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THE LIMIT BEHAVIOR OF SOLUTIONS TO THE RADIATIVE TRANSFER EQUATION IN A SYSTEM OF SEMITRANSPARENT BODIES AS THE ABSORPTION AND SCATTERING COEFFICIENTS TEND TO INFINITY

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We consider the boundary value problem describing radiative transfer in a system of semitransparent bodies with the diffuse reflection and diffuse refraction conditions. Under the assumption that the absorption and scattering coefficients tend to infinity, we study the limit behavior of the solutions. Bibliography: 6 *titles.*

1 Introduction

Let $G = \bigcup\limits_{j=1}^{m} G_j$ be a system of bodies $G_j \subset \mathbb{R}^3$ such that $\overline{G}_i \cap \overline{G}_j$ for $i \neq j$. Assume that body G_j is a bounded domain with boundary ∂G_j of class $C^{1+\lambda_j}$, $0 < \lambda_j < 1$.

We assume that each body G_j is occupied by a semitransparent (under radiation) optically homogeneous material with the extinction $\beta_j = s_j + \varkappa_j$, scattering $s_j \geqslant 0$, absorption $\varkappa_j > 0$ and refraction $k_j \geqslant 1$.

Propagation of a monochromatic radiation in G is described by the integro-differential radiative transfer equation

$$\omega \cdot \nabla I + \beta I = s\mathscr{S}(I) + \varkappa k^2 F, \quad (\omega, x) \in D. \tag{1.1}$$

The sought function $I(\omega, x)$ is defined on the set $D = \Omega \times G$, where $\Omega = \{\omega \in \mathbb{R}^3 \mid |\omega| = 1\}$ is the unit sphere in $\mathbb{R}^3$ (the sphere of directions) and is interpreted as the intensity of the radiation at the point $x \in G$ propagating along the direction $\omega \in \Omega$.

In Equation (1.1), $\omega \cdot \nabla I = \sum\limits_{i=1}^{3} \omega_i \dfrac{\partial}{\partial x_i} I$ is the derivative of I along the direction ω. Here, $\mathscr{S}$ denotes the operator

$$\mathscr{S}(I)(\omega, x) = \frac{1}{4\pi} \int\limits_{\Omega} I(\omega', x)\, d\omega', \quad (\omega, x) \in D.$$

Translated from *Problemy Matematicheskogo Analiza* **125**, 2023, pp. 5-16.

Furthermore, $\beta(x) = \beta_j$, $s(x) = s_j$, $\varkappa(x) = \varkappa_j$, $k(x) = k_j$ for $x \in G_j$, $1 \leqslant j \leqslant m$; $F = F(x)$ is the density of isotropic volume of the radiation sources.

In this paper, we assume that for each body G_j the extinction coefficient β_j tends to infinity, i.e., $\beta_j = 1/\varepsilon_j$, where $\varepsilon_j \to 0$. We introduce the *albedo* $\varpi(x) = \varpi_j = s_j/\beta_j$ for $x \in G_j$, $1 \leqslant j \leqslant m$ and assume that the values of ϖ_j are constant. We set $\varepsilon(x) = \varepsilon_j$ for $x \in G_j$, $1 \leqslant j \leqslant m$, and assume that $\varepsilon \to 0$, i.e., $\max\limits_{1 \leqslant j \leqslant m} \varepsilon_j \to 0$.

In the case under consideration, the absorption and scattering coefficients take the form $\varkappa = (1 - \varpi)/\varepsilon$, $s = \varpi/\varepsilon$ and tend to infinity as $\varepsilon \to 0$, whereas Equation (1.1) takes the form

$$\omega \cdot \nabla I_\varepsilon + \frac{1}{\varepsilon} I_\varepsilon = \frac{\varpi}{\varepsilon} \mathscr{S}(I_\varepsilon) + \frac{1 - \varpi}{\varepsilon} k^2 F_\varepsilon, \quad (\omega, x) \in D. \tag{1.2}$$

We denote by $n_j(x)$ the outward normal to the boundary ∂G_j at the point $x \in \partial G_j$. We introduce the sets

$$\Gamma_j^- = \{(\omega, x) \in \Omega \times \partial G_j \mid \omega \cdot n_j(x) < 0\}, \quad \Gamma^- = \bigcup_{j=1}^m \Gamma_j^-,$$

$$\Gamma_j^+ = \{(\omega, x) \in \Omega \times \partial G_j \mid \omega \cdot n_j(x) > 0\}, \quad \Gamma^+ = \bigcup_{j=1}^m \Gamma_j^+.$$

We denote by $I_\varepsilon|_{\Gamma^-}$ and $I_\varepsilon|_{\Gamma^+}$ the values (traces) of the solutions to Equation (1.2) on Γ^- and Γ^+ respectively.

Let ∂G_j be the diffuse reflecting and diffuse refracting surfaces. We denote by θ_j the coefficient $0 < \theta_j < 1$ characterizing the reflection properties of ∂G_j. We set $\theta(x) = \theta_j$ for $x \in \partial G_j$, $1 \leqslant j \leqslant m$.

For Equation (1.2) we consider the boundary condition of diffuse reflection and diffuse refraction of radiation

$$I_\varepsilon|_{\Gamma^-} = \mathscr{R}^-(I_\varepsilon|_{\Gamma^+}) + \mathscr{P}^-(J_\varepsilon), \quad (\omega, x) \in \Gamma^- \tag{1.3}$$

and the equation

$$J_\varepsilon = T[\mathscr{R}^+(J_\varepsilon) + \mathscr{P}^+(I_\varepsilon|_{\Gamma^+})] + J_{*,\varepsilon}, \quad (\omega, x) \in \Gamma^-, \tag{1.4}$$

describing the connection between the intensity J_ε of radiation falling on ∂G and the intensities of the reflected radiation $\mathscr{R}^+(J_\varepsilon)$, refracted radiation $\mathscr{P}^+(I_\varepsilon|_{\Gamma^+})$, and the radiation $J_{*,\varepsilon}$ coming from outside.

Here, $\mathscr{R}^-$ and $\mathscr{R}^+$ are the diffuse reflection operators, $\mathscr{P}^-$ and $\mathscr{P}^+$ are the diffuse refraction operators, and T is the translation operator. These operators will be defined in Section 2.

We note that Equation (1.4) is also necessary in the case where G consists of only one, but nonconvex body. If G is a single convex body, then $J_\varepsilon = J_{*,\varepsilon}$ and Equation (1.4) is not necessary.

The goal of the paper is to study the limit behavior of solutions $(I_\varepsilon, J_\varepsilon)$ to the problem (1.2)–(1.4) as $\varepsilon \to 0$. The limit behavior of the values $I_\varepsilon|_{\Gamma^+}$, $I_\varepsilon|_{\Gamma^-}$ of the solutions on Γ^+ and Γ^- is of a particular interest. In the simplified version, when G consists of only one body and Equation (1.4) is omitted, the limit behavior of the solution was studied in [1].

The paper is organized as follows. In Section 2, we introduce necessary function spaces and operators and recall some their properties. In Section 3, we recall properties of the spatially one-dimensional problem for the transfer equation. In Section 4, we study an auxiliary problem about the limit behavior of the solutions in the case where the set G is a single body. In Section 5, we establish the main result of the paper, the theorem about the limit behavior of solutions to the problem (1.2)–(1.4) as $\varepsilon \to 0$.

2 Function Spaces and Operators

Let Z be a set with a given measure $d\mu$, and let Z_1 be its measurable subset with respect to the measure $d\mu$. We denote by $L^p(Z_1; d\mu)$, $1 \leqslant p < \infty$, the Lebesgue space of functions f that are defined on Z_1, measurable with respect to the measure $d\mu$, and possess the finite norm

$$\|f\|_{L^p(Z_1;d\mu)} = \bigg(\int\limits_{Z_1} |f(z)|^p \, d\mu(z) \bigg)^{1/p}.$$

We note that $D = \Omega \times G = \bigcup\limits_{j=1}^{m} D_j$, where $D_j = \Omega \times G_j$.

We denote by $d\sigma(x)$ and $d\omega$ the measures on ∂G and in Ω induced by the Lebesgue measure in $\mathbb{R}^3$. We set

$$L^p(\partial G_j) = L^p(\partial G_j; d\sigma), \quad L^p(D_j) = L^p(D_j; d\omega dx), \quad 1 \leqslant j \leqslant m,$$
$$L^p(\partial G) = L^p(\partial G; d\sigma), \quad L^p(D) = L^p(D; d\omega dx),$$

introduce the following measures on Γ^- and Γ^+:

$$\widehat{d\Gamma}^-(\omega, x) = |\omega \cdot n_j(x)| \, d\omega d\sigma(x), \quad (\omega, x) \in \Gamma_j^-, \quad 1 \leqslant j \leqslant m,$$
$$\widehat{d\Gamma}^+(\omega, x) = \omega \cdot n_j(x) \, d\omega d\sigma(x), \quad (\omega, x) \in \Gamma_j^+, \quad 1 \leqslant j \leqslant m,$$

and define $\widehat{L}^p(\Gamma_j^\pm) = L^p(\Gamma_j^\pm; \widehat{d\Gamma}^\pm)$, $1 \leqslant j \leqslant m$, and $\widehat{L}^p(\Gamma^\pm) = L^p(\Gamma^\pm; \widehat{d\Gamma}^\pm)$.

On Γ^-, we define the function

$$\widehat{\tau}^+(\omega, x) = \sup\{t > 0 \mid x + s\omega \in G \quad \forall\, s \in (0, t)\}$$

and pay attention to the following formula [2, 4]:

$$\int\limits_{D_j} f(\omega, x) \, d\omega dx = \int\limits_{\Gamma_j^-} \Bigg[\int\limits_0^{\widehat{\tau}^+(\omega,x)} f(\omega, x + t\omega) \, dt \Bigg] \widehat{d\Gamma}^-(\omega, x) \quad \forall\, f \in L^1(D_j). \tag{2.1}$$

In what follows, we need the following property of $\widehat{\tau}^+$.

Lemma 2.1. *Assume that $\mu_0 \in (0, 1)$ and $1 \leqslant j \leqslant m$. Then there exists $\ell_j(\mu_0) > 0$ such that $\widehat{\tau}^+(\omega, x_0) \geqslant \ell_j(\mu_0)$ for all $(\omega, x_0) \in \Gamma_j^-$ such that $\omega \in \Omega_j^{-,\mu_0}(x_0) = \{\omega \in \Omega \mid \omega \cdot n_j(x_0) < -\mu_0\}$.*

Proof. We set $V_{r_0} = \{y' = (y_1, y_2) \in \mathbb{R}^2 \mid |y'| \leqslant r_0\}$. By the assumption $\partial G_j \in C^{1+\lambda_j}$, $0 < \lambda_j < 1$, for each point $x_0 \in \partial G_j$ there exists a Cartesian coordinate system with origin x_0, a basis e_1, e_2, $e_3 = n_j(x_0)$, a cylinder $\mathscr{C}(x_0) = \{x = x_0 + y_1 e_1 + y_2 e_2 + y_3 n_j(x_0) \mid |y'| \leqslant r_0, \ |y_3| <$

$r_0\}$, where $r_0 > 0$ is independent of x_0, and a function $\gamma_{x_0} \in C^{1+\lambda_j}(V_{r_0})$ depending on x_0 such that

$$G_j \cap \mathscr{C}(x_0) = \{x = x_0 + y_1 e_1 + y_2 e_2 + y_3 n_j(x_0) \mid |y'| \leqslant r_0,\ -r_0 < y_3 < \gamma_{x_0}(y')\}.$$

Furthermore, $\gamma_{x_0}(0,0) = 0$, $\nabla_{y'}\gamma_{x_0}(0,0) = (0,0)$ and

$$|\gamma_{x_0}(y')| \leqslant C|y'|^{1+\lambda_j} \quad \forall\, y' \in V_{r_0},$$

where C is a constant independent of x_0.

Assume that $(\omega, x_0) \in \Gamma_j^-$ and $\mu = \omega \cdot n_j(x_0) < -\mu_0$. Then $x_0 + s\omega = x_0 + y_1 e_1 + y_2 e_2 + y_3 n_j(x_0)$, where $y_1 = s(\omega \cdot e_1)$, $y_2 = s(\omega \cdot e_2)$, $y_3 = s\mu$. We note that $|y'| = s\sqrt{1-\mu^2}$ and

$$-r_0 < y_3 = s\mu < -Cs^{1+\lambda_j}(1-\mu^2)^{(1+\lambda_j)/2} = -C|y'|^{1+\lambda_j} \leqslant \gamma_{x_0}(y')$$

if

$$0 < s < \ell_j(\mu_0) = \min\Big\{r_0,\, \Big[\frac{\mu_0}{C(1-\mu_0^2)^{(1+\lambda_j)/2}}\Big]^{1/\lambda_j}\Big\}.$$

Consequently, $x_0 + s\omega \in G_j$ for all $s \in (0, \ell_j(\mu_0))$. Therefore,

$$\widehat{\tau}^+(\omega, x_0) = \sup\{t > 0 \mid x_0 + s\omega \in G \quad \forall\, s \in (0,t)\} \geqslant \ell_j(\mu_0) > 0.$$

The lemma is proved. □

2.1. The spaces $\widehat{\mathscr{W}^p}(D_j)$ and $\widehat{\mathscr{W}^p}(D)$. By the *weak derivative* in direction ω of a function $f \in L^1(D_j)$ we understand a function $w \in L^1(D_j)$, denoted by $w = \omega \cdot \nabla f$, which satisfies the identity

$$\int\limits_{D_j} [f(\omega,x)\,\omega\cdot\nabla\varphi(x) + w(\omega,x)\varphi(x)]\psi(\omega)\,d\omega dx = 0 \quad \forall\,\varphi \in C_0^\infty(G_j), \quad \forall\,\psi \in L^\infty(\Omega).$$

We denote by $\mathscr{W}^p(D_j)$, $1 \leqslant p < \infty$, the Banach space of functions $f \in L^p(D_j)$ possessing the weak derivatives $\omega \cdot \nabla f \in L^p(D_j)$, equipped with the norm

$$\|f\|_{\mathscr{W}^p(D_j)} = (\|f\|^p_{L^p(D_j)} + \|\,\omega\cdot\nabla f\|^p_{L^p(D_j)})^{1/p}.$$

We denote by $f|_{\Gamma_j^-}$ and $f|_{\Gamma_j^+}$ the traces of a function $f \in \mathscr{W}^p(D_j)$ on Γ_j^- and Γ_j^+ respectively.

In $\mathscr{W}^p(D_j)$, we introduce the linear manifold $\widehat{\mathscr{W}^p}(D_j) = \{f \in \mathscr{W}^p(D_j) \mid f|_{\Gamma_j^+} \in \widehat{L}^p(\Gamma_j^+)\}$. Let us list some properties of $f \in \widehat{\mathscr{W}^p}(D_j)$.

1. If $f \in \widehat{\mathscr{W}^1}(D_j)$, then $f|_{\Gamma_j^-} \in \widehat{L}^1(\Gamma_j^-)$ and

$$\int\limits_{D_j} \omega\cdot\nabla f\,d\omega dx = \int\limits_{\Gamma_j^+} f|_{\Gamma_j^+}\,\widehat{d\Gamma}^+ - \int\limits_{\Gamma_j^-} f|_{\Gamma_j^-}\,\widehat{d\Gamma}^-. \tag{2.2}$$

2. If $f \in \widehat{\mathscr{W}^p}(D_j)$, then $|f|^p \in \widehat{\mathscr{W}^1}(D_j)$, $f|_{\Gamma_j^-} \in \widehat{L}^p(\Gamma_j^-)$; moreover, $p|f|^{p-1}\operatorname{sgn} f\,(\omega\cdot\nabla f) = \omega\cdot\nabla|f|^p$ and

$$\int\limits_{D_j} \omega\cdot\nabla|f|^p\,d\omega dx = \|f|_{\Gamma_j^+}\|^p_{\widehat{L}^p(\Gamma_j^+)} - \|f|_{\Gamma_j^-}\|^p_{\widehat{L}^p(\Gamma_j^-)}. \tag{2.3}$$

Let $\widehat{\mathscr{W}^p}(D) = \{f \in L^p(D) \mid f \in \widehat{\mathscr{W}^p}(D_j),\ \ 1 \leqslant j \leqslant m\}$. We denote by $f|_{\Gamma^-}$ and $f|_{\Gamma^+}$ the traces of $f \in \mathscr{W}^p(D)$ on Γ^- and $f|_{\Gamma^+}$. For further properties of the functions $f \in \widehat{\mathscr{W}^p}(D_j)$, $f \in \widehat{\mathscr{W}^p}(D)$ and their traces $f|_{\Gamma_j^\pm}$, $f|_{\Gamma^\pm}$ we refer to [3, 4].

2.2. The reflection and refraction operators. Let $x \in \partial G_j$, $1 \leqslant j \leqslant m$. We set

$$\Omega_j^+(x) = \{\omega \in \Omega \mid \omega\cdot n_j(x) > 0\}, \quad \Omega_j^-(x) = \{\omega \in \Omega \mid \omega\cdot n_j(x) < 0\}.$$

We define the operators $\mathscr{M}^+ : \widehat{L}^p(\Gamma^+) \to L^p(\partial G)$, $\mathscr{M}^- : \widehat{L}^p(\Gamma^-) \to L^p(\partial G)$ and $\mathscr{M}_j^+ : \widehat{L}^p(\Gamma_j^+) \to L^p(\partial G_j)$, $\mathscr{M}_j^- : \widehat{L}^p(\Gamma_j^-) \to L^p(\partial G_j)$, $1 \leqslant j \leqslant m$, by

$$\mathscr{M}^+(\varphi)(x) = \mathscr{M}_j^+(\varphi)(x) = \frac{1}{\pi}\int\limits_{\Omega_j^+(x)} \varphi(\omega,x)\,\omega\cdot n_j(x)\,d\omega, \quad x \in \partial G_j, \quad 1 \leqslant j \leqslant m,$$

$$\mathscr{M}^-(\psi)(x) = \mathscr{M}_j^-(\psi)(x) = \frac{1}{\pi}\int\limits_{\Omega_j^-(x)} \psi(\omega,x)\,|\omega\cdot n_j(x)|\,d\omega, \quad x \in \partial G_j, \quad 1 \leqslant j \leqslant m.$$

We note that $\mathscr{M}_j^+(1) = 1$, $\mathscr{M}_j^-(1) = 1$ and

$$\|\mathscr{M}_j^+\|_{\widehat{L}^p(\Gamma^+)\to L^p(\partial G_j)} = 1/\pi, \quad \|\mathscr{M}_j^-\|_{\widehat{L}^p(\Gamma^-)\to L^p(\partial G_j)} = 1/\pi. \tag{2.4}$$

We introduce the diffuse reflection operators $\mathscr{R}^- : \widehat{L}^p(\Gamma^+) \to \widehat{L}^p(\Gamma^-)$, $\mathscr{R}_j^- : \widehat{L}^p(\Gamma_j^+) \to \widehat{L}^p(\Gamma_j^-)$, $\mathscr{R}^+ : \widehat{L}^p(\Gamma^-) \to \widehat{L}^p(\Gamma^+)$, $\mathscr{R}_j^+ : \widehat{L}^p(\Gamma_j^-) \to \widehat{L}^p(\Gamma_j^+)$ and the diffuse refraction operators $\mathscr{P}^- : \widehat{L}^p(\Gamma^-) \to \widehat{L}^p(\Gamma^-)$, $\mathscr{P}_j^- : \widehat{L}^p(\Gamma_j^-) \to \widehat{L}^p(\Gamma_j^-)$, $\mathscr{P}^+ : \widehat{L}^p(\Gamma^+) \to \widehat{L}^p(\Gamma^+)$, $\mathscr{P}_j^+ : \widehat{L}^p(\Gamma_j^+) \to \widehat{L}^p(\Gamma_j^+)$ by the formulas

$$\begin{aligned}
&\mathscr{R}^-(\varphi)(\omega,x) = \mathscr{R}_j^-(\varphi)(\omega,x) = \theta_j\mathscr{M}_j^+(\varphi), \quad (\omega,x) \in \Gamma_j^-, \quad 1 \leqslant j \leqslant m,\\
&\mathscr{R}^+(\psi)(\omega,x) = \mathscr{R}_j^+(\psi)(\omega,x) = \theta_j\mathscr{M}_j^-(\psi), \quad (\omega,x) \in \Gamma_j^+, \quad 1 \leqslant j \leqslant m,\\
&\mathscr{P}^-(\psi)(\omega,x) = \mathscr{P}_j^-(\psi)(\omega,x) = (1-\theta_j)k_j^2\mathscr{M}_j^-(\psi), \quad (\omega,x) \in \Gamma_j^-, \quad 1 \leqslant j \leqslant m,\\
&\mathscr{P}^+(\varphi)(\omega,x) = \mathscr{P}_j^+(\varphi)(\omega,x) = \frac{1-\theta_j}{k_j^2}\mathscr{M}_j^+(\varphi), \quad (\omega,x) \in \Gamma_j^+, \quad 1 \leqslant j \leqslant m.
\end{aligned}$$

Here, $\varphi \in \widehat{L}^p(\Gamma^+)$, $\psi \in \widehat{L}^p(\Gamma^-)$.

2.3. The translation operator T. We recall the definition and some properties of the translation operator (or the shooting operator) T [2, 4].

Let $(\omega,x) \in \Gamma^-$. We define the ray $\ell^-(\omega,x) = \{x - t\omega \mid t > 0\}$ and introduce the set

$$\overset{*}{\Gamma}{}^- = \{(\omega,x) \in \Gamma^- \mid \ell^-(\omega,x) \cap \overline{G} = \varnothing\}.$$

Let $(\omega, x) \in \Gamma^- \setminus \overset{*}{\Gamma}{}^-$. Then the ray $\ell^-(\omega, x)$ intersects $\overline{G}$. We set

$$\tau^-(\omega, x) = \inf\,\{t > 0 \mid x - t\omega \in \overline{G}\}, \quad X^-(\omega, x) = x - \tau^-(\omega, x)\omega.$$

It is clear that $X^-(\omega, x)$ is a point of the boundary ∂G from which the point $x \in \partial G$ is "visible" in the direction ω and $\tau^-(\omega, x) > 0$ is the distance between the points x and $X^-(\omega, x)$.

We introduce the set

$$\widetilde{\Gamma}^- = \{(\omega, x) \in \Gamma^- \setminus \overset{*}{\Gamma}{}^- \mid (\omega, X^-(\omega, x)) \in \Gamma^+\}$$

and define the operator T by the formula

$$T\varphi(\omega, x) = \begin{cases} \varphi(\omega, X^-(\omega, x)), & (\omega, x) \in \widetilde{\Gamma}^-, \\ 0, & (\omega, x) \in \Gamma^- \setminus \widetilde{\Gamma}^-. \end{cases}$$

As known, $T : \widehat{L}^p(\Gamma^+) \to \widehat{L}^p(\Gamma^-)$ for all $p \geqslant 1$; moreover, $\|T\|_{\widehat{L}^p(\Gamma^+)\to\widehat{L}^p(\Gamma^-)} \leqslant 1$.

3 One-Dimensional Problem

The limit properties of the solutions to the problem under consideration are closely connected with the values at $\tau = 0$ of the solutions to the following problem with one spatial variable which describes radiative transfer in the half-space:

$$-\mu\frac{d\psi_j(\mu, \tau)}{d\tau} + \psi_j(\mu, \tau) = \frac{\varpi_j}{2}\int_{-1}^{1} \psi_j(\mu', \tau)\, d\mu', \quad \mu \in [-1, 0) \cup (0, 1], \quad \tau \in [0, +\infty), \tag{3.1}$$

$$\psi_j(\mu, 0) = 2\theta_j \int_0^1 \psi_j(\mu', 0)\mu'\, d\mu' + 1 - \theta_j, \quad \mu \in [-1, 0), \tag{3.2}$$

$$\psi_j(\mu, +\infty) = 0, \quad \mu \in (0, 1]. \tag{3.3}$$

The properties of the solution to this problem which will be used below were studied in [1]. By (3.2), the value $\psi_{0,j} = \psi_j(\mu, 0)$ is independent of $\mu \in [-1, 0)$. From [1] it follows that

$$1 - \theta_j < \psi_{0,j} \leqslant \frac{(1 - \theta_j)(2 - \varpi_j)}{2 - \varpi_j - \theta_j\varpi_j} < 1. \tag{3.4}$$

Furthermore,

$$0 < \psi_j(\mu, 0) \leqslant \frac{\varpi_j}{2 - \varpi_j}\psi_{0,j}, \quad 0 < \mu \leqslant 1. \tag{3.5}$$

On ∂G, we introduce a piecewise constant function ψ_0 with the values $\psi_0(x) = \psi_{0,j}$ for $x \in \partial G_j$, $1 \leqslant j \leqslant m$, and define the function $\widehat{\psi}_+$ on Γ^+ by

$$\widehat{\psi}_+(\omega, x) = \widehat{\psi}_{+,j}(\omega, x) = \psi_j(\omega \cdot n_j(x), 0), \quad (\omega, x) \in \Gamma_j^+, \quad 1 \leqslant j \leqslant m.$$

We note that

$$\mathscr{M}_j^+(\widehat{\psi}_{+,j}) = \frac{1}{\pi} \int\limits_{\Omega_j^+(x)} \psi_j(\omega \cdot n_j(x), 0)\, d\omega = 2 \int\limits_0^1 \psi_j(\mu', 0)\mu'\, d\mu'.$$

Therefore, from (3.2) it follows that $\theta_j \mathscr{M}_j^+(\widehat{\psi}_{+,j}) = \psi_{0,j} - (1-\theta_j)$, $1 \leqslant j \leqslant m$. Hence

$$\mathscr{M}^+(\widehat{\psi}_+) = 1 - \frac{1-\psi_0}{\theta}, \quad \mathscr{M}^+(1-\widehat{\psi}_+) = \frac{1-\psi_0}{\theta}. \tag{3.6}$$

4 Auxiliary Problem

We consider the auxiliary problem

$$\omega \cdot \nabla I_\varepsilon + \frac{1}{\varepsilon} I_\varepsilon = \frac{\varpi}{\varepsilon} \mathscr{S}(I_\varepsilon) + \frac{1-\varpi}{\varepsilon} k^2 F, \quad (\omega, x) \in D, \tag{4.1}$$

$$I_\varepsilon|_{\Gamma^-} = \mathscr{R}^-(I_\varepsilon|_{\Gamma^+}) + (1-\theta)g, \quad (\omega, x) \in \Gamma^-, \tag{4.2}$$

where $F \in W^{1,1}(G)$, $g \in L^1(\partial G)$.

By a *solution* to the problem (4.1), (4.2) we mean a function $I_\varepsilon \in \widehat{\mathscr{W}}^1(D)$ satisfying Equation (4.1) almost everywhere in D and the boundary condition (4.2) almost everywhere on Γ^-. From [3, 4] it follows that the solution to this problem exists and is unique. By the boundary condition (4.2), the value $I_\varepsilon|_{\Gamma^-}$ is independent of ω.

The problem (4.1), (4.2) naturally splits into m independent problems

$$\omega \cdot \nabla I_{\varepsilon,j} + \frac{1}{\varepsilon_j} I_{\varepsilon,j} = \frac{\varpi_j}{\varepsilon_j} \mathscr{S}(I_{\varepsilon,j}) + \frac{1-\varpi_j}{\varepsilon_j} k_j^2 F_j, \quad (\omega, x) \in D_j, \tag{4.3}$$

$$I_{\varepsilon,j}|_{\Gamma_j^-} = \mathscr{R}_j^-(I_{\varepsilon,j}|_{\Gamma_j^+}) + (1-\theta_j)g_j, \quad (\omega, x) \in \Gamma_j^-. \tag{4.4}$$

Here, $I_{\varepsilon,j}$ is the restriction of I_ε on D_j, F_j is the restriction of F on G_j, and g_j is the restriction of g on ∂G_j, $1 \leqslant j \leqslant m$. We denote by $F_j|_{\partial G_j}$ the trace of F_j on ∂G_j and by $F|_{\partial G}$ the trace of F on ∂G coinciding with $F_j|_{\partial G_j}$ for $x \in \partial G_j$.

4.1. The operators $\mathscr{A}_\varepsilon$ and $\mathscr{A}_{\varepsilon,j}$. We introduce the operator $\mathscr{A}_\varepsilon : L^1(\partial G) \to \widehat{L}^1(\Gamma^+)$ mapping a function $g \in L^1(\partial G)$ to the trace $I_\varepsilon|_{\Gamma^+}$ on Γ^+ of the solution to the problem (4.1), (4.2) with $F = 0$. We also introduce the operator $\mathscr{A}_{\varepsilon,j}$ mapping a function $g_j \in L^1(\partial G_j)$ to the trace $I_{\varepsilon,j}|_{\Gamma_j^+}$ on Γ_j^+ of the solution to the problem

$$\omega \cdot \nabla I_{\varepsilon,j} + \frac{1}{\varepsilon_j} I_{\varepsilon,j} = \frac{\varpi_j}{\varepsilon_j} \mathscr{S}(I_{\varepsilon,j}), \quad (\omega, x) \in D_j, \tag{4.5}$$

$$I_{\varepsilon,j}|_{\Gamma_j^-} = \mathscr{R}_j^-(I_{\varepsilon,j}|_{\Gamma_j^+}) + (1-\theta_j)g_j, \quad (\omega, x) \in \Gamma_j^-, \tag{4.6}$$

i.e., the problem (4.3), (4.4) with $F_j = 0$.

We note that $\mathscr{A}_\varepsilon(g)(\omega, x) = \mathscr{A}_{\varepsilon,j}(g_j)(\omega, x)$, $(\omega, x) \in \Gamma_j^+$, $1 \leqslant j \leqslant m$.

From [1] it follows that

$$\mathscr{A}_\varepsilon(g) \to \widehat{\psi}_+ g \quad \text{in} \quad \widehat{L}^1(\Gamma^+), \quad \varepsilon \to 0. \tag{4.7}$$

Lemma 4.1. *Let $I_{\varepsilon,j}$ be a solution to the problem* (4.5), (4.6). *Then*

$$(1-\theta_j)\int\limits_{\Gamma_j^+} I_{\varepsilon,j}|_{\Gamma_j^+}\, \widehat{d}\Gamma^+ + \frac{1-\varpi_j}{\varepsilon_j}\int\limits_{D_j} I_{\varepsilon,j}\, d\omega dx = (1-\theta_j)\pi\int\limits_{\partial G_j} g_j\, d\sigma. \tag{4.8}$$

Proof. Integrating (4.5) over D_j, using formula (2.2), and taking into account that

$$\int\limits_{D_j} \mathscr{S}(I_{\varepsilon,j})\, d\omega dx = \int\limits_{D_j} I_{\varepsilon,j}\, d\omega dx,$$

we arrive at the identity

$$\int\limits_{\Gamma_j^+} I_{\varepsilon,j}|_{\Gamma_j^+}\, \widehat{d}\Gamma^+ + \frac{1-\varpi_j}{\varepsilon_j}\int\limits_{D_j} I_{\varepsilon,j}\, d\omega dx = \int\limits_{\Gamma_j^-} I_{\varepsilon,j}|_{\Gamma_j^-}\, \widehat{d}\Gamma^-.$$

From (4.6) it follows that

$$\int\limits_{\Gamma_j^-} I_{\varepsilon,j}|_{\Gamma_j^-}\, \widehat{d}\Gamma^- = \int\limits_{\Gamma_j^-} [\mathscr{R}_j^-(I_{\varepsilon,j}|_{\Gamma^+}) + (1-\theta_j)g_j]\, \widehat{d}\Gamma^- = \theta_j\int\limits_{\Gamma_j^+} I_{\varepsilon,j}|_{\Gamma_j^+}\, \widehat{d}\Gamma^+ + (1-\theta_j)\pi\int\limits_{\partial G_j} g_j\, d\sigma.$$

Thus, the equality (4.8) holds. The lemma is proved. □

Lemma 4.2. *The following estimate holds*:

$$\varlimsup_{\varepsilon_j\to 0} \|\mathscr{A}_{\varepsilon,j}\|_{L^1(\partial G_j)\to\widehat{L}^1(\Gamma_j^+)} \leqslant \varpi_j\pi. \tag{4.9}$$

Proof. We represent a solution to the problem (4.5), (4.6) in the form $I_{\varepsilon,j} = I_{\varepsilon,j}^+ - I_{\varepsilon,j}^-$, where $I_{\varepsilon,j}^+$ and $I_{\varepsilon,j}^-$ are the solutions corresponding to $g_j^+ = \max\{g_j, 0\}$ and $g_j^- = \max\{-g_j, 0\}$ instead of g_j. It is clear that $I_{\varepsilon,j}^+ \geqslant 0$ and $I_{\varepsilon,j}^- \geqslant 0$. We note that the equality (4.8) for $I_{\varepsilon,j}^+$ takes the form

$$(1-\theta_j)\|I_{\varepsilon,j}^+|_{\Gamma_j^+}\|_{\widehat{L}^1(\Gamma_j^+)} + \frac{1-\varpi_j}{\varepsilon_j}\|I_{\varepsilon,j}^+\|_{L^1(D_j)} = (1-\theta_j)\pi\|g_j^+\|_{L^1(\partial G_j)}. \tag{4.10}$$

Let us estimate $\|I_{\varepsilon,j}^+\|_{L^1(D_j)}$ from below. Since $\mathscr{S}(I_{\varepsilon,j}^+) \geqslant 0$ and $\mathscr{R}_j^-(I_{\varepsilon,j}^+|_{\Gamma_j^+}) \geqslant 0$, we have $I_{\varepsilon,j}^+ \geqslant \underline{I}_{\varepsilon,j}^+$, where $\underline{I}_{\varepsilon,j}^+$ is the solution to the problem

$$\omega\cdot\nabla\underline{I}_{\varepsilon,j}^+ + \frac{1}{\varepsilon}\underline{I}_{\varepsilon,j}^+ = 0, \quad (\omega, x)\in D_j, \tag{4.11}$$

$$\underline{I}_{\varepsilon,j}^+|_{\Gamma^-} = (1-\theta_j)g_j^+, \quad (\omega, x)\in\Gamma_j^-. \tag{4.12}$$

Formula (2.1) applied to $f = \underline{I}_{\varepsilon,j}^+$ yields the equality

$$\|\,\underline{I}_{\varepsilon,j}^+\|_{L^1(D_j)} = \int\limits_{\Gamma_j^-}\left[\int\limits_0^{\widehat{\tau}^+(\omega,x)} \underline{I}_{\varepsilon,j}^+(\omega, x+t\omega)\, dt\right]\widehat{d}\Gamma^-(\omega, x).$$

Using the explicit formula [4]–[6], for solutions to the problem (4.11), (4.12)

$$\underline{I}^+_{\varepsilon,j}(\omega, x + t\omega) = e^{-t/\varepsilon_j}(1-\theta_j)g_j^+(x),$$

we have

$$\|\underline{I}^+_{\varepsilon,j}\|_{L^1(D_j)} = \varepsilon_j \int\limits_{\partial G_j} \Bigg[\int\limits_{\Omega_j^-(x)} (1 - e^{-\widehat{\tau}^+(\omega,x)/\varepsilon_j})|\,\omega\cdot n_j(x)|\,d\omega\Bigg](1-\theta_j)g_j^+(x)\,d\sigma(x).$$

Let $0 < \mu_0 < 1$. Since $\widehat{\tau}^+(\omega,x) \geqslant \ell_j(\mu_0) > 0$ for $\omega \in \Omega_j^{-,\mu_0}(x)$ by Lemma 2.1, we get

$$\begin{aligned}\frac{1}{\varepsilon_j}\|\underline{I}^+_{\varepsilon,j}\|_{L^1(D_j)} &\geqslant \int\limits_{\partial G_j}\Bigg[\int\limits_{\Omega_j^{-,\mu_0}(x)} (1 - e^{-\ell_j(\mu_0)/\varepsilon_j})|\,\omega\cdot n_j(x)|\,d\omega\Bigg](1-\theta_j)g_j^+(x)\,d\sigma(x)\\ &= \int\limits_{\partial G_j} 2\pi \int\limits_{-1}^{-\mu_0} |\mu|\,d\mu(1 - e^{-\ell_j(\mu_0)/\varepsilon_j})(1-\theta_j)g_j^+(x)\,d\sigma(x)\\ &= \pi(1-\mu_0^2)(1 - e^{-\ell_j(\mu_0)/\varepsilon_j})(1-\theta_j)\|g_j^+\|_{L^1(\partial G_j)}.\end{aligned}$$

Thus, from (4.10) we obtain the estimate

$$\|I^+_{\varepsilon,j}|_{\Gamma_j^+}\|_{\widehat{L}^1(\Gamma_j^+)} \leqslant [1 - (1-\varpi_j)(1-\mu_0^2)(1 - e^{-\ell_j(\mu_0)/\varepsilon_j})]\pi\|g_j^+\|_{L^1(\partial G_j)}.$$

The following estimate is proved in a similar way:

$$\|I^-_{\varepsilon,j}|_{\Gamma_j^+}\|_{\widehat{L}^1(\Gamma_j^+)} \leqslant [1 - (1-\varpi_j)(1-\mu_0^2)(1 - e^{-\ell_j(\mu_0)/\varepsilon_j})]\pi\|g_j^-\|_{L^1(\partial G_j)}.$$

As a consequence,

$$\begin{aligned}\|\mathscr{A}_{\varepsilon,j}(g_j)\|_{\widehat{L}^1(\Gamma_j^+)} = \|I_{\varepsilon,j}|_{\Gamma_j^+}\|_{\widehat{L}^1(\Gamma_j^+)} &= \|I^+_{\varepsilon,j}|_{\Gamma_j^+}\|_{\widehat{L}^1(\Gamma_j^+)} + \|I^-_{\varepsilon,j}|_{\Gamma_j^+}\|_{\widehat{L}^1(\Gamma_j^+)}\\ &\leqslant [1 - (1-\varpi_j)(1-\mu_0^2)(1 - e^{-\ell_j(\mu_0)/\varepsilon_j})]\pi\|g_j\|_{L^1(\partial G)}.\end{aligned}$$

Hence

$$\|\mathscr{A}_{\varepsilon,j}\|_{L^1(\partial G_j)\to\widehat{L}^1(\Gamma_j^+)} \leqslant [1 - (1-\varpi_j)(1-\mu_0^2)(1 - e^{-\ell_j(\mu_0)/\varepsilon_j})]\pi$$

which implies

$$\overline{\lim_{\varepsilon_j\to 0}}\, \|\mathscr{A}_{\varepsilon,j}\|_{L^1(\partial G_j)\to\widehat{L}^1(\Gamma_j^+)} \leqslant [1 - (1-\varpi_j)(1-\mu_0^2)]\pi.$$

Using the arbitrariness of the choice of μ_0, we obtain the estimate (4.9). □

Corollary 4.1. *The following estimate holds*:

$$\overline{\lim_{\varepsilon\to 0}}\, \|\mathscr{A}_{\varepsilon}\|_{L^1(\partial G)\to\widehat{L}^1(\Gamma^+)} \leqslant \max_{1\leqslant j\leqslant m} \varpi_j\,\pi. \tag{4.13}$$

Corollary 4.2. *If $g_\varepsilon, g \in L^1(\partial G)$ and $g_\varepsilon \to g$ in $L^1(\partial G)$ as $\varepsilon \to 0$, then*

$$\mathscr{A}_\varepsilon(g_\varepsilon) \to \widehat{\psi}_+ g \quad in \quad \widehat{L}^1(\Gamma^+), \quad \varepsilon \to 0. \tag{4.14}$$

Corollary 4.2 follows from (4.7) and (4.13).

4.2. The operators $\mathscr{B}_\varepsilon$ and $\mathscr{B}_{\varepsilon,j}$. We introduce the operator $\mathscr{B}_\varepsilon$ mapping a function $F \in W^{1,1}(G)$ to the trace $I_\varepsilon|_{\Gamma^+}$ on Γ^+ of the solution to the problem (4.1), (4.2) with $g = 0$. We also introduce the operator $\mathscr{B}_{\varepsilon,j}$ mapping a function $F_j \in W^{1,1}(G_j)$ to the trace $I_{\varepsilon,j}|_{\Gamma_j^+}$ on Γ_j^+ of the solution to the problem

$$\omega \cdot \nabla I_{\varepsilon,j} + \frac{1}{\varepsilon_j} I_{\varepsilon,j} = \frac{\varpi_j}{\varepsilon_j}\mathscr{S}(I_{\varepsilon,j}) + \frac{1-\varpi_j}{\varepsilon_j} k_j^2 F_j, \quad (\omega, x) \in D_j, \tag{4.15}$$

$$I_{\varepsilon,j}|_{\Gamma_j^-} = \mathscr{R}_j^-(I_{\varepsilon,j}|_{\Gamma_j^+}), \quad (\omega, x) \in \Gamma_j^-, \tag{4.16}$$

i.e., the problem (4.3), (4.4) with $g_j = 0$. We recall that F_j is the restriction of F on G_j.

It is clear that $\mathscr{B}_\varepsilon(F)(\omega, x) = \mathscr{B}_{\varepsilon,j}(F_j)(\omega, x)$, $(\omega, x) \in \Gamma_j^+$, $1 \leqslant j \leqslant m$. From [1] it follows that

$$\mathscr{B}_\varepsilon(F) \to (1 - \widehat{\psi}_+) k^2 F|_{\partial G} \quad \text{in} \quad \widehat{L}^1(\Gamma^+), \quad \varepsilon \to 0. \tag{4.17}$$

Lemma 4.3. *Let $F_\varepsilon, F \in W^{1,1}(G)$, and let $F_\varepsilon|_{\partial G} \to F|_{\partial G}$ in $L^1(\partial G)$ as $\varepsilon \to 0$. Assume that for every $1 \leqslant j \leqslant m$ there exists $p_j > 1$ such that*

$$\nabla F_{\varepsilon,j} \in L^{p_j}(G_j), \quad \varepsilon_j^{1-1/p_j} \|\nabla F_{\varepsilon,j}\|_{L^{p_j}(G_j)} \to 0, \quad \varepsilon_j \to 0. \tag{4.18}$$

Then

$$\mathscr{B}_\varepsilon(F_\varepsilon) \to (1 - \widehat{\psi}_+) k^2 F|_{\partial G} \quad in \quad \widehat{L}^1(\Gamma^+). \tag{4.19}$$

Proof. It suffices to show that $\mathscr{B}_{\varepsilon,j}(F_{\varepsilon,j}) \to (1 - \widehat{\psi}_{+,j}) k_j^2 F_j|_{\partial G_j}$ in $\widehat{L}^1(\Gamma_j^+)$ for all $1 \leqslant j \leqslant m$.

We express the solution to the problem (4.15), (4.16) in the form $I_{\varepsilon,j} = k_j^2 F_{\varepsilon,j} + I_{\varepsilon,j}^{(1)} + I_{\varepsilon,j}^{(2)}$, where $I_{\varepsilon,j}^{(1)}$ is the solution to the problem (4.5), (4.6) with $g_j = -k_j^2 F_{\varepsilon,j}|_{\partial G_j}$, and $I_{\varepsilon,j}^{(2)}$ is the solution to the problem

$$\omega \cdot \nabla I_{\varepsilon,j}^{(2)} + \frac{1}{\varepsilon_j} I_{\varepsilon,j}^{(2)} = \frac{\varpi_j}{\varepsilon_j}\mathscr{S}(I_{\varepsilon,j}^{(2)}) - k_j^2(\omega \cdot \nabla F_{\varepsilon,j}), \quad (\omega, x) \in D_j, \tag{4.20}$$

$$I_{\varepsilon,j}^{(2)}|_{\Gamma_j^-} = \mathscr{R}_j^-(I_{\varepsilon,j}^{(2)}|_{\Gamma_j^+}), \quad (\omega, x) \in \Gamma_j^-. \tag{4.21}$$

It is clear that $\mathscr{B}_{\varepsilon,j}(F_{\varepsilon,j}) = k_j^2 F_{\varepsilon,j}|_{\partial G_j} + \mathscr{A}_{\varepsilon,j}(-k_j^2 F_{\varepsilon,j}|_{\partial G_j}) + I_{\varepsilon,j}^{(2)}|_{\Gamma_j^+}$, where $\mathscr{A}_{\varepsilon,j}(-k_j^2 F_{\varepsilon,j}|_{\partial G_j}) \to -\widehat{\psi}_{+,j} k_j^2 F_j|_{\partial G_j}$ in $\widehat{L}^1(\Gamma_j^+)$ by Corollary 4.2. Therefore, it suffices to prove that $I_{\varepsilon,j}^{(2)}|_{\Gamma_j^+} \to 0$ in $\widehat{L}^1(\Gamma_j^+)$.

Since $\omega \cdot \nabla F_{\varepsilon,j} \in L^{p_j}(D_j)$, from [3, 4] it follows that $I_{\varepsilon,j}^{(2)} \in \widehat{\mathscr{W}}^{p_j}(D_j)$.

Multiplying (4.20) by $|I_{\varepsilon,j}^{(2)}|^{p_j-1} \operatorname{sgn} I_{\varepsilon,j}^{(2)}$ and using (2.3), we obtain the inequality

$$\frac{1}{p_j}\omega \cdot \nabla |I_{\varepsilon,j}^{(2)}|^{p_j} + \frac{1}{\varepsilon_j}|I_{\varepsilon,j}^{(2)}|^{p_j} \leqslant \frac{\varpi_j}{\varepsilon_j}\mathscr{S}(|I_{\varepsilon,j}^{(2)}|)|I_{\varepsilon,j}^{(2)}|^{p_j-1} + k_j^2|\nabla F_\varepsilon||I_{\varepsilon,j}^{(2)}|^{p_j-1}, \quad (\omega, x) \in D_j.$$

Integrating over D_j and taking into account that $\|\mathscr{S}(|I^{(2)}_{\varepsilon,j}|)\|_{L^{p_j}(D_j)} \leqslant \|I^{(2)}_{\varepsilon,j}\|_{L^{p_j}(D_j)}$, we have

$$\begin{aligned}
&\frac{1}{p_j}\|I^{(2)}_{\varepsilon,j}|_{\Gamma_j^+}\|^{p_j}_{\widehat{L}^{p_j}(\Gamma_j^+)} + \frac{1}{\varepsilon_j}\|I^{(2)}_{\varepsilon,j}\|^{p_j}_{L^p(D_j)} \\
&\leqslant \frac{1}{p_j}\|I^{(2)}_{\varepsilon,j}|_{\Gamma_j^-}\|^{p_j}_{\widehat{L}^{p_j}(\Gamma_j^-)} + \frac{\varpi_j}{\varepsilon_j}\|I^{(2)}_{\varepsilon,j}\|^{p_j}_{L^{p_j}(D_j)} + k_j^2(4\pi)^{1/p_j}\|\nabla F_\varepsilon\|_{L^{p_j}(G_j)}\|I^{(2)}_{\varepsilon,j}\|^{p_j-1}_{L^{p_j}(D_j)}.
\end{aligned} \tag{4.22}$$

By (4.21),

$$\|I^{(2)}_{\varepsilon,j}|_{\Gamma_j^-}\|_{\widehat{L}^{p_j}(\Gamma_j^-)} = \|\mathscr{R}_j^-(I^{(2)}_{\varepsilon,j}|_{\Gamma_j^+})\|_{\widehat{L}^{p_j}(\Gamma_j^-)} \leqslant \theta_j\|I^{(2)}_{\varepsilon,j}|_{\Gamma_j^+}\|_{\widehat{L}^{p_j}(\Gamma_j^+)}.$$

Therefore, from (4.22) we obtain the inequality

$$\begin{aligned}
&\frac{1-\theta_j^{p_j}}{p_j}\|I^{(2)}_{\varepsilon,j}|_{\Gamma_j^+}\|^{p_j}_{\widehat{L}^{p_j}(\Gamma_j^+)} + \frac{1-\varpi_j}{\varepsilon_j}\|I^{(2)}_{\varepsilon,j}\|^{p_j}_{L^p(D_j)} \\
&\leqslant \frac{1}{p_j}\Big(\frac{\varepsilon_j}{1-\varpi_j}\Big)^{p_j-1}k_j^{2p_j}4\pi\|\nabla F_\varepsilon\|^{p_j}_{L^{p_j}(G_j)} + \frac{p_j-1}{p_j}\frac{1-\varpi_j}{\varepsilon_j}\|I^{(2)}_{\varepsilon,j}\|^{p_j}_{L^{p_j}(D_j)}
\end{aligned}$$

which implies

$$(1-\theta_j^{p_j})\|I^{(2)}_{\varepsilon,j}|_{\Gamma_j^+}\|^{p_j}_{\widehat{L}^{p_j}(\Gamma_j^+)} + \frac{1-\varpi_j}{\varepsilon_j}\|I^{(2)}_{\varepsilon,j}\|^{p_j}_{L^p(D_j)} \leqslant \Big(\frac{\varepsilon_j}{1-\varpi_j}\Big)^{p_j-1}k_j^{2p_j}4\pi\|\nabla F_\varepsilon\|^{p_j}_{L^{p_j}(G_j)} \to 0$$

as $\varepsilon_j \to 0$. Consequently, $I^{(2)}_{\varepsilon,j}|_{\Gamma_j^+} \to 0$ in $\widehat{L}^1(\Gamma_j^+)$ as $\varepsilon_j \to 0$. The lemma is proved. □

4.3. The operators $\mathscr{C}_\varepsilon$ and $\mathscr{C}_{\varepsilon,j}$. We define operators $\mathscr{C}_\varepsilon : \widehat{L}^1(\Gamma^-) \to \widehat{L}^1(\Gamma^+)$ and $\mathscr{C}_{\varepsilon,j} : \widehat{L}^1(\Gamma_j^-) \to \widehat{L}^1(\Gamma_j^+)$ by $\mathscr{C}_\varepsilon = \mathscr{R}^+ + \mathscr{P}^+\mathscr{A}_\varepsilon k^2\mathscr{M}^-$ and $\mathscr{C}_{\varepsilon,j} = \mathscr{R}_j^+ + \mathscr{P}_j^+\mathscr{A}_{\varepsilon,j}k_j^2\mathscr{M}_j^-$. We note that $\mathscr{C}_\varepsilon(J)(\omega,x) = \mathscr{C}_{\varepsilon,j}(J_j)(\omega,x)$, $(\omega,x) \in \Gamma_j^+$, $1 \leqslant j \leqslant m$, forall $J \in \widehat{L}^1(\Gamma^-)$, where J_j is the restriction of J on Γ_j^-.

Lemma 4.4. *The following estimate holds*:

$$\varlimsup_{\varepsilon_j\to 0}\|\mathscr{C}_{\varepsilon,j}\|_{\widehat{L}^1(\Gamma_j^-)\to\widehat{L}^1(\Gamma_j^+)} \leqslant \theta_j + (1-\theta_j)\varpi_j. \tag{4.23}$$

Proof. Taking into account that

$$\|\mathscr{R}_j^+\|_{\widehat{L}^1(\Gamma_j^-)\to\widehat{L}^1(\Gamma_j^+)} \leqslant \theta_j, \quad \|\mathscr{P}_j^+\|_{\widehat{L}^1(\Gamma_j^+)\to\widehat{L}^1(\Gamma_j^+)} \leqslant \frac{1-\theta_j}{k_j^2}, \quad \|\mathscr{M}_j^-\|_{\widehat{L}^1(\Gamma_j^-)\to L^1(\partial G_j)} = \frac{1}{\pi},$$

we have

$$\|\mathscr{C}_{\varepsilon,j}\|_{\widehat{L}^1(\Gamma_j^+)\to\widehat{L}^1(\Gamma_j^+)} \leqslant \theta_j + \frac{1-\theta_j}{\pi}\|\mathscr{A}_{\varepsilon,j}\|_{L^1(\partial G_j)\to\widehat{L}^1(\Gamma_j^+)}.$$

By the estimate (4.9), we obtain (4.23). The lemma is proved. □

Corollary 4.3. *The following estimate holds*:

$$\varlimsup_{\varepsilon\to 0}\|\mathscr{C}_\varepsilon\|_{\widehat{L}^1(\Gamma^-)\to\widehat{L}^1(\Gamma^+)} \leqslant \max_{1\leqslant j\leqslant m}[\theta_j + (1-\theta_j)\varpi_j]. \tag{4.24}$$

4.4. The operator $\mathscr{R}^+_{\rm lim}$. We introduce the operator $\mathscr{R}^+_{\rm lim} : \widehat{L}^1(\Gamma^-) \to \widehat{L}^1(\Gamma^+)$ by the equality $\mathscr{R}^+_{\rm lim}(J)(\omega, x) = \theta_{\rm lim}(x)\mathscr{M}^-(J)(x)$, $(\omega, x) \in \Gamma^+$, where

$$\theta_{\rm lim} = \theta + (1-\theta)\Big(1 - \frac{1-\psi_0}{\theta}\Big).$$

From (3.4) it follows that

$$0 < 1 - \frac{1-\psi_0}{\theta} \leqslant \frac{(1-\theta)\varpi}{(1-\theta)\varpi + 2(1-\varpi)} < \varpi.$$

Thus, $\theta < \theta_{\rm lim} \leqslant \theta + (1-\theta)\varpi < 1$. Consequently,

$$\|\mathscr{R}^+_{\rm lim}\|_{\widehat{L}^1(\Gamma^-)\to\widehat{L}^1(\Gamma^+)} \leqslant \overline{\theta}_{\rm lim} = \max_{1\leqslant j\leqslant m} [\theta_j + (1-\theta_j)\varpi_j] < 1. \tag{4.25}$$

Lemma 4.5. *Let $\varphi_\varepsilon, \varphi \in \widehat{L}^1(\Gamma^-)$, and let $\varphi_\varepsilon \to \varphi$ in $\widehat{L}^1(\Gamma^-)$ as $\varepsilon \to 0$. Then*

$$\mathscr{C}_\varepsilon(\varphi_\varepsilon) \to \mathscr{R}^+_{\rm lim}(\varphi) \quad in \quad \widehat{L}^1(\Gamma^-), \quad \varepsilon \to 0. \tag{4.26}$$

Proof. By (4.14) and the first formula in (3.6), we have

$$\mathscr{C}_\varepsilon(\varphi_\varepsilon) = \mathscr{R}^+(\varphi_\varepsilon) + \mathscr{P}^+\mathscr{A}_\varepsilon k^2\mathscr{M}^-(\varphi_\varepsilon) = \theta\mathscr{M}^-(\varphi_\varepsilon) + (1-\theta)\mathscr{M}^+\mathscr{A}_\varepsilon\mathscr{M}^-(\varphi_\varepsilon)$$

$$\to \theta\mathscr{M}^-(\varphi) + (1-\theta)\mathscr{M}^+(\widehat{\psi}_+)\mathscr{M}^-(\varphi) = \Big[\theta + (1-\theta)\Big(1 - \frac{1-\psi_0}{\theta}\Big)\Big]\mathscr{M}^-(\varphi) = \mathscr{R}^-_{\rm lim}(\varphi)$$

in $\widehat{L}^1(\Gamma^-)$. The lemma is proved. □

5 Theorem on Limit Behavior of Solutions

We recall that our goal is to study the limit behavior of the solutions to the problem

$$\omega\cdot\nabla I_\varepsilon + \frac{1}{\varepsilon}I_\varepsilon = \frac{\varpi}{\varepsilon}\mathscr{S}(I_\varepsilon) + \frac{1-\varpi}{\varepsilon}k^2F_\varepsilon, \quad (\omega, x) \in D, \tag{5.1}$$

$$I_\varepsilon|_{\Gamma^-} = \mathscr{R}^-(I_\varepsilon|_{\Gamma^+}) + \mathscr{P}^-(J_\varepsilon), \quad (\omega, x) \in \Gamma^-, \tag{5.2}$$

$$J_\varepsilon = T[\mathscr{R}^+(J_\varepsilon) + \mathscr{P}^+(I_\varepsilon|_{\Gamma^+})] + J_{*,\varepsilon}, \quad (\omega, x) \in \Gamma^-, \tag{5.3}$$

as $\varepsilon \to 0$ where $F_\varepsilon \in W^{1,1}(G)$, $J_{*,\varepsilon} \in \widehat{L}^1(\Gamma^-)$. By a *solution* to this problem we understand a couple of functions $(I_\varepsilon, J_\varepsilon) \in \widehat{\mathscr{W}}^1(D)$ that satisfy Equation (5.1) almost everywhere in D, the boundary condition (5.2) and Equation (5.3) almost everywhere on Γ^-. The existence and uniqueness of a solution to the problem (5.1)–(5.3) follow from [3, 4].

Theorem 5.1. *Assume that $F \in W^{1,1}(G)$, $J_* \in \widehat{L}^1(\Gamma^-)$, $F_\varepsilon|_{\partial G} \to F|_{\partial G}$ in $L^1(\partial G)$, and $J_{*,\varepsilon} \to J_*$ in $\widehat{L}^1(\Gamma^-)$ as $\varepsilon \to 0$. If for all $1 \leqslant j \leqslant m$ the condition* (4.18) *holds, then the solutions $(I_\varepsilon, J_\varepsilon)$ to the problem* (5.1)–(5.3) *possess the following limit properties as $\varepsilon \to 0$:*

$$I_\varepsilon \to k^2F \quad in \quad L^1(D), \tag{5.4}$$

$$I_\varepsilon|_{\Gamma^+} \to \widehat{\psi}_+k^2\mathscr{M}^-(J_{\rm lim}) + (1-\widehat{\psi}_+)k^2F|_{\partial G} \quad in \quad \widehat{L}^1(\Gamma^+), \tag{5.5}$$

$$I_\varepsilon|_{\Gamma^-} \to \psi_0k^2\mathscr{M}^-(J_{\rm lim}) + (1-\psi_0)k^2F|_{\partial G} \quad in \quad L^1(\partial G), \tag{5.6}$$

$$J_\varepsilon \to J_{\rm lim} \quad in \quad \widehat{L}^1(\Gamma^+), \tag{5.7}$$

where $J_{\rm lim}$ is a solution to the equation

$$J_{\rm lim} = T\mathscr{R}^+_{\rm lim}(J_{\rm lim}) + T[(1-\theta_{\rm lim})F|_{\partial G}] + J_*, \quad (\omega, x) \in \Gamma^-. \tag{5.8}$$

Proof. From the estimate (4.25) it follows that

$$\|T\mathscr{R}^+_{\lim}\|_{\widehat{L}^1(\Gamma^-)\to\widehat{L}^1(\Gamma^-)} \leqslant \|\mathscr{R}^+_{\lim}\|_{\widehat{L}^1(\Gamma^-)\to\widehat{L}^1(\Gamma^+)} \leqslant \overline{\theta}_{\lim} < 1.$$

Therefore, a solution to Equation (5.8) exists, is unique, and is expressed by the convergent Neumann series in $\widehat{L}^1(\Gamma^-)$

$$J_{\lim} = \sum_{k=0}^{\infty}(T\mathscr{R}^+_{\lim})^k(T[(1-\theta_{\lim})F|_{\partial G}] + J_*). \tag{5.9}$$

Since $\mathscr{P}^- = (1-\theta)k^2\mathscr{M}^-$, we have

$$I_\varepsilon|_{\Gamma^+} = \mathscr{A}_\varepsilon k^2\mathscr{M}^-(J_\varepsilon) + \mathscr{B}_\varepsilon(F_\varepsilon). \tag{5.10}$$

Hence $\mathscr{P}^+(I_\varepsilon|_{\Gamma^+}) = \mathscr{P}^+\mathscr{A}_\varepsilon k^2\mathscr{M}^-(J_\varepsilon) + \mathscr{P}^+\mathscr{B}_\varepsilon(F_\varepsilon)$. Substituting this formula into (5.3), we obtain the equation

$$J_\varepsilon = T\mathscr{C}_\varepsilon(J_\varepsilon) + T\mathscr{P}^+\mathscr{B}_\varepsilon(F_\varepsilon) + J_{*,\varepsilon}, \quad (\omega, x) \in \Gamma^-, \tag{5.11}$$

for J_ε. We recall that $\mathscr{C}_\varepsilon = \mathscr{R}^+ + \mathscr{P}^+\mathscr{A}_\varepsilon k^2\mathscr{M}^-$. Using the estimate (4.24), we find

$$\varlimsup_{\varepsilon\to 0}\|T\mathscr{C}_\varepsilon\|_{\widehat{L}^1(\Gamma^-)\to\widehat{L}^1(\Gamma^-)} \leqslant \varlimsup_{\varepsilon\to 0}\|\mathscr{C}_\varepsilon\|_{\widehat{L}^1(\Gamma^-)\to\widehat{L}^1(\Gamma^+)} \leqslant \overline{\theta}_{\lim} < 1.$$

Consequently, for sufficiently small ε the solution J_ε to Equation (5.11) is represented by the Neumann series convergent in $\widehat{L}^1(\Gamma^-)$

$$J_\varepsilon = \sum_{k=0}^{\infty}(T\mathscr{C}_\varepsilon)^k[T\mathscr{P}^+\mathscr{B}_\varepsilon(F_\varepsilon) + J_{*,\varepsilon}],$$

where we can pass term-by-term to the limit as $\varepsilon \to 0$. Thus, using (4.19) and (4.26), we find

$$J_\varepsilon \to J_{\lim} = \sum_{k=0}^{\infty}(T\mathscr{R}^+_{\lim})^k(T\mathscr{P}^+[(1-\widehat{\psi}_+)k^2F|_{\partial G}] + J_*) \quad \text{in} \quad \widehat{L}^1(\Gamma^-).$$

Taking into account that, in view of the second formula in (3.6),

$$\mathscr{P}^+[(1-\widehat{\psi}_+)k^2F|_{\partial G}] = (1-\theta)\mathscr{M}^+(1-\widehat{\psi}_+)F|_{\partial G} = (1-\theta)\frac{1-\psi_0}{\theta} = 1-\theta_{\lim},$$

we see that $J_{\lim}$ coincides with the solution (5.9) to Equation (5.8). Passing to the limit in the identities (5.10) and (5.2) and taking into account (4.14) and (4.19), we find

$$I_\varepsilon|_{\Gamma^+} = \mathscr{A}_\varepsilon k^2\mathscr{M}^-(J_\varepsilon) + \mathscr{B}_\varepsilon(F_\varepsilon) \to \widehat{\psi}_+k^2\mathscr{M}^-(J_{\lim}) + (1-\widehat{\psi}_+)k^2F|_{\partial G} \quad \text{in} \quad \widehat{L}^1(\Gamma^+),$$

$$\begin{aligned} I_\varepsilon|_{\Gamma^-} &= \mathscr{R}^-(I_\varepsilon|_{\Gamma^+}) + \mathscr{P}^-(J_\varepsilon) \to \mathscr{R}^-(\widehat{\psi}_+k^2\mathscr{M}^-(J_{\lim})) + \mathscr{R}^-((1-\widehat{\psi}_+)k^2F|_{\partial G}) + \mathscr{P}^-(J_{\lim}) \\ &= [\theta\mathscr{M}^+(\widehat{\psi}_+) + (1-\theta)]k^2\mathscr{M}^-(J_{\lim}) + \theta\mathscr{M}^+(1-\widehat{\psi}_+)k^2F|_{\partial G} \\ &= \psi_0k^2\mathscr{M}^-(J_{\lim}) + (1-\psi_0)k^2F|_{\partial G} \quad \text{in} \quad L^1(\partial G). \end{aligned}$$

By [1, Lemma 4.2] for all $1 \leqslant j \leqslant m$ the following estimate holds:

$$\|I_{\varepsilon,j} - k_j^2F_{\varepsilon,j}\|_{L^1(D_j)} \leqslant \frac{\varepsilon_jk_j^2}{1-\varpi_j}[4\pi\|\nabla F_{\varepsilon,j}\|_{L^1(G_j)} + (1-\theta_j)\pi\|\mathscr{M}_j^-(J_\varepsilon) - F_{\varepsilon,j}|_{\partial G_j}\|_{L^1(\partial G_j)}],$$

which implies $I_\varepsilon \to k^2F$ in $L^1(D)$ as $\varepsilon \to 0$. The theorem is proved. □

Remark 5.1. In the case $F_\varepsilon = F$, i.e., if the right-hand side of (5.1) is independent of ε, the assumption (4.18) is not required. This assumption was used only to justify that $\mathscr{B}_\varepsilon(F_\varepsilon) \to (1-\widehat{\psi}_+)k^2F$ in $\widehat{L}^1(\Gamma^+)$. In this case, it suffices to use (4.17).

Grant Support

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Declarations

Data availability This manuscript has no associated data.

Ethical Conduct Not applicable.

Conflicts of interest The authors declare that there is no conflict of interest.

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