 k_3};$$

(vii) *up to a permutation,*

$$k_1 < -\left|\frac{k_2 + k_3}{2}\right|.$$

REFERENCES

1. P. Bueken and M. Djorić, "Three-dimensional Lorentz metrics and curvature homogeneity of order one," *Ann. Glob. Anal. Geom.*, **18**, 85–103 (2000).
2. G. Calvaruso, "Homogeneous structures on three-dimensional Lorentzian manifolds," *J. Geom. Phys.*, **57**, 1279–1291 (2007).
3. G. Calvaruso, "Pseudo-Riemannian' 3-manifolds with prescribed distinct constant Ricci eigenvalues," *Differ. Geom. Appl.*, **26**, 419–433 (2008).
4. G. Calvaruso and O. Kowalski, "On the Ricci operator of locally homogeneous Lorentzian 3-manifolds," *Cent. Eur. J. Math.*, **7**, No. 1, 124–139 (2009).
5. L. A. Cordero and P. E. Parker, "Left-invariant Lorentzian metrics on 3-dimensional Lie groups," *Rend. Mat.*, **17**, 129–155 (1997).
6. O. P. Gladunova and D. N. Oskorbin, "Application of computer algebra systems to the study of the spectrum of the curvature operator on metric Lie groups," *Izv. Altaisk. Univ.*, **1-1 (77)**, 19–23 (2013).
7. O. Kowalski, "Nonhomogeneous Riemannian' 3-manifolds with distinct constant Ricci eigenvalues," *Nagoya Math. J.*, **132**, 1–36 (1993).
8. O. Kowalski and S. Nikcevic, "On Ricci eigenvalues of locally homogeneous Riemann 3-manifolds," *Geom. Dedic.*, No. 1, 65–72 (1996).
9. J. Mikeš, E. Stepanova, A. Vanžurová, et al., *Differential Geometry of Special Mappings*, Palacký University, Olomouc, Czech Republic (2015).
10. J. Mikeš and A. Vanžurová, "Reconstruction of an affine connection in generalized Fermi coordinates," *Bull. Malays. Math. Sci. Soc.*, **40**, No. 1, 205–213 (2017).
11. J. Milnor, "Curvature of left invariant metric on Lie groups," *Adv. Math.*, **21**, 293–329 (1976).
12. D. N. Oskorbin and E. D. Rodionov, "On the spectrum of the curvature operator of a three-dimensional Lie group with a left-invariant Riemannian metric," *Doklady Mathematics*, **87**, No. 3, 307–309 (2013).
13. E. D. Rodionov, V. V. Slavskii, and L. N. Chibrikova, "Locally conformally homogeneous pseudo-Riemannian' spaces," *Sib. Adv. Math.*, **17**, No. 3, 186–212 (2007).

COMPLIANCE WITH ETHICAL STANDARDS

Conflict of interests. The authors declare no conflict of interest.

Funding. This work was supported by the Russian Foundation for Basic Research (project No. 18-31-00033 mol_a).

Financial and non-financial interests. The authors have no relevant financial or non-financial interests to disclose.

S. V. Klepikova
Altai State University, Barnaul, Russia
E-mail: klepikova.svetlana.math@gmail.com
O. P. Khromova
Altai State University, Barnaul, Russia
E-mail: khromova.olesya@gmail.com