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ENUMERATION OF LABELED SERIES-PARALLEL TRICYCLIC GRAPHS

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***Abstract*.** A series-parallel graph is a graph that does not contain a complete graph with four vertices as a minor. An explicit formula for the number of labeled series-parallel tricyclic graphs with a given number of vertices is obtained, and the corresponding asymptotics for the number of such graphs with a large number of vertices is found. We prove that under a uniform probability distribution, the probability that the labeled tricyclic graph is a series-parallel graph is asymptotically equal to 13/15.

***Keywords and phrases*:** enumeration, labeled graph, series-parallel graph, asymptotics, probability.

***AMS Subject Classification*:** 05C30

1. Introduction.

Definition 1 (see [1])**.** A graph is said to be *series-parallel* if it does not contain a subdivision of the complete graph K_4.

Definition 2. The *cyclomatic number* of a connected graph is the difference between the number of edges and the number of vertices increased by one. A *k-cyclic graph* is a graph with cyclomatic number k.

Series-parallel graphs are used for constructing reliable communication networks (see [6]).

Asymptotics for the numbers of labeled connected and 2-connected series-parallel graphs with a large number of vertices were found in [1]. Labeled series-parallel connected and 2-connected graphs were listed according to the number of vertices in [10]. The numbers of labeled series-parallel tricyclic and tetracyclic 2-connected graphs with a given number of vertices were found in [12] and [9], respectively.

In this paper, we obtain an explicit formula for the number of labeled, connected, series-parallel, tricyclic graphs with a given number of vertices and find an asymptotics for the number of such graphs with a large number of vertices. We prove that under a uniform probability distribution, the probability that a labeled tricyclic graph is a series-parallel graph is asymptotically equal to 13/15.

2. Enumeration of graphs.

We consider undirected, simple connected graphs.

Theorem 1. *The number $SP(n,3)$ of labeled, connected, series-parallel, tricyclic graphs with n vertices, $n \geq 5$, is expressed by the formula*

$$SP(n,3) = \frac{(n-1)!}{24} \sum_{i=0}^{n-5} \Bigg(\binom{i+2}{2} \frac{n^{n-i-5}}{2(n-i-7)!} + \binom{i+4}{4} \left(\frac{12n^{n-i-5}}{(n-i-6)!} - \frac{13n^{n-i-6}}{(n-i-7)!} + \frac{4n^{n-i-7}}{(n-i-8)!} \right) + \binom{i+6}{6} \left(\frac{70n^{n-i-5}}{(n-i-5)!} - \frac{127n^{n-i-6}}{(n-i-6)!} + \frac{98n^{n-i-7}}{(n-i-7)!} - \frac{38n^{n-i-8}}{(n-i-8)!} + \frac{6n^{n-i-9}}{(n-i-9)!} \right) \Bigg). \quad (1)$$

Proof. For the number $S(n,k)$ of labeled, connected, k-cyclic graphs with n vertices, the following expression was obtained in [8]:

$$S(n,k) = \frac{(n-1)!}{nk!} [z^{-1}] e^{nz} Y_k \Big(n1!B_1'(z),\ n2!B_2'(z),\ \dots,\ nk!B_k'(z) \Big) z^{-n}, \quad (2)$$

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where $[z^{-1}]$ is the formal residue operator (see [2]), $B_k(z)$ is the exponential generating function for the number of labeled k-cyclic blocks, and $Y_k(x_1, \dots, x_k)$ are the partition polynomials (Bell polynomials). These polynomials can be expressed by the formulas (see [7])

$$Y_k(x_1, \dots, x_k) = \sum_{\pi(k)} \frac{k!}{m_1! \dots m_k!} \left(\frac{x_1}{1!}\right)^{m_1} \dots \left(\frac{x_k}{k!}\right)^{m_k},$$

where the summation is performed over all partitions $\pi(k)$ of the number k, i.e., over all nonnegative solutions $(m_1, m_2, \dots, m_k)$ of the equation $m_1 + 2m_2 + \dots + km_k = k$, $m_i \geq 0$, $i = 1, \dots, k$.

Now we assume that the numbers $S(n,k)$ and the functions $B_k(z)$ belong to the class of labeled series-parallel graphs. The formula (2) may be invalid for the subclass of connected graphs (see [10]).

Definition 3 (see [3])**.** A class of graphs is said to be *block stable* if a graph belongs to this class if and only if every block of this graph also belongs to this class.

For a block stable class of graphs, the formula (2) is valid (see [10]). It is known that the class of series-parallel graphs is a block stable class of graphs (see [3]).

Since (see [7]) $Y_3(x_1, x_2, x_3) = x_1^3 + 3x_1x_2 + x_3$ and $x_i = ni! B_i'(z)$, we have

$$SP(n,3) = \frac{(n-1)!}{6n} [z^{-1}] e^{nz} \Big(n^3 (B_1'(z))^3 + 6n^2 B_1'(z) B_2'(z) + 6n B_3'(z) \Big) z^{-n}.$$

A unicyclic block is a simple cycle (series-parallel graph), and hence $B(n,1) = (n-1)/2$. All bicyclic blocks are series-parallel graphs, and in [14] and [12], respectively, the following formulas were found:

$$B(n,2) = \frac{n!(n-3)(n+2)}{24}, \quad B(n,3) = \frac{n!(n-3)(n-4)}{5760}(3n^3 + 36n^2 + 71n + 50).$$

Thus, we have

$$B_1(z) = \sum_{n=3}^{\infty} \frac{1}{2}(n-1)! \frac{z^n}{n!}, \quad B_1'(z) = \frac{z^2}{2(1-z)},$$

$$B_2(z) = \frac{z^4(3-2z)}{12(1-z)^3}, \quad B_2'(z) = \frac{12z^3 - 13z^4 + 4z^5}{12(1-z)^4},$$

$$B_3(z) = \frac{z^5(28 - 47z + 28z^2 - 6z^3)}{48(1-z)^6}, \quad B_3'(z) = \frac{70z^4 - 127z^5 + 98z^6 - 38z^7 + 6z^8}{24(1-z)^7}.$$

Summation and differentiation of the series for $B_2(z)$ and $B_3(z)$ were performed using the Maple software package.

Using the well-known expansion from [7, p. 141], we obtain

$$(1-z)^{-p-1} = \sum_{i=0}^{\infty} \binom{i+p}{p} z^i,$$

$$SP(n,3) = \frac{(n-1)!}{6} [z^{-1}] e^{nz} \times$$
$$\times \left(\frac{n^2 z^6}{8(1-z)^3} + \frac{nz^2(12z^3 - 13z^4 + 4z^5)}{4(1-z)^5} + \frac{70z^4 - 127z^5 + 98z^6 - 38z^7 + 6z^8}{4(1-z)^7} \right) z^{-n}$$
$$= \frac{(n-1)!}{6} [z^{-1}] \sum_{p=0}^{\infty} \frac{n^p z^p}{p!} \times$$
$$\times \left(\frac{n^2}{8} \sum_{i=0}^{\infty} \binom{i+2}{2} z^{i+6-n} + \frac{12n}{4} \sum_{i=0}^{\infty} \binom{i+4}{4} z^{i+5-n} - \frac{13n}{4} \sum_{i=0}^{\infty} \binom{i+4}{4} z^{i+6-n} + \right.$$

$$+\frac{4n}{4}\sum_{i=0}^{\infty}\binom{i+4}{4}z^{i+7-n}+\frac{70n}{4}\sum_{i=0}^{\infty}\binom{i+6}{6}z^{i+4-n}-\frac{127n}{4}\sum_{i=0}^{\infty}\binom{i+6}{6}z^{i+5-n}+$$

$$+\frac{98n}{4}\sum_{i=0}^{\infty}\binom{i+6}{6}z^{i+6-n}-\frac{38n}{4}\sum_{i=0}^{\infty}\binom{i+6}{6}z^{i+7-n}+\frac{6n}{4}\sum_{i=0}^{\infty}\binom{i+6}{6}z^{i+8-n}\Bigg)=$$

$$=\frac{(n-1)!}{6}\sum_{i=0}^{n-5}\Bigg(\frac{n^{n-i-5}}{8(n-i-7)!}\binom{i+2}{2}+\frac{12n^{n-i-5}}{4(n-i-6)!}\binom{i+4}{4}-\frac{13n^{n-i-6}}{4(n-i-7)!}\binom{i+4}{4}+$$

$$+\frac{4n^{n-i-7}}{4(n-i-8)!}\binom{i+4}{4}+\frac{70n^{n-i-5}}{4(n-i-5)!}\binom{i+6}{6}-\frac{127n^{n-i-6}}{4(n-i-6)!}\binom{i+6}{6}+$$

$$+\frac{98n^{n-i-7}}{4(n-i-7)!}\binom{i+6}{6}-\frac{38n^{n-i-8}}{4(n-i-8)!}\binom{i+6}{6}+\frac{6n^{n-i-9}}{4(n-i-9)!}\binom{i+6}{6}\Bigg). \quad (3)$$

The upper limit in the sum has been replaced by $n-5$ since the corresponding factorials in the denominator vanish for $i > n-5$. The proof is complete. □

In the following table, we present the numbers $SP(n,3)$ calculated by Theorem 1 and the Maple software:

n	5	6	7	8	9	10
$SP(n,3)$	70	4275	190995	7832440	317391480	13111660800

3. Asymptotics and probability.

Lemma. *Let $U(a,b,z)$ be the Tricomi confluent hypergeometric function. Then the following expansions holds*:

$$\frac{e^{nz}}{(1-z)^m}=\sum_{p=0}^{\infty}\frac{n^{p+m}}{p!}U(m,m+p+1,n)z^p.$$

Proof. Recall the following well-known expansion:

$$(1+z)^{\alpha}e^{-tz}=\sum_{p=0}^{\infty}L_p^{\alpha-p}(z)t^p$$

(see [4, p. 706]), where L_p^k are Laguerre polynomials. In our case $t=-z$, $\alpha=-m$, $z=n$, and

$$\frac{e^{nz}}{(1-z)^m}=\sum_{p=0}^{\infty}(-1)^pL_p^{-m-p}(n)z^p.$$

Using the relationship

$$U(-n,b;z)=(-1)^n n!\,L_n^{b-1}(z) \quad \text{or} \quad L_n^a(z)=\frac{(-1)^n}{n!}U(-n,a+1;z)$$

between the Laguerre polynomials with the Tricomi confluent hypergeometric function (see [5, p. 584]), we obtain

$$\frac{e^{nz}}{(1-z)^m}=\sum_{p=0}^{\infty}\frac{1}{p!}U(-p,-p-m+1;n)z^p.$$

Using the transformation formula $U(a,b;z)=z^{1-b}U(1+a-b,2-b;z)$ (see [5, p. 584]), we obtain the required fact. □

Theorem 2. *The number $SP(n,3)$ of labeled connected series-parallel tricyclic graphs with n vertices has the following asymptotics as $n\to\infty$*

$$SP(n,3)\sim\frac{13}{384}\sqrt{\frac{\pi}{2}}\,n^{n+5/2}.$$

Proof. Using Lemma 3, we obtain from (3)

$$SP(n,3) = \frac{(n-1)!}{6}[z^{-1}]\frac{n^2z^6}{8}\sum_{p=0}^{\infty}\frac{n^{p+3}}{p!}U(3,p+4;n)z^{p-n}+$$
$$+\frac{nz^2}{4}(12z^3-13z^4+4z^5)\sum_{p=0}^{\infty}\frac{n^{p+5}}{p!}U(5,p+6;n)z^{p-n}$$
$$+\frac{1}{4}(70z^4-127z^5+98z^6-38z^7+6z^8)\sum_{p=0}^{\infty}\frac{n^{p+7}}{p!}U(7,p+8;n)z^{p-n}$$
$$=\frac{(n-1)!}{6}\Big(\frac{n^{n-2}}{8(n-7)!}U(3,n-3;n)+\frac{12n^n}{4(n-6)!}U(5,n;n)-\frac{13n^{n-1}}{4(n-7)!}U(5,n-1;n)$$
$$+\frac{4n^{n-2}}{4(n-8)!}U(5,n-2;n)+\frac{70n^{n+2}}{4(n-5)!}U(7,n+3;n)-\frac{127n^{n+1}}{4(n-6)!}U(7,n+2;n)$$
$$+\frac{98n^n}{4(n-7)!}U(7,n+1;n)-\frac{38n^{n-1}}{4(n-5)!}U(7,n;n)+\frac{6n^{n-2}}{4(n-9)!}U(7,n-1;n).$$

If a and m are fixed and $n\to\infty$, we have the asymptotics

$$U(a,n-m,n)\sim\frac{\sqrt{\pi}}{(2n)^{a/2}\Gamma\big(\frac{a+1}{2}\big)}$$

(see [11]). Since $(n+k)!/n!\sim n^k$ as $n\to\infty$ for fixed k and

$$SP(n,3)\sim\frac{n^{n+4}\sqrt{\pi}}{48(2n)^{3/2}\Gamma(2)}+\frac{n^{n+5}\sqrt{\pi}}{2(2n)^{5/2}\Gamma(3)}-\frac{13n^{n+5}\sqrt{\pi}}{24(2n)^{5/2}\Gamma(3)}+\frac{n^{n+5}\sqrt{\pi}}{6(2n)^{5/2}\Gamma(3)}$$
$$+\frac{70n^{n+6}\sqrt{\pi}}{24(2n)^{7/2}\Gamma(4)}-\frac{127n^{n+6}\sqrt{\pi}}{24(2n)^{7/2}\Gamma(4)}+\frac{98n^{n+6}\sqrt{\pi}}{24(2n)^{7/2}\Gamma(4)}-\frac{38n^{n+6}\sqrt{\pi}}{24(2n)^{7/2}\Gamma(4)}+\frac{n^{n+6}\sqrt{\pi}}{4(2n)^{7/2}\Gamma(4)}$$
$$\sim\sqrt{\frac{\pi}{2}}n^{n+5/2}\left(\frac{1}{96}+\frac{1}{16}-\frac{13}{192}+\frac{1}{48}+\frac{35}{576}-\frac{127}{1152}+\frac{49}{576}-\frac{19}{576}+\frac{1}{192}\right)=\frac{13}{384}\sqrt{\frac{\pi}{2}}\,n^{n+5/2}.$$

The proof is complete. □

On the set of labeled tricyclic graphs with n vertices, introduce the uniform probability distribution.

Corollary. *The probability P of a labeled tricyclic graph to be a series-parallel graph is asymptotically equal to* 13/15.

Proof. Let $f(n,n+2)$ be the number of labeled connected graphs with n and $n+2$ edges (tricyclic graphs). E. Wright found the following asymptotics as $n\to\infty$ (see [13]):

$$f(n,n+k)\sim\rho_k n^{n+(3k-1)/2},\quad \rho_k=\frac{\sqrt{\pi}\sigma_k}{2^{(3k-1)/2}\Gamma(3k/2+1)},\quad \sigma_2=\frac{15}{16}.$$

Therefore, we have

$$f(n,n+2)\sim\frac{\sqrt{\pi}\sigma_2}{2^{5/2}\Gamma(4)}n^{n+5/2}=\frac{15\sqrt{\pi}}{384\sqrt{2}}n^{n+5/2}$$

and hence

$$P=\frac{SP(n,3)}{f(n,n+2)}\sim\frac{13}{15}$$

as $n\to\infty$. The proof is complete. □

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COMPLIANCE WITH ETHICAL STANDARDS

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