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PROPERTIES OF TOPOLOGICAL PARTITIONS AND MAPPINGS OF TOPOLOGICAL GROUPS

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Abstract. In this paper, we examine topological partitions of topological spaces that arise in connection with continuous mappings of topological spaces. The content of the paper is closely related to such classical fundamental concepts of general topology as compactness, homogeneity, and Čech completeness. New facts related to these concepts are obtained.

Keywords and phrases: topological partition, Čech-complete space, k-covering mapping, dyadic compactum, factor topology, topological group.

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1. Introduction. In the theory of general topological spaces (also called general or "set-theoretic" topology) and in topological algebra, so-called topological partitions naturally appear; their mathematical structure is more general than the nature of topological spaces. The notion of a topological partition arose a long time ago. This notion is based of the concept of a continuous mapping, which plays a major role in comparing topological spaces. It seems that topological partitions will play an increasingly important role in the theory of topological spaces and in topological algebra. In this paper, we take several steps in this direction: we introduce a number of notions related immediately to topological partitions and outline new questions. At the same time, the content of the paper is closely related to many classical fundamental concepts of general topology, namely, compactness, homogeneity, and completeness in the sense of Čech; we also obtain new facts related to these concepts.

In general, the notation and terminology used are the same as in [8, 11]. However, we recall that the *tightness* of a topological space X is countable if for each set $A \subset X$ and each point x in the closure of the set A, there is a countable subset B of the set A whose closure contains the point x. The *cardinality* of a set X is denoted by $|X|$. If the assumption of separability in a space is not explicitly stated, then this space should be considered Hausdorff. Unless otherwise stated, all topological spaces are assumed to be Hausdorff spaces. A Tychonoff space X is said to be *Čech complete* if it is a G_δ-type set in some of its compact Hausdorff extensions. We also recall that a continuous mapping f of a space X onto a space Y is called a *k-covering* (or *compactly covering*) mapping if for each compactum $F \subset Y$, there exists a compactum $K \subset X$ such that $f(K) = F$ (see [1, 3]. Here and below, by a "compactum" we mean a compact Hausdorff space.

Topological partitions naturally arise in topological algebra. Recall that a group G with a topology $\mathcal{T}$ is called a *left topological group* if the left shift $l_x : G \to G$ defined by the formula $l_x(y) = xy$ for each $y \in G$, is a homeomorphism of the space $(G, \mathcal{T})$ onto itself for every $x \in G$. More general examples of topological partitions are related to arbitrary continuous mappings.

In this paper, we discuss some directions of topological algebra developed in the classical works of V. V. Uspensky [16, 17], M. M. Choban [9, 10], E. Michael [13, 14], Jan van Mill [15], etc. (see also [4, 7]). Recall that dyadic compacts are continuous images of the Cantor cubes D^τ, where D is the discrete two-set space. A compactum X is called a *Dugundji compact* if for each zero-dimensional compactum Z

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and its closed subset A, any continuous mapping f of the space A into the space X can be extended to a continuous mapping of the compactum Z into the compactum X (see [16, 17]).

The main results of the paper are, in particular, Theorems 4.3, 4.4, 4.9, and 4.11 and Corollary 4.6. They concern the properties of compacta, which are continuous images of topological groups under continuous closed or open mappings.

2. Preliminaries. A *partition* of a set X or a topological space $(X, \mathcal{T})$ is a covering γ of the set X by nonempty, pairwise disjoint subsets. A *topological partition* of a topological space $(X, \mathcal{T})$ is s pair $[(X, \mathcal{T}), \gamma]$, where γ is a partition of the set X. Thus, a topological partition $[(X, \mathcal{T}), \gamma]$ can be identified with the triple $[X, \mathcal{T}, \gamma]$. In this case, one can introduce a topology $\mathcal{T}_\gamma$ on the set γ; it is called the factor topology on the partition γ (see [11]).

Below we introduce a number of properties of topological partitions $[(X, \mathcal{T}), \gamma]$ immediately in terms of properties of the space $(X, \mathcal{T})$ and the partition γ . These properties are not always properties of the factor topology, but are closely related to the latter.

We present a few definitions of this kind. A topological partition $[(X, \mathcal{T}), \gamma]$ is called a *T_1-partition* if each element P of the family γ is a closed set in the space $(X, \mathcal{T})$. A subset A of a set X is said to be *γ-separated* or *separated in the topological partition* $\mathcal{Z} = [(X, \mathcal{T}), \gamma]$ if no two points from A are contained in the same element of the family γ.

A subset $A \subset X$ is called a *saturated subset* of a topological partition $\mathcal{Z} = [(X, \mathcal{T}), \gamma]$ if it contains each element of the partition γ that it intersects with.

A subset A of a set X is called a *covering subset* of a topological partition $\mathcal{Z} = [(X, \mathcal{T}), \gamma]$ if it intersects with each element' of the partition γ. Subsets A and B of a set X are said to be *γ-similar* (or simply *similar*) if an element P of the family γ intersects with A if and only if P intersects with B.

A topological partition $[(X, \mathcal{T}), \gamma]$ is said to be *α_1-compact* if there exists a compact subspace K of the space X such that $K \cap P \neq \varnothing$ for each $P \in \gamma$.

We say that a topological partition $[(X, \mathcal{T}), \gamma]$ is *sensely compact* if there exists a covering subset A of the set X such that each infinite subset of the set A has a complete accumulation point in the space X.

A topological partition $[(X, \mathcal{T}), \gamma]$ is said to be *α_3-compact* if for any infinite γ-separated subset A of the set X, there exists a γ-separated subset B of the set X such that A and B are γ-similar and B possesses a complete accumulation point in $(X, \mathcal{T})$.

A topological partition $[(X, \mathcal{T}), \gamma]$ is said to be *factor compact* if the factor space $(\gamma, \mathcal{T}_\gamma)$ is compact. Similarly, a topological partition $[(X, \mathcal{T}), \gamma]$ is called a *factor Hausdorff space* if the factor space $(\gamma, \mathcal{T}_\gamma)$ is a Hausdorff space. As usually, the *cardinality* of a topological partition $[(X, \mathcal{T}), \gamma]$ is the cardinality of the family γ.

In the following assertion, we state three clear sufficient conditions of the compactness of the factor space $(\gamma, \mathcal{T}_\gamma)$.

Proposition 2.1. *If a topological partition $[(X, \mathcal{T}), \gamma]$ is α_1-compact (respectively, densely compact or α_3-compact), then the factor space $(\gamma, \mathcal{T}_\gamma)$ is compact (i.e., the topological partition $[(X, \mathcal{T}), \gamma]$ is factor compact).*

We omit the simple proof of this assertion for all three cases.

The concept of homogeneity that arises in connection with the concept of topological partitions is also of great interest. Recall that a topological space is said to be *homogeneous* if each of its points can be translated into another point by a homeomorphism. A similar definition for topological partitions is as follows: A topological partition $[(X, \mathcal{T}), \gamma]$ is said to be *homogeneous* if for any $P_1, P_2 \in \gamma$, there exists a homeomorphism g of the space X onto itself such that $g(P_1) = P_2$ and $g(P) \in \gamma$ for each $P \in \gamma$. The following assertion holds.

Proposition 2.2. *If a topological partition $[(X, \mathcal{T}), \gamma]$ is homogeneous, then the factor space $(\gamma, \mathcal{T}_\gamma)$ is homogeneous.*

A task is to find out additional restrictions under which the converse assertion is valid. The following specific question about topological partitions deserves attention: Does a homogeneous, finite topological partition exist on the disk?

The notion of a homogeneous topological partition of a space $(X, \mathcal{T})$ allows one to give a new definition of the notion of the homogeneity of the space $(X, \mathcal{T})$ itself. Namely, we say that a space $(X, \mathcal{T})$ is *potentially τ-homogeneous*, where τ is a finite or infinite cardinal number, if there exists a topological partition $\mathcal{Z} = [(X, \mathcal{T}), \gamma]$ on $(X, \mathcal{T})$ of cardinality τ. Moreover, if γ can be inscribed in an arbitrary finite open covering of the space $(X, \mathcal{T})$, then $(X, \mathcal{T})$ is called a *completely τ-homogeneous*.

Topological partitions are naturally related to the notion of dimension. We say that a topological partition $\mathcal{Z} = [(X, \mathcal{T}), \gamma]$ has dimension 0 (or is ind-zero-dimensional) if the space $(X, \mathcal{T})$ possesses a base $\mathcal{B}$ consisting of sets whose saturations in $\mathcal{Z}$ is closed in the space $(X, \mathcal{T})$. Moreover, the *saturation of a set $A \subset X$ in a topological partition $\mathcal{Z} = [(X, \mathcal{T}), \gamma]$* is the set $\bigcup\{P \in \gamma : P \cap A \neq \varnothing\}$.

3. Some results. Assume that $(G, \mathcal{T})$ if a left topological group and H is a subgroup of the group G. The set $\{xH : x \in G\}$ of all left cosets xH of the group G is a disjunct covering γ_H of the group G by nonempty sets xH. Thus, $[(G, \mathcal{T}), \gamma_H]$ is a topological partition; it is denoted by G_H and is called a topological partition *generated* on G by the subgroup H.

If the set $\{xH : x \in G\}$ of all left cosets xH of a left topological group G with respect to a subgroup H is chosen as the set of points of a new topological space endowed with the factor topology (see [11]), then the space obtained in this way is called the *left coset space* (or the *left factor space*) of the left topological group G with respect to the subgroup H; notation G/H (see [8, Theorem 1.5.1]). The natural mapping of the set G onto the set G/H is usually denoted by q or q_H.

Theorem 3.1. *If G is a Lindelöf closed left topological group with countable tightness and H is a closed subgroup of the group G such that the factor space G/H is a compact Hausdoreff space, then $|G/H| \leq 2^\omega$.*

Proof. Since the tightness does not increase under factor mappings, we conclude that the tightness of the factor space G/H is countable. However, the space G/H is homogeneous as a coset space of any left topological group (see [8, Theorem 1.5.1]). De la Vega's theorem (see [18]) states that the cardinality of any homogeneous compactum of countable tightness does not exceed 2^ω. Therefore, $|G/H| \leq 2^\omega$. □

Theorem 3.2. *Assume that G is a Lindelöf left topological group of countable tightness and let the continuum hypothesis CH be adopted. We also assume that H is a closed subgroup of the group G such that the left factor space G/H is a compact Hausdorff space. Then G/H satisfies the first countability axiom.*

Proof. By Theorem 3.1 we have $|G/H| \leq 2^\omega$. Since the coset space G/H is a compact Hausdorff space, the Čech–Pospíšil theorem implies that G/H satisfies the first countability axiom at least at one point (see [11]). The homogeneity of the space G/H implies that G/H satisfies the first countability axiom at all points. □

The above arguments show that the following more general assertion holds. We say that a space X is an *FSC-space* if each free sequence in X is countable. It is well known that any Lindelöf space of countable tightness is an FSC-space (see [4]). Moreover, we have the following assertion.

Proposition 3.3. *If a space X is the union of a countable family of Lindelöf spaces of countable tightness, then X is an FSC-space.*

The proof is standard and we omit it. Now we see that the following two assertions are obvious.

Theorem 3.4. *If G is a left topological group, which is an FSC-space, and H is a closed subgroup of the group G such that the factor space G/H is a compact Hausdorff space, then $|G/H| \leq 2^{\omega}$.*

Theorem 3.5. *Assume that G is a left topological group, which is an FSC-space. Let the continuum hypothesis CH be adopted. Also, we assume that H is a closed subgroup of the group G such that the factor space G/H is a compact Hausdorff space. Then G/H satisfies the first countability axiom.*

4. Groups that are complete in the Čech sense and their images. Results obtained above can be applied to the study of continuous images of topological groups that are complete in the Čech sense (briefly, Čech-complete groups).

Proposition 4.1. *Assume that G is a left topological group, which is a Čech-complete space, and H is a closed subgroup of the group G such that the left coset space G/H is a Hausdorff space. The space G/H is compact if and only if there exists a compact subspace K of the space G such that $q(K) = G/H$, where q is the factor mapping of the topological group G onto the left factor space G/H.*

Proof. If such a compactum K exists, then G/H is compact since $q(K) = G/H$ and q is continuous.

Now we assume that G/H is compact. Since the mapping $q : G \to G/H$ is open and continuous and the space G is Čech-complete, we conclude that there exists a compact subspace K of the space G such that $q(K) = G/H$ (see [3, Theorem 1.2]. □

Corollary 4.2. *Assume that G is a Čech-complete, left topological group and H is a closed subgroup of the group G such that the left coset space G/H is a Hausdorff space. Then the following conditions are pairwise equivalent*:

(1) *the factor space G/H is compact*;
(2) *the topological partition $[(G, \mathfrak{T}), \gamma_H]$, where $\gamma_H = \{xH : x \in G\}$, is α_3-compact*;
(3) *the topological partition $[(G, \mathfrak{T}), \gamma_H]$, where $\gamma_H = \{xH : x \in G\}$, is densely compact*;
(4) *the topological partition $[(G, \mathfrak{T}), \gamma_H]$, where $\gamma_H = \{xH : x \in G\}$, is α_1-compact.*

Corollary 4.2 follows from Propositions 2.1 and 4.1.

In the study of topological partitions, we must not restrict ourselves to topological groups and their coset spaces. Note that each (surjective) mapping f of a topological space X onto a set Y generates a topological partition $\gamma_f = \{f^{-1}(y) : y \in Y\}$ of the space X into the preimages of points of Y under f. We denote this partition by $[X, \mathfrak{T}, f]$, where $\mathfrak{T}$ is the topology on X.

Theorem 4.3. *Assume that G is a Čech-complete subspace of a topological group M, which is a G_δ-type set in M, and f is a continuous open mapping of the space G onto a Hausdorff space Y. Then an arbitrary compactum $F \subset Y$ is contained in some dyadic compactum $F_1 \subset Y$. In particular, if the space Y itself is a compactum, then this compactum is dyadic.*

Proof. As is known, each continuous open mapping f of a Čech-complete space X onto a Hausdorff space Y is a compactly covering mapping, i.e., for each compactum $P \subset Y$, there exists a compactum $B \subset X$ such that $f(B) = P$ (see [3, Theorem 1.2]). This means that, within the conditions of Theorem 4.3, there exists a compactum $K \subset G$ such that $f(K) = F$. However, G is a space of countable type since G is Čech-complete (see [2]). Therefore, there exists a compactum $K_0 \subset G$ of the type G_δ in G such that $K \subset K_0$. Since G is a set of the type G_δ in M and K_0 is a set of the type G_δ in G, we see that K_0 is a set of the type G_δ in the topological group M. Clearly, $F \subset f(K_0)$. Choban proved (see [9]) that each compactum of the type G_δ is a topological group is dyadic. Therefore, K_0 is a dyadic compactum. Then $f(K_0)$ is also dyadic compactum. Now we set $F_1 = f(K_0)$. □

The following assertion is similar to Theorem 4.3.

Theorem 4.4. *Assume that $(G, \mathcal{T})$ is a topological group and a subspace $P \subset G$ is a set of the type G_δ in $(G, \mathcal{T})$ and a cirrus space. We also assume that a Hausdorff space Y is the continuous image of the space P and a compactum $F \subset Y$ is the image of a compact $K \subset P$ under f. Then there exists a dyadic compactum $F_0 \subset Y$ such that $F \subset F_0$.*

The proof of Theorem 4.4 is similar to the proof of Theorem 4.3. Theorem 4.3 easily implies that following well-known fact (see [9]).

Corollary 4.5. *Assume that G is a Čech-complete topological group and H is a closed subgroup of the group G such that the left factor space G/H is a compact space. Then G/H is a dyadic compactum and the factor mapping $q : G \to G/H$ is a compactly covering mapping.*

Indeed, it was proved in [9] that each compact coset space of a Čech-complete topological group can be represented as a coset space of an ω-narrow of a topological group. Uspenskii proved (see [16]) that if an ω-narrow topological group acts continuously and transitively on a compactum F then F is a Dugunji compactum.

Corollary 4.6. *Assume that G is a Čech-complete subspace of a topological group M, which is a set of the type G_δ in M, and f is a continuous open mapping of the space G onto a Hausdorff space Y of countable tightness. Then any arbitrary compact $F \subset Y$ is metrizable. In particular, if the space Y itself is a compactum, then the compactum Y is metrizable.*

Proof. By Theorem 4.3, the compactum F is contained in some dyadic compactum $F_1 \subset Y$. The tightness of the compactum F_1 is countable since it does not exceed the tightness of the space Y. It remains to apply the well-known fact: each dyadic compactum of countable tightness is metrizable (see [7]). We conclude that the compactum F_1 is metrizable and, therefore, the compactum F is metrizable. □

In particular, Corollary 4.6 is applicable to Čech-complete topological groups. In this connection, we introduce the following notion. A topological space X is said to be *G_δ-encircled* if there exists a topological group G containing X as a subspace, which is a set of the type G_δ in G.

It was proved in [5] that if G is a topological group and f is a continuous mapping of the space G onto a Hausdorff space F satisfying the first countability axiom such that $f(K) = F$, where K is a compact subspace of the space G, then F is separable and metrizable. This assertion generalizes a classical theorem proved by Esenin-Volpin (see [11, 12]): each dyadic compactum F satisfying the first countability axiom is metrizable. However, as far as the author knows, the following questions remain open.

Problem 4.7. *Assume that G is a Lindelöf topological group of countable tightness and F is a compactum satisfying the first countability axiom, which is the continuous image of the space G. Is the compactum F metrizable?*

Problem 4.8 (see [5])**.** *Assume that G is a topological group and K is a compact subspace of it. We also assume that f is a continuous mapping of the space G onto the compactum F such that $f(K) = F$. Is the compactum F dyadic?*

We note that Theorem 4.3 and Corollary 4.6 can be partially extended to continuous closed mappings.

Theorem 4.9. *Assume that G is a Čech-complete topological group and f is a continuous closed mapping of the space G onto a Hausdorff space Y. Then an arbitrary compactum $F \subset Y$ is contained in some dyadic compactum $F_1 \subset Y$. In particular, if the space Y itself is a compactum, then this compactum is dyadic.*

Proof. It is well known that each Čech-complete topological group is paracompact (see [8]). E. Michael proved that each continuous closed mapping f of a paracompact Hausdorff space X onto an arbitrary topological space Y is a compactly covering mapping (see [14]). Therefore, the following assertion holds.

Proposition 4.10. *Each continuous closed mapping of a Čech-complete topological group onto an arbitrary space is a compactly covering mapping.*

This implies that there exists a compactum K lying in G such that $f(K) = F$. Next, we argue as in the proof of Theorems 4.3. □

Theorem 4.11. *Assume that G is a Čech-complete topological group and f is a continuous closed mapping of the space G onto a Hausdorff space Y of countable tightness. Then any compactum $F \subset Y$ is metrizable. In particular, if the space Y itself is a compactum, then it is metrizable.*

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